MA1101R Linear Algebra I
Help Sheet for Final Examinations

Disclaimer: This help sheet does not contain everything. Please note that any form of error found in this help sheet is solely due to my own human error, and not committed on purpose in order to "snake". Should you make use of this formula sheet, please be sure to spend some time checking it to make sure that it is accurate.

- *Joshua of House Ursaia*

Properties of transpose matrices	
$(A^T)^T = A$	$(cA)^T = cA^T$
$(A+B)^T = A^T + B^T$	$(AB)^T = B^T A^T$
Properties of inverse matrices	
$(A^{-1})^{-1} = A$	$(cA)^{-1} = \frac{1}{c}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$	$(AB)^{-1} = B^{-1}A^{-1}$

Elementary Matrices		
$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
cR_i	$R_i \leftrightarrow R_j$	$R_i + cR_j$
$\det(EA) = c\det(A)$	$\det(EA) = -\det(A)$	$\det(EA) = \det(A)$

Ways to find a determinant of a matrix	
Matrices with zero row / column	$det(\boldsymbol{A}) = 0$
2×2 matrices	$det(\boldsymbol{A}) = ad - bc$
3×3 matrices	(Equation omitted)
Triangular matrices	$det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$
Cofactor expansions Use if rows or columns have many zero entries.	Along the ith row: $det(\boldsymbol{A}) = a_{i1}\boldsymbol{A_{i1}} + a_{i2}\boldsymbol{A_{i2}} + \cdots + a_{in}\boldsymbol{A_{in}}$ Along the jth column: $det(\boldsymbol{A}) = a_{1j}\boldsymbol{A_{1j}} + a_{2j}\boldsymbol{A_{2j}} + \cdots + a_{nj}\boldsymbol{A_{nj}}$ $\boldsymbol{A_{ij}} = (-1)^{i+j}\, det\,(\boldsymbol{M_{ij}})$, where $\boldsymbol{M_{ij}}$ is the submatrix obtained from A by deleting the ith row and jth column.
Elementary matrices	$det(\boldsymbol{EA}) = det(\boldsymbol{E})det(\boldsymbol{A})$
$det(\boldsymbol{A}) = det(\boldsymbol{A^T})$	$det(c\boldsymbol{A}) = c^n det(\boldsymbol{A})$
$det(\boldsymbol{AB}) = det(\boldsymbol{A})det(\boldsymbol{B})$	$det(\boldsymbol{A^{-1}}) = \frac{1}{det(\boldsymbol{A})}$

Ways to find the inverse of a matrix

1. Find the reduced row-echelon form of $(A|I)$, which is $(I|A^{-1})$.
2. When given a polynomial function of the matrix A, manipulate the equation such that an I appears at one side of the equation. Then $A(\cdots) = I$.
3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
4. $\boldsymbol{A^{-1}} = \frac{1}{det(A)} adj(\boldsymbol{A})$, where $adj(\boldsymbol{A}) = (A_{ji})_{n\times n}$.
   If $\boldsymbol{A}$ is invertible, then $adj(\boldsymbol{A})$ is invertible.

Solving a linear system using Cramer's Rule

Given $\boldsymbol{Ax} = \boldsymbol{b}$, then $x = \frac{1}{det(A)}\begin{pmatrix} det(\boldsymbol{A_1}) \\ \vdots \\ det(\boldsymbol{A_n}) \end{pmatrix}$, where $\boldsymbol{A_j}$ is obtained by replacing its jth column with $\boldsymbol{b}$.

Geometric objects		
Line in $\mathbb{R}^2$	Implicit	$\{(x,y) \mid ax + by = c\}$
	Explicit	$\{(x_0, y_0) + t(a,b) \mid t \in \mathbb{R}\}$
Plane in $\mathbb{R}^3$	Implicit	$\{(x,y,z) \mid ax + by + cz = d\}$
	Explicit	$\{(x_0, y_0, z_0) + s(a_1, b_1, c_1) + t(a_2, b_x, c_2) \mid s, t \in \mathbb{R}\}$
Line in $\mathbb{R}^3$	Implicit	$\{(x,y,z) \mid a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2\}$
	Explicit	$\{(x_0, y_0, z_0) + t(a,b,c) \mid t \in \mathbb{R}\}$

Checking if a vector v is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$

1. Suppose that $\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$.
2. Treat $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ and $\boldsymbol{v}$ as column vectors and set up an augmented matrix $(\boldsymbol{A}|\boldsymbol{v}) = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k}|\boldsymbol{v})$.
3. Perform Gaussian Elimination on the augmented matrix.

