Integrales con cambio a coordenadas polares

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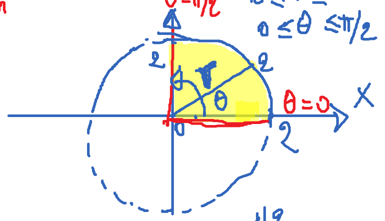
$$\iint_R F(x,y)\,dx\,dy = \iint_S F(x(u,v),\, y(u,v))\cdot |J(u,v)|\,du\,dv \quad ; \quad J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$u \downarrow r \qquad v \downarrow \theta$

Ejemplo 1.

Calcular la integral doble $\iint_D e^{-(x^2+y^2)}\,dA$ ($dy\,dx$, $dx\,dy$), donde D es la región en el primer cuadrante acotado por el círculo $x^2+y^2=4$ y los ejes coordenados.

Solución



$0 \le r \le 2$
$0 \le \theta \le \pi/2$

- $x = r\cos\theta$
- $y = r\,\text{sen}\,\theta$
- $x^2+y^2 = 4$

$(r\cos\theta)^2 + (r\,\text{sen}\,\theta)^2 = 4$

$r^2 = 4$

$r = 2$

$$J(r,\theta)=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta\end{vmatrix} = r$$

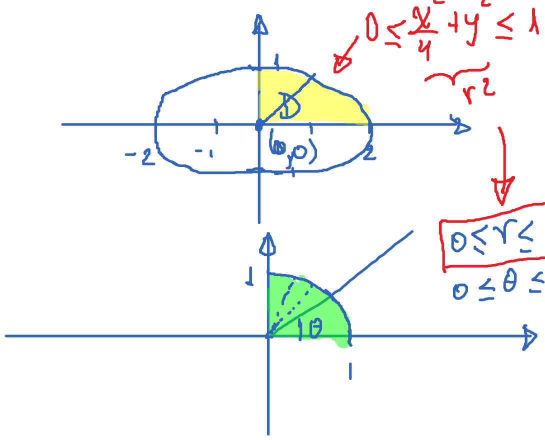
$$\iint_D e^{-(x^2+y^2)}\,dA = \int_0^{\pi/2}\int_0^2 -\frac{1}{2}e^{-r^2}(-2r)\,dr\,d\theta = -\frac{1}{2}\int_0^{\pi/2} e^{-r^2}\Big|_0^2\,d\theta = -\frac{1}{2}\int_0^{\pi/2} e^{-4}-1\,d\theta$$

$$= -\frac{1}{2}(e^{-4}-1)\int_0^{\pi/2} d\theta = -\frac{1}{2}(e^{-4}-1)\frac{\pi}{2} = \frac{\pi}{4}(1-e^{-4})$$

Ejemplo 2.

Calcular $\iint_D xy\,dx\,dy$, donde D es un dominio limitado por la elipse $\frac{x^2}{4}+y^2=1$ y situado en el primer cuadrante.

Solución



$0 \le \underbrace{\frac{x^2}{4}+y^2}_{r^2} \le 1$

$\boxed{0 \le r \le 1}$

$0 \le \theta \le \pi/2$

- $\left(\frac{x}{2}\right)^2 + (y)^2 = 1$

En coordenadas polares

$\frac{x}{2} = r\cos\theta \longrightarrow x = 2r\cos\theta$

$y = r\,\text{sen}\,\theta \longrightarrow y = r\,\text{sen}\,\theta$

- $(r\cos\theta)^2 + (r\,\text{sen}\,\theta)^2 = 1$

$r^2 = 1$

$r = 1 \;,\; 0 \le \theta \le \pi/2$

- $$J(r,\theta)=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta\end{vmatrix} = 2r\cos^2\theta + 2r\,\text{sen}^2\theta = 2r$$

$$\cdot\ J(r,\theta)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix} & \\ \operatorname{sen}\theta & r\cos\theta\end{vmatrix}=2r\cos^2\theta+2r\operatorname{sen}^2\theta=2r$$

$$\cdot\ \iint_D xy\,dx\,dy=\int_0^{\pi/2}\int_0^1 2r\cos\theta\cdot r\operatorname{sen}\theta\cdot 2r\,dr\,d\theta=4\int_0^{\pi/2}\int_0^1 r^3\operatorname{sen}\theta\cdot\cos\theta\,dr\,d\theta$$

$$=4\int_0^{\pi/2}\frac{r^4}{4}\operatorname{sen}\theta\cdot\cos\theta\Big|_0^1 d\theta=\int_0^{\pi/2}(\operatorname{sen}\theta)\cos\theta\,d\theta=\frac{\operatorname{sen}^2\theta}{2}\Big|_0^{\pi/2}$$

