Optimality of Decentralized Lockdown Strategies for the SIRS Model with Vaccinations

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***Abstract*—We consider an epidemic model formed by N elements, where $N < \infty$. More precisely, we analyze the susceptible-infected-recovered-susceptible (SIRS) with vaccinations, an extension of the susceptible-infected-recovered (SIR) model where the susceptible population can be vaccinated and a recovered element is again susceptible after a random time. In our model, susceptible elements can avoid getting the infection with some probability (i.e., with a lockdown probability). We assume that each infected element incurs a cost per unit of time and the susceptible population incurs a cost that is decreasing and linear on the lockdown probability. We investigate a non-cooperative game where each player is an element of the population that can select its lockdown probability and aims to minimize its expected cost. Our first contribution consists of formulating the best response lockdown strategy of one player to the strategy of the rest of the players as a Markov Decision Process, which combined with a simple fixed-point algorithm, allows us to compute a solution to this decentralized setting (i.e., a symmetric Nash equilibrium of the game under analysis). We also formulate the centralized problem of finding the global optimum of this model, i.e., the lockdown strategy that minimizes the cost of the whole population, as a Markov Decision Process. We establish some analytical results on the structure of the solution of the centralized and the decentralized problems. Furthermore, our numerical results show that both strategies have a switching curve. We also conclude that the derived Nash equilibria and the global optimum are very similar, i.e. the decentralized architecture is very close to optimal. Finally, we conclude that the global optimum strategy confines more than the solution of the decentralized system (i.e. than the Nash equilibrium).**

***Index Terms*—SIRS model; Symmetric Nash equilibria; Markov Decision Process; Efficiency;**

I. Introduction

A. Motivation

The susceptible-infected-recovered (SIR) model is one of the simplest and most studied stochastic models. It allows us to model the dynamics of a virus spread in complex networks. The recent COVID-19 pandemic has shown the great importance of carrying out research on this topic. A crucial aspect in this context is to obtain analytical results about how the dynamics of the population change according to the diverse actions that can be selected, for instance, according to how the population gets confined, how it gets vaccinated, or how to allocate the resources of a hospital to provide service to the infected population.

If we consider that each of the elements of the population can make self-interested decisions, then non-cooperative game theory becomes a crucial tool for analyzing the performance of decentralized epidemic models. The solution to a non-cooperative game is known as the Nash equilibrium, which is defined as the set of strategies of the players such that none of them can benefit from a unilateral deviation. A natural question in non-cooperative games is to know whether the Nash equilibria are efficient, i.e., if the performance of Nash equilibria is equal to the strategy that minimizes the cost of the whole population. In the positive case, one can conclude that decentralized and self-interested decisions lead to an optimal performance setting.

B. Contributions

We consider an epidemic model in continuous time with a finite number of elements N. Our model is an extension of the SIR model, in which the susceptible population gets vaccinated at some rate and the recovered population becomes susceptible again after a random time. We consider that susceptible elements can choose the probability of avoiding to get the infection (i.e., its confinement probability). We assume that there is a cost per unit of time associated to each of the elements of the infected population. Moreover, we assume that each element of the susceptible population incurs a cost, which is decreasing and linear on the confinement probability.

We formulate a non-cooperative game where each element of the population is a player. We consider that each player can choose its confinement probability and aims to minimize its expected cost, i.e., the sum of the cost of being infected plus the cost of choosing the lockdown strategy. We formulate a Markov Decision Process to find the best-response strategy of this game and, use a fixed-point algorithm, to compute a symmetric Nash equilibrium of the game.

We also consider the problem of finding the global optimum of this problem, i.e., of determining the lockdown strategy that must follow all the elements of the population so that the expected cost of the population is minimized. The solution to this is found by formulating this problem as a Markov Decision Process.

Our first contribution consists of providing several structural analytical results on both strategies. For instance, we show that, when the number of infected elements is zero, the Nash equilibrium and the global optimum consist of being completely exposed to the infection. Then, we present numerical experiments that show that both strategies under consideration have a switching curve, and we study the shape of the obtained switching curves. We also study the efficiency of Nash equilibria by comparing the performance of the system under the Nash equilibrium and the global optimum. We observe that both strategies are very similar, but the proportion of states where it is optimal to be confined is less for the Nash equilibrium. This implies that the solution to the decentralized setting (i.e. the Nash equilibrium) confines less than the global optimum.

C. Related Work

The SIR model was introduced in [8] and it considers that each of the elements of the population belongs to one of the following three groups: susceptible, infected, and recovered. This model has a very large number of applications and, therefore, it has been studied from different perspectives in the last century; however, due to the COVID-19 epidemic, the interest of researchers in this model has increased a lot recently. In this section, we discuss lockdown strategies in non-cooperative games and optimization problems. A full overview of epidemic models is provided in [12] and recent works about COVID-19 in [4].

Some researchers analyze the existence of lockdown strategies that consist of a Nash equilibrium. For example, the authors in [6] formulate a non-cooperative game in an epidemic model with asymptomatic infections where agents can choose social distancing, vaccination, and testing. Moreover, mean field games have been formulated to study how the population takes social distancing measures [13] and how the players choose the transition rates from states [2].

Other authors have been interested in studying the optimal lockdown policy considering variations of the SIR model: dividing the population into groups of different ages [1], considering contacts between populations of different regions [9], splitting the susceptible population into two groups (that is, the confined and those that are not confined) [10] and including deaths [3]. The authors in [7] show that the SIR model fits the data of COVID-19 during the lockdown. Mean field theory has been used in [15] to determine optimal confinement policies. Other works formulate the problem of finding the optimal lockdown policy using Markov Decision Processes in complex epidemic models [11], [14] and solve the problem numerically. In our work, we present partial analytical results of the optimal lockdown strategy and we compare it with the Nash equilibrium.

