Exercise 4

a) ~~[illegible]~~ $P(\mu) = \frac{1}{Z} e^{-\beta H(\mu)}$, $\mu = \{\vec{r}_i, \vec{p}_i\}$.

$$\Rightarrow P(\{\vec{r}_i, \vec{p}_i\}) = \frac{1}{Z} \exp\left[-\beta \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} - \beta \sum_{i=1}^{N} m g z_i\right]$$

$$= \frac{1}{Z} \exp\left(-\beta \frac{p_1^2}{2m} - \beta m g z_1\right) \exp\left(-\beta \frac{p_2^2}{2m} - \beta m g z_2\right) \cdots \exp\left(-\beta \frac{p_N^2}{2m} - \beta m g z_N\right)$$

All particles are equal $\Rightarrow$ all particles have same probability distribution.

Can find for instance $P(\vec{r}_1, \vec{p}_1)$ by integrating over $d^3r_2 \ldots d^3r_N\, d^3p_2 \ldots d^3p_N$.

and we are left with $P(\vec{r}, \vec{p}) = C \exp\left[-\beta \frac{\vec{p}^{\,2}}{2m} - \beta m g z\right]$.

The probabilities must sum up to 1.

$$1 = \int d^3r\, d^3p\, C \exp\left[-\frac{\beta \vec{p}^{\,2}}{2m} - \beta m g z\right] = C \cdot A \int_0^L e^{-\beta m g z} \left(\int_{-\infty}^{\infty} dp_x\, e^{-\frac{\beta p_x^2}{2m}}\right)^3$$

$$= C A \left(\frac{\pi 2m}{\beta}\right)^{3/2} \left(-\frac{1}{\beta m g} e^{-\beta m g z}\right)\Big|_0^L$$

$$= C A \left(\frac{2m\pi}{\beta}\right)^{3/2} \left(1 - e^{-\beta m g L}\right) \frac{1}{\beta m g} \Rightarrow C = \frac{\beta m g}{A} \left(\frac{\beta}{2m\pi}\right)^{3/2} \left(1 - e^{-\beta m g L}\right)^{-1}$$

$$= \sqrt{\frac{\beta^5 g^2}{8A^2 m \pi^3}} \left(1 - e^{-\beta m g L}\right)^{-1}.$$

b) $P_p(\vec{p}) = \int d^3r\, P(\vec{r}, \vec{p})$

$$= C e^{-\frac{\beta \vec{p}^{\,2}}{2m}} \cdot A \int_0^L dz\, e^{-\beta m g z} = C \frac{A}{\beta m g} \left(1 - e^{-\beta m g L}\right) e^{-\frac{\beta \vec{p}^{\,2}}{2m}} = \left(\frac{\beta}{2m\pi}\right)^{3/2} e^{-\frac{\beta \vec{p}^{\,2}}{2m}}$$

$$\left\langle \frac{\vec{p}^{\,2}}{2m} \right\rangle = \frac{1}{2m} \left(\frac{\beta}{2m\pi}\right)^{3/2} \int d^3p \left(p_x^2 + p_y^2 + p_z^2\right) \exp\left(-\frac{\beta}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)\right)$$

$$= \frac{3}{2m} \left(\frac{\beta}{2m\pi}\right)^{3/2} \int dp_x\, dp_y\, dp_z\, p_x^2 \exp\left(-\frac{\beta}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)\right)$$

$$= \frac{3}{2m} \left(\frac{\beta}{2m\pi}\right)^{3/2} \left(\frac{2m\pi}{\beta}\right) \int_{-\infty}^{\infty} dp_x\, p_x^2 e^{-\frac{\beta p_x^2}{2m}} = \frac{3}{2m} \sqrt{\frac{\beta}{2m\pi}} \frac{\sqrt{\pi}}{2} \left(\frac{2m}{\beta}\right)^{3/2} = 3 \frac{1}{\beta} \cdot \frac{1}{2} = \frac{3}{2} k_B T.$$

c) $P_r(\vec{r}) = \frac{P(\vec{r},\vec{p})}{P_p(\vec{p})} = \frac{\beta m g}{A}(1-e^{-\beta m g L})^{-1} e^{-\beta m g z}$

$$\langle V(\vec{r})\rangle = \langle m g z\rangle = \frac{\beta m^2 g^2}{A}\int d^3r \, \frac{z\, e^{-\beta m g z}}{1-e^{-\beta m g L}} = \frac{\beta m^2 g^2}{1-e^{-\beta m g L}} \frac{1}{(\beta m g)^2}\left(1-(\beta m g L+1)e^{-\beta m g L}\right)$$

$$= \frac{1-(\beta m g L+1)e^{-\beta m g L}}{\beta(1-e^{-\beta m g L})}$$

When $k_B T \gg mgL \Rightarrow mgL\beta \ll 1$. Let $\beta = x$, $a = mgL$.

