CAMBIO DE VARIABLES EN INTEGRALES TRIPLES

Sean R y S las regiones correspondientes bajo la transformación T uno a uno del espacio UVW al espacio XYZ, donde las funciones coordenadas de T son

$$x = f(u,v,w), \qquad y = g(u,v,w), \qquad z = h(u,v,w)$$

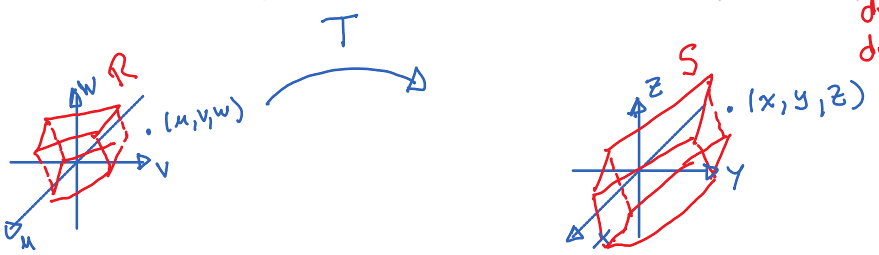
El jacobiano de la transformación T es:

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Fórmula para el cambio de variables en integrales triples

$$\iiint_S F(x,y,z)\,dzdydx = \iiint_R F(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \boxed{dudvdw}$$

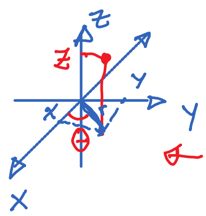
dvdudw
dwdudv



INTEGRALES TRIPLES EN COORDENADAS CILINDRICAS

$x = rcos(\theta)$; $y = rsen(\theta)$; z=z ; Jacobiano $J(r,\theta,z) = r$

$$\iiint_D f(x,y,z)dV = \iiint_U f(rcos(\theta), rsen(\theta), z) r dzdrd\theta$$



Coordenadas polares en el plano XY

Obs: Las variables pueden cambiar de papeles de acuerdo como se presente la región de integración

Ejemplo 1. GUIA CVV PAG 135 (13)

Calcular $\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz$, donde D es el sólido limitado por $z = \sqrt{x^2+y^2}$, $z = 1$.

plano

Rpta. $\pi/6$.

$Z^2 = X^2 + Y^2$

Cono

$\sqrt{x^2+y^2} \leq z \leq 1$

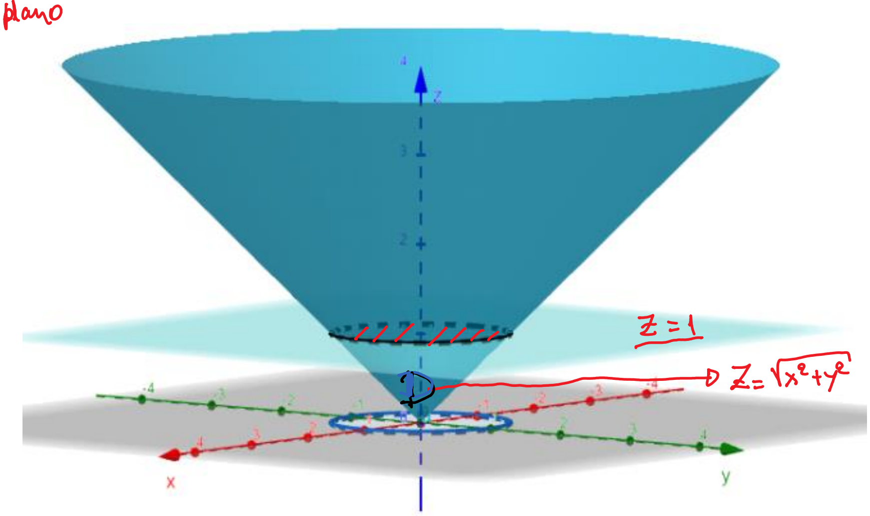
$r \leq z \leq 1$

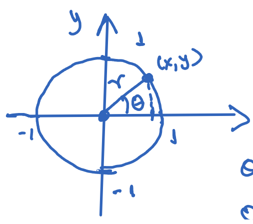
• Intersección:

$z = \sqrt{x^2+y^2} \wedge z = 1$

$\sqrt{x^2+y^2} = 1$

$x^2 + y^2 = 1$





$x = r\cos\theta$

$y = r\sin\theta$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$$\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_r^1 \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_r^1 r^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^2 z\right]_r^1 dr\,d\theta \qquad J(r,\theta,z)=r$$

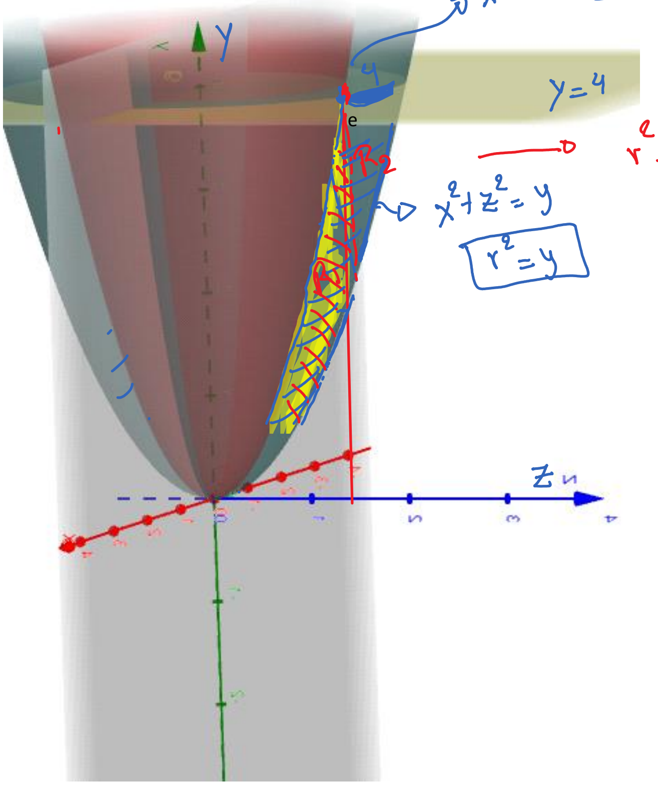
$$= \int_0^{2\pi}\int_0^1 (r^2-r^3)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{3}-\frac{1}{4}\right)d\theta = \frac{1}{12}\,\theta\Big|_0^{2\pi} = \frac{\pi}{6} \;//$$

Ejemplo 2. Calcule el volumen del sólido limitado por $x^2+z^2 \le y$, $2x^2+2z^2 \ge y$, $y=4$.

Solución

- $x^2+z^2=y \to$ paraboloide
- $2x^2+2z^2=y \to$ paraboloide

$x^2+z^2 = \frac{y}{2}$



$r^2 \le y \le 4$

- Intersección:

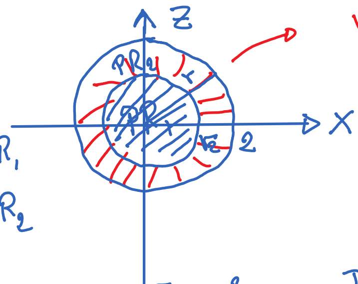
$* \; x^2+z^2=y \;\wedge\; y=4$

$x^2+z^2=4$

$* \; 2(x^2+z^2)=y \;\wedge\; y=4$

$x^2+z^2=\frac{4}{2}$

$x^2+z^2=2$



$\sqrt{2} \le r \le 2$
$0 \le \theta \le 2\pi$

PR_1: proy. de R_1

PR_2: proy. de R_2

$x = r\cos\theta$
$z = r\sin\theta$
$y = y$
$J(r,\theta,y)=r$

$$V_1 = \iiint_{R_1} 1\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{2r^2} r\,dy\,dr\,d\theta$$

$$V_1 = \int_0^{2\pi}\int_0^{\sqrt{2}}\left[ry\right]_{r^2}^{2r^2} dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta$$

