SOLUCIÓN EX PARCIAL 1

lunes, 24 de mayo de 2021 06:39

Considere la función $f(x,y) = \sqrt[3]{4 - x^2 - 4y^2}$

a. Determine su dominio y rango. Represente gráficamente su dominio.

b. Grafique las curvas de contorno y curvas de nivel de la función para k=-1,0,1. $\to z = K \in \text{Ran} f = [0,2]$

b. ¿La función es continua en (0,1)? Justifique su respuesta.

Solución

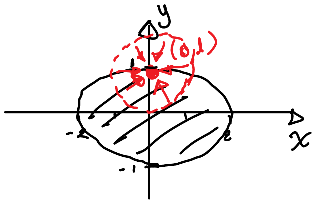
(a) Dominio :

$4 - x^2 - 4y^2 \geq 0$

$-x^2 - 4y^2 \geq -4$

$x^2 + 4y^2 \leq 4$

$\frac{x^2}{4} + y^2 \leq 1$



Rango:

$x^2 + 4y^2 \geq 0$

$-x^2 - 4y^2 \leq 0$

$0 \leq 4 - x^2 - 4y^2 \leq 4$

$0 \leq \sqrt{4 - x^2 - 4y^2} \leq 2$

$\text{Dom} f = \{ (x,y) \in \mathbb{R}^2 / \frac{x^2}{4} + y^2 \leq 1 \}$; $\text{Ran} f = [0,2]$

(b) $K = -1 \notin \text{Ran} f \Rightarrow$ no hay curvas de contorno $\Rightarrow$ no hay curva de nivel.

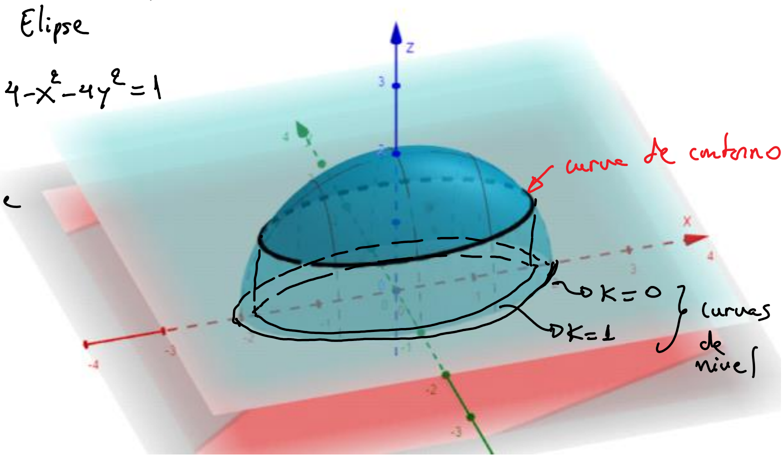
$K = 0 \in \text{Ran} f \Rightarrow \sqrt{4 - x^2 - 4y^2} = 0 \Rightarrow 4 - x^2 - 4y^2 = 0$

$\frac{x^2}{4} + y^2 = 1$ Elipse

$K = 1 \in \text{Ran} f \Rightarrow \sqrt{4 - x^2 - 4y^2} = 1 \Rightarrow 4 - x^2 - 4y^2 = 1$

$x^2 + 4y^2 = 3$

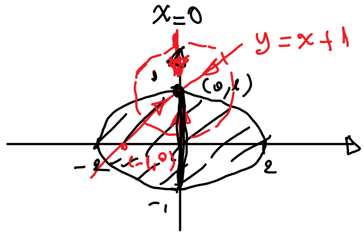
$\frac{x^2}{3} + \frac{y^2}{\frac{3}{4}} = 1$ Elipse

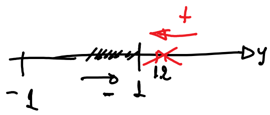


(c) ¿ f es continua en (0,1)?

(i) $f(0,1) = 0$ ✓

(ii) ¿$\exists \lim_{(x,y) \to (0,1)} f(x,y)$?





Camino eje y :

$\lim_{(x,y)\to(0,1)} \sqrt{4 - x^2 - 4y^2} = \lim_{\substack{x=0 \\ y \to 1}} \sqrt{4 - 0^2 - 4y^2} = \underbrace{\lim_{y \to 1} \sqrt{4 - 4y^2}}$

$\lim_{y \to 1^-} \sqrt{4 - 4y^2} = 0$

$\lim_{y \to 1^+} \sqrt{4 - 4y^2} \nexists$

$\therefore$ f no es continua en (0,1)

Determine los puntos donde la función es continua $f(x,y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$. Si presenta puntos de discontinuidad diga de que tipo es este.