Take limit $\lim_{x\to 0} \frac{1-(xa+1)e^{-ax}}{x(1-e^{-ax})} \underset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{-ae^{-ax}+a(xa+1)e^{-ax}}{1-e^{-ax}+xae^{-ax}}$

l'Hopital

$$= \lim_{x\to 0} \frac{a^2e^{-ax}+a^2e^{-ax}-a^2(xa+1)e^{-ax}}{ae^{-ax}+ae^{-ax}-xa^2e^{-ax}} = \frac{a^2}{2a} = \frac{1}{2}a.$$

So when $\beta\to 0$ $\langle V(\vec{r})\rangle \simeq \frac{1}{2}mgL$. Interpretation; Particles moves like there is no gravitational field with mean z value $\frac{L}{2}$.

When $T\to 0$, $\beta\to\infty$ so $\langle V(\vec{r})\rangle \simeq \frac{1}{\beta}\to 0$.

d) $Z = \frac{1}{N!\,h^{3N}}\int d^3r_1 \dots d^3r_N\, d^3p_1 \dots d^3p_N \exp\left(-\beta\left(\sum_{i=1}^{N}\frac{\vec{p}_i^{\,2}}{2m} + mgz_i\right)\right)$

$$= \frac{A^N}{N!\,h^{3N}}\Bigg[\underbrace{\int_0^L dz \exp(-\beta m g z)}_{\frac{1}{\beta m g}(1-e^{-\beta m g L})}\Bigg]^N \cdot \Bigg[\underbrace{\int_{-\infty}^{\infty} dp_x \exp\left(-\frac{\beta p_x^2}{2m}\right)}_{\sqrt{\frac{2\pi m}{\beta}}}\Bigg]^{3N}$$

$$= \frac{A^N}{N!\,h^{3N}}\frac{(1-e^{-\beta m g L})^N}{(\beta m g)^N}\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}N}.$$

e) $\langle H\rangle = -\frac{\partial}{\partial\beta}\ln(Z) = -\frac{\partial}{\partial\beta}\left\{N\ln(1-e^{-\beta m g L}) - N\ln(\beta m g) + \frac{3}{2}N(\ln(2\pi m)-\ln(\beta))\right\}$

$$= -\left[\frac{N}{1-e^{-\beta m g L}}\cdot mgL\,e^{-\beta m g L} - \frac{N}{\beta} - \frac{3}{2}N\frac{1}{\beta}\right]$$

$$= \frac{5N}{2\beta} - \frac{NmgL}{e^{\beta m g L}-1}.$$

f) $F = -k_B T \ln(Z)$ ~~…~~

$$\Rightarrow S = \frac{E-F}{T} = \frac{5}{2}k_B N - \frac{N m g L k_B \beta}{e^{\beta m g L}-1} + k_B \ln(Z)$$

$$\ln(Z) = \frac{3}{2}N\ln\left(\frac{2\pi m}{\beta}\right) + N\ln\left(\frac{(1-e^{-\beta m g L})A}{\beta m g}\right) - \underbrace{\ln N!}_{\simeq -N\ln N + N}$$

$$S = \frac{7}{2}k_B N - \frac{N m g L k_B \beta}{e^{\beta m g L}-1} + \frac{3}{2}k_B N \ln\left(\frac{2\pi m}{\beta}\right) + N k_B \ln\left(\frac{A(1-e^{-\beta m g L})}{\beta m g}\right) - N k_B \ln N.$$

$$= N k_B\left[2 + \frac{3}{2}\ln\left(\frac{2\pi e m}{\beta}\right) - \ln N + \ln\left(\frac{A(1-e^{-\beta m g L})}{\beta m g}\right) - \frac{m g L \beta}{e^{\beta m g L}-1}\right].$$

g) $k_B T \gg m g L \Rightarrow$ let $\beta \to 0$:

$$\lim_{\beta\to 0}\frac{1-e^{-\beta m g L}}{\beta m g} = \lim_{\beta\to 0}\frac{m g L}{m g} = L.$$

$$\lim_{\beta\to 0}\frac{m g L \beta}{e^{\beta m g L}-1} = \lim_{\beta\to 0}\frac{m g L}{m g L e^{m g L \beta}} = 1.$$

$$\Rightarrow S \simeq N k_B\Big[\underbrace{1 + \ln N}_{\text{constants.}} + \underbrace{\frac{3}{2}\ln(2\pi e m k_B T) + \ln(V)}_{\text{entropy for an ideal gas.}}\Big].$$

$k_B T \ll m g L \Rightarrow$ let $\beta \to \infty$:

~~$\lim_{\beta\to\infty}\frac{1-e^{-\beta m g L}}{\beta m g} = \lim_{\beta\to\infty} \ldots = 0.$~~

$$\lim_{\beta\to\infty}\frac{m g L \beta}{e^{\beta m g L}-1} = \lim_{\beta\to\infty}\frac{m g L}{m g L e^{\beta}} = 0.$$

$$\Rightarrow S \simeq N k_B\left[2 - \ln N + \frac{3}{2}\ln(2\pi e m k_B T) + \ln\left(\frac{A k_B T}{m g}\right)\right].$$

Effective volume $\frac{A k_B T}{m g} \ll A L = V.$