On the Performance of Dynamic Matching Models with Threshold-based Policies *

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1. INTRODUCTION

We consider a dynamic matching model where a bipartite graph determines the compatibility among two set of nodes, usually referred to as "supply" and "demand" nodes. Items arrive to these nodes following a stochastic process and are matched by pairs - and leave the system - according to a matching policy. Items that are not matched queue up until they are matched later on, which entails a holding cost.

Most of the literature in dynamic matching models considers First-Come-First-Serve or Match-the-Longest [1, 5] matching policies and characterizes the steady-state distribution, which is of a product-form. Recent literature, see for instance [2–4], studies bipartite dynamic matching models where a bipartite graph describes the compatibility constrains among customers and servers. In [4] the authors consider the N-shaped bipartite matching model and show that a threshold-based policy optimizes the mean holding cost. The authors also conjecture that an analogous policy may be optimal in a more generalized setting.

In this extended abstract, we aim at analyzing the performance of threshold-based policies for the W-shaped dynamic matching model. On the one hand, we characterize the steady-state of the model for small threshold values and show that this is of a product-form. Moreover, we show that the threshold value is of fundamental importance when optimizing the weighed mean holding cost of the system. On the other hand, for general threshold values, we consider an approximation of the model and study the behavior of its weighted holding cost with respect to the threshold value.

2. MODEL DESCRIPTION

We consider a discrete-time W-shaped dynamic matching model. The model consists of two sets of nodes, $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$, where items in nodes v_1 and v_3 are compatible with items in nodes v_4 and v_5 respectively, and items in node v_2 are compatible with items in any node in V_2. Unmatched items queue up at its corresponding node.

At each time slot, two items arrive simultaneously, one to a node in V_1 and another to a node in V_2. We denote by p_i the probability that an item arrives to node v_i for $i = 1, \ldots, 5$, where $p_1 + p_2 + p_3 = 1$ and $p_4 + p_5 = 1$. Moreover, we assume symmetric arrivals, i.e., $p_1 = p_3$ and $p_4 = p_5$. We denote by $\vec{Q} = (Q_1, \ldots, Q_5)$ the queue length vector in steady-state.

We consider the following threshold-based matching policy with threshold value $T \in \mathbb{N} \cup \{\infty\}$: first match all possible items in v_1 and v_3 with those in v_4 and v_5, respectively. Then, if the number of items in a V_2-node exceeds the threshold value T, these items are matched to items in v_2, up until there are $T-1$ items in both V_2-nodes. When $T = 1$, items in v_2 are immediately matched to any item in a V_2-node. Hence, the system behaves as if there was no threshold.

The Markovian state descriptor of this system is given by the queue length vector at each time slot, after matching all possible items following the threshold-based policy. We denote by $t_{s_1}^{s_2}$ the 1-step transition probability from state s_1 to s_2. Due to the symmetric arrivals and threshold-based policy, the associated Markov chain is symmetric w.r.t. queues v_1 and v_3, and queues v_4 and v_5. We use the latter symmetry to define a partition of the state space to apply lumpability, as defined in [6]. With some abuse of notation, we let $\vec{Q}$ denote the queue length vector in steady-state for the lumped Markov chain. Note that, $Q_1 = 0$ and $Q_2 + Q_3 = Q_4 + Q_5$, therefore $\big(Q_2, Q_3; Q_4\big)$ describes the state of the system.

We consider a linear cost function w.r.t. the queue lengths and symmetric costs for nodes v_1 and v_3 and for nodes v_4 and v_5. We aim at analyzing the weighted mean holding cost of the form $\mathbb{E}\big[C(Q)\big] = \mathbb{E}\big[Q_3\big] + c\mathbb{E}\big[Q_2\big]$. We assume $c \in [0, 1]$; in which case, it has been conjectured in [4] that, for this matching model, threshold-based policies are optimal.

3. EXACT ANALYSIS FOR T=1,2,3

In this section, we assume threshold values $T = 1, 2, 3$. Let us define the path from state x to state y through a sequence of states $(x_1, \ldots, x_n)$ as I_x^y, with $x_1 = x$ and $x_n = y$. We also define the set of all possible paths without cycles from state x to state y by $\mathcal{B}(x, y)$.

In the following, we provide several definitions required to characterize the product-form steady-state distribution of this model. Consider states s_0, s_f and y as well as the path from s_0 to y, $I_{s_0}^y$.

We denote by $p_y^{s_f}(I_{s_0}^y)$ the probability of all the possible paths without cycles from state y to s_f, and where the path from s_0 to y is $I_{s_0}^y$. This is given by

$$p_y^{s_f}(I_{s_0}^y) = \sum_{(x_1,\ldots,x_n)\in\mathcal{B}(y,s_f)} t_{x_1}^{x_2} \prod_{i=2}^{n-1} \delta\left(\frac{t_{x_i}^{x_{i+1}}}{p_{x_i}^{s_f}(I_{s_0}^{x_i})}, I_{s_0}^y\right), \quad (1)$$

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where the function δ is defined as

$$\delta\left(\frac{t_{x_i}^{x_{i+1}}}{p_{x_i}^{s_f}(I_{s_0}^{x_i})}, I_{s_0}^{y}\right) = \begin{cases} \frac{t_{x_i}^{x_{i+1}}}{p_{x_i}^{s_f}(I_{s_0}^{x_i})}, & \text{if } I_{s_0}^{y} \cap I_{y}^{x_i} = \{y\} \\ 1, & \text{if } I_{s_0}^{y} \cap I_{y}^{x_i} \neq \{y\} \end{cases}. \quad (2)$$

Formula (1) represents the probability of all possible paths from y to s_f, omitting the portions of possible paths that intersect with $I_{s_0}^{y}$. I.e., when a possible path from y to s_f reaches a state in $I_{s_0}^{y}$, that path is not expanded further.

