MATRICES

WHAT ARE MATRICES?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

i-th row

jth column

$A = (a_{ij})_{m \times n}$ — m: number of rows, n: number of columns

a_{ij}: (i,j)-entry of A.

Ans: A matrix is a rectangular array of numbers.

MATRIX OPERATIONS

A, B, C are matrices of the same size.

① Equality

$\boxed{A=B} \Rightarrow \underbrace{m=p, n=q}_{\text{same size}}$, and $\underbrace{a_{ij}=b_{ij} \text{ for all } i,j}_{\text{corresponding entries are equal}}$.

② Matrix Addition

$\boxed{A+B = (a_{ij}+b_{ij})_{m\times n}}$

③ Matrix Subtraction

$\boxed{A-B = (a_{ij}-b_{ij})_{m\times n}}$

④ Scalar Multiplication

$\boxed{cA = (ca_{ij})_{m\times n}}$

• Multiply all entries in matrix A by the scalar c.

$A \xrightarrow{cR_1} \xrightarrow{cR_2} \cdots \xrightarrow{cR_n} cA$ (n times)

$\boxed{\det(cA) = c^n \det(A)}$

(i) Commutative Law: $\boxed{A+B=B+A}$

(ii) Associative Law: $\boxed{A+(B+C)=(A+B)+C}$

Rules pertaining to zero:

- $A+0=0+A=A$
- $A-A=0$
- $0A=0$

Rules pertaining to scalars:

- $c(A+B)=cA+cB$
- $(c+d)A = cA+dA$
- $(cd)A = c(dA) + d(cA)$
- $c(AB)=(cA)B=A(cB)$

MATRIX MULTIPLICATION

Given $A=(a_{ij})_{m\times p}$, $B=(b_{ij})_{p\times n}$,

We can only compute AB when no. of columns of A = No. of rows in B.

$m\times n$ matrix

$$AB = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} au+bx & av+by & aw+bz \\ cu+dx & cv+dy & cw+dz \\ eu+fx & ev+fy & ew+fz \end{bmatrix}$$

$\boxed{\det(AB)=\det(A)\det(B)}$

(i,j)-entry $= a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

(i) Matrix Multiplication is NOT Commutative.

- AB: pre-multiplication of A to B
- BA: post-multiplication of A to B.

$\therefore$ $(AB)^2$ and A^2B^2 may be different

– When $AB=0$, it is not necessary that $A=0$ or $B=0$

(ii) Associative Law: $\boxed{A(BC)=(AB)C=ABC}$

(iii) Distribution Laws: $\boxed{A(B_1+B_2)=AB_1+AB_2}$ $\boxed{(C_1+C_2)A = C_1A+C_2A}$

POWERS

$A^n = \begin{cases} I & \text{if } n=0 \\ \underbrace{AA\cdots A}_{n \text{ times}} & \text{if } n\geq 1 \end{cases}$ — A: square matrix, n: nonnegative integer

(i) Law of Indices: $A^mA^n = A^{m+n}$

$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ times}}$

TRANSPOSES

$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}_{m\times n}$. Then $A^T = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}_{n\times m}$

• $A=A^T$ if the square matrix A is symmetric.

The transpose of A, is a matrix whose (i,j)-entry is a_{ji}.

- $\boxed{(A^T)^T=A}$
- $\boxed{(cA)^T=cA^T}$
- $\boxed{(A+B)^T=A^T+B^T}$
- $\boxed{(AC)^T=C^TA^T}$
- $\boxed{\det(A^T)=\det(A)}$

NOTATION

$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$, $a_i = [a_{i1}\ a_{i2} \cdots a_{ip}]$

– Notation where each entry represents a row.

$B = [b_1\ b_2 \cdots b_n]$, $b_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$

– Notation where every entry represents a column.

$$AB = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{bmatrix}$$

$a_ib_j = [a_{i1}\ a_{i2} \cdots a_{ip}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$ which is the (i,j)-entry of AB.

– The i^{th} row of A pre-multiply to the j^{th} column of B.

$AB = A[b_1\ b_2 \cdots b_n] = [Ab_1\ Ab_2 \cdots Ab_n]$

- Ab_j is the j^{th} column of AB

$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1B \\ a_2B \\ \vdots \\ a_mB \end{bmatrix}$ — a_iB is the ith row of AB.