D. Organization

The rest of the paper is organized as follows. In Section II, we describe the model under study in this article as well as the optimization problems we investigate. In Section III, we present how we have formulated Markov Decision Processes to deal with the optimization problems under study. In Section IV, we provide the efficiency results of this work. Finally, in Section V, we present the main conclusions of this article.

II. MODEL DESCRIPTION

A. The SIRS model with vaccinations

We consider a population of N homogeneous elements that evolve in continuous time. Each of the elements can be in one of the following states: susceptible (S), infected (I) or recovered (R).

The dynamics of one element is a continuous-time Markov chain that can be described as follows. An element encounters another element at a rate γ. If a susceptible element encounters an infected element, then it becomes infected. An infected element gets recovered at rate ρ. A recovered individual cannot be infected until it gets susceptible again at rate β. Moreover, with a vaccination rate α, a susceptible individual becomes recovered directly without being infected.

We uniformize the continuous-time Markov chain with a uniformization constant $\Omega < (N(\gamma+\rho+\beta+\alpha))^{-1}$ and obtain a discrete-time Markov chain.

In this model, susceptible elements follow a confinement strategy $\pi : \mathbb{N} \to [0,1]$, where $\pi(t)$ indicates the confinement probability of that element at time t. More precisely, when $\pi(t)=1$, susceptible elements are completely exposed to the epidemic at time t, whereas when $\pi(t)=0$ they are protected from taking the infection, i.e., they are confined.

B. Social optimum

Let us consider the dynamics of the SIRS model with vaccinations described previously. We consider that there is a confinement cost that applies to each susceptible element. We assume that the confinement cost of a susceptible element at time t is $c_L-\pi(t)$, where $c_L \geq 1$. We also consider that there is a cost associated to each infected element; more precisely, each infected element leads to a cost of $c_I > 0$ per unit of time.

Thus, if $\bar{M}_S(t)$ and $\bar{M}_I(t)$ denote the number of susceptible and infected elements, respectively, at time t among the population of size N, the global cost of the system when all elements follow the same confinement strategy π is:

$$W(\pi) = \sum_{t=0}^{\infty} \delta^t \left((c_L - \pi(t))\bar{M}_S(t) + c_I \bar{M}_I(t)\right) \tag{GLOBAL-COST}$$

where $\delta \in (0,1)$ is the discount factor.

The social optimum (or global optimum) is the confinement strategy that minimizes the cost along the whole population, that is the confinement strategy π^{opt} that satisfies

$$\pi^{\text{opt}} \in \arg\min_{\pi} W(\pi)$$

In Section III-B, we formulate this problem as a Markov Decision Process and show that it can be solved using the Bellman equations.

C. Symmetric Nash equilibrium

We now formulate a non-cooperative game the model presented in Section II-A. Let us pick one element of the population, that we call it Player 0. We consider that Player 0 can choose his confinement strategy $\pi^0(t)$, where $\pi^0(t) \in [0,1]$ for all $t \in [0,\infty)$. Moreover, the rest of the players, except for Player 0 will follow the confinement strategy π. Let $x_S^{\pi^0,\pi}(t)$ and $x_I^{\pi^0,\pi}(t)$ be the indicator functions for Player 0 being susceptible and infected, respectively, at time t if it follows confinement strategy π^0 and the rest of the players follow strategy π. Then, the expected individual cost of Player 0 is given by

$$C_0(\pi^0,\pi) = \sum_{t=0}^{\infty} \delta^t \left((c_L - \pi^0(t)) x_S^{\pi^0,\pi}(t) + c_I x_I^{\pi^0,\pi}(t) \right),$$

where $\delta \in (0,1)$.

The Best Response of Player 0 is a strategy that minimizes its cost for a given strategy of the rest of the population π, that is,

$$\text{BR}(\pi) = \arg\min_{\pi^0} C(\pi^0,\pi). \tag{BR}$$

A symmetric Nash equilibrium for this game is defined as a fixed point strategy for the Best Response operator, i.e., π^{sne} is a symmetric Nash equilibrium when

$$\pi^{sne} \in \text{BR}(\pi^{sne}).$$

In this work, we apply the following fixed-point algorithm to compute a symmetric Nash equilibrium: $\pi_{k+1} = BR(\pi_k)$, with π_0 an arbitrary initial strategy for all players except for one. Clearly, when this algorithm converges, i.e., when $\pi_{k+1} = \pi_k$, a symmetric Nash equilibrium is found.

In Section III-A, we formulate the problem of finding the Best Response as a Markov Decision Process. This will be used to analyze the efficiency of Nash equilibria, i.e., to compare the performance of symmetric Nash equilibria with the optimal performance.

III. Markov Decision Process formulation

In this section, we address the problem of finding the best-response strategy as well as the global optimum of this model. In Section III-A, we formulate the problem of finding the best response as a Markov Decision Process and, in Section III-B, we find the solution of the global optimum by formulating the problem as a Markov Decision Process.

A. Markov Decision Process formulation for (BR)

To obtain the Best Response strategy of Player 0 to the confinement strategy π, we formulate the problem as a Markov Decision Process.

