Aplicaciones de integrales triples

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Aplicaciones:

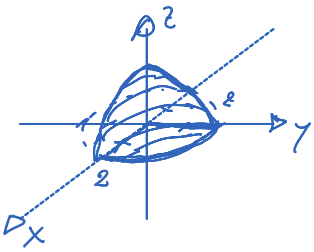
1. Si $F(x,y,z)=1$, entonces $V=\iiint\limits_D dV$ es el volumen del sólido D.
2. Si $\rho=\rho(x,y,z)$ es la densidad, entonces $m=\iiint\limits_D \rho(x,y,z)\,dV$, es la masa del sólido D.
3. Las integrales $M_{xy}=\iiint\limits_D z\,\rho(x,y,z)\,dV$, $M_{zx}=\iiint\limits_D y\,\rho(x,y,z)\,dV$, $M_{yz}=\iiint\limits_D x\,\rho(x,y,z)\,dV$ son los momentos de primer orden del sólido.
4. Las coordenadas del centro de masa de D son $\overline{x}=\dfrac{M_{yz}}{m}$, $\overline{y}=\dfrac{M_{xz}}{m}$, $\overline{z}=\dfrac{M_{xy}}{m}$.
   Si $\rho=Constante$, el centro de masa se llama centroide del sólido.

Ejemplos

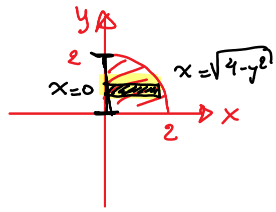
1. Evaluar la integral $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\,dz\,dx\,dy$ usando coordenadas esféricas.

Solución

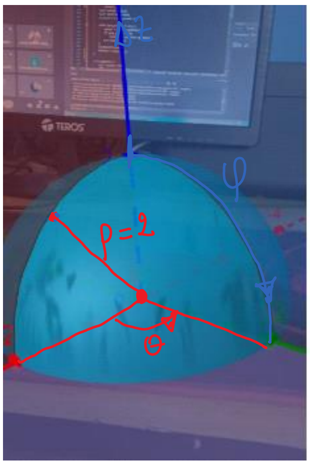
Región de integración D: $\begin{cases} 0\le y\le 2 \\ 0\le x\le\sqrt{4-y^2} \\ 0\le z\le\sqrt{4-x^2-y^2}\end{cases}$ $\left.\begin{matrix}\\ \\ \end{matrix}\right]$ Plano xy



- $z=\sqrt{4-x^2-y^2}$
- $z=0$
- Intersección



- $x=\sqrt{4-y^2} \Rightarrow x^2=4-y^2 \Rightarrow x^2+y^2=4$
- $z=\sqrt{4-x^2-y^2} \Rightarrow z^2=4-x^2-y^2 \Rightarrow x^2+y^2+z^2=4$ esfera



$0\le\theta\le\pi/2$

$0\le\varphi\le\pi/2$

$0\le\rho\le 2$

$$\boxed{x^2+y^2+z^2=\rho^2}$$

$x=\rho sen(\varphi)\cos\theta,\; y=\rho sen(\varphi) sen\theta,\; z=\rho\cos(\varphi),\; J(\rho,\varphi,\theta)=\rho^2 sen(\varphi)$

$$\iiint\limits_D \frac{1}{x^2+y^2+z^2}\,dz\,dx\,dy=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \frac{1}{\cancel{\rho^2}}\cdot\cancel{\rho^2}sen(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2} \rho\, sen\varphi\Big|_0^2\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2} 2sen\varphi\,d\varphi\,d\theta=\int_0^{\pi/2} -2\cos\varphi\Big|_0^{\pi/2}\,d\theta$$

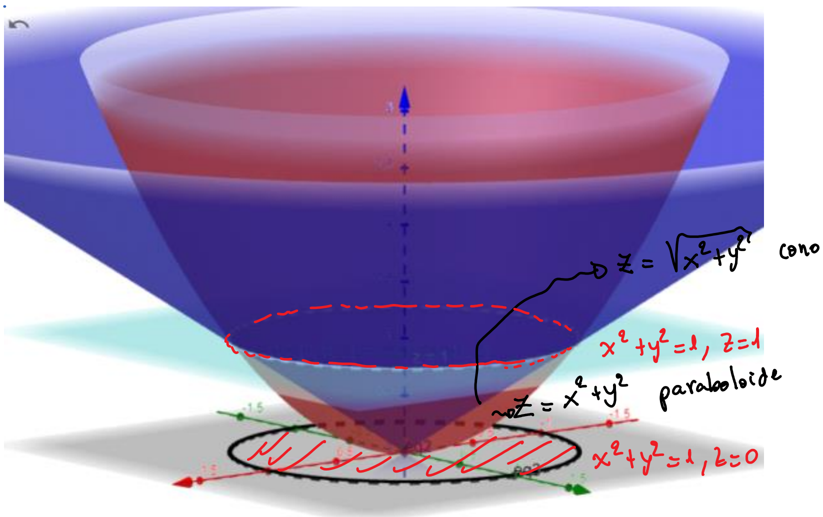
$$=\int_0^{\pi/2} -2\cancelto{0}{\cos(\pi/2)}+2\cancelto{1}{\cos(0)}\,d\theta=2\,\theta\Big|_0^{\pi/2}=\pi$$

2. Encontrar el centro de masa del sólido dentro del paraboloide $x^2+y^2=z$ y fuera del

cono $x^2+y^2=z^2$, p es la densidad de volumen constante K slug/p^3.

