Multiplicadores de Lagrange

miércoles, 23 de junio de 2021 07:00

Ejercicios: Hallar los valores máximos y mínimos de la función $z = e^{-x^2-y^2}(2x^2+3y^2)$ en el círculo $x^2+y^2 \leq 4$

Solución

1° $\text{Dom} f = \mathbb{R}^2$

2° P.C.

$$\begin{cases} \frac{\partial z}{\partial x} = e^{-x^2-y^2}\cdot(-2x)(2x^2+3y^2) + e^{-x^2-y^2}\cdot(4x) = 0 \\ \frac{\partial z}{\partial y} = e^{-x^2-y^2}\cdot(-2y)(2x^2+3y^2) + e^{-x^2-y^2}\cdot(6y) = 0 \end{cases} \Rightarrow \begin{matrix} e^{-x^2-y^2}\cdot(-2x)(2x^2+3y^2-2) = 0 \\ e^{-x^2-y^2}\cdot(-2y)(2x^2+3y^2-3) = 0 \end{matrix}$$

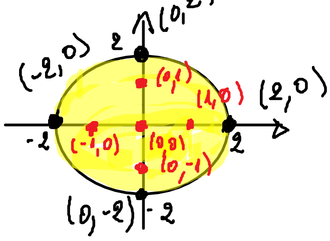
$$\Rightarrow \begin{cases} x=0 \quad \vee \quad 2x^2+3y^2-2=0 \\ y=0 \quad \vee \quad 2x^2+3y^2-3=0 \end{cases}$$

$\Rightarrow$ Si $x=0 \wedge y=0 \Rightarrow (0,0)$

Si $x=0 \wedge 2x^2+3y^2-3=0 \Rightarrow y=\pm 1$
$\Rightarrow (0,1)$; $(0,-1)$

Si $2x^2+3y^2-2=0 \wedge y=0 \Rightarrow x=\pm 1$
$\Rightarrow (1,0)$; $(-1,0)$

Si $2x^2+3y^2-2=0 \wedge 2x^2+3y^2-3=0 \Rightarrow$ no hay solución



3° $\min/\max \; z = e^{-x^2-y^2}(2x^2+3y^2)$
s.a. $x^2+y^2-4=0$

$\min/\max \; z = f(x,y)$
s.a. $g(x,y)=0$

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

Función de Lagrange $L(x,y,\lambda) = e^{-x^2-y^2}(2x^2+3y^2) + \lambda(x^2+y^2-4)$

i) P.C

$$\begin{cases} \frac{\partial L}{\partial x} = e^{-x^2-y^2}(-2x)(2x^2+3y^2-2) + \lambda(2x) = 0 & (1) \\ \frac{\partial L}{\partial y} = e^{-x^2-y^2}\cdot(-2y)(2x^2+3y^2-3) + \lambda(2y) = 0 & (2) \\ \frac{\partial L}{\partial \lambda} = x^2+y^2-4=0 & (3) \end{cases}$$

y

$-x$

De (1) y (2) :

$$\begin{cases} e^{-x^2-y^2}(-2xy)(2x^2+3y^2-2) + 2xy\lambda = 0 \\ e^{-x^2-y^2}(2xy)(2x^2+3y^2-3) - 2xy\lambda = 0 \end{cases} +$$

$$-2xy\,e^{-x^2-y^2}(1) = 0$$

$xy = 0 \Rightarrow x = 0 \vee y = 0$

En (3) Si $x=0 \wedge \cancel{x^2}+y^2-4=0$
$\Rightarrow (0,2)$; $(0,-2)$

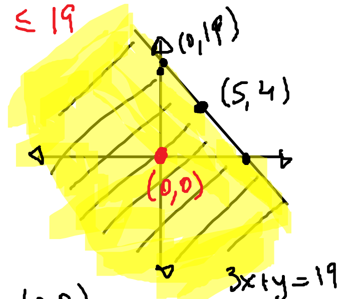
Si $y=0 \wedge x^2+\cancel{y^2}-4=0$
$\Rightarrow (2,0)$; $(-2,0)$

4° $f(0,0) = 0 \leftarrow$ valor mín abs
$f(-1,0) = 2e^{-1} \approx 0.74$
$f(1,0) = 2e^{-1}$

$f(0,1) = 3e^{-1}$
$f(0,-1) = 3e^{-1} \approx 1.1 \rightarrow$ valor máx abs.
$f(0,2) = 12e^{-4}$

$f(0,-2) = 12e^{-4} \approx 0.22$
$f(2,0) = 8e^{-4}$
$f(-2,0) = 8e^{-4} \approx 0.15$

Ejercicio 2.
Determinar el máximo de $f(x,y) = 6xy - 3x^2 - 4y^2$ si $3x + y \le 19$



$\underbrace{3x+y-19}_{g(x,y)} = 0$

Solución:

1° $Dom f = \mathbb{R}^2$

2° P.C. $\begin{cases} \dfrac{\partial f}{\partial x} = 6y - 6x = 0 \\ \dfrac{\partial f}{\partial y} = 6x - 8y = 0 \end{cases} \Rightarrow \begin{array}{l} -6x + 6y = 0 \\ 6x - 8y = 0 \\ \hline \quad -2y = 0 \end{array}$ $\boxed{\begin{array}{l} y = 0 \\ x = 0 \end{array}} \Rightarrow (0,0)$