We consider the state of the system as a triplet (X, M_S, M_I), where $X \in \{S, I, R\}$ is the state of Player 0, and M_S and M_I are the number of susceptible and infected players, respectively, among the rest of the population. It immediately follows that $M_S + M_I \le N-1$ since the population size is N. The action is π^0, which is the confinement probability of Player 0. We assume that there exists $d \in \mathbb{N}$ such that $\pi^0 \in A_0$, where $A_0 = \{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\}$. Following what we said in Section II-C, when Player 0 gets infected, it incurs a cost per unit of time c_I and, when Player 0 selects the strategy π^0, it incurs a cost of $c_L - \pi^0$. If all the players except for Player 0 follow the confinement strategy π, then the best-response of Player 0 to π is the solution of the following Bellman equations:

$$\begin{aligned}
V(S,M_S,M_I) = {} & \min_{\pi^0(M_S,M_I)\in A_0} (c_L - \pi^0(M_S,M_I)) \\
& + \delta(p_I V(I,M_S,M_I) \\
& + p_V V(R,M_S,M_I) \\
& + \mathbb{1}_{\{M_S\ge 1\}} q_V V(S,M_S-1,M_I) \\
& + \mathbb{1}_{\{M_S\le N-2\}} q_S V(S,M_S+1,M_I) \\
& + \mathbb{1}_{\{M_S\ge 1, M_I\le N-2\}} q_I \\
& \quad V(S,M_S-1,M_I+1) \\
& + \mathbb{1}_{\{M_I\ge 1\}} q_R V(S,M_S,M_I-1) \\
& + \hat{p_S} V(S,M_S,M_I)), \qquad \text{(1a)} \\
V(I,M_S,M_I) = {} & c_I \\
& + \delta(\mathbb{1}_{\{M_S\ge 1, M_I\le N-2\}} q_I' \\
& \quad V(I,M_S-1,M_I+1) \\
& + \mathbb{1}_{\{M_S\ge 1\}} q_V V(I,M_S-1,M_I) \\
& + \mathbb{1}_{\{M_S+M_I\le N-2\}} q_S \\
& \quad V(I,M_S+1,M_I) \\
& + p_R V(R,M_S,M_I) \\
& + \mathbb{1}_{\{M_I\ge 1\}} q_R V(I,M_S,M_I-1) \\
& + \hat{p_I} V(I,M_S,M_I)), \qquad \text{(1b)} \\
V(R,M_S,M_I) = {} & \delta(p_S V(S,M_S,M_I) \\
& + \mathbb{1}_{\{M_S+M_I\le N-2\}} q_S \\
& \quad V(R,M_S+1,M_I) \\
& + \mathbb{1}_{\{M_S\ge 1\}} q_V V(R,M_S-1,M_I) \\
& + \mathbb{1}_{\{M_S\ge 1, M_I\le N-2\}} q_I \\
& \quad V(R,M_S-1,M_I+1) \\
& + \mathbb{1}_{\{M_I\ge 1\}} q_R V(R,M_S,M_I-1) \\
& + \hat{p_R} V(R,M_S,M_I)), \qquad \text{(1c)}
\end{aligned}$$

for $M_S, M_I = 0,\dots,N-1,\ M_S+M_I \le N-1$, where

$$\begin{aligned}
p_I &= \Omega\gamma\pi^0(M_S,M_I)M_I/(N-1) \\
p_R &= \Omega\rho \\
p_S &= \Omega\beta \\
p_V &= \Omega\alpha \\
q_I &= \Omega\gamma M_S \pi(M_S,M_I)M_I/(N-1) \\
q_I' &= \Omega\gamma M_S \pi(M_S,M_I)(M_I+1)/(N-1) \\
q_R &= \Omega M_I \rho \\
q_S &= \Omega\beta(N-1-M_S-M_I) \\
q_V &= \Omega\alpha M_S \\
\hat{p_S} &= 1 - p_I - q_I - q_R - q_S - p_V - q_V \\
\hat{p_I} &= 1 - p_R - q_I' - q_R - q_S - q_V \\
\hat{p_R} &= 1 - p_S - q_S - q_I - q_R - q_V
\end{aligned}$$

Let us first comment on the probability transitions related to the movement of Player 0. The transition probability p_I is the probability that, when Player 0 is susceptible, it becomes infected. When Player 0 is susceptible, it is vaccinated with probability p_V, in which case it moves to the recovered state. The transition probability p_R is the probability that Player 0 gets recovered from the infection (i.e., the probability that Player 0 moves from the infected state to the recovered state), and p_S of becoming susceptible when it is in the recovered state.

We now focus on the transition probabilities of the rest of the players. When Player 0 is either susceptible or recovered, the probability that a different player (i.e., one of the rest of the players) is infected is q_I. For the same event, but with Player 0 being infected, the transition probability is q_I'. Both transition probabilities, q_I and q_I', depend on the fixed confinement strategy π followed by all players different from Player 0.

For any state of Player 0, q_R is the probability that a player different from Player 0 gets recovered from an infection (i.e., it moves from the infected state to the recovered state) in the next time-step. Analogously, q_S is the probability that a player that is not Player 0 becomes susceptible after being recovered, and q_V is the probability that a player different to Player 0 gets vaccinated (i.e., it moves from the susceptible state to the recovered state).

Finally, $\hat{p_S}$ (resp. $\hat{p_I}$ and $\hat{p_R}$) is the probability that, when Player 0 is susceptible (resp. infected and recovered), none of the players (Player 0 or the rest of the players) change its state.

For any (M_S, M_I), let π^{BR} be the solution of the Bellman equations defined in (1). Thus, for M_S and M_I positives such that $M_s, M_I \geq 0$, and $M_S + M_I \leq N-1$,, we have that:

$$
\begin{aligned}
\pi^{BR}(M_S, M_I) = \underset{\pi^0(M_S,M_I)\in A_0}{\arg\min} \; & (c_L - \pi^0(M_S, M_I)) \\
& + \delta(p_I V(I, M_S, M_I) \\
& + p_V V(R, M_S, M_I) \\
& + \mathbb{1}_{\{M_S\geq 1\}} q_V V(S, M_S-1, M_I) \\
& + \mathbb{1}_{\{M_S\leq N-2\}} q_S V(S, M_S+1, M_I) \\
& + \mathbb{1}_{\{M_S\geq 1, M_I\leq N-2\}} q_I \\
& \quad V(S, M_S-1, M_I+1) \\
& + \mathbb{1}_{\{M_I\geq 1\}} q_R V(S, M_S, M_I-1) \\
& + \hat{p_S} V(S, M_S, M_I)).
\end{aligned} \tag{2}
$$

From the equations (1), we can extract the following properties of a strategy that solves the Bellman equations, or in other words, the Best Response strategy when the other $N-1$ players follow strategy π.

