Derivadas de orden superior

A-H Sea $z = g(x^2 + y^2)$, donde g es una función real de variable real, dos veces derivable. Demuestre que

$$y\frac{\partial^2 z}{\partial x^2} - x\frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial x} = 0$$

Derivadas Parciales de orden superior

1) $z = g(x^2+y^2)$ — g es una función real de variable real, dos veces derivable

Demostrar

$$y \frac{\delta^2 z}{\delta x^2} - x \cdot \frac{\delta^2 z}{\delta y \delta x} - \frac{\delta z}{\delta x} = 0$$

1) • $\frac{\delta z}{\delta x} = g'(x^2+y^2) \cdot 2x$

• $\frac{\delta^2 z}{\delta^2 x} = g''(x^2+y^2) \cdot 2x \cdot 2x + g'(x^2+y^2) \cdot 2$

$\frac{\delta^2 z}{\delta^2 x} = 4g''(x^2+y^2)x^2 + 2g'(x^2+y^2)$

• $\frac{\delta^2 z}{\delta y \delta x} = g''(x^2+y^2)\, 2y \cdot 2x + g'(x^2+y^2)\, 0$

$\frac{\delta^2 z}{\delta y \delta x} = 4g''(x^2+y^2)xy$

Rpta. No se comprobó la igualdad

$\rightarrow y\left(4g''(x^2+y^2)x^2 + 2g'(x^2+y^2)\right) - x\left(4g''(x^2+y^2)xy\right) - g'(x^2+y^2)2x = 0$

$\cancel{4 g''(x^2+y^2)x^2y} + 2g'(x^2+y^2)y - \cancel{4g''(x^2+y^2)x^2y} - 2g'(x^2+y^2)x \neq 0$

C-F / B-G Sea $u = \varphi(x - at) + \psi(x + at)$, donde φ y ψ son dos funciones reales de variable real, dos veces derivables. Demuestre que u es solución de la ecuación de calor

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Demostrar:

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta z}{\delta y^2} = 2 \qquad \delta$$

2) $u = \varphi(x - at) + \psi(x + at)$ ψ y φ son funciones reales de variable real, dos veces derivable. Demostrar que u es solución de la ecuación de calor

$$\frac{\delta^2 u}{\delta t^2} = a^2 \cdot \frac{\delta^2 u}{\delta x^2}$$

$$\frac{\delta u}{\delta x} = (\varphi\ x\ at)' + (\psi\ (x\ \ at))'$$

(1°) $$\frac{\delta u}{\delta x} = \varphi'(x - at) + \psi''(x + at)$$

(2°) $$\frac{\delta^2 u}{\delta x^2} = (\varphi'(x - at))' + (\psi'(x + at))'$$

$$\frac{\delta^2 u}{\delta x^2} = \varphi''(x - at) + \psi''(x + at)$$

(3) $\frac{\delta u}{\delta t} = (\varphi(x-at))' + (\psi(x+at))'$

$\frac{\delta u}{\delta t} = -a\varphi'(x-at) + a\psi'(x+at)$

(4) $\frac{\delta^2 u}{\delta t^2} = (-a\,\varphi'(x-at))' + (a\,\psi'(x+at))'$

$\frac{\delta^2 u}{\delta t^2} = a^2\varphi''(x-at) + a^2\psi''(x+at)$

Probando:

$\frac{\delta^2 u}{\delta t^2} = a^2 \cdot \frac{\delta^2 u}{\delta x^2}$

$a^2\varphi''(x-at) + a^2\psi''(x+at) = a^2\left[\varphi''(x-at) + \psi''(x+at)\right]$

$a^2\varphi''(x-at) + a^2\psi''(x+at) = a^2\varphi''(x-at) + a^2\psi''(x+at)$

Rpta. Se comprobo la igualdad de la ecuación.

D-E Sea $z = x\varphi(x+y) + y\psi(x-y)$, donde φ y ψ son dos funciones reales de variable real, dos veces derivables. Demuestre que

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2\frac{\partial^2 z}{\partial x \partial y}$$

3. $z = x\,\varphi(x+y) + y\,\psi(x-y)$, φ y ψ son funciones reales de variable real, dos veces derivables, demostrar que:

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 2\,\frac{\delta^2 z}{\delta x \delta y}$$

(1) $\dfrac{\delta z}{\delta x} = (x\varphi(x+y))' + (y\,\psi(x-y))'$

$\dfrac{\delta z}{\delta x} = \varphi(x+y) + x(\varphi'(x+y)) + y(\psi'(x-y))$

$\dfrac{\delta z}{\delta x} = \varphi(x+y) + x\,\varphi'(x+y) + y\,\psi'(x-y)$

(2) $\dfrac{\delta^2 z}{\delta x^2} = (\varphi(x+y))' + (x\,\varphi'(x+y))' + (y\,\psi'(x-y))'$