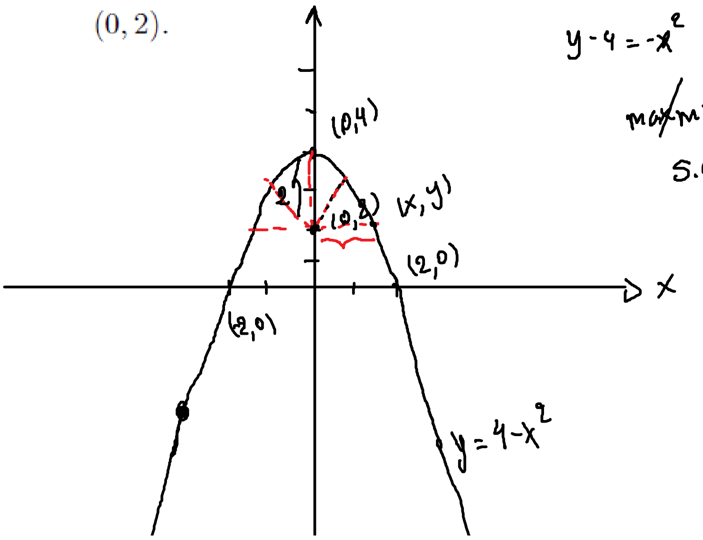
3° $L(x,y,\lambda) = 6xy - 3x^2 - 4y^2 + \lambda(3x + y - 19)$

i) P.C. $\begin{cases} \dfrac{\partial L}{\partial x} = 6y - 6x + 3\lambda = 0 & (1) \\ \dfrac{\partial L}{\partial y} = 6x - 8y + \lambda = 0 & (2) \\ \dfrac{\partial L}{\partial \lambda} = 3x + y - 19 = 0 & (3) \end{cases} \Rightarrow$ $\lambda = \dfrac{-6y + 6x}{3} = -2y + 2x$, $\lambda = -6x + 8y$ $\Rightarrow$ $-2y + 2x = -6x + 8y$, $8x = 10y$, $\boxed{x = \frac{5}{4}y}$

En (3) $\quad 3\left(\frac{5}{4}y\right) + y - 19 = 0 \Rightarrow \frac{19}{4}y - 19 = 0 \Rightarrow \boxed{\begin{array}{l} y = 4 \\ x = 5 \end{array}}$

4° $f(0,0) = 0$ ✓ $\Rightarrow$ valor máx abs
$f(5,4) = 120 - 75 - 64 = -19$
$f(0,19) = -1444$
$\rightarrow$ sobre la recta $3x + y = 19$

Ejercicio
3. Hallar los puntos de la gráfica de $y = 4 - x^2$ que están más próximos y más alejados del punto $(0, 2)$.



$y - 4 = -x^2$

~~max~~ min $D(x,y) = (x-0)^2 + (y-2)^2 = x^2 + (y-2)^2$

s.a. $\underbrace{y + x^2 - 4}_{g(x,y)} = 0$

i) $L(x,y,\lambda) = D(x,y) + \lambda g(x,y)$
$L(x,y,\lambda) = x^2 + (y-2)^2 + \lambda(y + x^2 - 4)$

ii) P.C. $\begin{cases} \dfrac{\partial L}{\partial x} = 2x + 2x\lambda = 0 & (1) \\ \dfrac{\partial L}{\partial y} = 2(y-2) + \lambda = 0 & (2) \\ \dfrac{\partial L}{\partial \lambda} = y + x^2 - 4 = 0 & (3) \end{cases}$

$$\frac{\partial L}{\partial y} = 2(y-2) + \ldots$$

$$\frac{\partial L}{\partial \lambda} = y + x^2 - 4 = 0 \quad (3)$$

$$\Rightarrow \begin{cases} 2x(1+\lambda) = 0 \\ -2(y-2) = \lambda \\ y + x^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \vee \lambda = -1 \\ \lambda = -2(y-2) \\ y + x^2 - 4 = 0 \end{cases} \Rightarrow$$

$*$ Si $\boxed{x=0} \wedge \lambda = -2(y-2) \wedge y + x^2 - 4 = 0$

$\Rightarrow \lambda = -2(y-2) \wedge \boxed{y=4}$

$\boxed{\lambda = -4}$ $\Rightarrow (0,4)$

$*$ Si $\lambda = -1 \wedge \lambda = -2(y-2) \wedge y + x^2 - 4 = 0$

$\Rightarrow 1 = 2(y-2) \wedge y + x^2 - 4 = 0$

$y = 5/2 \Rightarrow x^2 - 4 + \frac{5}{2} = 0$

$x^2 = 3/2$

$\hookrightarrow x = \pm\sqrt{\frac{3}{2}}$

$\Rightarrow \left(\sqrt{3/2}, 5/2\right); \left(-\sqrt{3/2}, 5/2\right)$

$$D(x,y) = \underset{0}{x^2} + (\underset{4}{y}-2)^2$$

iii) $D(0,4) = 4$

$D(\sqrt{3/2}, 5/2) = \left(\sqrt{\frac{3}{2}}\right)^2 + \left(\frac{5}{2} - 2\right)^2 = \frac{3}{2} + \frac{1}{4} = 7/4$

$D(-\sqrt{3/2}, 5/2) = \left(-\sqrt{\frac{3}{2}}\right)^2 + \left(\frac{5}{2} - 2\right)^2 = \frac{3}{2} + \frac{1}{4} = 7/4$

$\Big\}$ Valores mínimos absolutos

$D(2,0) = 4 + 4 = 8$

$\hookrightarrow$ un punto cualquiera de la parábola