MA1521 Calculus for Computing
Help Sheet for Final Examinations

Disclaimer: Any form of error found in this help sheet is solely due to my own human error, and not committed on purpose in order to "snake". If you use this formula sheet, please check through at least once to ensure that all the formulae are indeed correct.

\- *Joshua of House Ursaia*

Rules on Functions

Sum or difference of f and g	$(f \pm g)(x) = f(x) \pm g(x)$
Product of f and g	$(fg)(x) = f(x)g(x)$
Quotient of f by g	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ *provided* $g(x) \neq 0$
Composition of functions	$(f \circ g)(x) = f(g(x))$

Rules on Limits

Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L'$.

Sum or difference of limits	$\lim_{x\to a}(f \pm g)(x) = L \pm L'$
Product of limits	$\lim_{x\to a}(fg)(x) = L\,L'$
Quotient of limits	$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{L}{L'}$ *provided* $L' \neq 0$
Scalar multiplication of limits	$\lim_{x\to a} kf(x) = kL$ for any real number k.

Derivative of a function f at a point $x = a$

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a} = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

Rules for Differentiation

Linearity	$(kf)'(x) = kf'(x)$ $(f \pm g)'(x) = f'(x) + g'(x)$
Product Rule	$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Chain Rule	$(f \circ g)'(x) = f'(g(x))g'(x) = (f' \circ g)(x)g'(x)$

General Differentiation Formulae

$$\frac{d}{dx}x^n = nx^{n-1}$$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

Exponential and Logarithmic Functions

$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\frac{d}{dx}(a^x) = a^x \ln a$	$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Parametric Differentiation

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$	$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

Trigonometric Formulae

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	$a^2 = b^2 + c^2 - 2bc\cos A$
$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

θ	0 0	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	U

L'Hospital's Rule

$\frac{0}{0}$ form	$f(x_0) = 0$ *and* $g(x_0) = 0$	$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$
$\frac{\infty}{\infty}$ form	$f(x)$ and $g(x)$ both approach ∞ as $x \to x_0$	

Concavity Test

$f(x)$ is *concave down* on the interval I:	$f''(x) < 0$
$f(x)$ is *concave up* on the interval I:	$f''(x) > 0$

A point c is a **point of inflection** of the function f if f is continuous at c, and:
The graph of f changes from concave up (or down) before c to concave down (or up) after c.

Increasing and Decreasing Functions

f is *increasing* on the interval I:	$f'(x) < 0$
f is *decreasing* on the interval I:	$f'(x) < 0$

Local Extremes

Points where a function f can have an **extreme value** are:

- **Critical points:** Interior points where $f'(x) = 0$ or $f'(x)$ does not exist.
- End points of the domain of f.

	First Derivative Test	Second Derivative Test
$f(c)$ is a *local maximum:*	$f'(x) > 0$ before $x = c$ $f'(x) < 0$ after $x = c$	$f'(c) = 0$ *and* $f''(c) < 0$
$f(c)$ is a *local minimum:*	$f'(x) < 0$ before $x = c$ $f'(x) > 0$ after $x = c$	$f'(c) = 0$ *and* $f''(c) > 0$

General Integral Formulae

The function $F(x)$ is an **antiderivative** of the function $f(x)$ if $F'(x) = f(x)$ for all x in the domain f.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Trigonometric Functions

$\int \sin kx\,dx = -\frac{\cos kx}{k} + C$	$\int \cos kx\,dx = \frac{\sin kx}{k} + C$
$\int \tan kx\,dx = -\frac{\ln\lvert\cos kx\rvert}{k} + C$	$\int \cot kx\,dx = -\frac{\ln\lvert\sin kx\rvert}{k} + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x\tan x\,dx = \sec x + C$	$\int \csc x\cot x\,dx = -\csc x + C$

Rules for Indefinite Integrals

Scalar Multiplication	$\int kf(x)\,dx = k\int f(x)dx$ $\int -f(x)\,dx = -\int f(x)dx$
Sum and Difference of Integrals	$\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)dx$

Rules for Definite Integrals

Upper and lower limits are equal	$\int_a^a f(x)dx = 0$
Swapping lower limits with upper limits	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
Scalar Multiplication	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$ $\int_a^b -f(x)dx = -\int_a^b f(x)dx$
Sum and Difference of Integrals	$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
<u>Comparing two definite integrals:</u> If $f(x) \geq g(x)$ on $[a, b]$,	$\int_a^b f(x)dx \geq \int_a^b g(x)dx$
<u>Combining two definite integrals:</u> If f is continuous along the interval joining a, b and c,	$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Fundamental Theorem of Calculus