Proposition 1. *Let π^{BR} be the solution to equations* (1). *Then, for any $M_S = 0, 1, \dots N-1$,*

$$\pi^{BR}(M_S, 0) = 1.$$

Proof. Consider the equation (1a). If $M_I = 0$, then $p_I = 0$, which is the only transition probability in which the optimization term π^0 arises. Thus, we have for $M_S = 0, 1, \dots, N-1$, that

$$
\begin{aligned}
\pi^{BR}(M_S, 0) = \underset{\pi^0(M_S,0)\in A_0}{\arg\min} \; & (c_L - \pi^0(M_S, 0)) \\
& + \delta(p_V V(R, M_S, 0) \\
& + \mathbb{1}_{\{M_S\geq 1\}} q_V V(S, M_S-1, 0) \\
& + \mathbb{1}_{\{M_S\leq N-2\}} q_S V(S, M_S+1, 0) \\
& + \mathbb{1}_{\{M_S\geq 1\}} q_I V(S, M_S-1, 1) \\
& + (1 - p_V - q_V - q_S - q_I) V(S, M_S, 0)),
\end{aligned}
$$

We now note that the above expression can be alternatively written as

$$\pi^{BR}(M_S, 0) = \underset{\pi^0(M_S,0)\in A_0}{\arg\min} \; (-\pi^0(M_S, 0)),$$

for $M_S = 0, 1, \dots, N-1$. As a result,we conclude that $\pi^{BR}(M_S, 0) = 1$ for any $M_S = 0, \dots, N-1$.

□

The above result says that, when the rest of the players, different from Player 0 are not infected, then the Best Response of Player 0 is to be completely exposed (i.e. $\pi^{BR} = 1$).

The last result says that the Best Response of Player 0 to the confinement strategy π is either to confine or to be completely exposed (i.e., $\pi^{BR} \in \{0, 1\}$.)

Proposition 2. *Let π^{BR} be the solution to equations* (1). *Then, for any M_S, M_I with $M_S, M_I = 0, 1, \dots, N-1$ and $M_S + M_I \leq N-1$,*

$$\pi^{BR}(M_S, M_I) \in \{0, 1\}$$

Proof. The result follows immediately because the minimization problem is linear in $\pi^0(M_S, M_I)$ and $A_0 = \{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\}$. □

B. Markov Decision Process formulation of the solution of (GLOBAL-COST)

We now focus on the social optimum strategy. We model the problem as a Markov Decision Process, similar to what we did with the Best Response strategy computation. Let us consider $(\bar{M_S}, \bar{M_I}), 1 \leq \bar{M_S}, \bar{M_I} \leq N$ with $\bar{M_S} + \bar{M_I} \leq N$ the states that represent the number of susceptible and infected elements, respectively. We know that π^{opt} is the solution of the following Bellman equations: for $\bar{M_S}, \bar{M_I} = 0, 1, \dots, N, \; \bar{M_S} + \bar{M_I} \leq N$, we have that

$$
\begin{aligned}
V(\bar{M_S}, \bar{M_I}) = \min_{\pi(\bar{M_S},\bar{M_I})\in A_0} \; & (c_L - \pi(\bar{M_S}, \bar{M_I}))\bar{M_S} + c_I \bar{M_I} \\
& + \delta \left(\mathbb{1}_{\{\bar{M_S}\geq 1\}} w_V V(\bar{M_S}-1, \bar{M_I}) \right. \\
& + \mathbb{1}_{\{\bar{M_S}\geq 1, \bar{M_I}\leq N-1\}} w_I V(\bar{M_S}-1, \bar{M_I}+1) \\
& + \mathbb{1}_{\{\bar{M_I}\geq 1\}} w_R V(\bar{M_S}, \bar{M_I}-1) \\
& + \mathbb{1}_{\{\bar{M_S}\leq N-1, \bar{M_S}+\bar{M_I}\leq N-1\}} w_S V(\bar{M_S}+1, \bar{M_I}) \\
& \left. + \hat{w} V(\bar{M_S}, \bar{M_I}) \right),
\end{aligned} \tag{3}
$$

with transition probabilities

$$\begin{aligned} w_V &= \Omega\alpha\bar{M}_S \\ w_I &= \Omega\gamma\bar{M}_S\pi(\bar{M}_S,\bar{M}_I)\bar{M}_I/(N-1) \\ w_R &= \Omega\rho\bar{M}_I \\ w_S &= \Omega\beta(N-\bar{M}_S-\bar{M}_I) \\ \hat{w} &= 1-w_V-w_I-w_R-w_S \end{aligned}$$

We now describe the probability transitions of the system. The transition probability w_V is the probability that an element is vaccinated and reaches the recovered state. The probability w_I is the probability that a susceptible element becomes infected. This probability depends on the confinement strategy π. The probability that one element gets recovered from infection is w_R, and the probability that a recovered element becomes susceptible again is w_S. Finally, $\bar{w}$ is the probability that no changes in the state of any element occur.

From the Bellman equations for $V(\bar{M}_S,\bar{M}_I)$, we extract the following properties of the social optimum confinement strategy.