$\dfrac{\delta^2 z}{\delta x^2} = \varphi'(x+y) + \varphi'(x+y) + x\,\varphi''(x+y) + y\,\psi''(x-y)$

$\dfrac{\delta^2 z}{\delta x^2} = 2\varphi'(x+y) + x\,\varphi''(x+y) + y\,\psi''(x-y)$

(3) $\dfrac{\delta z}{\delta y} = (x\,(\varphi(x+y)))' + (y\,\psi(x-y))'$

$\dfrac{\delta z}{\delta y} = x\,\varphi'(x+y) + \psi(x-y) - y\,\psi'(x-y)$

(4). $\frac{\delta^2 z}{\delta y^2} = (x\varphi'(x+y))' + (\psi(x-y))' - (y\psi'(x-y))'$

$\frac{\delta^2 z}{\delta y^2} = x\varphi''(x+y) - \psi'(x-y) - (\psi'(x-y) - y\psi''(x-y))$

$\frac{\delta^2 z}{\delta y^2} = x\varphi''(x+y) - \psi'(x-y) - \psi'(x-y) + y\psi''(x-y)$

$\frac{\delta^2 z}{\delta y^2} = x\varphi''(x+y) - 2\psi'(x-y) + y\psi''(x-y)$

(5). $\frac{\delta^2 z}{\delta x\,\delta y} = \frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right)$

$\frac{\delta^2 z}{\delta x\,\delta y} = \frac{\delta}{\delta x}\left(x\varphi'(x+y) + \psi(x-y) - y\psi'(x-y)\right)$

$\frac{\delta^2 z}{\delta x\,\delta y} = (x\varphi'(x+y))' + (\psi(x-y))' - (y\psi'(x-y))'$

$\frac{\delta^2 z}{\delta x\,\delta y} = \varphi'(x+y) + x\varphi''(x+y) + \psi'(x-y) - y\psi''(x-y)$

→ Comprobando

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 2\frac{\delta^2 z}{\delta x\,\delta y}$$

$2\varphi'(x+y) + x\varphi''(x+y) + y\psi''(x-y) + x\varphi''(x+y) - 2\psi'(x-y) + y\psi''(x-y) =$
$2(\varphi'(x+y) + x\varphi''(x+y) + \psi'(x-y) - y\psi''(x-y))$

$2y\psi''(x-y)' - 2\psi'(x-y) = 2\psi'(x-y) - 2y\psi''(x-y)$

Rpta: No se comprueba la igualdad

Regla de la cadena

D-E Mostrar que $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$ para $w = f(x, y)$, $x = u - v$ y $y = v - u$

1) Mostrar que $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$, para $w = f(x, y)$

$x = u - v$
$y = v - u$

$$\Rightarrow \qquad w = f(x, y)$$

$$x = u - v$$
$$y = v - u$$

$$(1) \quad \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$(2) \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial v} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

Comprobando

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

$$\frac{\partial w}{\partial x} - \cancel{\frac{\partial w}{\partial y}} + \left(-\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial w}{\partial y}}\right) = 0$$

$$0 = 0$$

C-F Mostrar que $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$ para $w = (x-y)sen(y-x)$, $x = u - v$ y $y = v - u$

2) Mostrar $\frac{\delta w}{\delta u} + \frac{\delta w}{\delta v} = 0$

- $w = (x-y)\ sen\ (y-x)$
- $x = u - v$
- $y = v - u$

$$\Rightarrow \frac{\delta w}{\delta u} = \frac{\delta w}{\delta x} \cdot \frac{\delta x}{\delta u} + \frac{\delta w}{\delta y} \cdot \frac{\delta y}{\delta u} \; ; \; \frac{\delta w}{\delta v} = \frac{\delta w}{\delta x} \cdot \frac{\delta x}{\delta v} + \frac{\delta w}{\delta y} \cdot \frac{\delta y}{\delta v}$$

1. $\frac{\delta w}{\delta x} = \left((x-y)\ sen\ (y-x)\right)'$

$$\frac{\delta w}{\delta x} = sen(y-x) + (x-y)\cos(y-x)(-1)$$

$$\frac{\delta w}{\delta x} = sen\ (y-x) - (x-y)\cos(y-x)$$

2. $\frac{\delta w}{\delta y} = \left((x-y)\ sen\ (y-x)\right)'$

$$\frac{\delta w}{\delta y} = -1(sen(y-x)) + (x-y)\cos(y-x)$$

$$\frac{\delta w}{\delta y} = -sen(y-x) + (x-y)\cos(y-x)$$

$$\Rightarrow \frac{\delta w}{\delta u} = \frac{\delta w}{\delta x} \cdot 1 + \frac{\delta w}{\delta y}(-1)$$