We denote by $p_{s_0}^{s_f}$ the probability of all the possible paths without cycles from state s_0 to s_f, such that $t_{s_0}^{s_f} > 0$ and $t_{s_f}^{s_0} > 0$. This is given by

$$p_{s_0}^{s_f} = \sum_{(x_1,\ldots,x_n)\in\mathcal{B}(s_0,s_f)} t_{x_1}^{x_2} \prod_{i=2}^{n-1} \frac{t_{x_i}^{x_{i+1}}}{p_{x_i}^{s_f}(I_{s_0}^{x_i})}.$$

THEOREM 1. *Let s_0 and s_f be any two adjacent states, i.e., $t_{s_0}^{s_f} > 0$ and $t_{s_f}^{s_0} > 0$. Then, the following relation satisfies the balance equations of the system:*

$$p_{s_0}^{s_f}\pi(s_0) = p_{s_f}^{s_0}\pi(s_f).$$

Theorem 1 states that a balance property is satisfied between any two adjacent states.

From Theorem 1 we fully characterize the stationary distribution, for $T = 1, 2, 3$, and observe that it is of product-form. We also obtain explicit expressions of the weighted mean holding costs of this model. The order relation among the weighted mean holding costs depends on the holding weight c and arrival probability p_1, as shown in Figure 1. Observe that both for small values of c and large values of p_1, either $T = 2$ or $T = 3$ are optimal. Indeed, when p_1 is close to $\frac{1}{2}$, $T = 3$ outperforms both $T = 1, 2$.

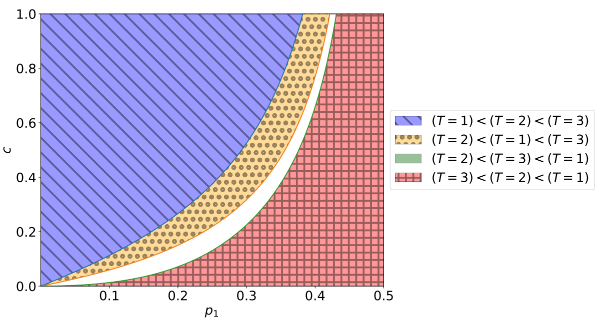

Figure 1: The order relation among the weighted mean holding costs w.r.t. the threshold value T, for $T = 1, 2, 3$.

4. APPROXIMATE ANALYSIS FOR T>3

In this section, we consider a slightly different matching policy: whenever the number of waiting items in nodes v_4 and v_5 is smaller than $T-1$, and a new item pair (v_2, v_4) or (v_2, v_5) arrives, this item pair is immediately matched and leaves the system, i.e., only (v_1, v_5), and (v_3, v_4) pair arrivals queue up.

Under this policy, one can easily solve the balance equations of the lumped Markov chain, because there are fewer transitions among the states:

$$\pi(q_2, q_3; q_4) = \begin{cases} \left(\frac{p_2}{p_1}\right)^{q_2} \pi(0,0;0), & q_4, q_5 \le T-1 \\ \left(\frac{p_1}{p_1+p_2}\right)^{q_4-T+1} \pi(0,0;0), & \text{otherwise} \end{cases}. \quad (3)$$

We obtain explicit expressions of the weighted mean holding cost of this modified model for any value of T. When $p_1 = p_2 = \frac{1}{3}$, the weighted mean holding cost is minimized at threshold values $T \le 3$, for any value of $c \in [0, 1]$. In Figure 2 (left), we observe that as c increases, i.e., the weight of node v_2 increases, a smaller threshold value T is optimal. However, as T increases, node v_2 items are less likely to be matched. Hence, a smaller weight on v_2 is fairer.

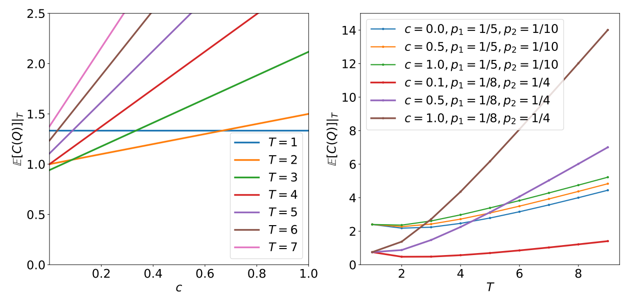

Figure 2: The weighted mean holding costs for $T = 1, \ldots, 7$ with respect to c when $p_1 = \frac{1}{3}$ (left); and the weighted mean holding costs for c and p_1, with respect to T (right).

When $p_1 \neq p_2$, the threshold value T that minimizes the weighted mean holding cost depends on both the value of c and p_1, as shown in Figure 2 (right). Moreover, when $p_1 > p_2$ the weighted mean holding cost increases unbounded as $T \to \infty$, for any $c \in [0, 1]$. The same is true when $p_1 < p_2$ and $c > 0$. However, when $p_1 < p_2$ and $c = 0$, the weighted mean holding cost approaches 0 as $T \to \infty$.

5. FUTURE WORK

In this extended abstract, we fully characterize the stationary distribution of the W-shaped dynamic matching model for thresholds values $T = 1, 2, 3$. Generalizing these results to any threshold value remains an open problem that we aim to tackle in the near future. Moreover, we believe that the balance property that we obtain is also applicable to a broader framework, such as order independent queues.

6. REFERENCES

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