Campos vectoriales e integrales de línea

Definición

Un campo vectorial definido en $U \subset \mathbb{R}^n$ región es una función vectorial

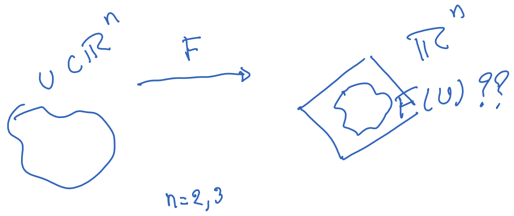
$$F: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x = (x_1, \dots, x_n) \longrightarrow F(x) = (F_1(x), \dots, F_n(x))$$

donde $F_i : U \subset \mathbb{R}^n \to \mathbb{R}$ son las funciones coordenadas de F.

Ejemplos

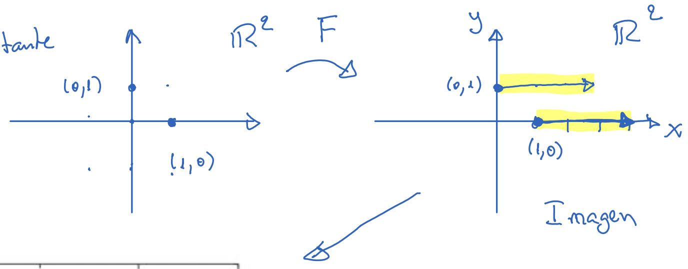
1. $F(x,y) = (3,0)$ ✓
2. $F(x,y) = (x+y,\ x-y)$ ✓
3. $F(x,y) = (y, x^2)$ ✓

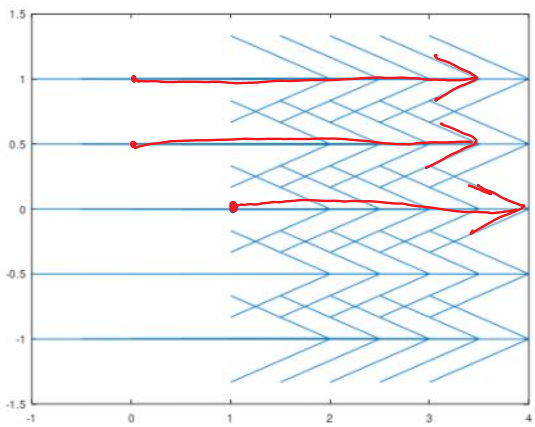


① $F(x,y) = (3,0)$ ✓ campo constante

$F(1,0) = (3,0)$

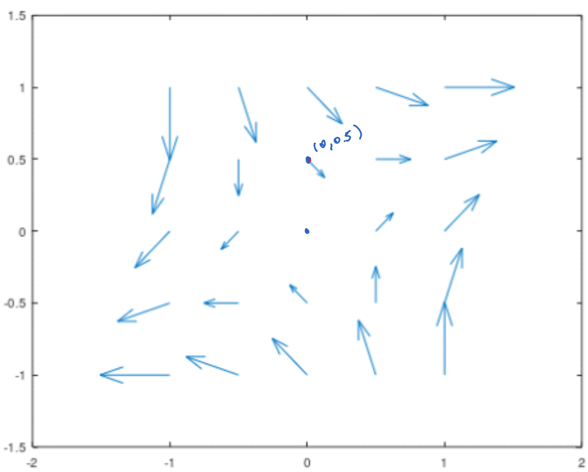
$F(0,1) = (3,0)$





$F(x,y) = (3,0)$

② $F(x,y) = (x+y,\ x-y)$

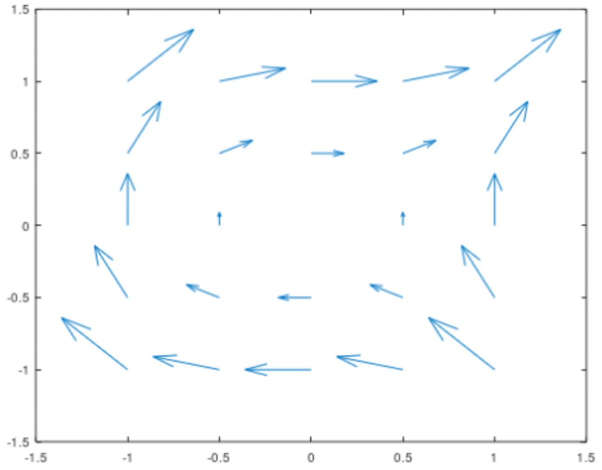


Imagen

$F(0,0) = (0,0)$

$F(0,0.5) = (0.5, -0.5)$

③ $F(x,y) = (y, x^2)$



Integrales de Línea

Sea $F: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $F = (\overset{\downarrow}{F_1}, \dots, \overset{\downarrow}{F_n})$ un campo continuo y sea $\lambda: [a,b] \longrightarrow U$ un camino de clase C^1. La integral de línea del campo F a lo largo del camino λ es denotada por

$$\int_\lambda F d\lambda \qquad \text{ó} \qquad \int_\lambda F_1(x)dx_1 + \dots + F_n(x)dx_n$$