$$\frac{\delta w}{\delta u} = \frac{\delta w}{\delta x} - \frac{\delta w}{\delta y}$$

$$\frac{\delta w}{\delta u} = \operatorname{Sen}(y-x) - (x-y)\cos(y-x) - \left[-\operatorname{sen}(y-x) + (x-y)\cos(y-x)\right]$$

$$\frac{\delta w}{\delta u} = 2\operatorname{sen}(y-x) - 2(x-y)\cos(y-x)$$

$$\frac{dw}{dv} = \frac{\delta w}{\delta x}\cdot\frac{dx}{dv} + \frac{\delta w}{\delta y}\cdot\frac{\delta y}{dv}$$

$$\frac{\delta w}{\delta v} = \frac{\delta w}{\delta x}\cdot(-1) + \frac{\delta w}{\delta y}\cdot 1$$

$$\frac{\delta w}{\delta v} = -\frac{\delta w}{\delta x} + \frac{\delta w}{\delta y}$$

$$\frac{\delta w}{\delta v} = -\left(\operatorname{sen}(y-x) - (x-y)\cos(y-x)\right) + \left(-\operatorname{sen}(y-x) + (x-y)\cos(y-x)\right)$$

$$\frac{\delta w}{\delta v} = -2\operatorname{sen}(y-x) + 2(x-y)\cos(y-x)$$

Probando:

$$\Rightarrow \quad \frac{\delta w}{\delta u} + \frac{\delta w}{\delta v} = 0$$

$$\cancel{2\operatorname{sen}(y-x)} - \cancel{2(x-y)\cos(y-x)} + \left(-\cancel{2\operatorname{sen}(y-x)} + \cancel{2(x-y)\cos(y-x)}\right) = 0$$

$$0 = 0$$

Rpta: Se comprueba la igualdad

4.

4) $F(x,y) = f(x+3y, 2x-y)$

$f: \mathbb{R}^2 \to \mathbb{R}$

- $\nabla f(0,0)$: $(4, -3)$
- Determinar la derivada direccional en el origen con dirección del vector $= (1,1)$

$F(x,y) = f(u, v)$

$u = x + 3y$

$v = 2x - y$

$$\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right)$$

(1) $$\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f(x_0,y_0)}{\partial u} \cdot \frac{\partial u}{\partial x}(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial v} \cdot \frac{\partial v}{\partial x}(x_0,y_0)$$

$$4 = \frac{\partial f}{\partial u}(0,0) + \frac{\partial f}{\partial v}(0,0)$$

(2) $$\frac{\partial f}{\partial y}(x_0,y_0) = \frac{\partial f}{\partial u}(x_0,y_0) \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y}(x_0,y_0)$$

$$= 3 = 3\frac{\partial f}{\partial u}(0,0) - \frac{\partial f}{\partial v}(0,0)$$

$$\frac{\partial f}{\partial v}(0,0) = \frac{15}{4}$$

$$\frac{\partial f}{\partial u}(0,0) = \frac{1}{4}$$

$$\nabla F(x_0,y_0) = \left(\frac{\delta F}{\delta x}(x_0,y_0), \frac{\delta F}{\delta y}(x_0,y_0) \right)$$

$$\cdot \frac{\delta F}{\delta x}(x_0,y_0) = \frac{\delta F}{\delta u}(x_0,y_0) \cdot \frac{\delta u}{\delta x}(x_0,y_0) + \frac{\delta F}{\delta v}(x_0,y_0)\frac{\delta v}{\delta x}(x_0,y_0)$$

$$\frac{\delta F}{\delta x}(x_0,y_0) = \frac{dF}{du}(x_0,y_0) + 2\frac{dF}{dv}(x_0,y_0)$$

$$\frac{\delta F}{\delta x}(x_0,y_0) = \frac{1}{4}\frac{dF}{df} + \frac{30}{4}\frac{dF}{df}$$

$$\frac{\delta F}{\delta x}(0,0) = \frac{31}{4}$$

$$\cdot \frac{\delta F}{\delta y}(x_0,y_0) = \frac{\delta F}{du}(x_0,y_0)\frac{du}{dy}(x_0,y_0) + \frac{\delta F}{dv}(x_0,y_0)\frac{dv}{dy}(x_0,y_0)$$

$$\frac{\delta F}{dy}(x_0,y_0) = 3\frac{dF}{du}(x_0,y_0) + \left(-\frac{\delta F}{dv}(x_0,y_0)\right)$$

$$\frac{\delta F}{\delta y}(x_0,y_0) = 3 \cdot \frac{1}{4} \cdot \frac{dF}{df} + \left(-\frac{15}{4}\frac{\delta f}{\delta f}\right)$$