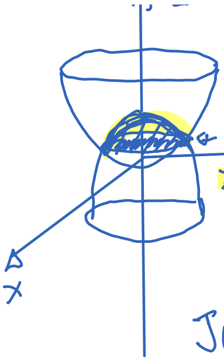
$V_1 = 2\pi$

$$V_2 = \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{r^2}^{4} r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_{\sqrt{2}}^{2} r(4-r^2)\,dr\,d\theta$$

$$V_2 = \int_0^{2\pi}\int_{\sqrt{2}}^{2} 4r-r^3\,dr\,d\theta = \int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_{\sqrt{2}}^{2} dr = \int_0^{2\pi} 8-4-(4-1)\,dr = 2\pi$$

$V_T = V_1+V_2 = 4\pi\,u^3 \;//$

3. Hallar el volumen del sólido en $\mathbb{R}^3$ limitado por las gráficas de las superficies $z=x^2+4y^2-2$ y $z=2-x^2-4y^2$.



$z = x^2+4y^2-2$

y

$z = 2-x^2-4y^2$

x

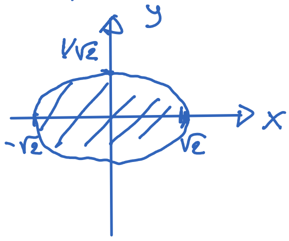
$z = x^2+4y^2-2 \quad \wedge \quad z = 2-x^2-4y^2$

$x^2+4y^2-2 = 2-x^2-4y^2$

$2x^2+8y^2-4=0$

$\Leftarrow \quad \frac{x^2}{4}+y^2=\frac{1}{2}$

$\frac{x^2}{2}+\frac{y^2}{\frac{1}{2}}=1$

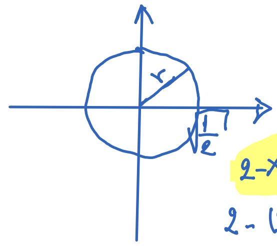


y, $\sqrt{2}$, x, $-\sqrt{2}$, $\sqrt{2}$

$$\boxed{\begin{aligned}\frac{x}{2} &= r\cos\theta\\ y &= r\,\text{sen}\,\theta\end{aligned}}$$

$$J(r,\theta)=\begin{vmatrix}2\cos\theta & -2r\,\text{sen}\,\theta\\ \text{sen}\,\theta & r\cos\theta\end{vmatrix}=2r$$

$$\Rightarrow \left(\frac{x}{2}\right)^2+y^2=\frac{1}{2} \Rightarrow (r\cos\theta)^2+(r\,\text{sen}\,\theta)^2=\frac{1}{2} \Rightarrow r^2=1/2 \Rightarrow r=\sqrt{\frac{1}{2}}$$



r, $\sqrt{\frac{1}{2}}$

$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{\frac{1}{2}}$

$2-x^2-4y^2 \geq z \geq x^2+4y^2-2$

$2-(2r\cos\theta)^2-4(r\,\text{sen}\,\theta)^2 \geq z \geq (2r\cos\theta)^2+4(r\,\text{sen}\,\theta)^2-2$

$2-4r^2 \geq z \geq 4r^2-2$

$$V=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_{4r^2-2}^{2-4r^2} 2r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}2r\left(2-4r^2-4r^2+2\right)dr\,d\theta=\int_0^{2\pi}\int_0^{1/\sqrt{2}}2r\left(4-8r^2\right)dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{1/\sqrt{2}}8r-16r^3\,dr\,d\theta=8\int_0^{2\pi}\frac{r^2}{2}-\frac{2r^4}{4}\Big|_0^{1/\sqrt{2}}d\theta=8\int_0^{2\pi}\frac{1}{4}-\frac{2}{16}\,d\theta=8\cdot\frac{1}{8}\cdot 2\pi=2\pi$$