Solución integrales triples y línea

Dado el campo de fuerza $\vec{F} = (y^3 + 1, 3xy^2 + 1)$

a. ¿Es $\vec{F}$ conservativo? Si es conservativo halle la función potencial.

b. Halle el trabajo realizado para mover la partícula a lo largo de la circunferencia completa

c. Halle el trabajo realizado para mover un partícula desde el punto (0; 0) al (2; 0) a lo largo de la semicircunferencia $(x-1)^2 + y^2 = 1$ con $y \geq 0$

Solución :

a) Veamos $F(x,y) = (y^3+1, 3xy^2+1) \Rightarrow P(x,y) = y^3+1$, $Q(x,y) = 3xy^2+1$

Se tiene

$$\frac{\partial P}{\partial y} = 3y^2 = \frac{\partial Q}{\partial x}$$

$\therefore$ F es un campo conservativo

Ahora hallemos $f : \mathbb{R}^2 \to \mathbb{R}$ tal que $\underset{(*)}{\frac{\partial f}{\partial x} = P}$ y $\underset{(**)}{\frac{\partial f}{\partial y} = Q}$ $\quad (\nabla f = F)$

$(*)$ $\frac{\partial f}{\partial x} = y^3+1 \Rightarrow \int \frac{\partial f}{\partial x} dx = \int y^3+1\, dx \Rightarrow f(x,y) = xy^3 + x + h(y)$

$(**)$ $\frac{\partial f}{\partial y} = 3xy^2+1 \Rightarrow 3xy^2 + h'(y) = 3xy^2 + 1 \Rightarrow h'(y) = 1 \Rightarrow h(y) = y + C$

$\Rightarrow f(x,y) = xy^3 + x + y + C$ es función potencial de F.

b) $W = \oint_C F(x,y)d\lambda$; C : circunferencia completa

Como F es conservativo $\Rightarrow W = \oint_C F(x,y)d\lambda = 0$ puesto que C es curva cerrada

c) $W = \int_{(0,0)}^{(2,0)} F(x,y)\, d\lambda = f(2,0) - f(0,0) = 2 + C - (C) = 2$

Sea C la curva de intersección entre el plano $x - z = 1$ y el elipsoide $x^2 + 2y^2 + z^2 = 1$ recorrida en sentido antihorario vista desde la parte superior de z. Calcular $\oint_C \frac{1}{2}y^2dx + zdy + xdz$

Solución

Intersección : $x - z = 1 \ \wedge \ x^2 + 2y^2 + z^2 = 1$

$x - 1 = z \ \to \ x^2 + 2y^2 + (x-1)^2 = 1$

$x^2 + 2y^2 + x^2 - 2x + \cancel{1} = \cancel{1}$

$2x^2 - 2x + 2y^2 = 0 \ \to \ 2\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + 2y^2 = 0$

$\Rightarrow \ 2\left(x - \frac{1}{2}\right)^2 + 2y^2 = \frac{1}{2}$

$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \ \to \ x - \frac{1}{2} = \frac{1}{2}\cos t$, $0 \leq t \leq 2\pi$

$y = \frac{1}{2}\operatorname{sen} t$

$z = x - 1 = \frac{1}{2}\cos t + \frac{1}{2} - 1 = \frac{1}{2}\cos t - \frac{1}{2}$

$\lambda(t) = \left(\frac{1}{2}\cos t + \frac{1}{2}, \frac{1}{2}\operatorname{sen} t, \frac{1}{2}\cos t - \frac{1}{2}\right)$; $0 \leq t \leq 2\pi$

$\lambda'(t) = \left(-\frac{\operatorname{sen} t}{2}, \frac{\cos t}{2}, -\frac{\operatorname{sen} t}{2}\right)$

$\Rightarrow \oint_C \frac{1}{2}y^2 dx + z\,dy + x\,dz = \int_0^{2\pi} \frac{1}{2}\cdot\frac{1}{4}\operatorname{sen}^2 t \cdot\left(-\frac{\operatorname{sen} t}{2}\right)dt + \left(\frac{1}{2}\cos t - \frac{1}{2}\right)\frac{\cos t}{2}\,dt + \left(\frac{1}{2}\cos t - \frac{1}{2}\right)\left(-\frac{\operatorname{sen} t}{2}\right)dt$

$= \int_0^{2\pi} \left(-\frac{1}{16}(1 - \cos^2 t)\operatorname{sen} t + \frac{1}{4}\left(\frac{1 + \cos 2t}{2}\right) - \frac{1}{4}\cos t - \frac{1}{4}\operatorname{sen} t \cos t + \frac{1}{4}\operatorname{sen} t\right)dt$

