ST2131 Probability
Help Sheet for Final Examinations

Given r experiments:	
Number of Outcomes (each experiment i has n_i possible outcomes)	$n_1 n_2 \cdots n_r$
Given n distinct objects:	
Number of Permutations	$n!$
Number of Combinations (of r items that we choose from the objects)	$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$
Number of Divisions (of respective sizes $n_1, n_2, \cdots, n_r$)	$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\,n_2! \cdots n_r!}$
Given n objects, not necessarily distinct:	
Number of Permutations (if among the objects, $n_1, n_2, \cdots, n_r$ of them are alike)	$\frac{n!}{n_1!\,n_2! \cdots n_r!}$

Probability Formulae	
Null Event $\emptyset$	$P(\emptyset) = 0$
Union (of a finite sequence of mutually exclusive events)	$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$
Complement Event A^c	$P(A^c) = 1 - P(A)$
Subset A of an event B, ie. $A \subset B$	$P(A) + P(BA^c) = P(B)$
	$P(A) \leq P(B)$
Union (of any two events)	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
For a sample space $S = \{s_1, s_2, \cdots, s_n\}$ with a finite number of **equally likely outcomes**	$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Number\ of\ outcomes\ in\ S}$
	$P(s_i) = \frac{1}{\lvert S \rvert}$

Inclusion-Exclusion Principle

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i_1 < i_2 \leq n} P(A_{i_1} A_{i_2}) + \cdots + (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} P(A_{i_1} \cdots A_{i_r}) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n)$$

The notation $\sum_{1 \leq i_1 < \cdots < i_r \leq n} P(A_{i_1} \cdots A_{i_r})$represents the sum of the probabilities of all possible intersections of $A_{i_1} \cdots A_{i_r}$. For example, when n=4,

$$\sum_{1 \leq i_1 < i_2 \leq 4} P(A_{i_1} A_{i_2}) = P(A_1A_2) + P(A_1A_3) + P(A_1A_4) + P(A_2A_3) + P(A_2A_4) + P(A_3A_4)$$

Continuity of Probability

For a sequence of events $\{E_n\}, n \geq 1$,		$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$
Increasing Sequence	$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots$	$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$
Decreasing Sequence	$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots$	$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$

Partition of a Sample Space

The events $A_1, A_2, \cdots, A_n$ are said to partition the sample space S if they are both:

Mutually Exclusive,	$A_iA_j = \emptyset$ when $i \neq j$	**and Collectively Exhaustive,**	$\bigcup_{i=1}^{n} A_i = S$

Baye's Laws	
Conditional Probability	$P(B\mid A) = \frac{P(BA)}{P(A)}$
Baye's First Law	$P(B) = P(B\mid A_1)P(A_1) + \cdots + P(B\mid A_n)P(A_n)$
Baye's Second Law	$P(A_i\mid B) = \frac{P(B\mid A_i)P(A_i)}{P(B\mid A_1)P(A_1) + \cdots + P(B\mid A_n)P(A_n)}$
Independent Events	
Two Independent Events	$P(AB) = P(A)P(B)$
Conditional Probability	$P(A\mid B) = P(A)$

If A and B are independent, then A^c and B, A and B^c, A^c and B^c are also independent.

Events $A_1, A_2, \cdots, A_n$ are said to be **independent** if, for every sub-collection of events $A_{i_1}, A_{i_2}, \cdots, A_{i_r}$, $P(A_{i_1}A_{i_2} \cdots A_{i_r}) = P(A_{i_1}) \cdots P(A_{i_r})$

Probability Mass Functions (PMF)

$$p_X(x) = \begin{cases} P(X = x), & x = x_1, x_2, \cdots \\ 0, & \text{otherwise} \end{cases}$$

1. $p_X(x_i) \geq 0$ for $i = 1, 2, \cdots$
2. $p_X(x) \geq 0$ for all other values of x.
3. $\sum_{i=1}^{\infty} p_X(x_i) = 1$
4. The values of x_i for which the PMF is strictly positive corresponds with the support of X.