Inconsistent → $\mathbf{v}$ is not linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$.

Checking if a vector v is in span(S)

Just check if the vector $\boldsymbol{v}$ is a linear combination of all the vectors in the set S.

Checking if span(S) $= \mathbb{R}^n$

Method 1: Check number of vectors in S

$$k < n \Rightarrow span(\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}) \neq \mathbb{R}^n$$

Method 2: Check for zero row in row-echelon form

1. Treat $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ as column vectors and set up a matrix $\boldsymbol{A} = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k})$.
2. Perform Gaussian Elimination on $\boldsymbol{A}$ to obtain the row-echelon form R.

R has a zero row → $span(\boldsymbol{S}) \neq \mathbb{R}^n$.
R has no zero row → $span(S) = \mathbb{R}^n$.

Checking if $span(S_1) \subseteq span(S_2)$

Given two linear spans, $S_1 = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$, $S_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$.

1. Set up a "super augmented matrix", with vectors from S_2 on the left side and vectors from S_1 on the right side, i.e. $(\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_m}|\boldsymbol{u_1}|\boldsymbol{u_2}|\cdots|\boldsymbol{u_k})$.
2. Perform Gaussian Elimination on the augmented matrix.

Inconsistent → $span(S_1) \nsubseteq span(S_2)$.

Checking if $span(S_1) = span(S_2)$

Check if $span(S_1) \subseteq span(S_2)$ and $span(S_2) \subseteq span(S_1)$.

Removing "redundant vectors" in a linear span

If $\boldsymbol{u_k}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}$, then

$$span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}\} = span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}, \boldsymbol{u_k}\}$$

Subspaces

$V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if there $\exists \boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k} \in \mathbb{R}^n$, such that

$$V = span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}.$$

Checking if $V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$:

For V to be a subspace of $\mathbb{R}^n$, all three of the following must hold:

- $\boldsymbol{0} \in V$
- For any $\boldsymbol{v} \in V$ and $c \in \mathbb{R}$, $c\boldsymbol{v} \in V$.
- For any $\boldsymbol{u} \in V$ and $\boldsymbol{v} \in V$, $\boldsymbol{u} + \boldsymbol{v} \in V$.

Should any of the above not hold, V is not a subspace of $\mathbb{R}^n$.

Finding the solution space of a linear system

The solution space of $Ax = b$ is the solution set of $Ax = 0$, and it is a subspace of $\mathbb{R}^n$.

Linear Independence

When a group of $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ are linearly independent,

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{0} \Rightarrow c_1, c_2, \cdots, c_k = 0$$

Determining if a set is linearly independent

Method 1: Finding number of solutions

Given a set $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\} \subseteq \mathbb{R}^n$,

1. Treat $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ as column vectors and set up an augmented matrix $(\boldsymbol{A}|\boldsymbol{0}) = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k}|\boldsymbol{0})$.
2. Perform Gaussian Elimination on $(\boldsymbol{A}|\boldsymbol{0})$.

Linear system has a non-trivial solution → S is linearly dependent.
Linear system has only trivial solution → S is linearly independent.

Method 2: Checking number of vectors in S

As long as number of distinct vectors in the set,
$k > n$ → S is linearly dependent.

Comparing two subsets of $\mathbb{R}^n$ to check for linear independence

Given two finite sets S_1, S_2 such that $S_1 \subseteq S_2$,

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent.
S_2 is linearly independent $\Rightarrow S_1$ is linearly independent.

Checking if a $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ is a basis for a set V or $\mathbb{R}^n$

Check any two of the following conditions:

1. S is linearly independent.
2. $span(S) = V$
3. $|S| = n$

Finding coordinate vector $(\boldsymbol{w})_s$ relative to a basis S

1. Suppose that $\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$.
2. Treat $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ and $\boldsymbol{w}$ as column vectors and set up an augmented matrix $(\boldsymbol{A}|\boldsymbol{w}) = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k}|\boldsymbol{w})$.
3. Perform Gaussian Elimination on $(\boldsymbol{A}|\boldsymbol{w})$ and solve the linear system $\boldsymbol{Ax} = \boldsymbol{w}$.