Solución

masa : $m = \iiint\limits_D \rho(x,y,z)\,dV$



• Intersección:

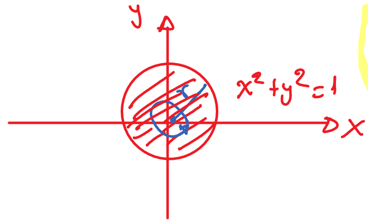
$x^2+y^2=z \quad \wedge \quad x^2+y^2=z^2$

$z=z^2$

$0=z^2-z=z(z-1)$

$\Rightarrow z=0 \quad \vee \quad z=1$

• Si $z=1 \Rightarrow x^2+y^2=1,\ z=1$



$x=r\cos\theta$
$y=r\,\text{sen}\,\theta$
$z=z$

$$D\left\{\begin{array}{l} 0\le r\le 1\\ 0\le\theta\le 2\pi\\ \underbrace{r^2}_{\text{paraboloide}}\le z\le \underbrace{r}_{\text{cono}}\end{array}\right.$$

$$\underset{\text{masa}}{m} = \iiint\limits_D \overset{\text{densidad}}{K}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} K\cdot \overset{\text{Jacobiano}}{r}\cdot dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 Krz\Big|_{r^2}^{r} dr\,d\theta = \int_0^{2\pi}\int_0^1 Kr(r-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 Kr^2-Kr^3\,dr\,d\theta = \int_0^{2\pi} \frac{Kr^3}{3}-\frac{Kr^4}{4}\Big|_0^1 d\theta = \frac{K}{12}\int_0^{2\pi} d\theta = \frac{2\pi K}{12} = \frac{\pi K}{6}$$

• Las integrales $M_{xy}=\int\int\limits_D\int z\,\rho(x,y,z)\,dV$, $M_{zx}=\int\int\limits_D\int y\,\rho(x,y,z)\,dV$, $M_{yz}=\int\int\limits_D\int x\,\rho(x,y,z)\,dV$ son los momentos de primer orden del sólido.

• Las coordenadas del centro de masa de D son $\overline{x}=\dfrac{M_{yz}}{m}$, $\overline{y}=\dfrac{M_{xz}}{m}$, $\overline{z}=\dfrac{M_{xy}}{m}$.
Si $\rho = Constante$, el centro de masa se llama centroide del sólido.

Así

$$\cdot\ M_{yz} = \iiint\limits_D xK\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\cos\theta\cdot K\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 Kr^2\cos\theta\, z\Big|_{r^2}^{r} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 Kr^2\cos\theta\,(r-r^2)\,dr\,d\theta = K\int_0^{2\pi}\int_0^1 (r^3-r^4)\cos\theta\,dr\,d\theta = K\int_0^{2\pi}\left(\frac{r^4}{4}-\frac{r^5}{5}\right)\cos\theta\Big|_0^1 d\theta$$

$$= K\int_0^{2\pi}\frac{1}{20}\cos\theta\,d\theta = \frac{K}{20}\,\text{sen}\,\theta\Big|_0^{2\pi} = \frac{K}{20}\left(\cancel{\text{sen}(2\pi)}^{\,0}-\cancel{\text{sen}(0)}^{\,0}\right) = 0$$

$$\Rightarrow \overline{x} = \frac{M_{yz}}{m} = \frac{0}{\frac{\pi K}{6}} = 0$$

$$\cdot\ M_{xz} = \iiint\limits_D yK\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,\text{sen}\,\theta\cdot K\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 K\,\text{sen}\,\theta\cdot r^2 z\Big|_{r^2}^{r} dr\,d\theta$$

$$\cdot\ M_{xz} = \iiint_D yK\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,\text{sen}\,\theta . K . r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 K\,\text{sen}\,\theta . r^2 z \Big|_{r^2}^{r} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 K\,\text{sen}\,\theta . r^2(r-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 K\,\text{sen}\,\theta\,(r^3-r^4)\,dr\,d\theta = \int_0^{2\pi} K\,\text{sen}\,\theta\left(\frac{r^4}{4}-\frac{r^5}{5}\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{K\,\text{sen}\,\theta}{20}\,d\theta = \frac{K}{20}(-\cos\theta)\Big|_0^{2\pi} = \frac{K}{20}\left(-\cos(2\pi)+\cos(0)\right) = 0$$

$$\Rightarrow \bar{y} = \frac{M_{xz}}{m} = 0$$

$$\cdot\ M_{xy} = \iiint_D zK\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} zKr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{z^2}{2}Kr\Big|_{r^2}^{r} dr\,d\theta = \int_0^{2\pi}\int_0^1\left(\frac{r^2-r^4}{2}\right)Kr\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\frac{K}{2}\left(r^3-r^5\right)dr\,d\theta = \int_0^{2\pi}\frac{K}{2}\left(\frac{r^4}{4}-\frac{r^6}{6}\right)\Big|_0^1 d\theta = \frac{K}{24}\int_0^{2\pi}d\theta = \frac{K(2\pi)}{24} = \frac{K\pi}{12}$$

$$\Rightarrow \bar{z} = \frac{M_{xy}}{m} = \frac{\frac{K\pi}{12}}{\frac{K\pi}{6}} = \frac{1}{2}\ //$$