Revealing memory effects without solving the master equation [New J. Phys. 20 073012 (2018)]

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Abstract

We study and compare the sensitivity of multiple non-Markovianity indicators for a qubit subjected to general phase-covariant noise. For each of the indicators, we derive analytical conditions to detect the dynamics as non-Markovian. We present these conditions as relations between the time-dependent decay rates for the general open system dynamics and its commutative and unital subclasses. These relations tell directly if the dynamics is non-Markovian w.r.t. each indicator, without the need to explicitly solve the master equation. Moreover, with a shift in perspective, we show that if one assumes only the general form of the master equation, measuring the non-Markovianity indicators gives us directly non-trivial information on the relations between the unknown decay rates.

Phase-covariant dynamics

The master equation used describes a qubit undergoing general phase covariant noise. The most general form of such master equation is

$$\frac{d\rho(t)}{dt} = -i\frac{\omega(t)}{2}[\sigma_z,\rho(t)] + \frac{\gamma_1(t)}{2}L_1(\rho(t)) + \frac{\gamma_2(t)}{2}L_2(\rho(t)) + \frac{\gamma_3(t)}{2}L_3(\rho(t))\,, \tag{1}$$

where the operators L_i are

$$L_1(\rho(t)) = \sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+,\rho(t)\}\,, \tag{2}$$

$$L_2(\rho(t)) = \sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho(t)\}\,, \tag{3}$$

$$L_3(\rho(t)) = \sigma_z\rho(t)\sigma_z - \rho(t)\,. \tag{4}$$

with Pauli operators $\sigma_x, \sigma_y, \sigma_z$ and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. The general solution for the qubit dynamics is

$$\Phi_t(\rho(0)) = \rho(t) = \begin{pmatrix} 1-P_1(t) & \alpha(t) \\ \alpha^*(t) & P_1(t) \end{pmatrix}, \tag{5}$$

where

$$P_1(t) = e^{-\Gamma(t)}[G(t)+P_1(0)]\,,\quad \alpha(t) = \alpha(0)e^{i\Omega(t)-\Gamma(t)/2-\tilde{\Gamma}(t)}\,,\quad \Omega(t) = \int_0^t \omega(t)d\tau\,,$$

$$\Gamma(t) = \int_0^t (\gamma_1(\tau)+\gamma_2(\tau))/2d\tau\,,\quad \tilde{\Gamma}(t) = \int_0^t \gamma_3(\tau)d\tau\,,\quad G(t) = \int_0^t \gamma_2(\tau)e^{\Gamma(\tau)}d\tau\,.$$

Indicators of non-Markovianity

The indicators considered here:

- **Entropy production rate:** Change in relative entropy between the initial and evolved state. Non-Markovianity indicated by negative rate of change in entropy.
- **Purity:** Non-Markovianity indicated by positive change in purity.
- **Trace distance** Non-Markovianity indicated by positive change in trace distance between two states.
- **Bloch volume measure:** Bloch volume refers to the set of dynamically accessible states in the Bloch sphere representation. Non-Markovianity indicated by positive change in this volume.
- **Singular-/eigenvalues:** Non-Markovianity indicated by increasing singular-/eigenvalues of the map.
- **l_1-measure of coherences:** Sum of off-diagonal elements of the density matrix. Non-Markovianity indicated by positive change in l_1 measure, that is increase in coherences.
- **Relative entropy of coherences (REC):** A coherence measure defined using the relative entropy of the system. Non-Markovianity indicated by positive change in REC, that is increase in coherences.

For details, see New J. Phys. **20** 073012 (2018).

Results

- Method for detecting non-Markovian dynamics without optimization calculations commonly seen in measures of non-Markovianity.
- Multiple different indicators considered. Results reveal a hierarchy in detection between indicators.
- Conditions very simple and depend only on the properties of the decay rates.
- Conditions can be used to reverse-engineer non-Markovian/Markovian master equations.

By defining $\gamma_1(t)+\gamma_2(t) \equiv \gamma'(t)$, the results can be illustrated as regions in the $\{\gamma'(t),\gamma_3(t)\}$ -space. The (commutative) example dynamics are chosen so that $\gamma_3(t)$ has negative values. By the GKSL-theorem, with a master equation in the Lindblad form, the negativity of a decay rate indicates non-Markovianity. However, here we see that some of the indicators used detects non-Markovianity at a later point, or not at all.