We obtain $(\boldsymbol{w})_s = (c_1 \quad c_2 \quad \cdots \quad c_k)$.

Properties of coordinate vectors

1. $(\boldsymbol{w})_S = \boldsymbol{0} \Rightarrow \boldsymbol{w} = \boldsymbol{0}$
2. $(c\boldsymbol{w})_S = c(\boldsymbol{w})_S$
3. $(\boldsymbol{u})_S + (\boldsymbol{v})_S = (\boldsymbol{u} + \boldsymbol{v})_S$

Finding transition matrix P from bases S to T

Given bases $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$

1. Set up an augmented matrix, with vectors from S_2 on the left side and vectors from S_1 on the right side, $(\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k}|\boldsymbol{u_1} \quad \boldsymbol{u_2} \quad \cdots \quad \boldsymbol{u_k})$
2. Perform Gauss-Jordan Elimination to obtain $(\boldsymbol{I}|\boldsymbol{P})$.

P is the transition matrix from S to T, where $(\boldsymbol{w})_T = \boldsymbol{P}(\boldsymbol{w})_S$ and $(\boldsymbol{w})_S = \boldsymbol{P^{-1}}(\boldsymbol{w})_T$.

Finding the dimension of vector space V

The dimension of V, dim(V), is the number of vectors in S, a basis for V.

Finding the dimension of a solution space of a linear system

Let V be a solution space of $\boldsymbol{Ax} = \boldsymbol{b}$, ie. the solution set of $\boldsymbol{Ax} = \boldsymbol{0}$.

1. Find R, the row-echelon form of A.

The number of pivot columns of R, equal to the number of arbitrary parameters in the solution, is equal to the dimension of V.

Finding the basis for the row space of an $m \times n$ matrix A

The row space of A is a subspace for $\mathbb{R}^n$.
Perform Gaussian Elimination on A. The nonzero rows of R are linearly independent. The set of nonzero rows of R form the basis of the row space of A.

Row space of $\mathbf{AB} \subseteq$ Row space of $\mathbf{B}$

Finding the basis for the column space of an $m \times n$ matrix A

The column space of A is a subspace for $\mathbb{R}^m$.
Perform Gaussian Elimination on A. The pivot columns of R form the basis for the column space of R. The columns in A that correspond to the pivot columns in R form the basis for the column space of A.

Column space of $\mathbf{AB} \subseteq$ Column space of $\mathbf{A}$

Checking if a linear system is consistent based on the column space

The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent iff b lies in the column space of A.

Finding a basis for a given spanning set

Given a vector space $V = span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}\}$, we want to find a basis for V.

Method 1: Row Space Method

1. View $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}$ as row vectors, and form a matrix $\begin{pmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_m} \end{pmatrix}$.
2. Perform Gaussian Elimination on that matrix to find R. The nonzero rows of R forms the basis for V.

Method 2: Column Space Method

1. View $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_m}$ as column vectors, and form a matrix $(\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_m})$.
2. Perform Gaussian Elimination on that matrix to find R.
3. Identify the pivot columns of R.
4. The corresponding $\boldsymbol{v_j}$ form the basis for V.

Extending a Linearly Independent Set to form a Basis for $\mathbb{R}^n$.

Given a linearly independent set $S = \{\boldsymbol{v_1}, \cdots, \boldsymbol{v_k}\}$, we want to extend it to form a basis for $\mathbb{R}^n$.

1. View $\boldsymbol{v_1}, \cdots, \boldsymbol{v_k}$ as row vectors and form a matrix $\begin{pmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_k} \end{pmatrix}$.
2. Perform Gaussian Elimination on that matrix to form R.
3. Identify the pivot columns of R.
4. Add rows to R such that all columns in R are pivot columns.
5. All the vectors now form a basis for $\mathbb{R}^n$.

Finding the Rank of a Matrix

The rank of a matrix A is the dimension (ie. number of vectors in the basis) of the row space of A, which is equal to the dimension of the column space of A.

$$Rank(\boldsymbol{A}) = Rank(\boldsymbol{A^T})$$

Finding the Nullspace of a Matrix

The nullspace of A is the solution space of the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$. The nullspace of A is equal to the nullspace of its row-echelon form R.

Finding the Nullity of a Matrix

The nullity of A is the dimension of the nullspace of A, which is equal to the nullity of R. It is simply the number of non-pivot columns of R.