Proposition 3. *Let π^{opt} be the solution to equations* (3)*. Then, for any $M_S=0,1,\dots N$,*

$$\pi^{opt}(M_S,0)=1.$$

Proof. Consider the equation (3). If $M_I=0$, then $w_I=0$, which results:

$$\begin{aligned} \pi^{opt} = &\operatorname*{arg\,min}_{\pi(\bar{M}_S,0)\in A_0} \ (c_L-\pi(\bar{M}_S,0))\bar{M}_S \\ &+\delta\left(\mathbb{1}_{\{\bar{M}_S\geq 1\}}w_V V(\bar{M}_S-1,0)\right. \\ &+\mathbb{1}_{\{\bar{M}_S\leq N-1\}}w_S V(\bar{M}_S+1,0) \\ &\left.+(1-w_V-w_S)V(\bar{M}_S,0)\right), \qquad M_S=0,1,\dots,N \\ = &\operatorname*{arg\,min}_{\pi(\bar{M}_S,0)\in A_0} \ (-\pi(\bar{M}_S,0)), \qquad M_S=0,1,\dots,N \end{aligned}$$

where the second equality holds since w_V and w_S do not depend on π^0. Therefore, for any $M_S=0,\dots,N$, it holds that $\pi^{\text{opt}}(M_S,0)=1$. □

The above result says that, when there are no infected elements, the optimal strategy is to be completely exposed (i.e., $\pi^{opt}=1$). The next result deals with the case where $M_S=0$ and states that $\pi^{opt}(0,M_I)\in A_0$, for any $M_I\in\{0,1,\dots,N\}$ (i.e., every confinement strategy is optimal when there are no susceptible elements).

Proposition 4. *Let π be a confinement policy for the* (GLOBAL-COST) *problem. Then, for any other policy π' with $\pi'(M_S,M_I)=\pi(M_S,M_I)$ for each $M_S\geq 1, M_I\geq 0$, where $M_S+M_I\leq N$,*

$$W(\pi)=W(\pi')$$

Proof. From equation (3), if $M_S=0$ the variable of the minimization problem, that is in the elements $(c_L-\pi(\bar{M}_S,\bar{M}_I))\bar{M}_S+c_I\bar{M}_I$ and w_I, is removed. Therefore, the cost of the global optimization problem is the same independent of the action taken by the policy in states where $M_S=0$. □

Our last result shows that π^{opt} is either to confine or to be completely exposed.

Proposition 5. *Let π^{opt} be the solution to equations* (3)*. Then, for any $\bar{M}_S,\bar{M}_I$ with $\bar{M}_S,\bar{M}_I=0,1,\dots,N$ and $\bar{M}_S+\bar{M}_I\leq N$,*

$$\pi^{opt}(\bar{M}_S,\bar{M}_I)\in\{0,1\}.$$

Proof. The result follows immediately because the minimization problem is linear in $\pi(M_S,M_I)$ and $A_0=\{0,\frac{1}{d},\dots,\frac{d-1}{d},1\}$. □

We would like to establish further analytical results for the solution of the considered problems to compare their performance and to study the optimality of the solution to the decentralized setting. Unfortunately, due to the complexity of the obtained equations, we have not been able to provide more analytical results, but in the next section, we present our numerical work as well as the conclusions we derive from it. The code to reproduce the experiments of the next section can be found at [5].

IV. Efficiency Analysis

In this section, we compare the cost of the Nash equilibria and the cost of the social optimum. We say that a Nash equilibrium is efficient when its cost coincides with the optimal cost (i.e., the cost of the social optimum).

We recall that π^{opt} is the global optimal strategy, i.e., the strategy that minimizes (GLOBAL-COST) and π^{sne} the symmetric Nash equilibrium strategy. By definition, we have that $W(\pi^{\text{opt}})\leq W(\pi^{\text{sne}})$. Thus, a symmetric Nash equilibrium is said to be efficient when $W(\pi^{sne})=W(\pi^{opt})$. In this work, we will say that a symmetric Nash equilibrium is efficient when both strategies coincide for all the states.

According to Proposition 2 and Proposition 5, we know that the global optimum and a symmetric Nash equilibrium are either not to confine or to be completely exposed to the epidemic, i.e., $\pi^{opt}\in\{0,1\}$ and $\pi^{sne}\in\{0,1\}$. Despite this simplification, given the difficulty of the Bellman equations, we have not been able to obtain an analytical solution to π^{sne} and π^{opt}. As a consequence, we have obtained π^{sne} and π^{opt} by solving (1) and (3) using value iteration for a wide range of parameters, from which we have obtained the conclusions that we discuss in this section.

The main difficulty in the comparison of π^{opt} and π^{sne} is that the domain of both strategies is different. Indeed, the domain of π^{opt} is

$$\bar{S}=\{(\bar{M}_S,\bar{M}_I)\in\mathbb{N}^2:\bar{M}_S+\bar{M}_S\leq N\},$$

whereas the domain of π^{sne} is $S=\{(M_S,M_I)\in\mathbb{N}^2:M_S+M_S\leq N-1\}$ (note that the domain of π^{sne} coincides with the domain of π^{BR}). To overcome this difficulty, we perform the efficiency analysis in two different manners: (a) by graphical visualization of the obtained results and (b) by comparing the proportion of states for which the solution is to be completely exposed.

An important conclusion of our work is that, even though both strategies are very similar, the symmetric Nash equilibrium policy is more exposed to the epidemic (or equivalently, the global optimum is more confined). From our numerical experiments, we also conclude that both strategies have a switching curve. We now present and discuss some illustrations that are representative of the general pattern. In the following plots, we represent with a green point the states where the global optimum or the Nash equilibrium is to be completely exposed (that is, $\pi^{sne} = 1$ or $\pi^{opt} = 1$) and with a red cross when $\pi^{sne} = 0$ or $\pi^{opt} = 0$.