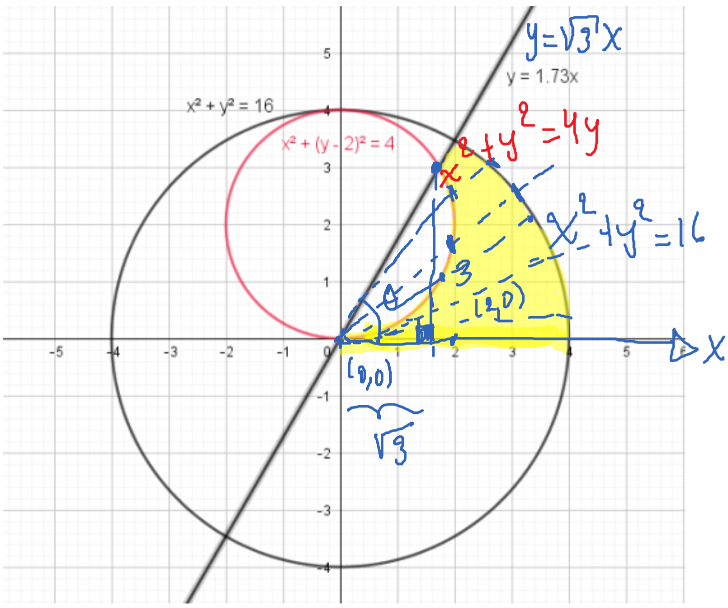
$$=\frac{1}{2}$$

$$\boxed{\int (f(x))^n f'(x)\,dx=\frac{(f(x))^{n+1}}{n+1}+C}$$

Ejemplo 3.

Halle el área de la región $R=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 16,\ x^2+y^2\ge 4y,\ y\ge 0,\ y\le\sqrt{3}x\}$

Solución



$$\tan\theta=\frac{3\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\sqrt{3}$$

Cambio a coordenadas polares

$x=r\cos\theta$
$y=r\operatorname{sen}\theta$ $\Rightarrow$ $J(r,\theta)=r$

- $x^2+y^2=16 \Rightarrow (r\cos\theta)^2+(r\operatorname{sen}\theta)^2=16$

$r^2=16$

$\boxed{r=4}$

- $x^2+y^2=4y$

$(r\cos\theta)^2+(r\operatorname{sen}\theta)^2=4r\operatorname{sen}\theta$

$r^2=4r\operatorname{sen}\theta$

$\boxed{r=4\operatorname{sen}\theta}$

- $\boxed{4\operatorname{sen}\theta\le r\le 4}$

- $y=0 \Rightarrow \theta=0$
- $y=\sqrt{3}x \Rightarrow r\operatorname{sen}\theta=\sqrt{3}\,r\cos\theta \Rightarrow \tan\theta=\sqrt{3}$

$\Rightarrow \theta=\pi/3$

$\boxed{0\le\theta\le\pi/3}$

$$\iint_R 1\,dx\,dy=\int_0^{\pi/3}\int_{4\operatorname{sen}\theta}^{4} r\,dr\,d\theta=\int_0^{\pi/3}\frac{r^2}{2}\Big|_{4\operatorname{sen}\theta}^{4}d\theta=\int_0^{\pi/3}8-8\operatorname{sen}^2\theta\,d\theta=8\int_0^{\pi/3}\cos^2\theta\,d\theta=8\int_0^{\pi/3}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=8\left[\frac{\theta}{2}+\frac{1}{4}\operatorname{sen}2\theta\right]_0^{\pi/3}=8\left(\frac{\pi}{6}+\frac{1}{4}\operatorname{sen}(2\pi/3)\right)=8\left(\frac{\pi}{6}+\frac{1}{8}\sqrt{3}\right)$$