Dimension Theorem

$$\text{Rank}(\mathbf{A}) + \text{Nullity}(\mathbf{A}) = n = \text{Number of columns of } \mathbf{A}$$

Finding a general solution for a linear system

A general solution to the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ can be expressed as

(a particular solution of $\mathbf{Ax} = \mathbf{b}$) + (a general solution of $\mathbf{Ax} = \mathbf{0}$)

Certain vector properties	
Norm of $\boldsymbol{v}$	$\lVert\boldsymbol{v}\rVert = \sqrt{{v_1}^2 + \cdots + {v_n}^2}$ $\lVert\boldsymbol{v}\rVert^2 = \boldsymbol{v} \cdot \boldsymbol{v}$
Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$	$d(\boldsymbol{u}, \boldsymbol{v}) = \lVert\boldsymbol{u} - \boldsymbol{v}\rVert$ $= \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$
Dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$	$\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + \cdots + u_nv_n$ $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u^T}\boldsymbol{v}$

	Take note that the $(i,j) - entry$ of a matrix AB is $\boldsymbol{a_i}^T\boldsymbol{b_j} = \boldsymbol{a_i} \cdot \boldsymbol{b_j}$.
Angle between $\boldsymbol{u}$ and $\boldsymbol{v}$	$\theta = cos^{-1}(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\lVert\boldsymbol{u}\rVert\lVert\boldsymbol{v}\rVert}), 0 \le \theta \le \pi$

Properties of dot product

Commutativity	$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
Distributivity	$(\boldsymbol{u}+\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$ $\boldsymbol{w} \cdot (\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v}$
Scalar Multiplication	$(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$ $\lVert c\boldsymbol{v}\rVert = \lvert c\rvert\lVert\boldsymbol{v}\rVert$
Cauchy-Schwarz Inequality	$\lvert\boldsymbol{u} \cdot \boldsymbol{v}\rvert \le \lVert\boldsymbol{u}\rVert\lVert\boldsymbol{v}\rVert$
Triangle Inequality	$\lVert\boldsymbol{u}+\boldsymbol{v}\rVert \le \lVert\boldsymbol{u}\rVert + \lVert\boldsymbol{v}\rVert$ $d(\boldsymbol{u},\boldsymbol{w}) \le d(\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{v},\boldsymbol{w})$
$\boldsymbol{v} \cdot \boldsymbol{v} \ge \boldsymbol{0}$ and $\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{0} \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$	

Finding coordinate vector $(w)_S$ relative to an orthogonal basis S

If $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is an orthogonal basis for the vector space V, then for any vector $\boldsymbol{w} \in V$,

$$(\boldsymbol{w})_S = (\frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}}, \cdots, \frac{\boldsymbol{w} \cdot \boldsymbol{u_k}}{\boldsymbol{u_k} \cdot \boldsymbol{u_k}})$$

$$\boldsymbol{w} = (\frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}})\boldsymbol{u_1} + \cdots + (\frac{\boldsymbol{w} \cdot \boldsymbol{u_k}}{\boldsymbol{u_k} \cdot \boldsymbol{u_k}})\boldsymbol{u_k}$$

Finding coordinate vector $(w)_T$ relative to an orthonormal basis T

If T $= \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ is an orthonormal basis for the vector space V, then for any vector $\boldsymbol{w} \in V$,

$$(\boldsymbol{w})_T = (\boldsymbol{w} \cdot \boldsymbol{v_1}, \cdots, \boldsymbol{w} \cdot \boldsymbol{v_k})$$

$$\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + \cdots + (\boldsymbol{w} \cdot \boldsymbol{v_k})\boldsymbol{v_k}$$

Finding projection p of a vector onto a vector space

Let $\boldsymbol{w} \in \mathbb{R}^n$ be a vector, andV be a vector subspace of $\mathbb{R}^n$.

$$\boldsymbol{n} = \boldsymbol{w} - \boldsymbol{p}$$

- $\boldsymbol{p}$ is the projection of vector $\boldsymbol{w}$ onto vector space V. It exists and is unique.
- $\boldsymbol{n}$ is a normal vector to V, i.e. a vector orthogonal to V.

