Derivadas parciales de orden superior

miércoles, 26 de mayo de 2021 06:30

Definición. Las derivadas de segundo orden de la función $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ se obtienen a partir de las $\overbrace{\text{derivadas}}^{f}$ parciales de primer orden $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x+h, y) - \frac{\partial f}{\partial x}(x, y)}{h}$$

$$f_x = \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \lim_{h \to 0} \frac{f_y(x, y+h) - f_y(x, y)}{h}$$

✓ $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}$$

✓ $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(x, y+h) - \frac{\partial f}{\partial x}(x, y)}{h}$$

$f(x,y)$
- $\frac{\partial f}{\partial x}(x,y)$
  - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$
    - f_{xxx}
    - f_{yxx}
  - $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$
    - f_{xyx}
    - f_{yyx}
- $\frac{\partial f}{\partial y}(x,y)$
  - $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
    - f_{xxy}
    - f_{yxy}
  - $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$
    - f_{xyy}
    - f_{yyy}

Teorema (Teorema de Schwarz)

Sea $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ una función, D conj. abierto. Si las derivadas $\frac{\partial^2 f}{\partial x \partial y}$ y $\frac{\partial^2 f}{\partial y \partial x}$ existen y son continuas en $D \Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Ejemplos: Guia DAM pág 46 Cap 2

· 64. Demuestre que cada una de las funciones dadas satisface la ecuación $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (llamada "Ecuación de Laplace").

(a) $f(x,y) = x^3 - 3xy^2$

✓ (b) $f(x,y) = e^{x^2-y^2}(\cos 2xy + sen 2xy)$

(a)

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial f}{\partial y} = -6xy$$

$\odot$ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x$

· $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -6y$] =

· $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -6y$

$\odot$ $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -6x$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 6x = 0 \quad /\!/\!/$$

(b) • $\frac{\partial f}{\partial x} = e^{x^2-y^2} \cdot 2x \cdot (\cos 2xy + \text{sen } 2xy) + e^{x^2-y^2} \cdot (-\text{sen } 2xy \cdot 2y + \cos 2xy \cdot 2y)$

• $\frac{\partial f}{\partial x} = 2x\, e^{x^2-y^2}(\cos 2xy + \text{sen } 2xy) + 2y\, e^{x^2-y^2}(-\text{sen } 2xy + \cos 2xy)$

· $\odot$ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \cancel{2e^{x^2-y^2}(\cos 2xy} + \text{sen } 2xy) + \underline{(2x)^2 e^{x^2-y^2}(\cos 2xy + \text{sen } 2xy)} + \overbrace{2x e^{x^2-y^2}}(-\text{sen } 2xy \cdot 2y + \cos 2xy \cdot 2y)$

∂x

- $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = 2e^{x^2-y^2}(\cos 2xy + \operatorname{sen} 2xy) + (2x)^2 e^{x^2-y^2}(\cos 2xy + \operatorname{sen} 2xy) + 2xe^{x^2-y^2}(-\operatorname{sen} 2xy \cdot 2y + \cos 2xy \cdot 2y)$

$+ 2ye^{x^2-y^2} \cdot (2x)(-\operatorname{sen} 2xy + \cos 2xy) + 2ye^{x^2-y^2}(-\cos 2xy \cdot 2y - \operatorname{sen} 2xy \cdot 2y)$

- $\dfrac{\partial f}{\partial y} = e^{x^2-y^2} \cdot (-2y)(\cos 2xy + \operatorname{sen} 2xy) + e^{x^2-y^2} \cdot (-\operatorname{sen} 2xy \cdot 2x + \cos 2xy \cdot 2x) = -2y e^{x^2-y^2}(\cos 2xy + \operatorname{sen} 2xy) + 2x e^{x^2-y^2}(-\operatorname{sen} 2xy + \cos 2xy)$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = -2e^{x^2-y^2}(\cos 2xy + \operatorname{sen} 2xy) + 4y^2 \cdot e^{x^2-y^2}(\cos 2xy + \operatorname{sen} 2xy) - 2ye^{x^2-y^2}(-\operatorname{sen} 2xy \cdot 2x + \cos 2xy \cdot 2x)$

$+ 2xe^{x^2-y^2}(-2y)(-\operatorname{sen} 2xy + \cos 2xy) + 2xe^{x^2-y^2} \cdot (-\cos 2xy \cdot 2x - \operatorname{sen} 2xy \cdot 2x)$

$\Rightarrow \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ //.

