Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4212 Finite Difference Method
Spring 2021

Semester Project Part 1

In the first part of the semester project, the aim is to implement finite difference methods for: (1) boundary value problems; (2) parabolic equations; (3) elliptic equations; (4) hyperbolic equations, and finite element methods for (5) boundary value problems for understanding how the actual computation works.

$\boxed{1}$ Consider a function $u(x)$ defined on $[0,1]$. We want to solve the following Poisson equation with a Neumann boundary condition

$$u_{xx} = f(x),\ u(0) = \alpha,\ u_x(1) = \sigma,$$

on equidistant points

$$x_0 = 0,\ x_1 = \frac{1}{M+1}, ...,\ x_M = \frac{M}{M+1},\ x_{M+1} = 1.$$

a) We construct the second order method for this problem (cf.the case 3 in 3.1.2): $A_h\boldsymbol{U} = \boldsymbol{f}$ where

$$A_h = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 1 & -2 & 1 \\ 0 & \dots & -\frac{h}{2} & 2h & -\frac{3h}{2} \end{pmatrix},\ \boldsymbol{U} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \\ U_{M+1} \end{pmatrix},\ \boldsymbol{f} = \begin{pmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ \vdots \\ f(x_M) \\ \sigma \end{pmatrix}.$$

Now let $f(x) = \cos(2\pi x) + x, \alpha = 0$ and $\sigma = 0$. First, solve this problem on a sheet of paper (we call this solution “analytical solution” and denote it $u(x)$). Then implement the above finite difference method and solve the problem numerically (we call this solution “numerical solution” and denote the corresponding solution in grid point x_i as U_i). The discrete ℓ_2-norm for a vector $\boldsymbol{V} \in \mathbb{R}^N$ and the continuous L_2-norm for function $v(x) \in L_2(\Omega)$ are defined as follows:

$$\|\boldsymbol{V}\|_2 := \sqrt{\frac{1}{N}\sum_{i=1}^{N}(V_i)^2} \tag{1}$$

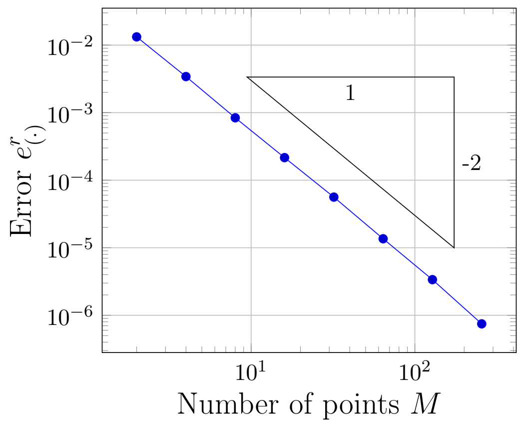
$$\|v(x)\|_2 := \sqrt{\int_\Omega v^2(x)\, d\,\Omega}. \tag{2}$$

We want to see that the numerical solution converges to the analytical solution. Consider the relative errors e_ℓ and e_{L_2} in terms of increasing M (decreasing $h \to 0$):

$$e_\ell^r := \frac{\|\boldsymbol{u} - \boldsymbol{U}\|_2}{\|\boldsymbol{u}\|_2} \tag{3}$$

$$e_{L_2}^r := \frac{\|u(x) - U(x)\|_2}{\|u(x)\|_2}. \tag{4}$$

Make a "log-log" plot of e_ℓ^r, and $e_{L_2}^r$ (both x and y axes are logarithmically scaled) to see the order of convergence. The plot should look like the following:



Note that you do not need to make the triangle in the graph (this might take some time to figure out how to), but we can provide a TikZ code to produce similar graphs where you only need to prepare your numerical results as a text file (e.g., .dat, .txt,...). You do not have to use it, this is just one option.

b) Modify your code for different boundary conditions:

$$u(0) = 1,\ u(1) = 1.$$

Do the same procedure as above: derive the analytical solution, implement a finite difference method for this problem, make a convergence plot of the error e_M. What order of convergence do you expect? Do you get the expected order of convergence?

c) Consider the same problem but with a Neumann boundary condition on both sides:

$$u_x(0) = 0,\ u_x(1) = \frac{1}{2}\,.$$

What is the issue here? Do you have a remedy for handling such kind of problems?

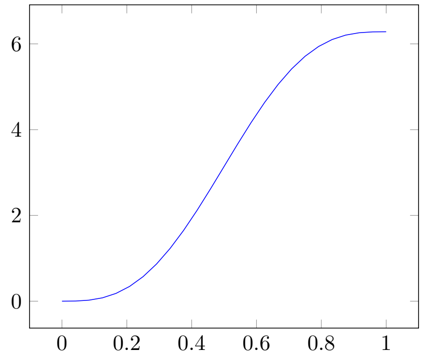
d) Use the following function $u(x) = e^{-\frac{1}{\epsilon}(x-\frac{1}{2})^2}$ as a manufactured solution for a boundary value problem on $\Omega = (0, 1)$ with Dirichlet boundary conditions at both ends:

$$u_{xx} = f(x) \text{ in } \Omega$$

Investigate both first and second order methods by presenting error plots for the discrete ℓ_2-norm and the continuous L_2-norm (use suitable interpolation functions) using uniform mesh refinement (UMR), then do adaptive mesh refinement (AMR) to optimize the accuracy using as few mesh point as possible.