If a function of x is the upper limit, and a constant is the lower limit:	$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$
If both upper and lower limits are constants:	$\int_a^b f(x)dx = F(b) - F(a)$

Integration by Substitution

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Integration by Parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Area Between Two Curves

$$A = \int_a^b [f_2(x) - f_1(x)]dx = \int_c^d [g_2(y) - g_1(y)]dy$$

Volume of Solids of Revolution

Revolution about the x-axis	$V = \int_a^b \pi[f(x)]^2dx$
Revolution about the y-axis	$V = \int_c^d \pi[g(y)]^2dy$

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$s_n = \frac{a(1-r^n)}{1-r}$	$s_\infty = \frac{a}{1-r}$

Series converges if $\lvert r\rvert < 1$. Series diverges if $\lvert r\rvert \geq 1$.

Rules on Series

Suppose that $\sum a_n = A$ and $\sum b_n = B$.

Sum Rule	$\sum(a_n + b_n) = A + B$
Difference Rule	$\sum(a_n - b_n) = A - B$
Constant Multiple Rule	$\sum(ka_n) = kA$

Ratio Test

Suppose that $\sum a_n$ is a series and let $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \rho$.

Series converges	$\rho < 1$
Series diverges	$\rho > 1$
No conclusion	$\rho = 1$

Power Series

About $x = 0$:	$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots = \sum_{n=0}^{\infty} c_nx^n$

	$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
About $x = a$:	$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$

Radius of Convergence

To determine the behavior of a series at infinity, apply ratio test to the series with its nth term. To do so, obtain $\left|\frac{u_{n+1}}{u_n}\right|$, *and find its limit when* $n \to \infty$.

- a is the centre of the power series, and h is known as the *radius of convergence* of the series.

$h = 0$	Series converges only at $x = a$, diverges for all $x \neq a$.
$h > 0$	Series converges for all $\lvert x-a\rvert < h$. Series diverges for all $x > a + h$ and $x < a - h$. Series may or may not diverge for $x = a + h$ or $x = a - h$.
h *is infinite*	Series converges for all x.

Given a series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, the integrated and differentiated series also converges for all $|x-a| < h$.

Taylor Series of *f* at *a*

$$f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor Series of various functions

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$	$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

Vectors in Three-Dimensional Space

Magnitude of a vector $\boldsymbol{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$	$\lVert\boldsymbol{v_1}\rVert = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2}$ $\lVert c\boldsymbol{v_1}\rVert = \lvert c\rvert\lVert\boldsymbol{v_1}\rVert$ $\boldsymbol{v_1} \cdot \boldsymbol{v_1} = \lVert\boldsymbol{v_1}\rVert^2$
Angle between two vectors $\boldsymbol{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$	$\cos\theta = \frac{\boldsymbol{v_1} \cdot \boldsymbol{v_2}}{\lVert\boldsymbol{v_1}\rVert\lVert\boldsymbol{v_2}\rVert}$

The **unit vector** of **w** has the same direction as **w** and is defined as $\frac{1}{\|\boldsymbol{w}\|}\boldsymbol{w}$.

Partial Differentiation

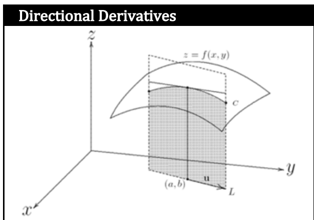
For a function $z = f(x,y)$,

Treat y as a constant and differentiate $f(x,y)$ with respect to x only.	$f_x = \frac{\partial z}{\partial x}$
Treat x as a constant and differentiate $f(x,y)$ with respect to y only.	$f_y = \frac{\partial z}{\partial y}$
For most functions in practice, this holds.	$f_{xy}(a,b) = f_{yx}(a,b)$

Chain Rule as applied to Partial Differentiation

For a function $w = f(x,y,z)$, where $x = x(t), y = y(t), z = z(t)$.	$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$
For a function $w = f(x,y,z)$, where $x = x(s,t), y = y(s,t), z = z(s,t)$	$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$ $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$

Directional Derivatives



The **directional derivative** of $f(x,y)$ at the point (a,b) in the direction of a unit vector $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is

$$D_u f(a,b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

$D_u f(a,b)$ gives the gradient of the tangent line to the curve C at the point (a,b).