Linear system $Ax=b$ | Linear system $Cx=d$

same solutions

$\Uparrow$ row equivalent

augmented matrix $(A|b)$ | augmented matrix $(C|d)$

LINEAR SYSTEMS

Given a system of linear equations:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=b_m \end{cases}$$

$\boxed{Ax=b}$

...... Rewritten in matrix form:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\text{Coefficient matrix } (A)}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{Variable Matrix } (x)} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\text{Constant Matrix } (b)}$$

$$x_1\overset{c_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + x_2\overset{c_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \ldots + x_n\overset{c_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where $A=[c_1\ c_2 \cdots c_n]$, where c_j is the j^{th} column of A.

ie. $x_1c_1+x_2c_2+\ldots+x_nc_n=b$ or $\sum_{j=1}^{n} x_jc_j=b$.

The matrix u is said to be a solution to the linear system $Ax=b$ if the equation is satisfied when we substitute $x=u$ into the equation, ie. $Au=b$.

TYPES OF MATRICES

① Column Matrix

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

② Row Matrix

$[2\ \ 1\ \ 0]$

$[1]$ – is a column, row, square, scalar, symmetric & triangular matrix.

③ Square Matrix — Square matrix of order n.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

a_{ii}: Diagonal entry

a_{ij} where $i\neq j$: Non-diagonal entry

④ Diagonal Matrix

$a_{ij}=0$ whenever $i\neq j$.

⑤ Scalar Matrix

$a_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ c & \text{if } i=j \text{ for a constant } c. \end{cases}$

⑥ Identity Matrix

I_n denotes an identity matrix of order n.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

⑦ Zero Matrix

$0_{m\times n}$ denotes zero matrix of size $m\times n$

$0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

⑧ Symmetric Matrix

A square matrix is symmetric if: $a_{ij}=a_{ji}$ for all i,j.

– Also, $\boxed{A=A^T}$.

$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

⑨ Triangular Matrix: $\boxed{\det(A)=a_{11}a_{22}\cdots a_{nn}}$ $\Rightarrow$ Given a square matrix A, we can use elementary row operations to transform A into a triangular matrix and compute the determinant accordingly.

(i) Upper Triangular

$a_{ij}=0$ whenever $i>j$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(ii) Lower Triangular

$a_{ij}=0$ whenever $i<j$.

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}$

Compiled by: Joshua Chew Jian Xiang @josuaaah.

INVERSES OF SQUARE MATRICES

A is a square matrix of order n.
A is invertible if $\exists$ a square matrix B s.t.:

$AB = I$ and $BA = I$.
$\rightarrow$ B is the inverse of A. $B = A^{-1}$

A is singular if it has no inverse: $\boxed{\det(A) = 0}$.
$\rightarrow$ A is invertible iff $\boxed{\det(A) \neq 0}$.

(i) Cancellation Laws for Matrix Multiplication

$AB_1 = AB_2 \Rightarrow B_1 = B_2$

$C_1A = C_2A \Rightarrow C_1 = C_2$

Applicable only if A is invertible. May not hold if A is singular.

(ii) Uniqueness of Inverses
If B and C are inverses of a square matrix A, then $B = C$.

If A is invertible, then these are also invertible:
- Scalar product: cA
- Transpose: A^T
- Inverse: A^{-1}
- Product with another invertible matrix: AB
- Power: A^n

Take note:
$(cA)^{-1} = \frac{1}{c}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$
$(A^{-1})^{-1} = A$
$(AB)^{-1} = B^{-1}A^{-1}$

$\cdot\ (A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$

If A is singular $\rightarrow$ AB is singular. BA is singular.

Finding inverses:
Given A, an invertible $n\times n$ matrix.
Construct an $n\times 2n$ matrix $(A|I)$.

$$(A|I) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (I|A^{-1}).$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

ELEMENTARY MATRICES

Elementary Matrix:
A square matrix that can be obtained from an identity matrix by performing a single elementary row operation.
- All are invertible, & their inverses are also elementary matrices.