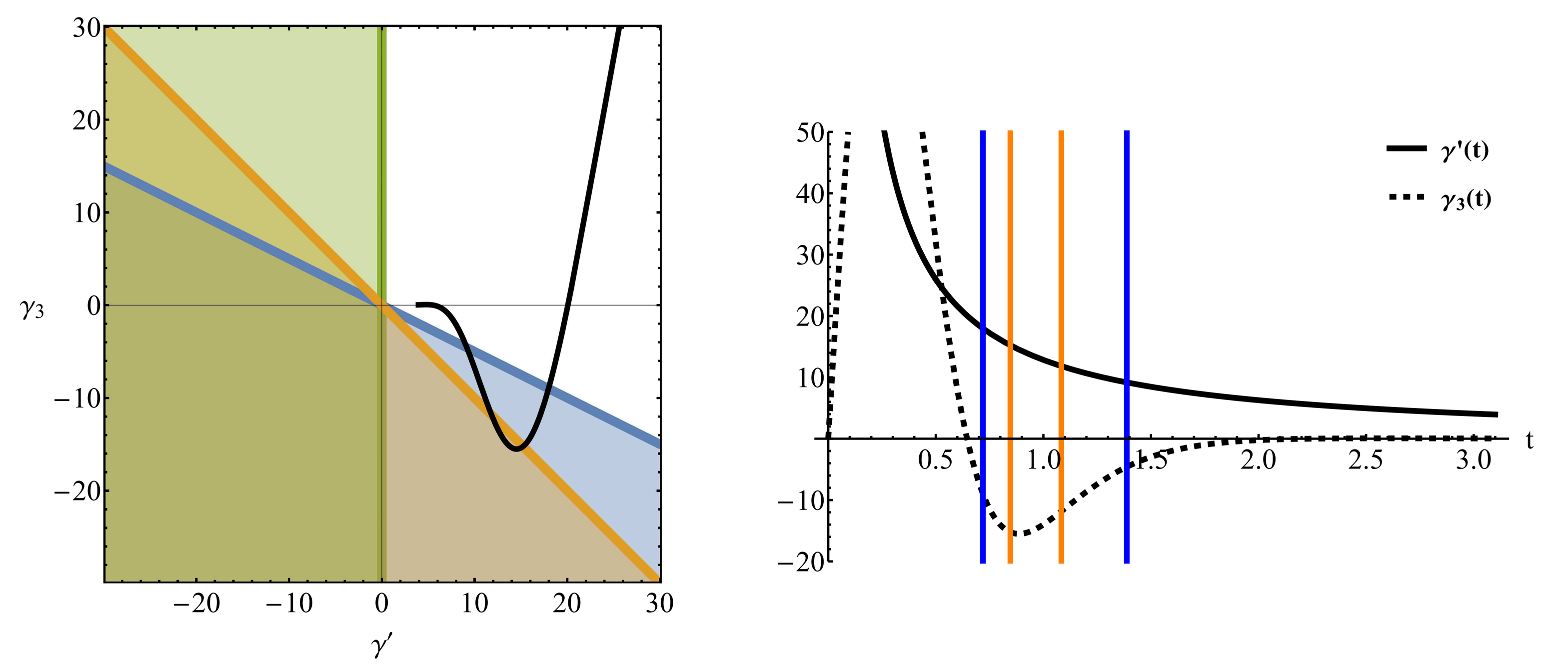

Fig. 1: On the left-hand side, the conditions from Table 1 are illustrated as a regionplot. The Black curve is the total dynamics of a chosen (commutative) example system. On the right-hand side, the individual dynamics of the decay rates are plotted, and the crossing of a region in the total dynamics is indicated by the vertical lines.

Results table

Indicator	General	Commutative, $\gamma_1(t)=\gamma(t)$, $\gamma_2(t)=\kappa\gamma(t)$	Unital, $\gamma_1(t)=\gamma_2(t)=\gamma(t)$
Entropy production $\sigma(\rho(t)) = -\frac{d}{dt}S(\rho(t)\vert\rho_0) < 0$	*	$(1+\kappa)\gamma(t)+4\gamma_3(t)<0$ $\gamma(t)<0$	$\gamma(t)+2\gamma_3(t)<0$ $\gamma(t)<0$
Purity $\frac{d}{dt}\mathcal{P}(t)>0$	*	*	$\gamma(t)+2\gamma_3(t)<0$ $\gamma(t)<0$
Trace distance $\lambda_{max}[\mathcal{D}(t)^T+\mathcal{D}(t)]>0$	$\gamma_1(t)+\gamma_2(t)+4\gamma_3(t)<0$ $\gamma_1(t)+\gamma_2(t)<0$	$(1+\kappa)\gamma(t)+4\gamma_3(t)<0$ $\gamma(t)<0$	$\gamma(t)+2\gamma_3(t)<0$ $\gamma(t)<0$
Bloch volume $\mathrm{tr}[\mathcal{D}(t)]>0$	$\gamma_1(t)+\gamma_2(t)+2\gamma_3(t)<0$	$(1+\kappa)\gamma(t)+2\gamma_3(t)<0$	$\gamma(t)+\gamma_3(t)<0$
Eigenvalues $\frac{d}{dt}\lambda_i(t)>0$	*	$(1+\kappa)\gamma(t)+4\gamma_3(t)<0$ $\gamma(t)<0$	$\gamma(t)+2\gamma_3(t)<0$ $\gamma(t)<0$
Singular values $\frac{d}{dt}s_i(t)>0$	*	*	$\gamma(t)+2\gamma_3(t)<0$ $\gamma(t)<0$
l_1-norm $\frac{d}{dt}C_{l_1}(t)>0$	$\gamma_1(t)+\gamma_2(t)+4\gamma_3(t)<0$	$(1+\kappa)\gamma(t)+4\gamma_3(t)<0$	$\gamma(t)+2\gamma_3(t)<0$

Table 1: Conditions for detecting non-Markovianity with different indicators and different classes of master equation 1. The commutative and unital results for entropy production, as well as purity, require specific choices of initial states. The upper condition involving $\gamma_3(t)$, was obtained with the initial state defined by $P_1(0)=\kappa/(\kappa+1)$ and $\alpha(0)\neq 0$, while the lower result uses initial state defined by $P_1(0)=1/(\kappa+1)$ and $\alpha(0)=0$.

$\lambda_i(t)$ and $s_i(t)$ are the i-th eigenvalue and singular value of the dynamical map in the Bloch vector representation. $\mathcal{D}(t)$ is the damping matrix of the map, defined as $\dot{\mathbf{r}}(t)=\mathcal{D}(t)\mathbf{r}(t)$, where $\mathbf{r}(t)$ is the Bloch vector. Cases indicated with $*$ could not be calculated due to limitations to applicability of specific indicators. REC was omitted, because we were unable to derive such simple condition for any case. However, numerically we found that REC is always weaker at detecting non-Markovianity than C_{l_1}.