$$\frac{\delta F(0,0)}{\delta y} = -\frac{12}{4}\frac{dF}{df}$$

$$\nabla F(0,0) = \left(\frac{31}{4}, -\frac{12}{4}\right)$$

$$v = (1,1)$$

$$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\frac{\delta f}{\delta v}(0,0) = \overrightarrow{\nabla f(0,0)} \cdot \vec{u}$$

$$\frac{\delta f}{\delta v}(0,0) = \left(\frac{31}{4}, -\frac{12}{4}\right) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\frac{\delta f}{\delta v}(0,0) = \frac{31\sqrt{2}}{8} - \frac{12\sqrt{2}}{8}$$

$$\frac{\delta f}{\delta v}(0,0) = \frac{19\sqrt{2}}{8}$$

Teorema de la función implicita

Dado el nivel cero de la función $F(x, y)$. Compruebe que esta función satisface las hipótesis del Teorema de la Función Implícita en el punto indicado (perteneciente al nivel cero). Obtenga la derivada de la función $y = f(x)$ en el punto dado

D-E $F(x, y) = x^2y + 3x^2 - 2y^2 - 2y = 0$, $P(1, 1)$

1) $F(x,y) = x^2y + 3x^2 - 2y^2 - 2y = 0, P(1,1)$

$\Rightarrow$ Condiciones

(1) $F(1,1) = 0$ Hipótesis

$F(1,1) = 1 + 3 - 2 - 2$

$F(1,1) = 0$ Satisface

(2) Tiene derivadas parciales continuas

$\frac{\delta F}{\delta x}, \frac{\delta F}{\delta y}$

- $\frac{\delta F}{\delta x} = 2yx + 6x$

- $\frac{\delta F}{\delta y} = x^2 - 4y - 2$

Son Continuas en todo su dominio

Satisface

(3) $\frac{\delta F}{\delta y}(1, 1) \neq 0$

$-5 \neq 0$ Satisface

(4) $y' = -\dfrac{\frac{\delta F(1,1)}{\delta x}}{\frac{\delta F(1,1)}{\delta y}} = -\dfrac{8}{(-5)} = \dfrac{8}{5}$

C-F $F(x,y) = y\ln(x^2+y^2) - 2xy = 0,\ P(0,1)$

2) $F(x,y) = y\ln(x^2+y^2) - 2xy = 0$

$P(0,1)$

$\Rightarrow$ Condiciones

(1) $F(0,1) = 0$ Hipótesis

$F(0,1) = \ln(1) - 0$

$F(0,1) = 0$ Satisface

(2) Tiene derivadas parciales continuas

$\frac{\delta F}{\delta x}, \frac{\delta F}{\delta y}$

- $\frac{\delta F}{\delta x} = y \frac{1}{x^2+y^2} \cdot 2x - 2y$

$\frac{\delta F}{\delta x} = \frac{2xy}{x^2+y^2} - 2y$

Son continuas en todo su dominio

- $\frac{\delta F}{\delta y} = \ln(x^2+y^2) + y \frac{1}{x^2+y^2} \cdot 2y - 2x$

$\frac{\delta F}{\delta y} = \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} - 2x$

Satisface

(3) $\frac{\delta F}{\delta y}(x_0, y_0) \neq 0$ Hipótesis

$\frac{\delta F}{\delta y}(0,1) = 0 + 2$

$\frac{\delta F}{\delta y}(0,1) = 2$

$2 \neq 0$ Satisface

(4) $y' = -\dfrac{\dfrac{\partial F}{\partial x}(0,1)}{\dfrac{\partial F}{\partial y}(0,1)} = -\dfrac{(-2)}{2} = 1$

$y'(0,1) = 1$ //

B-G $F(x,y) = x^y + y^x - 2xy = 0,\ P(2,2)$

3) $F(x,y) = x^y + y^x - 2xy = 0,\ P(2,2)$

$\Rightarrow$ Condiciones

(1) $F(2,2) = 0$ Hipótesis

$F(2,2) = 2^2 + 2^2 - 2.2.2$

$F(2,2) = 0$ Satisface

(2) Tiene derivadas parciales continuas $\frac{\delta F}{\delta x}, \frac{\delta F}{\delta y}$

- $\frac{\delta F}{\delta x} = y.x^{y-1} + y^x . \ln y - 2y$
- $\frac{\delta F}{\delta y} = x^y \ln x + x y^{x-1} - 2x$

Son continuas

Satisface

(3) $\frac{\delta F}{\delta y}(2,2) \neq 0$ Hipótesis

$4 \ln 2 + \cancel{4} - \cancel{4}$

$4 \ln 2 \neq 0$ Satisface

(4) $y' = - \dfrac{\frac{\delta F}{\delta x}(2,2)}{\frac{\delta F}{\delta y}(2,2)}$

$y'_{(2,2)} = - \dfrac{(\cancel{4} + \cancel{4 \ln 2} - \cancel{4})}{\cancel{4 \ln 2}}$

$y'(2,2) = -1$