$\boxed{\det(EA) = \det(E)\det(A)}$

① Multiplying a row by a constant:

$$\begin{array}{c} \\ E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & 0 \\ & & 1 & & \\ & & & k & & \\ & & & & 1 & \\ & 0 & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \leftarrow i^{th}\text{ row} \\ \uparrow i^{th}\text{ column} \end{array}$$

$\boxed{A \xrightarrow{kR_i} EA.}$ $\boxed{\det(EA) = k\det(A)}$

$$E^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & 0 \\ & & 1 & & \\ & & & \frac{1}{k} & & \\ & & & & 1 & \\ & 0 & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

$\boxed{A \xrightarrow{\frac{1}{k}R_i} E^{-1}A}$

② Interchange two rows:

$$\begin{array}{l} i^{th}\text{ row} \rightarrow \\ j^{th}\text{ row} \rightarrow \end{array} E = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ 0 & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$\boxed{\det(EA) = -\det(A)}$

$\boxed{A \xrightarrow{R_i \leftrightarrow R_j} EA}$, $\boxed{E^{-1} = E}$

③ Add a multiple of a row to another row:

$\boxed{A \xrightarrow{R_j + kR_i} EA}$ $\boxed{\det(EA) = \det(A)}$

The k is in the jth row (row to be changed) in the elementary matrix.

$$\begin{array}{cc} & \begin{array}{cc} j^{th}\text{ column} & i^{th}\text{ column} \end{array} \\ j^{th}\text{ row} \rightarrow & E = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & k & \\ & & & \ddots & \\ & & & 1 & \\ 0 & & & & \ddots \\ & & & & & 1 \end{bmatrix} \text{(if } i>j) \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} i^{th}\text{ column} & j^{th}\text{ column} \end{array} \\ j^{th}\text{ row} \rightarrow & E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & k & & 1 \\ 0 & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \text{(if } i<j) \end{array}$$

$\boxed{A \xrightarrow{R_j - kR_i} E^{-1}A}$

$$\begin{array}{cc} & \begin{array}{cc} j^{th}\text{ column} & i^{th}\text{ column} \end{array} \\ j^{th}\text{ row} \rightarrow & E^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & -k & \\ & & & \ddots & \\ & & & 1 & \\ 0 & & & & \ddots \\ & & & & & 1 \end{bmatrix} \text{(if } i>j) \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} i^{th}\text{ column} & j^{th}\text{ column} \end{array} \\ j^{th}\text{ row} \rightarrow & E^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & -k & & 1 \\ 0 & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \text{(if } i<j) \end{array}$$

A guide to elementary matrices: ***
$R_j + kR_i$ means E has k in the jth row and ith column.
kR_i means E has k in the ith row.
$R_i \leftrightarrow R_j$ means in the E you take out 1 in i^{th} and j^{th} rows and you terbalik.

A guide to determinants:
$R_j + kR_i$ means det. no change.
$R_i \leftrightarrow R_j$ means det. become negative.
kR_i means det. multiply by k.

Elementary Column Operations:
The effect of post-multiplication of an elementary matrix.
- kC_i
- $C_i \leftrightarrow C_j$
- $C_i + kC_j$

Effect on det is similar to ERO.

$A \xrightarrow{kC_i} AE$

Application to Augmented Matrices/Linear Systems:
If $(A|b)$ and $(C|d)$ are row equivalent, then:
$E_k\cdots E_2E_1(A|b) = (C|d)$
$\rightarrow (E_k\cdots E_2E_1A \mid E_k\cdots E_2E_1b) = (C|d)$
$\rightarrow E_k\cdots E_2E_1A = C$ and $E_k\cdots E_2E_1b = d$
$\rightarrow A = (E_k\cdots E_2E_1)^{-1}C$, $b = (E_k\cdots E_2E_1)^{-1}d$

1. $\boxed{A \text{ is invertible.}}$
$Ax = 0$
$A^{-1}Ax = A^{-1}0$
$Ix = 0$
$x = 0$

2. $\boxed{\text{The linear system } Ax = 0 \text{ has only the trivial solution.}}$
- Augmented matrix is $(A|0)$
- In REF, every column is a pivot column $\rightarrow$ REF has no zero row.

Conversely,
If the REF of A has at least one zero row, then A is singular.