65. Verifique si la función $z = sen(x^2 + y^2)$ satisface la ecuación

$$y\frac{\partial^2 z}{\partial x^2} - x\frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial y} = 0$$

- $\dfrac{\partial z}{\partial x} = \cos(x^2+y^2) \cdot 2x$
  - $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = -\operatorname{sen}(x^2+y^2)(2x)^2 + \cos(x^2+y^2) \cdot 2$
  - $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = -2x\operatorname{sen}(x^2+y^2) \cdot 2y$
- $\dfrac{\partial z}{\partial y} = \cos(x^2+y^2) \cdot 2y$

$\Rightarrow y\dfrac{\partial^2 z}{\partial x^2} - x\dfrac{\partial^2 z}{\partial y \partial x} - \dfrac{\partial z}{\partial y} = y\left(-4x^2\operatorname{sen}(x^2+y^2) + 2\cos(x^2+y^2)\right) - x\left(-4xy\operatorname{sen}(x^2+y^2)\right) - 2y\cos(x^2+y^2)$

$= -4x^2y\operatorname{sen}(x^2+y^2) + 2y\cos(x^2+y^2) + 4x^2y\operatorname{sen}(x^2+y^2) - 2y\cos(x^2+y^2) = 0$ //.

Ejercicio: Calcular las derivadas parciales de segundo orden en $(0,0)$ de

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Derivadas parciales de primer orden

- $(x,y) \neq (0,0) \Rightarrow f(x,y) = \dfrac{xy^3}{x^2+y^2}$
  - $\dfrac{\partial f}{\partial x} = \dfrac{y^3(x^2+y^2) - xy^3(2x)}{(x^2+y^2)^2} = \dfrac{y^5 - x^2y^3}{(x^2+y^2)^2}$
  - $\dfrac{\partial f}{\partial y} = \dfrac{3xy^2(x^2+y^2) - xy^3(2y)}{(x^2+y^2)^2} = \dfrac{3x^3y^2 + xy^4}{(x^2+y^2)^2}$

- $(x,y) = (0,0) \Rightarrow \dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{\frac{h \cdot 0^3}{h^2+0^2} - 0}{h} = 0$

$\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{\frac{0 \cdot h^3}{0^2+h^2} - 0}{h} = 0$

$\Rightarrow \dfrac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^5 - x^2y^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$; $\dfrac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{3x^3y^2 + xy^4}{(x^2+y^2)^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$

$\dfrac{\partial^2 f}{\partial x^2}(0,0) = \lim \dfrac{\frac{\partial f}{\partial x}(0+h,0) - \frac{\partial f}{\partial x}(0,0)}{} = \lim\limits_{h\to 0} \dfrac{\frac{0^5 - h^2 0^3}{(h^2+0^2)^2} - 0}{h} = 0$

$\left\{ \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) \right.$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0+h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{(h^2+0^2)^2}{h} = 0$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0,0+h) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3 0^3 h^2 + 0 h^4}{(0^2+h^2)^2} - 0}{h} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3h^3 0^2 + h 0^4}{(h^2+0^2)^2} - 0}{h} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,0+h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5 - 0^2 h^3}{(0^2+h^2)^2} - 0}{h} = \lim_{h\to 0} \frac{h^5}{h^5} = \lim_{h\to 0} 1 = 1$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

var función

Ejercicio: Cálculo avanzado USACH

Como ejemplo ilustrativo considere la función

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & si \;\; (x,y) \neq (0,0) \\ 0 & si \;\; (x,y) = (0,0) \end{cases}$$

y verifique que:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1, \quad \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$$

Observación: Use la definición y encuentre estos resultados.