Determine los puntos donde la función es continua $f(x,y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$. Si hay puntos donde es discontinua diga de que tipo son estos.

Solución

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2} & ,\ (x,y)\neq(0,0)\\ 0\ ✓ & ,\ (x,y)=(0,0)\ ✓\end{cases}$$

i) ¿f es continua en $(0,0)$?

- $f(0,0)=0$
- $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{x^2+xy+y^2}$

$$\lim_{\substack{x=0\\ y\to 0}}\frac{0}{y^2}=\lim_{y\to 0}0=0$$

$$\lim_{\substack{y=0\\ x\to 0}}\frac{0}{x^2}=0$$

$$\lim_{\substack{x=y\\ y\to 0}}\frac{y^2}{y^2+y^2+y^2}=\lim_{y\to 0}\frac{1}{3}=1/3$$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$\therefore$ f presenta discontinuidad inevitable en $(0,0)$.

$\therefore$ f es continua $\forall (x,y)\neq(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & ,\ (x,y)\neq(0,0)\\ 0\ ✓ & ,\ (x,y)=(0,0)\ ✓\end{cases}$$

i) ¿f es continua en $(0,0)$?

- $f(0,0)=0$ ✓
- $$\lim_{(x,y)\to(0,0)}\frac{x^2y^3}{2x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{x^2y^3}{(\underbrace{\sqrt{2}x}_{r})^2+y^2}=\lim_{r\to 0}\frac{\left(\frac{r\cos\theta}{\sqrt{2}}\right)^2(r\,\text{sen}\,\theta)^3}{r^2}=\lim_{r\to 0}\frac{r^3}{2}\overbrace{\cos^2\theta\,\text{sen}^3\theta}^{\text{acotado}}=0$$

$$\sqrt{2}x=r\cos\theta$$
$$y=r\,\text{sen}\,\theta$$

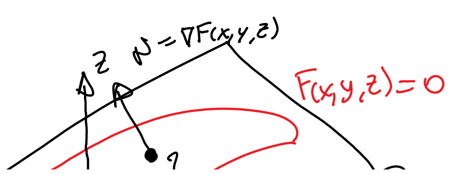
$\therefore$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$ así f es continua en $(0,0)$

$\therefore$ f es continua sobre $\mathbb{R}^2$.

Hallar la ecuación del plano tangente a la superficie $z^2-x^2-y^2+4x=0$ que sea perpendicular a la recta $L: x=3+4t; y=-2t; z=1+2t; t\in\mathbb{R}$.

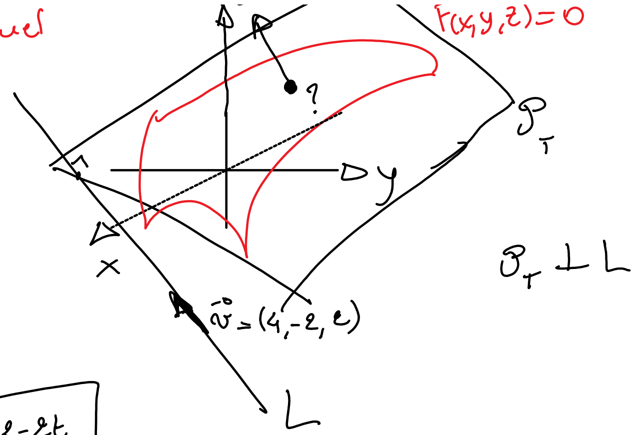
$$\left\{\begin{array}{l} F(x,y,z)=z^2-x^2-y^2+4x \\ w=0 \\ \text{hiperplano}\end{array}\right.$$

$\Rightarrow$ $z^2-x^2-y^2+4x=0$ superficie de nivel



$w = 0$ superficie de nivel

hiperplano



$\mathcal{P}_T \perp L$

. $\underset{\text{plano}}{\mathcal{P}_T} \perp \underset{\text{recta}}{L} \Rightarrow N \parallel \vec{v}$

$\Rightarrow \nabla F(x,y,z) = t\vec{v} \quad , \quad t \in \mathbb{R}$

$\Rightarrow (-2x+4, -2y, 2z) = t(4,-2,2)$

$\Rightarrow \begin{matrix} -2x+4 = 4t \\ -2y = -2t \\ 2z = 2t \end{matrix} \Rightarrow \begin{matrix} 2x = 4-4t \\ y = t \\ z = t \end{matrix} \Rightarrow \boxed{\begin{matrix} x = 2-2t \\ y = t \\ z = t \end{matrix}}$

