Cambio de variables en integrales triples

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CAMBIO DE VARIABLES EN INTEGRALES TRIPLES

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Sean R y S las regiones correspondientes bajo la transformación T uno a uno del espacio UVW al espacio XYZ, donde las funciones coordenadas de T son

$$x = f(u,v,w), \qquad y = g(u,v,w), \qquad z = h(u,v,w)$$

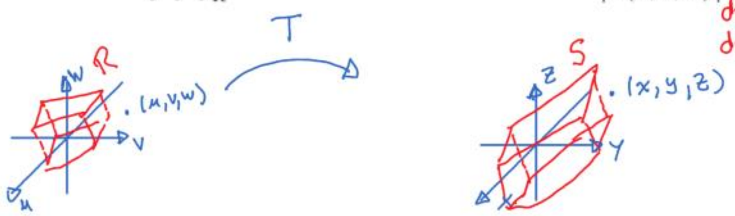
El jacobiano de la transformación T es:

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Fórmula para el cambio de variables en integrales triples

$$\iiint_S F(x,y,z)\,dzdydx = \iiint_R F(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \boxed{dudvdw}$$

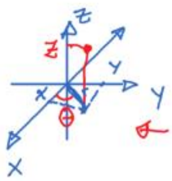
dvdudw
dwdudv



INTEGRALES TRIPLES EN COORDENADAS CILINDRICAS

$x = r\cos(\theta)\,;\ y = r\,sen(\theta);$ z=z ; Jacobiano $J(r,\theta,z) = r$

$$\iiint_D f(x,y,z)dV = \iiint_U f(r\cos(\theta), r\,sen(\theta), z)\,r\,dzdrd\theta$$



Coordenadas polares en el plano XY

Obs: Las variables pueden cambiar de papeles de acuerdo como se presente la región de integración

Ejemplo 1. GUIA CVV PAG 135 (13)

Calcular $\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz$, donde D es el sólido limitado por $z = \sqrt{x^2+y^2}$, $z = 1$. Rpta. $\pi/6$.

dz dy dx

plano

$Z^2 = x^2 + y^2$

cono plano

• $\sqrt{x^2+y^2} \le z \le 1$

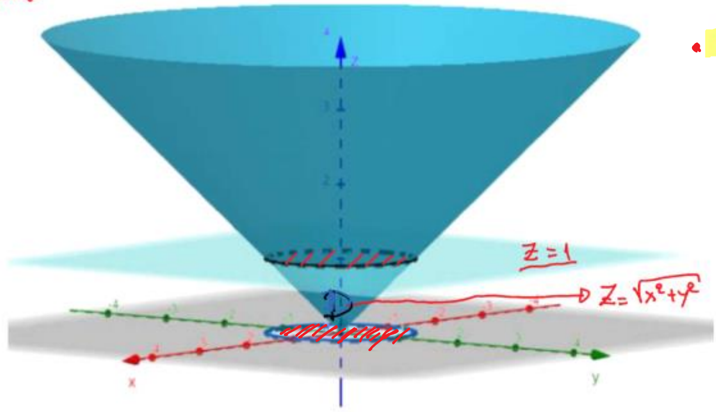
$r \le z \le 1$

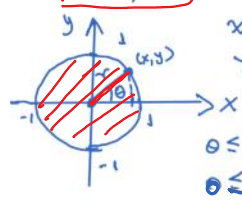
• Intersección:

$z = \sqrt{x^2+y^2} \wedge z = 1$

$\sqrt{x^2+y^2} = 1$

$\boxed{x^2+y^2=1}$





$x = r\cos\theta$
$y = r\,sen\theta$

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$x = r\cos\theta$, $y = r\,\text{sen}\,\theta$ $\longrightarrow$ $r \le z \le 1$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$J(r,\theta,z) = r$

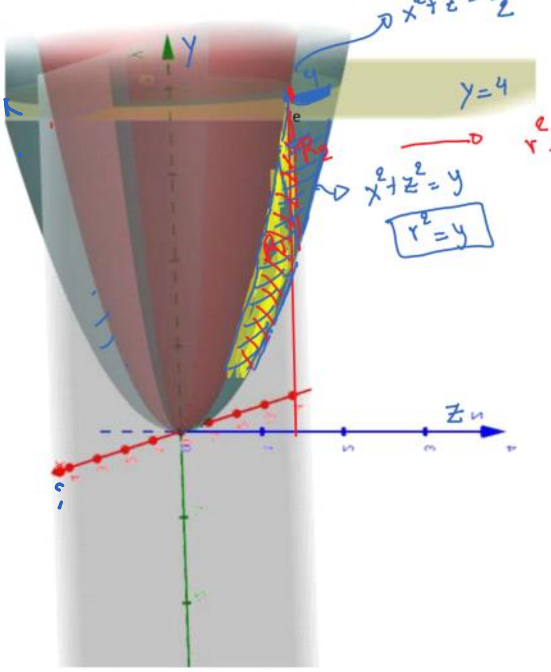
$$\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_r^1 \sqrt{(r\cos\theta)^2+(r\,\text{sen}\,\theta)^2}\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_r^1 r^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \Big[r^2 z\Big]_r^1 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{3} - \frac{1}{4}\right) d\theta = \frac{1}{12}\,\theta\Big|_0^{2\pi} = \frac{\pi}{6}$$

Ejemplo 2. Calcule el volumen del sólido limitado por $x^2+z^2 \le y$, $2x^2+2z^2 \ge y$, $y = 4$.