$\boldsymbol{p}$ is also considered the best approximation of $\boldsymbol{w}$ in V, because among all the vectors $\boldsymbol{v} \in V$, it has the shortest distance to $\boldsymbol{w}$, i.e.

$$\boldsymbol{d(u,p)} \le \boldsymbol{d(u,v)} \text{ for all } \boldsymbol{v} \in V$$

$$\boldsymbol{d(u,p)} = \boldsymbol{d(u,v)} \Rightarrow \boldsymbol{v} = \boldsymbol{p}$$

<u>Method 1: Linear Combination Method</u>

If $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ is an orthogonal basis forV, then the projection of w into V,

$$\boldsymbol{p} = (\frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}})\boldsymbol{u_1} + \cdots + (\frac{\boldsymbol{w} \cdot \boldsymbol{u_k}}{\boldsymbol{u_k} \cdot \boldsymbol{u_k}})\boldsymbol{u_k}$$

If T $= \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ is an orthonormal basis forV, then the projection of w into V,

$$\boldsymbol{p} = (\boldsymbol{w} \cdot \boldsymbol{v_1})\boldsymbol{v_1} + \cdots + (\boldsymbol{w} \cdot \boldsymbol{v_k})\boldsymbol{v_k}$$

<u>Method 2: Least-Squares Solution Method</u>

Suppose that $V = span\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_k}\}$.

1. View $\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_k}$ as column vectors. Construct a matrix $A = (\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_k})$.
2. Find a least squares solution, $\boldsymbol{u}$, to the linear system $\boldsymbol{Ax} = \boldsymbol{w}$. That is the solution of the linear system $\boldsymbol{A^TAx} = \boldsymbol{A^Tw}$.
3. The projection $\boldsymbol{p}$ of $\boldsymbol{w}$ inV is $\boldsymbol{p} = \boldsymbol{Au}$.

Constructing an orthogonal basis for a vector space

Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ be a basis (need not be orthogonal) for vector space V. Then we can construct $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$ as an orthogonal basis for V:

$$\boldsymbol{v_1} = \boldsymbol{u_1}$$

$$\boldsymbol{v_2} = \boldsymbol{u_2} - (\frac{\boldsymbol{u_2} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}})\boldsymbol{v_1}$$

$$\boldsymbol{v_k} = \boldsymbol{u_k} - (\frac{\boldsymbol{u_k} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}})\boldsymbol{v_1} - (\frac{\boldsymbol{u_k} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}})\boldsymbol{v_2} - \cdots - (\frac{\boldsymbol{u_k} \cdot \boldsymbol{v_{k-1}}}{\boldsymbol{v_{k-1}} \cdot \boldsymbol{v_{k-1}}})\boldsymbol{v_{k-1}}$$

Constructing an orthonormal basis from an orthogonal basis

From the orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$, we can construct $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \cdots, \boldsymbol{w_k}\}$ as an orthonormal basis for V.

$$\boldsymbol{w_i} = \frac{\boldsymbol{v_i}}{\lVert\boldsymbol{v_i}\rVert} \text{ for all } i = 1, \cdots, k$$

Finding a least squares solution to a linear system

Pre-multiply the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ at both sides by $\boldsymbol{A^T}$. We get $\boldsymbol{A^TAx} = \boldsymbol{A^Tb}$, and any solution to this equation will be the least-squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Checking if a matrix is an orthogonal matrix

A square matrix A is an orthogonal matrix if

$$\boldsymbol{A^T} = \boldsymbol{A^{-1}}$$

Properties of an orthogonal matrix

The columns of A form an **orthonormal basis** for $\mathbb{R}^n$, and the rows of A form an **orthonormal basis** for $\mathbb{R}^n$.

Take an orthonormal set, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$, and if you pre-multiply all the vectors in that set by an orthogonal matrix P, the resultant set $\{\boldsymbol{Pv_1}, \boldsymbol{Pv_2}, \cdots, \boldsymbol{Pv_k}\}$ is still an orthonormal set.

If sets S, T are orthonormal bases for $\mathbb{R}^n$, then the transition matrix from S to T, P is an orthogonal matrix. The transition matrix from T to S is simply $\boldsymbol{P^T} = \boldsymbol{P^{-1}}$.