3. $\boxed{\text{The reduced row echelon form of } A \text{ is an identity matrix.}}$

$\exists$ elementary matrices s.t.
$E_k\cdots E_2E_1A = I$, $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$

4. $\boxed{A \text{ can be expressed as a product of elementary matrices.}}$

elementary matrices are invertible

DETERMINANTS

Definition
For a square $n\times n$ matrix $A = (a_{ij})$,

$\boxed{M_{ij}}$ is an $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column.

$$\boxed{\det(A)} = \begin{cases} a_{11} & \text{if } n=1 \\ a_{11}\underline{A_{11}} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n>1 \end{cases}$$

where $\boxed{A_{ij}} = (-1)^{i+j}\det(M_{ij})$
$\rightarrow$ (i,j)-cofactor of A

determinant of A

$$\boxed{\det(A)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If A has:
- 2 identical rows
- 2 identical columns,
then $\det(A) = 0$.

2×2 Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$\boxed{\det(A) = ad - bc}$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

3×3 Matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

*Note: Do NOT generalise this method to higher order.

$\det(A) = aei + bfg + cdh - ceg - afh - bdi$

Cofactor Expansions: Basically finding $\det(A)$ using any given row/column

$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$ $\leftarrow$ cofactor expansion along the i^{th} row

$= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$ $\leftarrow$ cofactor expansion along the j^{th} column

ADJOINTS

Classical adjoint of A — (i,j)-cofactor of A

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\text{adj}(A).}$$

CRAMER'S RULE

Given a linear system $Ax = b$,
Let A_i be A but with the i^{th} column of A replaced by b:

$$A_i = \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-2} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

If A is invertible, then the system has only one solution:

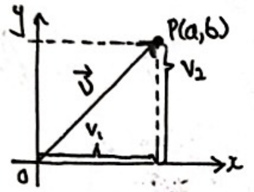
$$\boxed{x = \frac{1}{\det(A)}\begin{bmatrix} \det(A_1) \\ \det(A_2) \\ \vdots \\ \det(A_n) \end{bmatrix}}$$

Application:
$$x_1 = \frac{\det(A_1)}{\det(A)}$$
$$x_2 = \frac{\det(A_2)}{\det(A)}$$

VECTORS

VECTORS: THE BASICS

Given a vector $\overrightarrow{OP}$ in the xy-plane:



"Always view an n-vector as a single object, not n numbers"

Length $\|v\| = \sqrt{v_1^2 + v_2^2}$

Scalar multiple $cv = (cv_1, cv_2)$

Addition of two vectors, $u+v = (u_1+v_1, u_2+v_2)$ (just add the components correspondingly)

An n-vector / ordered n-tuple is

$v = (v_1, v_2, \dots, v_i, \dots, v_n)$

$v_i \in \mathbb{R}$ is the ith component of v.

EUCLIDEAN N-SPACE

Euclidean n-space refers to the set of all n-vectors of real numbers.

$$\boxed{\mathbb{R}^n = \{(v_1, v_2, \dots, v_n) \mid v_1, v_2, \dots, v_n \in \mathbb{R}\}}$$

- $\mathbb{R}^1$ is line. $\mathbb{R}^2$ is xy-plane. $\mathbb{R}^3$ is xyz-space.

Link to linear systems:

Implicit form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$A\underline{x} = b$ → x viewed as an n-vector, $x \in \mathbb{R}^n$.

Solution set of $Ax=b$ is a subset of $\mathbb{R}^n$.

- Implicit form:

$\left\{ \begin{array}{l|l} \text{General n-tuple} & \text{Conditions satisfied} \\ (u_1, u_2, \dots, u_n) & \text{by } u_1, u_2, \dots, u_n. \end{array} \right\}$

- Explicit form, aka general solution:

$\left\{ \begin{array}{l|l} \text{n-tuples in explicit form} & \text{range of the parameter} \\ (\frac{1}{2} - \frac{3}{2}t, -\frac{1}{2} + \frac{1}{2}t, t) & t \in \mathbb{R} \end{array} \right\}$

GEOMETRICAL INTERPRETATION

Line in $\mathbb{R}^2$: $\boxed{ax+by=c}$ a and b not both zero

Implicit form: $\{(x,y) \mid ax+by=c\}$

Explicit form: $\{(\frac{c-bt}{a}, t) \mid t \in \mathbb{R}\}$ if $a \neq 0$

$\{(s, \frac{c-as}{b}) \mid s \in \mathbb{R}\}$ if $b \neq 0$

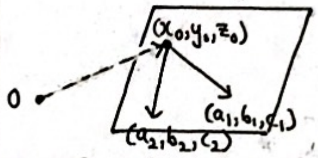
$\{(x_0 + ta, y_0 + tb) \mid t \in \mathbb{R}\}$

Plane in $\mathbb{R}^3$: $\boxed{ax+by+cz=d}$, a, b, c not all zero.