$\boldsymbol{P(a \leq X \leq b)}$	For $B = [a, b], P(X \in B) = \sum_{x \in B} p_X(x)$

Joint Probability Mass Functions

$p_{X,Y}(x, y) = P(X = x, Y = y)$	$p(x_1, x_2, \cdots, x_n) = P(X_1 = x_1, \cdots, X_n = x_n)$
$\boldsymbol{P(a \leq X \leq b)}$	$\sum_{a_1 < x \leq a_2} \sum_{b_1 < y \leq b_2} p_{X,Y}(x, y)$
$\boldsymbol{P(X \leq a, Y \leq b)}$	$\sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x, y)$
$\boldsymbol{P(X > a, Y > b)}$	$\sum_{x > a} \sum_{y > b} p_{X,Y}(x, y)$

Conditional Probability Mass Functions

$$p_{X\mid Y}(x\mid y) := P(X = x\mid Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Probability Density Functions (PDF)

X is a continuous random variable if there exists a non-negative function f_X defined for all real $x \in \mathbb{R}$, having the property that for any set B of real numbers,

$$P(X \in B) = \int_B f_X(x)\, dx$$

$\boldsymbol{P(a \leq X \leq b)}$	For $B = [a, b], P(X \in B) = \int_a^b f_x(x)\, dx$
$\boldsymbol{P(x - \frac{\epsilon}{2} \leq X \leq x + \frac{\epsilon}{2})}$	Considering an interval of length ϵ, $\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} f_x(x)\, dx \approx \epsilon f(x)$

Joint Probability Density Functions

$$f_{X,Y}(x, y) = P(X = x, Y = y)$$

$\boldsymbol{P(a \leq X \leq b)}$	$\int_A \int_B f_{X,Y}(x, y)\, dy\, dx = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$
$\boldsymbol{P(X \leq a, Y \leq b)}$	$F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx$

Transformations of Joint Random Variables

Suppose that $U = g(X, Y)$ and $V = g(X, Y)$, we can assume that X and Y are continuously distributed, and that you can uniquely solve X and Y in terms of U and V. Let $x = a(u, v)$, and $y = b(u, v)$. Then the joint PDF of U and V is given by

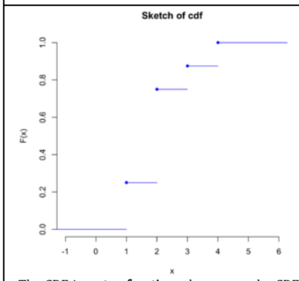
$$f_{U,V}(u, v) = f_{X,Y}(x, y)\lvert J(x, y)\rvert^{-1} \text{ where } J(x, y) := \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x} \neq 0.$$

Conditional Probability Density Functions

$$f_{X\mid Y}(x\mid y) := \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$\boldsymbol{P(X \in A \mid Y = y)}$	$\int_A f_{X\mid Y}(x\mid y)\, dx$

Cumulative Distributive Functions (DF)

Discrete Random Variables	Continuous Random Variables
$F_X(x) = P(X \leq x)$	
 The CDF is a **step function** where at x_i, the CDF takes a jump of size $p(x_i)$.	1. $P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f_x(x)\, dx = 1$ 2. $P(X = a) = F_x(a) - P(X < a) = \lim_{n\to\infty} F(b - \frac{1}{n})$. As a result, if the CDF is continuous at a, $P(X = a) = 0$. **Monotone Transformation Theorem:** Suppose that $g(X)$ is a strictly **monotonic** (increasing or decreasing) and **differentiable** function of x, then for random variable $Y = g(X)$, $f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left\lvert \frac{d}{dy} g^{-1}(y) \right\rvert, & if\ y = g(x)\ for\ some\ x \\ 0, & if\ y \neq g(x)\ for\ all\ x \end{cases}$

1. **Non-decreasing:** If $a < b$, then $F_X(a) \leq F_X(b)$.
2. **Limits:** $\lim_{b\to\infty} F_X(b) = 1$, $\lim_{b\to-\infty} F_X(b) = 0$
3. **Right-continuous:** For any $b \in \mathbb{R}$, $\lim_{x\to b^+} F_X(x) = F_X(b)$.