2 **a)** Consider the following heat equation on $x \in [0,1]$, $t > 0$ with a Neumann boundary condition,

$$u_t = u_{xx}, \quad u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = 2\pi x - \sin(2\pi x).$$



The initial condition $t = 0$

As before, consider equidistant points

$$x_0 = 0, \ x_1 = \frac{1}{M+1}, \ldots, \ x_M = \frac{M}{M+1}, \ x_{M+1} = 1,$$

to solve the equation.

Implement finite difference methods of order both 1 and 2. Use fictional nodes if you need. Explain the method you implemented, i.e., the linear system you are solving. For this problem, analytical solution is not available in a closed form, but we want to make a convergence plot. Therefore, compute a "reference solution" where sufficiently large number of points M^* (sufficiently small h^*) is used. Let us denote this reference solution at time t as $\boldsymbol{u}_{M^*}(t)$. Fix time $t > 0$, not too small, and construct a piecewise constant function from $\boldsymbol{u}_{M^*}(t)$, use this as an analytical solution to make convergence plots of the relative ℓ_2 error in terms of $M < M^*$:

$$e^r_\ell = \frac{\sqrt{\sum_{i=0}^{M+1} \frac{1}{M+2}(U_{M^*}(x_i) - U_M(x_i))^2}}{\sqrt{\sum_{i=0}^{M+1} \frac{1}{M+2}(U_{M^*}(x_i))^2}}.$$

Do both order 1 and 2 methods give you the correct order? Note that you need to calculate numerical solutions for the same fixed time t for different M.

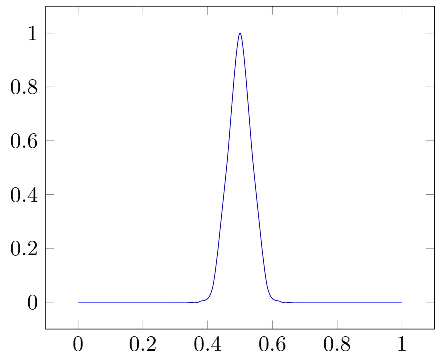
b) Choose an appropriate manufactured solution for an initial and boundary value problem on $\Omega(x,t) : x \in [0,1], t \in [0,T]$ where you choose the type of boundary conditions and compute the appropriate initial values:

$$u_t = u_{xx} \text{ in } \Omega$$

Investigate both first and second order methods in time by presenting error plots for the discrete ℓ_2-norm and the continuous L_2-norm (use appropriate interpolation functions) using uniform mesh refinement (UMR) in x- and t-directions, then do mesh refinement where $k = ch$ (c is a positive real constant) and $r = k/h^2$ is constant.

c) Now consider the following inviscid Burgers' equation on $x \in [0, 1]$, $t > 0$:

$$u_t = -uu_x, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = \exp(-400(x - 1/2)^2).$$



The initial condition $t = 0$

Look at Section 4.5 as a reference. For the time discretization, choose your own method. This equation itself is a hyperbolic equation, but a limit case of parabolic equation (Burgers' equation):

$$u_t = \varepsilon u_{xx} - uu_x, \quad \varepsilon \to 0.$$

This equation is known to have "breaking" (see this 2D example) where at some time point t^* the solution breaks and the unique solution does not exist as a function. Make a plot of the solution when the wave starts numerically breaking, and report that time t^*.

$\boxed{3}$ Consider the following 2D Laplace equation on the unit square $(x, y) \in [0, 1]^2 =: \Omega$,

$$u_{xx} + u_{yy} = 0, \quad u(x, y) = g(x, y) \text{ where } (x, y) \in \partial\Omega,$$

where the boundary condition is given by

$$\begin{aligned}
g(0, y) &= 0, \quad 0 \le y \le 1, \\
g(x, 0) &= 0, \quad 0 \le x \le 1, \\
g(1, y) &= 0, \quad 0 \le y \le 1, \\
g(x, 1) &= \sin(2\pi x), \quad 0 \le x \le 1.
\end{aligned}$$

a) By using separation of variables (i.e., assume that the solution can be written as multiplication of functions which only depend on one variable), derive the analytical solution.

b) Implement the five point formula for this problem. Using constant step sizes $h = 1/(M_x + 1)$ and $k = 1/(M_y + 1)$ for directions x and y, respectively. Make convergence plots for both discretization of directions x and y. What order of convergence do you observe?