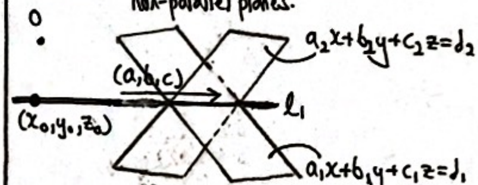
Implicit form: $\{(x,y,z) \mid ax+by+cz=d\}$

Explicit form: $\{(x_0, y_0, z_0) + s(a_1, b_1, c_1) + t(a_2, b_2, c_2) \mid s, t \in \mathbb{R}\}$

non-parallel vectors parallel to the plane.



Line in $\mathbb{R}^3$: Can be represented as an intersection of two non-parallel planes.



Implicit form: $\{(x,y,z) \mid a_1x + b_1y + c_1z = d_1 \text{ \& } a_2x + b_2y + c_2z = d_2\}$, a_i, b_i, c_i not all zero, planes are not parallel.

Explicit form: $\{(x_0 + ta, y_0 + tb, z_0 + tc) \mid t \in \mathbb{R}\}$

LINEAR COMBINATIONS AND SPANS

A linear combination of $v_1, v_2, \dots, v_k$ is

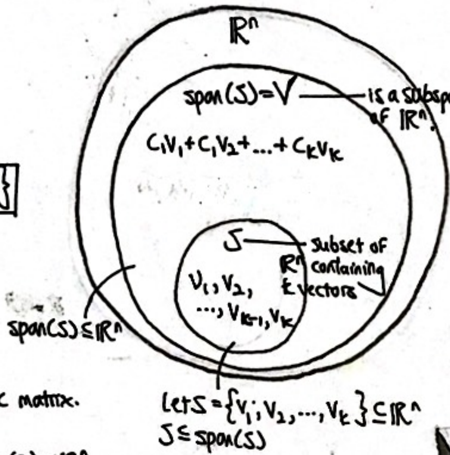
$$\boxed{c_1v_1 + c_2v_2 + \dots + c_kv_k, \text{ where } c_1, c_2, \dots, c_k \in \mathbb{R}}$$

To check if v is a linear combination of v_1 to v_k, suppose that $v = av_1 + bv_2 + \dots + kv_k$, then set up a linear system and perform Gaussian Elimination.

If linear system is inconsistent, then v is not an L.C. of v_1 to v_k.

The linear span of $S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ is the set of linear combinations of $v_1, v_2, \dots, v_k$, ie.

$$\boxed{\text{span}(S) = \{c_1v_1 + c_2v_2 + \dots + c_kv_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}}$$



Let $S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$

$S \subseteq \text{span}(S)$

When is $\text{span}(S) = \mathbb{R}^n$?

If number of vectors in S, $\boxed{k < n, \text{ then span}(S) \neq \mathbb{R}^n}$

* - View $v_1, v_2, \dots, v_k$ each as column vectors, and form a matrix $A = (v_1 \ v_2 \ \cdots \ v_k)$. Then perform Gaussian Elimination to get row-echelon form R, an $n \times k$ matrix.

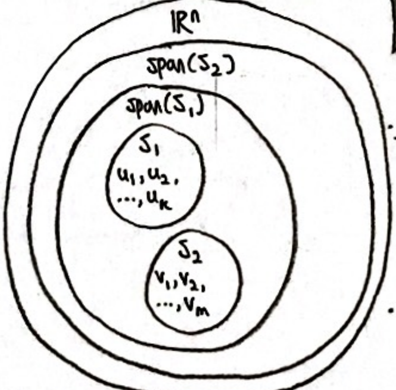
R will have at most k pivot columns

R has a zero row ⇒ system not always consistent ⇒ $\text{span}(S) \neq \mathbb{R}^n$.