Solución

- $x^2+z^2 = y \rightarrow$ paraboloide
- $2x^2+2z^2 = y \rightarrow$ paraboloide
  $x^2+z^2 = \frac{y}{2}$



$r^2 \le y \le 4$

- Intersección:

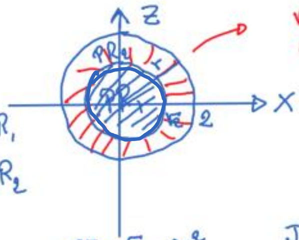
$*$ $x^2+z^2 = y \;\wedge\; y = 4$

$x^2+z^2 = 4$

$*$ $2(x^2+z^2) = y \;\wedge\; y = 4$

$x^2+z^2 = \frac{4}{2}$

$x^2+z^2 = 2$



$\sqrt{2} \le r \le 2$
$0 \le \theta \le 2\pi$

PR_1: proy. de R_1

PR_2: proy. de R_2

$x = r\cos\theta$
$z = r\,\text{sen}\,\theta$
$y = y$
$J(r,\theta,y) = r$

$$V_1 = \iiint_{R_1} 1\,dV = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{2r^2} r\,dy\,dr\,d\theta$$

$$V_1 = \int_0^{2\pi}\int_0^{\sqrt{2}} \Big[ry\Big]_{r^2}^{2r^2} dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta$$

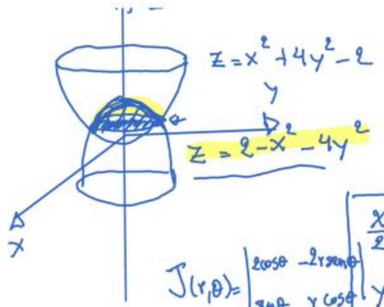
$$V_1 = 2\pi$$

$$V_2 = \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{r^2}^{4} r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_{\sqrt{2}}^{2} r(4-r^2)\,dr\,d\theta$$

$$V_2 = \int_0^{2\pi}\int_{\sqrt{2}}^{2} 4r - r^3\,dr\,d\theta = \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_{\sqrt{2}}^{2} d\theta = \int_0^{2\pi} 8-4-(4-1)\,d\theta = 2\pi$$

$$V_T = V_1 + V_2 = 4\pi\,u^3$$

3. Hallar el volumen del sólido en $\mathbb{R}^3$ limitado por las gráficas de las superficies $z = x^2+4y^2-2$ y $z = 2-x^2-4y^2$.



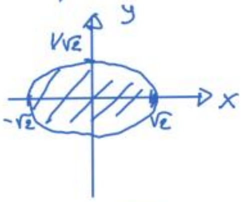
$z = x^2+4y^2-2 \;\wedge\; z = 2-x^2-4y^2$

$x^2+4y^2-2 = 2-x^2-4y^2$

$2x^2+8y^2-4=0$

$\Leftarrow \frac{x^2}{4}+y^2=\frac{1}{2}$

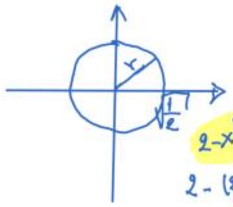
$\frac{x^2}{2}+\frac{y^2}{\frac{1}{2}}=1$



$\boxed{\begin{aligned}\frac{x}{2}&=r\cos\theta\\ y&=r\,\text{sen}\,\theta\end{aligned}}$

$J(r,\theta)=\begin{vmatrix}2\cos\theta & -2r\,\text{sen}\,\theta\\ \text{sen}\,\theta & r\cos\theta\end{vmatrix} = 2r$

$\Rightarrow \left(\frac{x}{2}\right)^2+y^2=\frac{1}{2} \Rightarrow (r\cos\theta)^2+(r\,\text{sen}\,\theta)^2=\frac{1}{2} \Rightarrow r^2=1/2 \Rightarrow r=\sqrt{\frac{1}{2}}$



$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{\frac{1}{2}}$

$2-x^2-4y^2 \ge z \ge x^2+4y^2-2$

$2-(2r\cos\theta)^2-4(r\,\text{sen}\,\theta)^2 \ge z \ge (2r\cos\theta)^2+4(r\,\text{sen}\,\theta)^2-2$

$2-4r^2 \ge z \ge 4r^2-2$

$$V=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{4r^2-2}^{2-4r^2} 2r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} 2r\left(2-4r^2-4r^2+2\right)dr\,d\theta = \int_0^{2\pi}\int_0^{1/\sqrt{2}} 2r\,(4-8r^2)\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{1/\sqrt{2}} 8r-16r^3\,dr\,d\theta = 8\int_0^{2\pi} \frac{r^2}{2}-\frac{2r^4}{4}\Big|_0^{1/\sqrt{2}}d\theta = 8\int_0^{2\pi}\frac{1}{4}-\frac{2}{16}\,d\theta = 8\cdot\frac{1}{8}\cdot 2\pi = 2\pi$$