In Figure 1, we consider $N = 30$, $\gamma = 0.7$, $\rho = 0.3$, $\alpha = 0.2$, $\beta = 0.2$, $c_L = 1$ and $c_I = 5.5$. Our first observation consists of noting that the Nash equilibrium strategy is more completely exposed than the global optimum. Indeed, the proportion of states for which the $\pi^{opt} = 0$ is equal to 0.658, whereas the proportion of states for which $\pi^{sne} = 0$ is equal to 0.594. These figures also show that there exists a switching curve for both strategies; indeed, if $\pi^{opt}(M_S, M_I) = 0$, then $\pi^{opt}(M_S, M_I + 1) = 0$, for $M_I + M_S \leq N - 1$, and if $\pi^{opt}(M_S, M_I) = 1$, then $\pi^{opt}(M_S, M_I - 1) = 1$, for $M_I > 0$ (likewise for π^{sne}).

We have carried out further experiments to understand better the shape of the switching curve of π^{sne} and π^{opt}. For this purpose, in Figure 2, we consider $\gamma = 0.34$ and $\rho = 0.7$ and the rest of the parameters as in Figure 1. From this illustration, we conclude that the switching curve of π^{sne} is increasing with respect to M_S, while the switching curve of π^{opt} is increasing with respect to M_S (when $M_S > 0$).

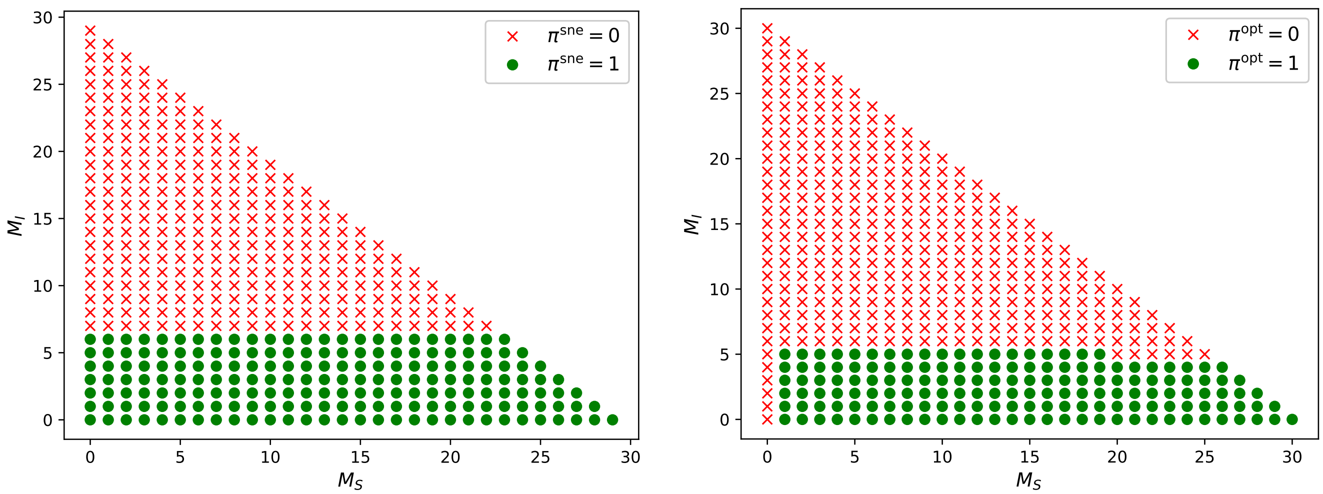
For the instance of Figure 2, proportion of states in which $\pi^{opt} = 0$ is 0.090, while the proportion of states in which $\pi^{sne} = 0$ is 0.082. This is the same behavior shown in the instance of Figure 1, that is, the social optimum policy confines more than the Nash equilibrium policy.

We now further analyze the difference in the behavior of the Nash Equilibrium policy π^{sne} and the social optimum policy π^{opt} for different configurations of the costs c_I and c_L. We do so by comparing how the proportion of confinement for each strategy (i.e., for the global optimum, the proportion of states where $\pi^{opt} = 0$ and, for the symmetric Nash equilibrium, the proportion of states where $\pi^{sne} = 0$) varies when the cost values change and the rest of the parameters are fixed. In Figure 3, we consider the same scenario as Figure 1, except for the costs c_I and c_L. More precisely, we consider the model under study in this article with the following parameters: $N = 30$, $\gamma = 0.7$, $\rho = 0.3$, $\alpha = 0.2$ and $\beta = 0.2$. In all these illustrations, the results related to the global optimum are represented with a dotted line, whereas the results that are associated with the symmetric Nash equilibrium with a solid line.

In Figure 3a, we show the evolution of the proportion of confinement for π^{sne} and π^{opt} as c_I varies from zero to 70. The comparison is made for different values of c_L, that is, for $c_L = 1$ (whose results are represented in blue), $c_L = 5$ (whose results are represented in orange), and $c_L = 20$ (whose results are represented in green). Our first observation is that each of the curves shown is non-decreasing with c_I and therefore, both strategies confine more if the infection cost increases and the rest of the parameters do not vary. For the three values of c_I, this illustration also shows that the curve of π^{sne} is never above its corresponding curve of π^{opt}, implying that in each of the cases, the Nash Equilibrium strategy confines less than or equal to the social optimum strategy. The difference in the proportion of confinement is zero when c_I is low, as both strategies do not confine, and have with the same proportion of confinement when c_I is high, as both strategies confine all the possible states in this case. We remark that the maximum possible proportion of confined states will always be below 1; this is in line with the result of Proposition 1 and of Proposition 3. When c_L is very large, the value of the cost c_I from which the proportion of confinement is not zero is higher. This means that a high lockdown cost leads to a strategy that prefers infection over confinement. For the analogous reason, we can observe that for the high values of c_L the confinement proportion reaches the maximum for higher values of c_I.