In relation to the partial derivatives of $f(x,y)$ at the point (a,b),

$$D_u f(a,b) = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$$

Suppose a point $\overline{\boldsymbol{p}}$ moves a small distance Δt along the direction of the unit vector $\boldsymbol{u}$ to a new position $\overline{\boldsymbol{q}}$. Then the increment in f,

$$\Delta f \approx [D_u f(\overline{\boldsymbol{p}})]\Delta t$$

Gradient Vector

The **gradient vector** of $f(x,y)$ point (a,b) is the vector

$$\nabla f(a,b) = f_x(a,b)\boldsymbol{i} + f_y(a,b)\boldsymbol{j}$$

In relation to the directional derivative of $f(x,y)$ at the point (a,b),

$$D_u f(a,b) = \nabla f(a,b) \cdot \boldsymbol{u}$$

The function f increases most rapidly in the direction $\nabla f(a,b)$, and decreases most rapidly in the direction $-\nabla f(a,b)$.

Finding Critical Points in Three-Dimensional Space

Given a function $f(x,y)$, the point (a,b) is a **critical point** if one of the conditions hold:

- $f_x(a,b) = 0$ and $f_y(a,b) = 0$; or
- $f_x(a,b)$ or $f_y(a,b)$ does not exist.

Classifying Critical Points using Second Derivative Test

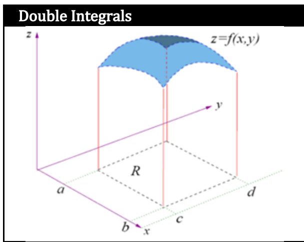
Having obtained a critical point (a,b), you must first compute

$$D = f_{xx}f_{yy} - {f_{xy}}^2$$

Local Minimum	$D > 0$ and $f_{xx}(a,b) > 0$
Local Maximum	$D > 0$ and $f_{xx}(a,b) < 0$
Saddle Point	$D < 0$
No Conclusion	$D = 0$

"Write lots of formulae into your cheat sheet, and if you don't know how to do a question, just scribble some random formula. We will see that you know what topic this is and give you some pity marks." – *Prof. Leung Pui Fai*

Double Integrals



$\iint_R f(x,y)dA$ is equal to the volume under the surface $z = f(x,y)$ and above the xy-plane over the region R.

Rules for Double Integrals

Sum and Difference of Double Integrals	$\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$
Scalar Multiplication of Double Integrals	$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$
<u>Comparing two double integrals:</u> If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$,	$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$
When you see a 1 inside the double integral:	$\iint_R dA = \iint_R 1\, dA = A(R)$, the area of R
<u>Combining two double integrals:</u>	$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$

Evaluation of Double Integrals

Expressing the region R as a **Type A Boundary**:

$$R: g_1(x) \leq y \leq g_2(x), a \leq x \leq b$$

Now, express the double integral as an **iterated integral**:

$$\iint_R f(x,y)dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx$$

Integrate $f(x,y)$ with respect to y first, treating x as a constant. Then, integrate the result with respect to x.

Expressing the region R as a **Type B Boundary**:

$$R: h_1(y) \leq x \leq h_2(y), c \leq y \leq d$$

Now, express the double integral as an **iterated integral**:

$$\iint_R f(x,y)dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy$$

Expressing the region R as a **Polar Rectangle**:

$$R: a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Now, express the double integral as an **iterated integral** by substituting $x = r\cos\theta, y = r\sin\theta$, and $dA = r\,dr\,d\theta$,

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Surface Area

Given a function $z = f(x,y)$, the surface area S that projects onto R is

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Separable First-Order Differential Equations

$$M(x) - N(y)\frac{dy}{dx} = 0$$

Simply separate the variables and integrate with respect to x.

$$M(x)dx = N(y)dy$$

$$\int M(x)\, dx = \int N(y)\, dy + c$$

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

We convert it into a separable form by setting $v = \frac{y}{x}$ $(\Rightarrow y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx})$.

Linear First-Order Ordinary Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

First find the **integrating factor**, and you do not have to have a constant of integration.

$$R(x) = e^{\int P(x)\, dx}$$

Next, write down the **general solution**, and you have to include a constant of integration.

$$y = \frac{1}{R(x)}\int R(x)Q(x)\, dx$$

Bernoulli Equations

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Convert this into a linear form by using a substitution to eliminate all y.

$$z = y^{1-n}$$

Solve the above using the algorithm for a linear first-order differential equation.

Radioactive Decay

Amount of substance	$\frac{d\mho}{dt} = -k\mho$	$\mho = \mho_0 e^{-kt}$
Half-Life	$\frac{\mho}{2} = \mho_0 e^{-kt_{\frac{1}{2}}}$	Half-life, $t_{\frac{1}{2}} = \frac{\ln 2}{k}$

Population Growth

Malthus Model	$\frac{dN}{dt} = kN$	$N = N_0 e^{kt}$
Logistic Model	$\frac{dN}{dt} = BN - sN^2$	$N = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1)e^{-Bt}}$
	$N_\infty = \frac{B}{s}$	Point of inflection: $N = \frac{B}{2s}$