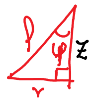
zIntegrales triples en Coordenadas esféricas

miércoles, 21 de julio de 2021 07:00

INTEGRALES TRIPLES EN COORDENADAS ESFERICAS

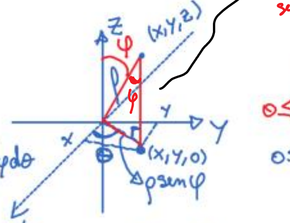
INTEGRALES TRIPLES EN COORDENADAS ESFERICAS



$\rho\cos\varphi = z$

$sen\varphi = \frac{r}{\rho} \to r = \rho sen\varphi$

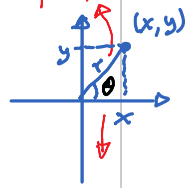
$x = \underbrace{\rho sen(\varphi)}_{radio}cos\theta,\ y = \underbrace{\rho sen(\varphi)}_{radio}sen\theta,\ z = \underbrace{\rho\cos(\varphi)},\ J(\rho,\varphi,\theta) = \rho^2 sen(\varphi)$



$\rho > 0$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$



$$\iiint_S f(x,y,z)\,dxdydz = \iiint_U f(\rho sen\varphi\cos\theta, \rho sen\varphi . sen\theta, \rho\cos\varphi)\,|J(\rho,\varphi,\theta)|\,d\rho d\varphi d\theta$$

x: $\cos\theta = \frac{x}{r}$

↪ $x = \rho sen\varphi\cos\theta$

y: $sen\theta = \frac{y}{r}$

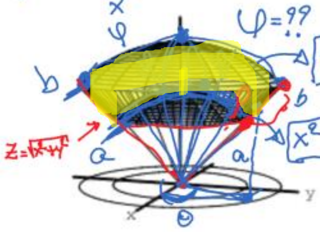
↪ $y = \rho sen\varphi sen\theta$

EJEMPLO 1:

Calcular el volumen del casquete esférico limitado por

$$\begin{array}{lrcl} \text{Esfera} & x^2+y^2+z^2 & = & a^2 \\ \text{Esfera} & x^2+y^2+z^2 & = & b^2 \\ \text{Cono} & x^2+y^2 & = & z^2, \end{array}$$

con $z \ge 0$, siendo $0 < a < b$.



$\boxed{x^2+y^2+z^2 = b^2}$

$\boxed{x^2+y^2+z^2 = a^2}$

SOLUCION

$x = \rho sen\varphi\cos\theta$

$y = \rho sen\varphi sen\theta$;

$z = \rho\cos\varphi$

$0 \le \theta \le 2\pi$

$a \le \rho \le b$

$0 \le \varphi \le \pi/4$

• $x^2+y^2+z^2 = a^2 \ \wedge\ z^2 = x^2+y^2$

$2z^2 = a^2$

$\rho\cos\varphi = z = \frac{1}{\sqrt{2}}a$

$a\cos\varphi = \frac{1}{\sqrt{2}}a \Rightarrow \boxed{\cos\varphi = \frac{1}{\sqrt{2}}}$

$\varphi = \pi/4$

• $x^2+y^2+z^2 = b^2 \ \wedge\ z^2 = x^2+y^2$

$2z^2 = b^2$

$b\cos\varphi = z = \frac{1}{\sqrt{2}}b \Rightarrow \boxed{\cos\varphi = \frac{1}{\sqrt{2}}}$

$\varphi = \pi/4$

$$\iiint_S 1\,dxdydz = \int_0^{2\pi}\int_0^{\pi/4}\int_a^b \rho^2 sen\varphi\, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3} sen\varphi \Big|_a^b d\varphi d\theta$$