We now consider the inverse situation as in the previous representation, i.e., we vary the cost of lockdown and study the proportion of confinement of both strategies. In Figure 3b, we show the evolution of the proportion of confinement of the two strategies as the lockdown cost increases from 0 to 70. We consider the following values of the cost of infection c_I: 15 (which we represent in blue), 30 (which we represent in orange), and 70 (which we represent in blue). We observe that the proportion of confinement of both strategies are now decreasing with c_L; indeed, the higher the lockdown cost, the lower the number of states in which the confinement will be applied. Moreover, we also observe that, as in the previous illustration, the proportion of confinement of π^{sne} is never above its corresponding curve for π^{opt}, driving to the same conclusion that Nash Equilibrium strategy confines less than the social optimum strategy.

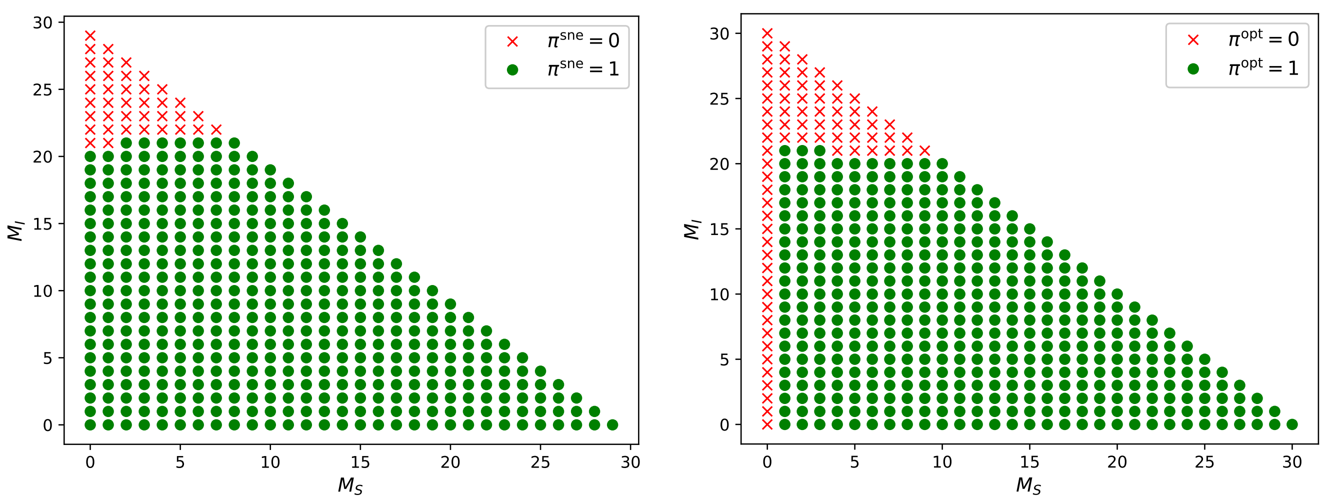
We analyze whether this observation generalizes to a more general setting. For this purpose, we randomly generated different scenarios by fixing $N = 30$ and considering random parameters among the rest of the parameters of the model. We have considered more than 400 scenarios and, for each of them, we compute the proportion of states in which $\pi^{sne} = 0$ minus the proportion of states in which $\pi^{opt} = 0$. We have checked that none of the scenarios satisfy the following conditions: (a) $\pi^{opt} = 0$ and $\pi^{sne} = 0$ for all the states, and (b) π^{opt} and π^{sne} get the maximum possible value (which is $\frac{N-1}{N+1}$ for any N, and for N=30, we get 0.934) for all the states. This has been done to focus only on non-trivial cases. In this experiments, we observe that the value of the proportion of states in which $\pi^{sne} = 0$ minus the proportion of states in which $\pi^{opt} = 0$ is always negative and they are, in most of cases, very close to zero. From these experiments, we conclude that these proportions are very close, but the decentralized setting (i.e., the Nash equilibria policy) confines less than the social optimum policy. We do not present an illustration of



(a) Nash equilibrium policy

(b) Social optimum policy

Fig. 1: Example of Nash equilibrium and social optimum policy for this model. Green dots represent completely exposed policy and red crosses confinement.



(a) Nash equilibrium policy

(b) Social optimum policy

Fig. 2: Example of Nash equilibrium and social optimum policy for $\gamma = 0.34$ and $\rho = 0.7$ and the rest of the parameters as in Figure 1. We observe that the proportion of states where it is optimal to confine increases in both instances.

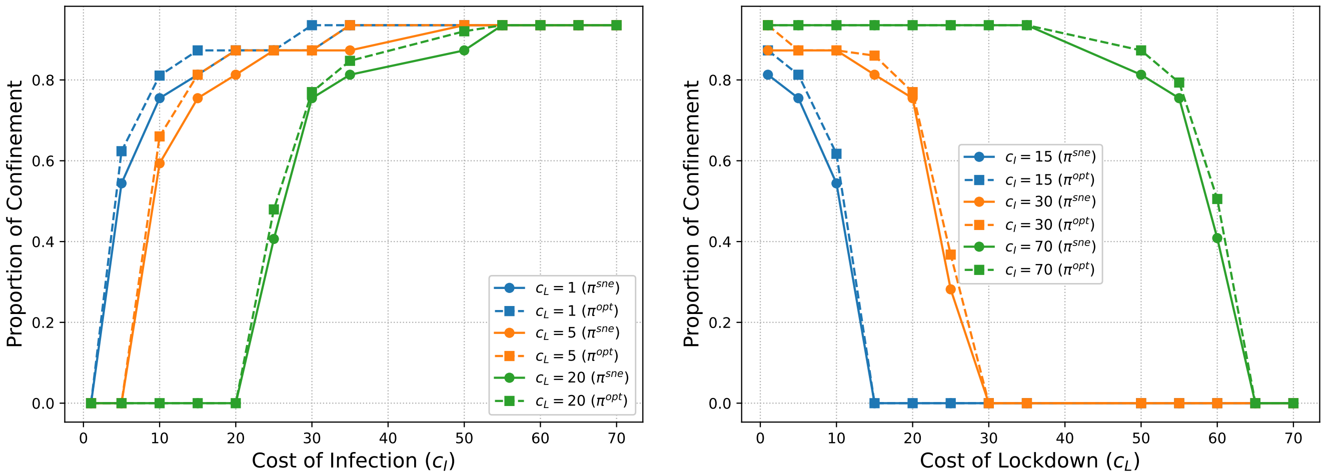
this set of experiments here due to lack of space.