$\boldsymbol{P(a \leq X \leq b)}$	$For\ B = [a, b], P(X \in B) = F_x(b) - F_x(a)$

Joint Distribution Functions (DF)

Discrete Random Variables	Continuous Random Variables
$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$	$F_{X,Y}(a, b) = P(X \leq a, Y \leq b)$ $= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx$
$F(x_1, x_2, \cdots, x_n) = P(X_1 \leq x_1, \cdots, X_n \leq x_n)$	

Conditional Distribution Functions (DF)

Discrete Random Variables	Continuous Random Variables
$F_{X\mid Y}(x\mid y) = P(X \leq x\mid Y = y) = \sum_{a \leq x} p_{X\mid Y}(a\mid y)$	$F_{X\mid Y}(x\mid y) = P(X \leq x \mid Y = y) = \int_{-\infty}^{x} F_{X\mid Y}(t\mid y)\, dt$

Interconversions between Probability Functions

PMF $\xrightarrow{\text{summation}}$ **CDF**	$F_x(x) = \sum_{y \leq x} p_x(y)$
CDF $\xrightarrow{\text{subtraction}}$ **PMF**	$p_x(x) = F_x(x) - F_x(x - 1)$
PDF $\xrightarrow{\text{integrate}}$ **CDF**	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
CDF $\xrightarrow{\text{differentiate}}$ **PDF**	$F'_x(x) = f_x(x)$
Joint CDF $\xrightarrow{\text{take limit}}$ **Marginal CDF**	$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x, y)$
	$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x, y)$
	$F(x_1, x_2) = \lim_{x_3\to\infty} F(x_1, x_2, x_3)$
Joint PDF $\xrightarrow{\text{double integral}}$ **Joint CDF**	$F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx$
Joint PMF $\xrightarrow{\text{summation}}$ **Marginal PMF**	$p_X(x) = P(X = x) = \sum_{y \in \mathbb{R}} p_{X,Y}(x, y)$
	$p_Y(y) = P(Y = y) = \sum_{x \in \mathbb{R}} p_{X,Y}(x, y)$
Joint PDF $\xrightarrow{\text{integral}}$ **Marginal PDF**	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$
Joint CDF $\xrightarrow{\text{2nd derivative}}$ **Joint PDF**	$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$
Condtional PMF $\rightarrow$ Joint PMF	$p_{X,Y}(x, y) = p_{X\mid Y}(x\mid y)p_Y(y) = p_{Y\mid X}(y\mid x)p_X(x)$
Condtional PMF $\rightarrow$ Marginal PMF	$p_{X\mid Y}(x\mid y) = p_X(x)$ if X and Y are independent
Condtional PDF $\rightarrow$ Marginal PDF	$f_{X\mid Y}(x\mid y) = f_X(x)$ if X and Y are independent

Equivalent Statements for Independent Random Variables

X and Y are independent	$X_1, \cdots, X_n$ are independent
$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$	
$p_{X,Y}(x, y) = p_X(x)p_Y(y)$	$p(x_1, x_2, \cdots, x_n) = p_{X_1}(x_1)p_{X_2}(x_2) \cdots p_{X_n}(x_n)$
$F_{X,Y}(x, y) = F_X(x)F_Y(y)$	$F(x_1, x_2, \cdots, x_n) = F_{X_1}(x_1)F_{X_2}(x_2) \cdots F_{X_n}(x_n)$

In addition, if the variables involved are **continuous**, X and Y are independent if and only if there exist functions $g, h: \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have $f_{X,Y}(x, y) = g(x)h(y)$.