R has no zero row ⇒ $\text{span}(S) = \mathbb{R}^n$.

1. $\boxed{0 \in \text{span}(S)}$, where 0 is the zero vector in $\mathbb{R}^n$.
2. $\boxed{v \in \text{span}(S) \Rightarrow cv \in \text{span}(S)}$ span(S) is closed under scalar multiplication.
3. $\boxed{u \in \text{span}(S) \text{ and } v \in \text{span}(S) \Rightarrow u+v \in \text{span}(S)}$ span(S) is closed under addition.

Given two subsets of $\mathbb{R}^n$, $S_1 = \{u_1, u_2, \dots, u_k\}$, $S_2 = \{v_1, v_2, \dots, v_m\}$.



$\boxed{\text{span}(S_1) \subseteq \text{span}(S_2) \Leftrightarrow \text{each } u_i \text{ is a linear combination of } v_1, v_2, \dots, v_m.}$

Redundant Vectors:

"redundant"

$\boxed{\text{If } v_k \text{ is a linear combination of } v_1, v_2, \dots, v_{k-1}, \text{ then span}\{v_1, v_2, \dots, v_{k-1}\} = \text{span}\{v_1, v_2, \dots, v_{k-1}, v_k\}}$

Equality of Spans: How to prove that $\boxed{\text{span}(S_1) = \text{span}(S_2)}$?

- You need to show that $\text{span}(S_1) \subseteq \text{span}(S_2)$, AND $\text{span}(S_2) \subseteq \text{span}(S_1)$

How to show?

①: $\text{span}(u_1, u_2, \dots, u_k) \subseteq \text{span}(v_1, v_2, v_3, \dots, v_m)$

$u_1, u_2, \dots, u_k \in \text{span}(v_1, v_2, \dots, v_m)$

Set up a "super augmented matrix":

$(v_1 \ v_2 \ \cdots \ v_m \mid u_1 \mid u_2 \mid \cdots \mid u_k)$

and then do Gaussian Elimination, and show that the augmented matrix is consistent.

Geometrical Interpretation:

span$\{u\}$: A line through the origin

span$\{u, v\} = \{su + tv \mid s, t \in \mathbb{R}\}$, s and u non-parallel: Plane through the origin.

span$\{3 \text{ non-coplanar vectors in } \mathbb{R}^3\}$: The entire $\mathbb{R}^3$ space.

SUBSPACES

- Definition of a subspace: A nonempty subset $V \subseteq \mathbb{R}^n$ is a subspace if $\forall \alpha, \beta \in \mathbb{R}, u, v \in V$, $\alpha u + \beta v \in V$.

$V \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if $\exists \ v_1, v_2, \dots, v_k \in \mathbb{R}^n$ such that $\boxed{V = \text{span}\{v_1, v_2, \dots, v_k\}}$.

- $\boxed{\{0\}}$ and $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$.

is the zero space $\{0\} = \text{span}\{0\}$.

To show that $V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$:

- Find $v_1, v_2, \dots, v_k \in \mathbb{R}^n$, and show that every $v = c_1v_1 + c_2v_2 + \dots + c_kv_k$ is in V.
- For $\mathbb{R}^2 / \mathbb{R}^3$: show that V represents a line / plane through the origin.

To show that $V \subseteq \mathbb{R}^n$ is NOT a subspace of $\mathbb{R}^n$: Choose 1 to show

- $0 \notin V$
- $c \in \mathbb{R}$ and $v \in V$ but $cv \notin V$ (violates closure under scalar multiplication)
- $u \in V$ and $v \in V$ but $u+v \notin V$ (violates closure under addition)

$\boxed{\text{The solution set of a homogeneous linear system of } n \text{ variables is a subspace of } \mathbb{R}^n.}$

- The solution set of a homogeneous linear system is called the solution space of the system.

∴ The solution set of a non-homogeneous linear system is NOT a subspace of $\mathbb{R}^n$.

Showing that V is subspace of $\mathbb{R}^n$: ① $0 \in V$ ② $u, v \in V \Rightarrow c_1u + c_2v \in V$ ③ $w \in V \Rightarrow cw \in V$

Compiled by: Joshua Chew Jian Xiang. @josuaaah