. Luego en la superficie : $z^2 - x^2 - y^2 + 4x = 0$

$t^2 - (2-2t)^2 - t^2 + 4(2-2t) = 0$

$\cancel{t^2} - 4 + \cancel{8t} - 4t^2 - \cancel{t^2} + 8 - \cancel{8t} = 0 \Rightarrow 4 - 4t^2 = 0 \Rightarrow t = \pm 1$

Así $\boxed{\mathcal{P}_T : N \cdot \big((x,y,z) - (x_0,y_0,z_0) \big) = 0}$

con $t = 1$: $(4,-2,2)\cdot(x-0, y-1, z-1) = 0 \Rightarrow 4x - 2y + \cancel{2} + 2z - \cancel{2} = 0$

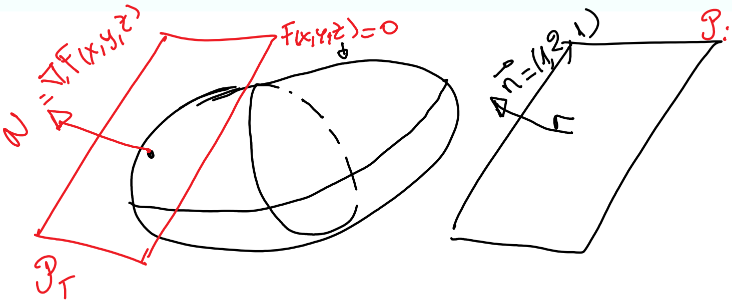
$\boxed{\mathcal{P}_{T_1} : 4x - 2y + 2z = 0}$ ✓

con $t = -1$: $(-4,2,-2)\cdot(x-4, y+1, z+1) = 0 \Rightarrow -4x + 16 + 2y + \cancel{2} - 2z - \cancel{2} = 0$

$\boxed{\mathcal{P}_{T_2} : -4x + 2y - 2z + 16 = 0}$ ✓

(*)

Hallar la ecuación del plano tangente al elipsoide $x^2 + y^2 + 2z^2 = 1$ que sea paralelo al plano $x + 2y + z = 1$.



$\mathcal{P}: x + 2y + z = 1$

$\boxed{F(x,y,z) = x^2 + y^2 + 2z^2 - 1}$

• $\mathcal{P}_T \parallel \mathcal{P} \Rightarrow \nabla F(x,y,z) \parallel \vec{n}$

$\Rightarrow (2x, 2y, 4z) = t(1,2,1) \Rightarrow \begin{matrix} 2x = t \\ 2y = 2t \\ 4z = t \end{matrix} \Rightarrow \boxed{\begin{matrix} x = t/2 \\ y = t \\ z = t/4 \end{matrix}}$

$\Rightarrow$ reemplazando en (*) : $x^2 + y^2 + 2z^2 = 1 \Rightarrow \left(\frac{t}{2}\right)^2 + t^2 + 2\left(\frac{t}{4}\right)^2 = 1$

$\Rightarrow \frac{2t^2}{2(4)} + \frac{2(4t^2)}{2(4)} + \frac{t^2}{8} = 1 \Rightarrow \frac{11t^2}{8} = 1 \Rightarrow t = \pm\sqrt{\frac{8}{11}}$

$\mathcal{P}_T : \nabla F(x,y,z) \cdot \big((x,y,z) - (x_0,y_0,z_0) \big) = 0$

$t = \sqrt{\frac{8}{11}} \Rightarrow \mathcal{P}_{T_1} : \sqrt{\frac{8}{11}}(1,2,1)\cdot\left(x - \frac{1}{2}\sqrt{\frac{8}{11}}, y - \sqrt{\frac{8}{11}}, z - \frac{1}{4}\sqrt{\frac{8}{11}}\right) = 0$

$x - \frac{1}{2}\sqrt{\frac{8}{11}} + 2y - 2\sqrt{\frac{8}{11}} + z - \frac{1}{4}\sqrt{\frac{8}{11}} = 0$

$x + 2y + z - \frac{11}{4}\sqrt{\frac{8}{11}} = 0$

$t = -\sqrt{\frac{8}{11}} \Rightarrow \mathcal{P}_{T_1} : -\sqrt{\frac{8}{11}}(1,2,1)\cdot\left(x + \frac{1}{2}\sqrt{\frac{8}{11}}, y + \sqrt{\frac{8}{11}}, z + \frac{1}{4}\sqrt{\frac{8}{11}}\right.$

$$x + 2y + z - \frac{11}{4}\sqrt{\frac{8}{11}} = 0$$

$$t = -\sqrt{\frac{8}{11}} \Rightarrow \wp_{T_1} : -\sqrt{\frac{8}{11}}(1,2,1) \cdot \left(x + \frac{1}{2}\sqrt{\frac{8}{11}},\ y + \sqrt{\frac{8}{11}},\ z + \frac{1}{4}\sqrt{\frac{8}{11}}\right) = 0$$