Expectation	**Discrete**	**Continuous**
$\boldsymbol{E(X)}$	$\sum_x x \cdot p_x(x)$	$\int_{-\infty}^{\infty} x f_X(x)\, dx$
$\boldsymbol{E[g(X)]}$	$\sum_i g(x_i) \cdot p_x(x_i) = \sum_x g(x) \cdot p_x(x)$	$\int_{-\infty}^{\infty} g(x) f_X(x)\, dx$
$\boldsymbol{E(aX + b)}$	$aE(X) + b$	
Expectations of Jointly Distributed Variables		
$\boldsymbol{E(X + Y)}$	$E(X) + E(Y)$	
$\boldsymbol{E(\sum_{i=1}^{n} X_i)}$	$\sum_{i=1}^{n} E(X_i)$	
$\boldsymbol{E[g(X, Y)]}$	$\sum_y \sum_x g(x, y)p_{X,Y}(x, y)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$
$\boldsymbol{E[g(X_1, X_2, \cdots, X_n)]}$	$\sum_{x_n} \cdots \sum_{x_2} \sum_{x_1} g(x_1, x_2, \cdots, x_n)p(x_1, x_2, \cdots, x_n)$	
$\boldsymbol{E[g(X, Y) + h(X, Y)]}$	$E[g(X, Y)] + E[h(X, Y)]$	
$\boldsymbol{E[g(X) + h(Y)]}$	$E[g(X)] + E[h(Y)]$	
Conditional Expectation		
$\boldsymbol{E[X\mid Y = y]}$	$\sum_x p_{X\mid Y}(x\mid y)$	$\int_{-\infty}^{\infty} x f_{X\mid Y}(x\mid y)\, dx$

$E[X\mid Y]$	A function of a random variable Y whose value at $Y=y$ is $E[X\mid Y=y]$. Simply replace y in the formula for $E[X\mid Y=y]$ with Y.	
$E[g(X)\mid Y=y]$	$\sum_x g(x)p_{X\mid Y}(x\mid y)$	$\int_{-\infty}^{\infty} g(x)\, f_{X\mid Y}(x\mid y)\, dx$
$E[\sum_{k=1}^n X_k\mid Y=y]$	$\sum_{k=1}^n E[X_k\mid Y=y]$	
$E[g(X)h(Y)\mid Y=y]$	$h(y)E[g(X)\mid Y=y]$	
$E[XY\mid Y=y]$	$yE[X\mid Y=y]$	
Properties of Expected Values		
Tail Sum Formula	For nonnegative integer-valued X, $E(X)=\sum_{k=1}^{\infty}P(X\ge k)=\sum_{k=0}^{\infty}P(X>k)$	For nonnegative continuous X, $E(X)=\int_0^{\infty}P(X>x)\,dx = \int_0^{\infty}P(X\ge x)\,dx$
Monotone Property	If $X\le Y$, then $E(X)\le E(Y)$.	
Independence	For any function $g,h:\mathbb{R}\to\mathbb{R}$, we have: $E[g(X)h(Y)]=E[g(X)]E[h(Y)]$	
Nonnegative Condition	If $g(x,y)\ge 0$ whenever $p_{X,Y}(x,y)>0$, then $E[g(X,Y)]\ge 0$.	If $g(x,y)\ge 0$ whenever $f_{X,Y}(x,y)>$ 0, then $E[g(X,Y)]\ge 0$.
Iterated Expectation Formula		
$E[X]=E[E(X\mid Y)]$	$\sum_y E(X\mid Y=y)P(Y=y)$ if Y is discrete	$\int_{-\infty}^{\infty}E(X\mid Y=y)\,f_Y(y)dy$ if Y is continuous
$P(A) = E[I_A] = E[E(I_A\mid Y)]$	$\sum_y E(I_A\mid Y=y)P(Y=y) = \sum_y P(A\mid Y=y)P(Y=y)$	$\int_{-\infty}^{\infty}E(I_A\mid Y=y)\,f_Y(y)\,dy = \int_{-\infty}^{\infty}P(A\mid Y=y)\,f_Y(y)\,dy$