$= \left[-\frac{1}{16}\left(\cos t + \frac{\cos^3 t}{3}\right) + \frac{1}{4}\left(\frac{t}{2} + \frac{\operatorname{sen}(2t)}{4}\right) - \frac{1}{4}\operatorname{sen} t - \frac{1}{4}\cdot\frac{\operatorname{sen}^2 t}{2} - \frac{\cos t}{4}\right]_0^{2\pi}$

$= \cancel{-\frac{1}{16}\left(1 + \frac{1}{3}\right)} + \frac{1}{4}(\pi) - \cancel{\frac{1}{4}} - \left(\cancel{-\frac{1}{16}\left(1 + \frac{1}{3}\right)} - \cancel{\frac{1}{4}}\right) = \frac{\pi}{4}$

$$= -\frac{1}{16}\cancel{\left(1+\frac{1}{3}\right)} + \frac{1}{4}(\pi) - \cancel{\frac{1}{4}} - \left(\cancel{-\frac{1}{16}\left(1+\frac{1}{3}\right)} - \cancel{\frac{1}{4}}\right) = \frac{\pi}{4} \text{ //}$$

Sea C la curva de intersección entre el plano $x+y+z=1$ y el cilindro $x^2+y^2=1$ recorrida en sentido antihorario vista desde la parte superior de z. Calcular $\oint_C xydx + yzdy + xzdz$.

Solución

Intersección : $x+y+z=1 \;\wedge\; x^2+y^2=1$

$$z = 1-x-y \;\wedge\; \underbrace{x^2+y^2=1}$$

$\lambda(t) = (\cos t, \operatorname{sen} t, 1-\cos t - \operatorname{sen} t)$; $0\le t\le 2\pi$

$\lambda'(t) = (-\operatorname{sen} t, \cos t, \operatorname{sen} t - \cos t)$;

$$\Rightarrow \oint_C xy\,dx + yz\,dy + xz\,dz = \int_0^{2\pi} \cos t\operatorname{sen} t(-\operatorname{sen} t)dt + \operatorname{sen} t(1-\cos t-\operatorname{sen} t)\cos t\,dt + \cos t(1-\cos t-\operatorname{sen} t)(\operatorname{sen} t-\cos t)dt$$

$$= \int_0^{2\pi} \overset{1}{-\operatorname{sen}^2 t\cos t} + \operatorname{sen} t\cos t - \cancel{\operatorname{sen} t\cos^2 t} - \overset{2}{\operatorname{sen}^2 t\cos t} - \cos^2 t + \cos^3 t + \cancel{\cos^2 t\operatorname{sen} t} + \cos t\operatorname{sen} t - \cos^2 t\operatorname{sen} t - \cos t\operatorname{sen}^2 t\,dt$$

$$= \int_0^{2\pi} -2\operatorname{sen}^2 t\cos t + 2\operatorname{sen} t\cos t - \frac{(1+\cos 2t)}{2} + (1-\operatorname{sen}^2 t)\cos t - \cos^2 t\operatorname{sen} t\,dt$$

$$= -\frac{3\operatorname{sen}^3 t}{3} + \operatorname{sen}^2 t - \frac{t}{2} - \frac{\operatorname{sen}(2t)}{4} + \operatorname{sen} t - \frac{\operatorname{sen}^3 t}{3} + \frac{\cos^3 t}{3}\Big|_0^{2\pi}$$

$$= -\frac{2\pi}{2} + \frac{1}{3} - \left(\frac{1}{3}\right) = -\pi \text{ //}$$

Calcular $\iiint_S x e^{(x^2+y^2+z^2)}\,dxdydz$, donde S es el sólido comprendido por $x^2+y^2+z^2 \ge 1$ y $x^2+y^2+z^2 \le 4$ en el primer octante.

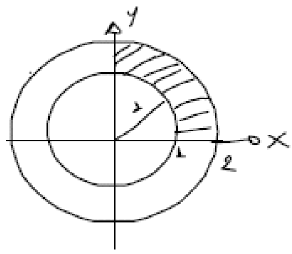
Solución

$x = \rho sen(\varphi)cos\theta,\; y = \rho sen(\varphi)sen\theta,\; z = \rho\cos(\varphi),\; J(\rho,\varphi,\theta) = \rho^2 sen(\varphi)$

- $x^2+y^2+z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$
- $x^2+y^2+z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$