$\boxed{4}$ Consider the following linearized Korteweg-deVries (KdV) equation for $x \in [-1, 1]$, $t > 0$,

$$u_t + (1 + \pi^2)u_x + u_{xxx} = 0, \; u(x, 0) = \sin(\pi x),$$

where the periodic boundary condition with period 2 is assumed:

$$u(x + 2, t) = u(x, t).$$

Note that due to this periodic boundary condition, we can consider that the function is defined on $\mathbb{R}$ but we only need to focus on one period $[-1, 1]$. Consider the grid point

$$x_0 = -1, \; x_1 = -1 + \frac{2}{M}, ..., \; x_M = 1.$$

a) Discretize the problem with central finite differences in space. For the temporal discretization, use both forward Euler and the trapezoidal rule (the Crank-Nicolson method). For the third derivative, use the following central difference approximation:

$$u_{xxx}|_{x=x_m} = \frac{u(x_{m+3}) - 3u(x_{m+1}) + 3u(x_{m-1}) - u(x_{m-3})}{8h^3} + \mathcal{O}\left(h^2\right).$$

Prove or disprove that these two discretization methods (finite difference-Euler, finite difference-trapezoidal) are Von Neumann stable (hint: see Section 5.9).

b) The analytical solution of this problem is given by $u(x, t) = \sin(\pi(x - t))$. Implement numerical methods explained above. Make the convergence plot of the discrete ℓ_2 error interms of spatial discretization M for the time $t = 1$ being fixed.

c) **[Optional]** When the periodic boundary condition is imposed, Fourier series expansion is one of the most useful tools. In the above setting, we can write

$$u(x, t) = \sum_{k\in\mathbb{Z}} \widehat{u}(k, t) \exp(\mathrm{i}\pi kx), \quad \widehat{u}(k, t) = \frac{1}{2}\int_{-1}^{1} u(x, t) \exp(-\mathrm{i}\pi kx)\,\mathrm{d}x,$$

where i is the imaginary unit. Due to the orthogonality and completeness of Fourier basis functions, we have

$$u_t(x, t) = \sum_{k\in\mathbb{Z}} \widehat{u}_t(k, t) \exp(\mathrm{i}\pi kx) = \sum_{k\in\mathbb{Z}} \left(-\mathrm{i}\pi k(1 + \pi^2) + \mathrm{i}\pi^3 k^3\right) \widehat{u}(k, t) \exp(\mathrm{i}\pi kx),$$

and

$$u(x, t) = \sum_{k\in\mathbb{Z}} \widehat{u}(k, 0) \exp\left(-\mathrm{i}\pi k(1 + \pi^2)t + \mathrm{i}\pi^3 k^3 t\right) \exp(\mathrm{i}\pi kx).$$

Prove that for any time $t > 0$,

$$\|u(x, t)\| := \sqrt{\frac{1}{2}\int_{-1}^{1} |u(x, t)|^2\,\mathrm{d}x} = \sqrt{\frac{1}{2}\int_{-1}^{1} |u(x, 0)|^2\,\mathrm{d}x},$$

in other words, the L_2 norm is conserved over time. Then numerically try to observe if this L_2 norm is numerically conserved by using the methods implemented above with your own choice of initial condition; make plots of the discrete ℓ_2 norm over time t ($x-$axis for time, $y-$axis for ℓ_2 norm).

$\boxed{5}$ **a)** Consider the Poisson equation in 1D:

$$-u_{xx} = f(x) \quad , \quad u(a) = d_1 \ , \ u(b) = d_2 \tag{5}$$

Assume that we have a uniform partition such that $x_n = a + nh$, where $h = (b-a)/N$ and $n \in [0, N]$. By discretizing the problem with linear finite elements, you get a linear equation system $\mathbf{Au} = \mathbf{f}$. Find the analytical expressions for $\mathbf{A}$ and $\mathbf{f}$.

b) Choose the following parameters with $0 \le x \le 1$:

$$f(x) = -2 \quad , \quad d_1 = 0 \ , \ d_2 = 1 \tag{6}$$

Choose $N = \{8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$ and make a loglog-plot of the L^2-error. Try AFEM (average $\alpha = 1.0$, maximum $\alpha = 0.7$) with $N = 20$.

c) Choose the following parameters with $-1 \le x \le 1$:

$$f(x) = -(40000x^2 - 200)e^{-100x^2} \quad , \quad d_1 = e^{-100} \ , \ d_2 = e^{-100} \tag{7}$$

Choose $N = \{8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$ and make a loglog-plot of the L^2-error. Try AFEM (average $\alpha = 1.0$, maximum $\alpha = 0.7$) with $N = 20$.

d) Choose the following parameters with $-1 \le x \le 1$:

$$f(x) = -(4000000x^2 - 2000)e^{-1000x^2} \quad , \quad d_1 = e^{-1000} \ , \ d_2 = e^{-1000} \tag{8}$$

Choose $N = \{8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$ and make a loglog-plot of the L^2-error. Try AFEM (average $\alpha = 1.0$, maximum $\alpha = 0.7$) with $N = 20$.

e) Choose the following parameters with $0 \le x \le 1$:

$$f(x) = \frac{2}{9}x^{-4/3} \quad , \quad d_1 = 0 \ , \ d_2 = 1 \tag{9}$$

Choose $N = \{8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$ and make a loglog-plot of the L^2-error. Try AFEM (average $\alpha = 1.0$, maximum $\alpha = 0.7$) with $N = 20$.