Variance	Discrete	Continuous
$var(X)$	$E(X-\mu)^2=E(X^2)-[E(X)]^2$	$\int_{-\infty}^{\infty}(x-E(X))^2 f_x(x)\,dx$
$var(aX+b)$	$a^2\ var(X)$	
Variance of Jointly Distrbuted Variables		
$var(\sum_{i=1}^n X_i)$	If X_is are not independent, $\sum_{i=1}^n var(X_i)+2\sum_{i<j}cov(X_i,X_j)$ If X_is are independent, $var(\sum_{i=1}^n X_i)=\sum_{i=1}^n var(X_i)$	
Properties of Variance		
Nonnegative	$var(X)\ge 0$	
Consequence	$E(X^2)\ge [E(X)]^2\ge 0$	
Degenerate Random Variables	$var(X)=0$ if and only if X is a **degenerate** random variable, ie. takes only one value, its mean.	

Covariance	Discrete	Continuous
$cov(X,Y)$	$E(X-\mu_X)(Y-\mu_Y)=E(XY)-E(X)E(Y)$	
$cov(X,X)$	$var(X)$	
$cov(Y,X)$	$cov(X,Y)$	
$cov(aX,bY)$	$ab\times cov(X,Y)$	
$cov(\sum_{i=1}^n a_iX_i,\sum_{j=1}^m b_jY_j)$	$\sum_{i=1}^n\sum_{j=1}^m a_ib_j\,cov(X_i,Y_j)$	
Independent Random Variables	$cov(X,Y)=0$	

Standard Deviation	Discrete	Continuous
σ_x	$\sqrt{var(X)}$	
$\sigma(aX+b)$	$\sigma(aX+b)=\lvert a\rvert\sigma(X)$	

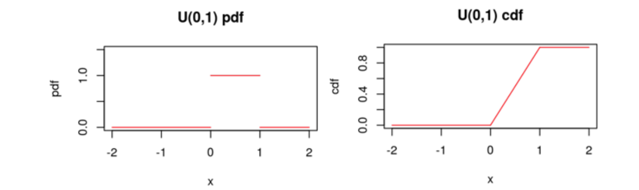
Discrete Random Variable		$P(X=k)$	$E(X)$	$var(X)$
Bernoulli $Be(p)$	$X=\begin{cases}1, & if\ success\\ 0, & if\ failure\end{cases}$	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$
Binomial $Bin(n,p)$	X is the number of successes in n Bernoulli(p) trials.	$\binom{n}{k}p^kq^{n-k}$	np	$np(1-p)$
Geometric $Geom(p)$	X is the number of Bernoulli(p) trials required to obtain the 1st success.	pq^{k-1}	$\frac{1}{p}$	$\frac{1-p}{p^2}$
	X' is the number of failures in the Bernoulli(p) trials in order to obtain 1st success, ie. $X=X'+1$.	pq^k	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
Negative Binomial $NB(r,p)$	X is the number of Bernoulli(p) trials required to obtain r successes. Note: $Geom(p)=NB(1,p)$.	$\binom{k-1}{r-1}p^rq^{k-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson $Poisson(\lambda)$	Eg. Number of misprints on a page, number of car accidents in a day.	$\frac{e^{-\lambda}\lambda^k}{k!}$	λ	λ
Hyper-geometric $H(n,N,m)$	Eg. Set of N balls, m red, $(N-m)$ blue. Choose n balls without replacement. X is the number of red balls in our sample.	$\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$	$\frac{nm}{N}$	$\frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1-\frac{nm}{N}\right]$