$> \quad 1\le\rho\le 2$

$0\le\theta\le\pi/2$



Primer octante $\to 0\le\varphi\le\pi/2$

$$\Rightarrow \int_1^2\int_0^{\pi/2}\int_0^{\pi/2} \rho\operatorname{sen}\varphi\cos\theta\, e^{\rho^2}\cdot\rho^2\operatorname{sen}\varphi\, d\theta d\varphi d\rho = \int_1^2\int_0^{\pi/2}\int_0^{\pi/2} \rho^3\operatorname{sen}^2\varphi\, e^{\rho^2}\cos\theta\, d\theta d\varphi d\rho$$

$$= \int_1^2\int_0^{\pi/2}\left[\rho^3\operatorname{sen}^2\varphi\, e^{\rho^2}\operatorname{sen}\theta\right]_0^{\pi/2} d\varphi d\rho = \int_1^2\int_0^{\pi/2} \rho^3\operatorname{sen}^2\varphi\, e^{\rho^2}\, d\varphi\, d\rho$$

$$= \int_1^2\int_0^{\pi/2} \rho^3 e^{\rho^2}\left(\frac{1-\cos 2\varphi}{2}\right)d\varphi d\rho = \int_1^2\left[\rho^3 e^{\rho^2}\left(\frac{\varphi}{2} - \frac{\operatorname{sen}2\varphi}{4}\right)\right]_0^{\pi/2} d\rho = \int_1^2 \rho^3 e^{\rho^2}\frac{\pi}{4}\,d\rho = \frac{\pi}{4}\int_1^2 \rho^3 e^{\rho^2} d\rho$$

$$= \frac{\pi}{4}\left[\frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}\right]_1^2$$

$$= \frac{\pi}{4}\left(2e^4 - \frac{1}{2}e^4 - \cancel{\frac{1}{2}e} + \cancel{\frac{1}{2}e}\right) = \frac{3\pi e^4}{8} \text{ //}$$

O.A.

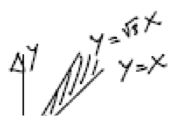
$u = \rho^2 \qquad dv = \frac{2}{2}\rho e^{\rho^2}$

$du = 2\rho\, d\rho \qquad v = \frac{1}{2}e^{\rho^2}$

$$\rho^2\frac{1}{2}e^{\rho^2} - \int \rho e^{\rho^2} d\rho = \frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}$$

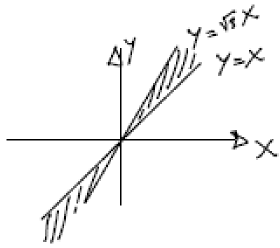
Calcule la integral triple $\iiint_\Omega \sqrt{x^2+y^2+z^2}\,dV$, donde Ω es el sólido acotado por las superficies

$1\le x^2+y^2+z^2\le 9,\;\; 0\le z\le\sqrt{x^2+y^2+z^2}$ y $x\le y\le\sqrt{3}x$.



Calcule la integral triple $\iiint_{\Omega} \sqrt{x^2+y^2+z^2}\, dV$, donde Ω es el sólido acotado por las superficies

$1 \le x^2+y^2+z^2 \le 9$, $\; 0 \le z \le \sqrt{x^2+y^2+z^2}$ y $x \le y \le \sqrt{3}x$.



Solución

$x = \rho sen(\varphi) cos\theta,\; y = \rho sen(\varphi) sen\theta,\; z = \rho\cos(\varphi),\; J(\rho,\varphi,\theta) = \rho^2 sen(\varphi)$

- $1 \le x^2+y^2+z^2 \le 9 \Rightarrow 1 \le \rho^2 \le 9 \Rightarrow 1 \le \rho \le 3$
- $0 \le z \le \sqrt{x^2+y^2+z^2} \Rightarrow \boxed{0 \le \rho\cos(\varphi) \le \rho} \Rightarrow 0 \le \cos(\varphi) \le 1 \Rightarrow 0 \le \varphi \le \pi/2$
- $x \le y \le \sqrt{3}x \Rightarrow \rho sen(\varphi)\cos\theta \le \rho sen(\varphi) sen\theta \le \sqrt{3}\rho sen(\varphi)\cos\theta \Rightarrow \cos\theta \le sen\theta \le \sqrt{3}\cos\theta \Rightarrow 1 \le \tan\theta \le \sqrt{3}$

$\Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{3}$

$$\Rightarrow \iiint_{\Omega} \sqrt{x^2+y^2+z^2}\, dV = 2\int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\int_1^3 \rho\cdot\rho^2 sen(\varphi)\, d\rho\, d\varphi\, d\theta = 2\int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\left[\frac{\rho^4}{4} sen(\varphi)\right]_1^3 d\varphi\, d\theta = 2\int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\frac{80}{4} sen\varphi\, d\varphi\, d\theta = 2\int_{\pi/4}^{\pi/3}\left[-\frac{80}{4}\cos(\varphi)\right]_0^{\pi/2} d\theta$$

$$= 2\int_{\pi/4}^{\pi/3}\frac{80}{4}\, d\theta = 2\left(\frac{80}{4}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right) = 2\left(\frac{80}{4}\left(\frac{\pi}{12}\right)\right) = \frac{40\pi}{12} = \frac{20\pi}{6} = \frac{10\pi}{3}$$