V. Conclusions and Future Work

We consider the SIRS model with vaccinations, which is a generalization of the traditional SIR model where the susceptible population can be vaccinated and the recovered population becomes susceptible after a random time. We assume that time is continuous and that there are $N < \infty$ elements. We study lockdown strategies from two different perspectives: (i) a decentralized setting (i.e., a non-cooperative game) where each element aims to minimize its own expected cost, and (ii) a centralized setting where the goal is to find the lockdown strategy that all the elements must follow so as to minimize the overall population expected cost. We formulate a Markov Decision Process to solve both problems, and we derive analytical results on the structure of the solution. For instance, we show that, when the number of infected elements is zero, the Nash equilibrium and the global optimum consist of being completely exposed to the infection. Moreover, we study numerically the derived solution to both problems and we provide the following conclusions:

- the symmetric Nash equilibria and the global optima strategies have a switching curve,
- the decentralized setting solution confines less than the solution to the centralized setting.

Our model presents several limitations that we plan to



(a) Proportion of confinement states by cost of infection c_I for different costs of lockdown c_L.

(b) Proportion of confinement states by cost of lockdown c_L for different costs of infection c_I.

Fig. 3: Comparison of the proportion of states confined by strategies π^{sne} and π^{opt} for the scenario with the parameters of Figure 1 with varying values for the cost of lockdown c_L and the cost of infection c_I.

address as future work. For instance, we are interested in providing analytical results about the observations we derive from the numerical experiments; for instance, the existence of a switching curve or that the proportion of confinement is less for the solution of the decentralized setting than for that of the centralized setting. Moreover, given that our approach suffers from the curse of dimensionality, we want to investigate numerical approaches that scale with the number of agents. We are also planning to consider real data (for instance, from the COVID-19 pandemic) to compare the real performance of a system with the performance of the solution of the decentralized problem and the centralized problem we study in this work. Another possible future research is considering studying the solution to both problems under analysis in this work using methods from machine learning, such as reinforcement learning; in this context, the goal would be to develop efficient algorithms that find the solution to these problems. Finally, an interesting future research is to consider more complex models to analyze whether the presented results generalize.

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References

[1] D. Acemoglu, V. Chernozhukov, I. Werning, M. D. Whinston, et al. *A multi-risk SIR model with optimally targeted lockdown*, volume 2020. National Bureau of Economic Research Cambridge, MA, 2020.

[2] A. Aurell, R. Carmona, G. Dayanikli, and M. Lauriere. Optimal incentives to mitigate epidemics: a stackelberg mean field game approach. *SIAM Journal on Control and Optimization*, 60(2):S294–S322, 2022.

[3] A. Borri, P. Palumbo, F. Papa, and C. Possieri. Optimal design of lockdown and reopening policies for early-stage epidemics through sir-d models. *Annual reviews in control*, 51:511–524, 2021.

[4] L. Cao and Q. Liu. Covid-19 modeling: A review. *ACM Computing Surveys*, 57(1):1–42, 2024.

[5] C. Carballo. Epidemic models with nash equilibria. https://github.com/ccarballolozano/epidemic-models-nash-equilibria, 2025.

[6] A. R. Hota, U. Maitra, E. Elokda, and S. Bolognani. Learning to mitigate epidemic risks: A dynamic population game approach. *Dynamic Games and Applications*, 13(4):1106–1129, 2023.

[7] A. Ianni and N. Rossi. Describing the covid-19 outbreak during the lockdown: fitting modified sir models to data. *The European Physical Journal Plus*, 135(11):885, 2020.

[8] W. O. Kermack and A. G. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character*, 115(772):700–721, 1927.

[9] S. Kichenassamy. Mathematical models of confinement and deconfinement. 2020.

[10] G. Nakamura, B. Grammaticos, and M. Badoual. Confinement strategies in a simple sir model. *Regular and Chaotic dynamics*, 25:509–521, 2020.

[11] A. Nasir, H. R. Baig, and M. Rafiq. Epidemics control model with consideration of seven-segment population model. *SN Applied Sciences*, 2:1–9, 2020.

[12] M. E. Newman. Networks: an introduction, 2010.

[13] S. Y. Olmez, S. Aggarwal, J. W. Kim, E. Miehling, T. Başar, M. West, and P. G. Mehta. Modeling presymptomatic spread in epidemics via mean-field games. In *2022 American control conference (ACC)*, pages 3648–3655. IEEE, 2022.

[14] L. Palopoli, D. Fontanelli, M. Frego, and M. Roveri. A markovian model for the spread of the sars-cov-2 virus. *Automatica*, 151:110921, 2023.

[15] V. Petrakova and O. Krivorotko. Comparison of two mean field approaches to modeling an epidemic spread. *Journal of Optimization Theory and Applications*, 204(3):39, 2025.