Sum of Independent Discrete Random Variables X and Y

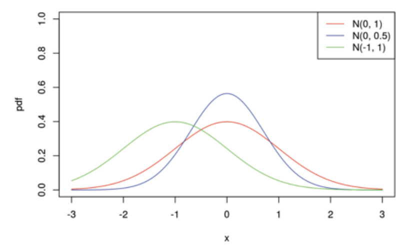
X	Y	X+Y
$X\sim Poisson(\lambda)$	$Y\sim Poisson(\mu)$	$X+Y\sim Poisson(\lambda+\mu)$
$X\sim Bin(n,p)$	$Y\sim Bin(m,p)$	$X+Y\sim Bin(n+m,p)$
$X\sim Geom(p)$	$Y\sim Geom(p)$	$X+Y\sim NB(2,p)$

Continuous	$f_X(x)$	$F_X(x)$	$E(X)$	$var(X)$
Uniform $X\sim U(a,b)$	$\begin{cases}\frac{1}{b-a}, & a<x<b\\ 0, & otherwise\end{cases}$	$\begin{cases}0, & if\ x<a\\ \frac{(x-a)}{(b-a)}, & if\ a\le x<b\\ 1, & if\ b\le x\end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal $X\sim N(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{(2\sigma^2)}}$		μ	σ^2
Standard Normal $Z\sim N(0,1)$		$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}u^2}du$	0	1
Exponential $X\sim Exp(\lambda)$	$\begin{cases}\lambda e^{-\lambda x}, & if\ x\ge 0\\ 0, & if\ x<0\end{cases}$	$\begin{cases}0, & if\ x\le 0\\ 1-e^{-\lambda x}, & if\ x>0\end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma $X\sim\Gamma(\alpha,\beta)$	$\begin{cases}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, & if\ x\ge 0\\ 0, & if\ x<0\end{cases}$ $\Gamma(\alpha)=\int_0^{\infty}e^{-y}y^{\alpha-1}dy$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$

Uniform Distributions, $X\sim U(a,b)$



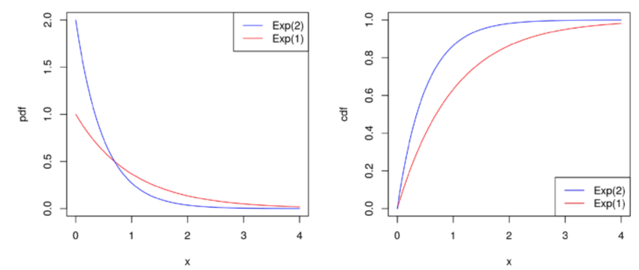
Normal Distributions, $X\sim N(\mu,\sigma^2)$



Moment Generating Function
$M_X(t)=exp(\mu t+\frac{\sigma^2t^2}{2})$
$M_{X+Y}(t) = exp([\mu_X+\mu_Y]t+\frac{[\sigma_X^2+\sigma_Y^2]t^2}{2})$
If $X\sim N(\mu,\sigma^2)$, then:
$\frac{X-\mu}{\sigma}\sim N(0,1)$
$X+Y\sim N(\mu_X+\mu_Y,\sigma_X^2+\sigma_Y^2)$
$\sum_{i=1}^n X_i\sim(\sum_{i=1}^n\mu_i,\sum_{i=1}^n\sigma_i^2)$
If $X\sim N(0,1)$, then:
$-X\sim N(0,1)$
$aX+b\sim N(b,a^2)$