Solving a linear system using QR-decomposition of the coefficient matrix

Given a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, **if** A is an $m \times n$ matrix whose columns are linearly independent, then you will be able to express A in the form

$$\boldsymbol{A} = \boldsymbol{QR}$$

- Q is an $m \times n$ orthogonal matrix.
- R is an invertible $n \times n$ upper triangular matrix.

$$(\boldsymbol{QR})\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{Q^TQRx} = \boldsymbol{Q^Tb}$$

$$\boldsymbol{Rx} = \boldsymbol{Q^Tb}$$

Then, you can solve for x by back-substitution.

Definition of Eigenvalues and Eigenvectors

$$\boldsymbol{Av} = \lambda \mathrm{v}$$

λ is an eigenvalue of A, and v is an eigenvector of A associated with λ. An eigenvector cannot be the zero vector.

Finding the eigenvalues of a matrix A

$$det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$$

$$det(\lambda\boldsymbol{I} - \boldsymbol{A}) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k}$$

The eigenvalues of A are precisely the roots of the equation above.

Finding eigenvalues for a triangular matrix

The eigenvalues of a triangular matrix is simply its diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$.

Finding the basis for an eigenspace of A

E_λ, the eigenspace of A associated to λ, is the nullspace of $(\lambda\boldsymbol{I} - \boldsymbol{A})$.

Determining if a Matrix is Diagonalizable

To show that A is diagonalizable, choose one to show:

- A has n distinct eigenvalues.
- $\lvert S_i\rvert = a(\lambda_i)$ for all i, i.e. A has n linearly independent eigenvectors.

<u>To show that A is not diagonalizable, choose one to show:</u>

- P is singular.
- $dim(E_i) < a(\lambda_i)$ for $some$ i, i.e. the total number of eigenvectors is less than the order of A.

Diagonalizing a Matrix

$\boldsymbol{P^{-1}AP} = \boldsymbol{D}$.

1. Solve $det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ to find the eigenvalues of A.
2. For each eigenvalue λ_i, find a basis S_i for the eigenspace E_{λ_i}.

The columns of P are the eigenvectors of A associated with these eigenvalues. The diagonal entries of D are the eigenvalues of A.

$$\mathbf{P} = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \ldots \quad \boldsymbol{v_n}),\ \boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Determining if a Matrix is Orthogonally Diagonalizable

A is orthogonally diagonalizable iff it is symmetric.

Orthogonally Diagonalizing a Matrix

$\boldsymbol{P^TAP}\ (= \boldsymbol{P^{-1}AP}) = \boldsymbol{D}$

1. Solve $det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ to find the distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$.
2. For each eigenvalue λ_i, find a basis S_{λ_i} for the eigenspace E_{λ_i}.
3. Use the Gram-Schmidt process to convert the basis S_{λ_i} to an orthonormal basis T_{λ_i}.

The columns of P are the eigenvectors of A associated with these eigenvalues. The diagonal entries of D are the eigenvalues of A.

$$\mathbf{P} = (\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \ldots \quad \boldsymbol{v_n})$$

Finding Powers of Matrices

$$\boldsymbol{A} = \boldsymbol{PDP^{-1}} = \boldsymbol{P}\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}\boldsymbol{P^{-1}}$$

$$\boldsymbol{A^m} = \boldsymbol{PD^mP^{-1}} = \boldsymbol{P}\begin{bmatrix} {\lambda_1}^m & 0 & 0 & 0 \\ 0 & {\lambda_2}^m & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {\lambda_n}^m \end{bmatrix}\boldsymbol{P^{-1}}$$

Using matrices to solve recurrence relations

Given a recurrence relation in the form $r_{n+1} = ar_n + br_{n-1}$, you want to find the closed formula for r_n in $terms$ of a, b and n.

1. Write down $\boldsymbol{x_n} = \binom{r_n}{r_{n+1}}$.
2. Write down a recurrence matrix A such that $\boldsymbol{x_n} = \boldsymbol{Ax_{n-1}}$.
3. Then find $\boldsymbol{P}, \boldsymbol{D^n}, \boldsymbol{P^{-1}}$ $\boldsymbol{such\ that}$ $\boldsymbol{x_n} = \boldsymbol{A^nx_0} = \boldsymbol{PD^nP^{-1}x_0}$.

Linear Transformations

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is such that $T(x) = \boldsymbol{A(x)}$, where A is the standard matrix for T, and it is unique.

