SOLUCION DEL SEGUNDO EXAMEN DE SIMULACRO

Se la ecuación $F(x,y,z) = z^3 + x^2 z + y^2 z + xy - 8 = 0$

a. Pruebe que en un entorno del punto $P(0,0,2)$ puede definirse la variable z como función $z = z(x,y)$ de las otras dos variables x e y.

b. Pruebe que (0,0) es un punto crítico de z(x,y) y clasifique.

c. Sea C la curva intersección de la superficie de ecuación $F(x,y,z) = 0$ con el plano de ecuación z =1 y sea $f(x,y) = x^2 + y^2$ . Halle los extremos relativos de f sujetos a la restricción C.

Solución

a) ✓ $F(0,0,2) = 0$

✓ $\frac{\partial F}{\partial z}(x,y,z) = 3z^2 + x^2 + y^2$ ; $\frac{\partial F}{\partial y}(x,y,z) = 2yz + x$ ; $\frac{\partial F}{\partial x}(x,y,z) = 2xz + y$ son continuas cerca de (0,0,2).

✓ $\frac{\partial F}{\partial z}(0,0,2) = 12 \neq 0$

$\Rightarrow$ por el T.F.I $z = f(x,y)$, $\forall (x,y) \in B_\delta(0,0)$

Además

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)} \quad ; \quad \frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)}$$

$$\frac{\partial z}{\partial x}(x,y) = -\frac{2xz + y}{3z^2 + x^2 + y^2} \quad ; \quad \frac{\partial z}{\partial y}(x,y) = -\frac{2yz + x}{3z^2 + x^2 + y^2}$$

b) $$\frac{\partial z}{\partial x}(0,0) = -\frac{\frac{\partial F}{\partial x}(0,0,2)}{\frac{\partial F}{\partial z}(0,0,2)} = -\frac{0}{12} = 0$$

$$\frac{\partial z}{\partial y}(0,0) = -\frac{\frac{\partial F}{\partial y}(0,0,2)}{\frac{\partial F}{\partial z}(0,0,2)} = -\frac{0}{12} = 0$$

$\Rightarrow$ (0,0) es punto crítico de z(x,y).

c) max/min $f(x,y) = x^2 + y^2$

s.a. $x^2 + y^2 + xy - 7 = 0$

• $L(x,y,\lambda) = x^2 + y^2 + \lambda(x^2 + y^2 + xy - 7)$

• $$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \lambda(2x + y) = 0 \\ \frac{\partial L}{\partial y} &= 2y + \lambda(2y + x) = 0 \end{aligned}\right\}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + xy - 7 = 0$$

$\lambda = -\frac{2x}{2x+y} = -\frac{2y}{2y+x}$

$-\cancel{4xy} - 2x^2 = -\cancel{4xy} - 2y^2$

$x^2 = y^2$

$x = y \ \vee \ x = -y$

• $x = -y \Rightarrow 2y^2 - y^2 - 7 = 0 \Rightarrow y = \pm\sqrt{7} \Rightarrow (\sqrt{7}, -\sqrt{7})$ ; $(-\sqrt{7}, \sqrt{7})$

- $x = y \Rightarrow 3y^2 - 7 = 0 \Rightarrow y = \pm\sqrt{\frac{7}{3}} \Rightarrow \left(\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right); \left(-\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$

- $f(\sqrt{7}, -\sqrt{7}) = 14$
- $f(-\sqrt{7}, \sqrt{7}) = 14$

$>$ valores máximos

- $f\left(\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right) = 14/3$
- $f\left(-\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right) = \frac{14}{3}$

$>$ valores mínimos

Sea $f : R^2 \to R$ una función con segundas derivadas parciales continuas. Considere la función $F(u,v) = f(uv, \frac{1}{2}(u^2 - v^2))$. Demuestre que

$(u^2+v^2)[(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2] = (\frac{\partial F}{\partial u})^2 + (\frac{\partial F}{\partial v})^2$

Solución

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\cdot v + \frac{\partial f}{\partial y}\cdot u$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x}\cdot u + \frac{\partial f}{\partial y}\cdot(-v)$$

$$\Rightarrow \left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 \cdot v^2 + \left(\frac{\partial f}{\partial y}\right)^2 \cdot u^2 + \cancel{\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}}uv + \left(\frac{\partial f}{\partial x}\right)^2 u^2 + \left(\frac{\partial f}{\partial y}\right)^2 v^2 - \cancel{\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}}uv$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 (u^2+v^2) + \left(\frac{\partial f}{\partial y}\right)^2 (u^2+v^2)$$

$$= \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right](u^2+v^2)$$