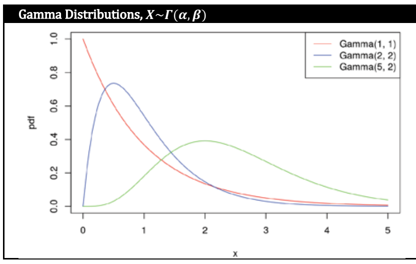
Properties of Normal Distributions	
Normalization	$P(a<X\le b)=P(\frac{a-\mu}{\sigma}<Z\le\frac{b-\mu}{\sigma})=\Phi(\frac{b-\mu}{\sigma})-\Phi(\frac{a-\mu}{\sigma})$
Half a graph	$P(Z\ge 0)=P(Z\le 0)=0.5$
Complement	$P(Z\le x)=1-P(Z>x)$ for $-\infty<x<\infty$
Symmetry	$P(Z\le -x)=P(Z\ge x)$ for $-\infty<x<\infty$
Quantiles	Suppose that $X\sim N(\mu,\sigma^2)$. Then if $P(Z\le z_q)=q$, then $P(X\le\sigma z_q+\mu)=q$. z_q is the **q-th quantile of Z**, and $\sigma z_q+\mu$ is the **q-th quantile of X**.

Exponential Distributions, $X\sim Exp(\lambda)$



Memoryless Property	$P(X>s+t\mid X>s)=P(X>t)$ for $s,t>0$

Gamma Distributions, $X\sim\Gamma(\alpha,\beta)$



α	$\Gamma(\alpha)$
1	$\int_0^{\infty}e^{-y}\,dy=1$
$\frac{1}{2}$	$\int_0^{\infty}e^{-y}y^{\frac{1}{2}-1}\,dy=\sqrt{\pi}$
n	$(n-1)\Gamma(n-1)=(n-1)!$
By Integration by Parts, $\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)$	
Note: $\Gamma(1,\lambda)=Exp(\lambda)$	

Moment Generating Function	Discrete	Continuous
$M_X(t)=E[e^{tX}]$	$\sum_x e^{tX}\cdot p_x(x)$	$\int_{-\infty}^{\infty}e^{tX}f_x(x)\,dx$
$E[X^n]$	$M_X{}^{n}(0):=\frac{d^n}{dt^n}M_X(t)\rvert_{t=0}$	
Independent Random Variables	If X and Y are independent, then $M_{X+Y}(t)=M_X(t)M_Y(t)$	
Test for Same Distribution	Suppose there exists $h>0$ such that $M_X(t)=M_Y(t),\forall t\in(-h,h)$. Then X and Y have the same distribution, ie. $F_x=F_y$; or $f_x=f_y$.	

Limit Theorems		
Let $X_1,X_2,\cdots$ be a sequence of **independent** and **identically distributed** random variables.		
Strong Law of Large Numbers If each has a finite mean of $\mu=E(X_i)$,	$\frac{X_1+X_2+\cdots+X_n}{n}\to\mu$ as $n\to\infty$	$P(\{\lim_{n\to\infty}\frac{X_1+X_2+\cdots+X_n}{n}=\mu\}) = 1$
Central Limit Theorem If each as a mean μ and variance σ^2,	$\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma\sqrt{n}}$ *tends to the* **standard normal** *as* $n\to\infty$ $\lim_{n\to\infty}P(\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma\sqrt{n}}\le x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}u^2}du$	
Normal Approximation $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$	$P(a\le\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma\sqrt{n}}\le b)=P(\frac{a-\mu}{\sigma/\sqrt{n}}\le\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\le\frac{b-\mu}{\sigma/\sqrt{n}})\approx\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}u^2}du$	
Normal Approximation to Binomial	Given a binomial distribution $X\sim Bin(n,p)$, the CLT yields that $\frac{X-np}{\sqrt{npq}}\approx Z\sim(0,1), or\ Bin(n,p)\approx N(np,npq)$ The above approximation is generally good for $np(1-p)\ge 10$. It is further improved if we incorporate a **continuity correction**:	
Continuity Correction		
$P(X=k)$	$P(k-\frac{1}{2}<X<k+\frac{1}{2})$	
$P(X\ge k)$	$P(X\ge k-\frac{1}{2})$	
$P(X\le k)$	$P(X\le k+\frac{1}{2})$	
$P(X>k)$	$P(X\ge k+1)=P(X\ge k+\frac{1}{2})$	
$P(X<k)$	$P(X\le k+1)=P(X\le k-\frac{1}{2})$	