Properties of Linear Transformations

1. $T(\boldsymbol{0}) = \boldsymbol{0}$
2. $T(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}) = c_1T(\boldsymbol{v_1}) + \cdots + c_kT(\boldsymbol{v_k})$

Showing that $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation

Just find a standard matrix A such that $T(x) = \boldsymbol{A}(x)$ for all $\boldsymbol{x}$ in $\mathbb{R}^n$. You can do so by performing a separation of parameters for the right-hand side of the given equation.

Showing that $T: \mathbb{R}^n \to \mathbb{R}^m$ is not a linear transformation

Show any of the following:

- $T(\boldsymbol{0}) \ne \boldsymbol{0}$
- $T(c\boldsymbol{v}) \ne cT(\boldsymbol{v})$
- $T(\boldsymbol{u}+\boldsymbol{v}) \ne T(\boldsymbol{u}) + T(\boldsymbol{v})$

Finding a standard matrix for T given a basis for $\mathbb{R}^n$

Given a basis $\boldsymbol{S} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$, if we know $T(\boldsymbol{v_1}), T(\boldsymbol{v_2}), \cdots, T(\boldsymbol{v_n})$, we are able to determine $T(\boldsymbol{v})$.

1. Every vector in $\mathbb{R}^n$ can be expressed as a unique linear combination $\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$.
2. Find a general form for $c_1, c_2, \cdots, c_n$. To do so, view $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}$ as column vectors, and solve the linear system $(\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_k}|\boldsymbol{v})$ by performing Gauss-Jordan Elimination to obtain an augmented matrix with I on the left-hand-side.
3. Notice that $T(\boldsymbol{v}) = c_1T(\boldsymbol{v_1}) + \cdots + c_kT(\boldsymbol{v_k})$.
4. Express the right hand size of the equation as $\boldsymbol{A}(\boldsymbol{v})$.

Finding a standard matrix for a composition

The standard matrix for $S \circ T$ is the

$$(Standard\ Matrix\ for\ S) \times (Standard\ Matrix\ for\ T).$$

Finding the range of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

<u>Method 1: Find images for all vectors in a basis</u>

1. Find any basis for $\mathbb{R}^n$, $\boldsymbol{S} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$.
2. Find the images $T(\boldsymbol{v_1}), T(\boldsymbol{v_2}), \cdots, T(\boldsymbol{v_n})$.

The range of T, $R(T) = span\{T(\boldsymbol{v_1}), T(\boldsymbol{v_2}), \cdots, T(\boldsymbol{v_n})\}$, and it is a subspace for $\mathbb{R}^m$.

<u>Method 2: Column space of the standard matrix</u>

R(T) is simply the column space of the standard matrix A.

Finding the rank of a linear transformation T

The rank of T is the dimensions of R(T). To find this, you will have to find the rank of the standard matrix A, or the dimensions of its column space. You will have to perform Gaussian Elimination on the matrix A and count the number of non-pivot columns.

Finding the kernel of a linear transformation T

The kernel of T is the set of all vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$. Ker(T) is a subspace of $\mathbb{R}^n$.

Ker(T) is simply the nullspace of A.

Finding the nullity of a linear transformation T

The nullity of T is simply the nullity of A.

Dimension Theorem for Linear Transformations

$rank(T) + nullity(T) = n$

Summary of properties for an $n \times n$ matrix A, $T: \mathbb{R}^n \to \mathbb{R}^n$

Matrix Property	A is invertible	A is singular
Determinant	$det(\boldsymbol{A}) \ne 0$	$det(\boldsymbol{A}) = 0$
Reduced row-echelon form	Identity matrix	Has a zero row
Homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$	Has only the trivial solution	Has non-trivial solutions
Linear system $\boldsymbol{Ax} = \boldsymbol{b}$	Has a unique solution	Has no solution or infinitely many solutions
Rows	Linearly independent	Linearly dependent
Columns	Linearly independent	Linearly dependent
Rank	$rank(\boldsymbol{A}) = n$	$rank(\boldsymbol{A}) < n$
Nullity	$nullity(\boldsymbol{A}) = 0$	$nullity(\boldsymbol{A}) > 0$
Eigenvalue	0 is not an eigenvalue	0 is an eigenvalue
Range of T_A	$R(T_A) = \mathbb{R}^n$	$R(T_A) \ne \mathbb{R}^n$
Kernel of T_A	$ker(T_A) = \{0\}$	$ker(T_A) \ne \{0\}$