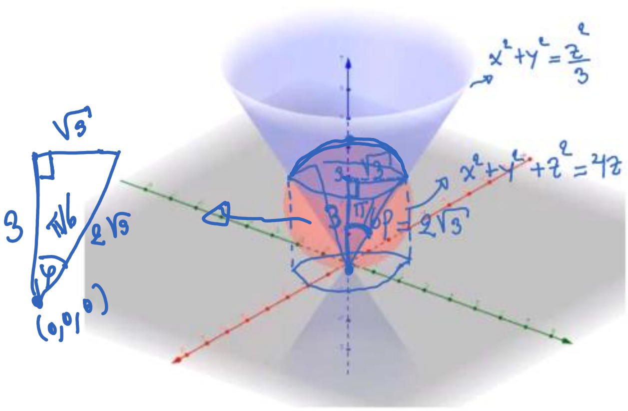
$$= \int_0^{2\pi}\int_0^{\pi/4}\left(\frac{b^3-a^3}{3}\right) sen\varphi\, d\varphi d\theta$$

$$= \int_0^{2\pi}\left(\frac{b^3-a^3}{3}\right)(-\cos\varphi)\Big|_0^{\pi/4} d\theta = \left(\frac{b^3-a^3}{3}\right)\Big(-\underbrace{\cos\left(\frac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}}} + \underbrace{\cos(0)}_{1}\Big)2\pi = \left(\frac{b^3-a^3}{3}\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)2\pi\, u^3$$

$$= \left(\frac{b^3-a^3}{3}\right)(2-\sqrt{2})\pi\, u^3$$

EJEMPLO 2:

Calcule el volumen del sólido limitado por $z^2 \ge 3x^2+3y^2$, $x^2+y^2+z^2 \le 4z$.



• Intersección

$\underbrace{x^2+y^2}= \frac{z^2}{3} \ \wedge\ x^2+y^2+z^2 = 4z$

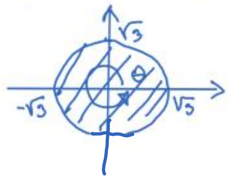
$\frac{z^2}{3} + \frac{3z^2}{3} - 4z = 0$

$\frac{4z^2}{3} - 4z = 0$

$4z\left(\frac{z}{3} - 1\right) = 0$

$\boxed{z=0}$, $\boxed{z=3}$

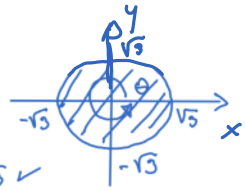
con $z = 3$: $x^2+y^2 = 3$



$\boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/6 \end{array}}$

$0 \le z \le \underline{\text{Esfera}}$

$0 \leq z \leq$ Esfera



$0 \leq \theta \leq 2\pi$

$0 \leq \varphi \leq \pi/6$

$\boxed{0 \leq \rho \leq 4\cos\varphi}$

- $x^2 + y^2 + z^2 = 4z$

$\rho^2 = 4\rho\cos\varphi$

$\boxed{\rho = 4\cos\varphi}$

$\varphi = \pi/6 \rightarrow \rho = 4\left(\frac{3}{2\sqrt{3}}\right) = 2\sqrt{3}$ ✓

$\varphi = 0 \rightarrow \rho = 4.1 = 4$ ✓

$$\iiint_S 1\, dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{4\cos\varphi} \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\left[\frac{\rho^3}{3}\operatorname{sen}\varphi\right]_0^{4\cos\varphi} d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6} \frac{64}{3}\cos^3\varphi \operatorname{sen}\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi} -\frac{64}{3}\cdot\frac{\cos^4\varphi}{4}\Bigg|_0^{\pi/6} d\theta = -\frac{64}{3}(2\pi)\left(\frac{\left(\cos\frac{\pi}{6}\right)^4}{4} - \frac{(\cos 0)^4}{4}\right) = -\frac{64}{3}(2\pi)\left(\frac{9}{64} - \frac{16}{64}\right) = -\frac{64}{3}(2\pi)\left(\frac{-7}{64}\right)$$

$$= \frac{14}{3}\pi u^3$$