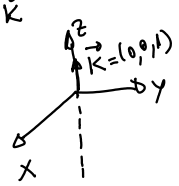
$$x + 2y + z + \frac{11}{4}\sqrt{\frac{8}{11}} = 0$$

Consideremos la función f definida por $f(x, y, z) = axy^2 + byz + cz^2x^3$. Si la derivada direccional de f en el punto $P(1, 2, -1)$, en la dirección paralela al eje z positivo es 64. Determine el valor de $a + b + c$.

Seleccione una:

- i. N.A.
- ii. 24
- iii. 18
- iv. 20
- ✓ v. 22

- $\frac{\partial f}{\partial \vec{K}}(1,2,-1) = 64 \Rightarrow \nabla f(1,2,-1) \cdot \vec{K} = 64$; $\nabla f(x,y,z) = (ay^2 + 3cz^2x^2,\ 2axy + bz,\ by + 2czx^3)$



$$\Rightarrow (4a + 3c,\ 4a - b,\ 2b - 2c) \cdot (0,0,1) = 64$$

$$\Rightarrow \boxed{b - c = 32}$$

- $\nabla f(1,2,-1) \parallel \vec{K} \Rightarrow (4a + 3c,\ 4a - b,\ 2b - 2c) = t(0,0,1) \Rightarrow \begin{cases} 4a + 3c = 0 \\ 4a - b = 0 \\ 2b - 2c = t \end{cases} \Rightarrow \begin{matrix} t = 64 \\ a = 6 \\ b = 24 \\ c = -8 \end{matrix}$

$$\Rightarrow \boxed{a + b + c = 22} //$$

Sea: $f(x, y) = \begin{cases} 2x - y & x \neq 0 \\ y & x = 0 \end{cases}$

Entonces:

Seleccione una:

- a. $\frac{\partial f}{\partial x}(0, 1) = 2;\ \frac{\partial f}{\partial y}(0, 1) = 1$
- b. $\frac{\partial f}{\partial x}(0, 1) = 2;\ \frac{\partial f}{\partial y}(0, 1) = \infty$
- c. $\frac{\partial f}{\partial x}(0, 1) = 2;\ \frac{\partial f}{\partial y}(0, 1) = -1$
- X d. $\frac{\partial f}{\partial x}(0, 1) = \infty;\ \frac{\partial f}{\partial y}(0, 1) = 1$

- $\frac{\partial f}{\partial x}(0,1) = \lim_{h \to 0} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{h \to 0} \frac{2h - 1 - 1}{h}$

$$= \lim_{h \to 0} \frac{2h - 2}{h} = \frac{-2}{0} = \infty$$

- $\frac{\partial f}{\partial y}(0,1) = \lim_{h \to 0} \frac{f(0,1+h) - f(0,1)}{h} = \lim_{h \to 0} \frac{\cancel{1} + h - \cancel{1}}{h} = 1$

Sea: $f(x, y) = \begin{cases} x - 5y & y \neq 0 \\ -x & y = 0 \end{cases}$

Entonces:

Seleccione una:

- a. $\frac{\partial f}{\partial x}(2, 0) = -1;\ \frac{\partial f}{\partial y}(2, 0) = -5$
- X b. $\frac{\partial f}{\partial x}(2, 0) = -1;\ \frac{\partial f}{\partial y}(2, 0) = \infty$
- c. $\frac{\partial f}{\partial x}(2, 0) = \infty;\ \frac{\partial f}{\partial y}(2, 0) = -5$
- d. $\frac{\partial f}{\partial x}(2, 0) = 1;\ \frac{\partial f}{\partial y}(2, 0) = -5$

- $\frac{\partial f}{\partial x}(2,0) = \lim_{h \to 0} \frac{f(2+h,0) - f(2,0)}{h}$

$$= \lim_{h \to 0} \frac{-(2+h) + 2}{h} = -1$$

- $\frac{\partial f}{\partial y}(2,0) = \lim_{h \to 0} \frac{f(2,0+h) - f(2,0)}{h}$

$$= \lim_{h \to 0} \frac{2 - 5h + 2}{h} = \frac{4}{0} = \infty$$