y se define

$$\int_\lambda \mathbf{F} d\lambda = \int_a^b F(\lambda(t)) \cdot \lambda'(t)\,dt = \int_a^b \Big(F_1(\lambda(t)), \dots, F_n(\lambda(t))\Big) \cdot (\lambda_1'(t), \dots, \lambda_n'(t))\,dt$$

producto interno

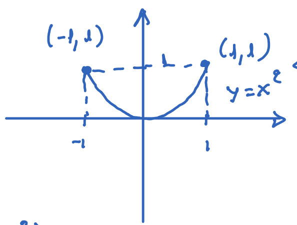
$$= \int_a^b F_1(\lambda(t))\lambda_1'(t) + F_2(\lambda(t))\lambda_2'(t) + \dots + F_n(\lambda(t))\lambda_n'(t)\,dt$$

Ejemplos

1. Calcular la integral de línea del campo vectorial $F(x,y) = \underbrace{(x^2 - 2xy,\ y^2 - 2xy)}$ a lo largo de la parábola $\boxed{y = x^2}$ desde $(-1,1)$ a $(1,1)$

Solución

$$\int_\lambda F d\lambda = \int_a^b F(\lambda(t)) \cdot \lambda'(t)\,dt$$



$x = t$
$y = t^2$

① Parametrizando la parábola

$\lambda(t) = (t, t^2)$; $-1 \le t \le 1$

$\lambda'(t) = (1, 2t)$

② $F(\lambda(t)) = F(t, t^2) = (t^2 - 2t^3,\ t^4 - 2t^3)$

③ $$\int_{-1}^{1} (t^2 - 2t^3,\ t^4 - 2t^3) \cdot (1, 2t)\,dt = \int_{-1}^{1} t^2 - 2t^3 + 2t^5 - 4t^4\,dt = \left[\frac{t^3}{3} - \frac{2t^4}{4} + \frac{2t^6}{6} - \frac{4t^5}{5}\right]_{-1}^{1}$$

$= -14/15$

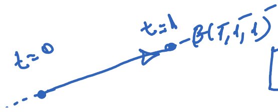
2. Calcular la integral de línea $\int yz\,dx + xy\,dy + xz\,dz$ entre los puntos $A(0,0,0)$ y $B(1,1,1)$ a lo largo de los sgtes caminos

a) Siguiendo el segmento rectilíneo $\vec{AB}$

b) La curva $C: x = y^2,\ z = 0$ desde A hasta $(1,1,0)$ y la recta $L: x=1, y=1, z=t$ desde $(1,1,0)$ hasta B.

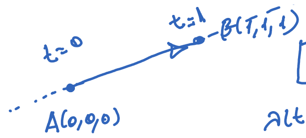
Solución

ⓐ $\int_\lambda yz\,dx + xy\,dy + xz\,dz$;



$\boxed{\lambda(t) = \overset{\downarrow}{A} + t\overset{\downarrow}{\vec{AB}}}$

(a) $\int_\lambda yz\,dx + xy\,dy + xz\,dz$;



$\boxed{F(x,y,z)=(yz,xy,xz)}$

$\boxed{\lambda(t)=A+t\vec{AB}}$

$\lambda(t)=(0,0,0)+t(1,1,1)$

• $\lambda(t)=(t,t,t)$; $0\le t\le 1$

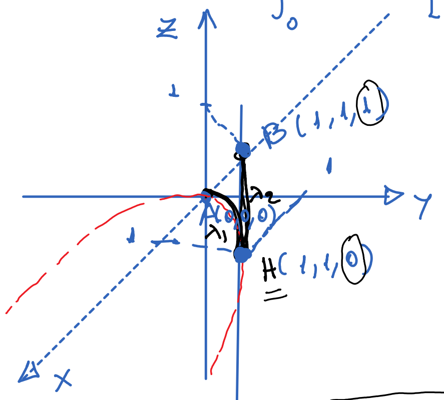
• $\lambda'(t)=(1,1,1)$

• $\int_\lambda yz\,dx + xy\,dy + xz\,dz = \int_0^1 (t^2\cdot 1 + t^2\cdot 1 + t^2\cdot 1)\,dt = \int_0^1 3t^2\,dt = \left[t^3\right]_0^1 = 1$

(b) $\int_{\lambda=\lambda_1\cup\lambda_2} yz\,dx + xy\,dy + xz\,dz =$

$\int_{\lambda_1} yz\,dx + xy\,dy + xz\,dz \; +$

$\int_{\lambda_2} yz\,dx + xy\,dy + xz\,dz$



• C: $\boxed{x=y^2,\ z=0}$

$\lambda=\lambda_1\cup\lambda_2$

$\lambda_1(t)=(t^2,t,0)$; $0\le t\le 1$

• $\lambda_2(t)=(1,1,t)$; $0\le t\le 1$

L: $\boxed{x=1,\ y=1,\ z=t}$, $t\in\mathbb{R}$

Tenemos que: $\lambda_1'(t)=(2t,1,0)$

$\boxed{\lambda_2'(t)=(0,0,1)}$

(i) $\int_{\lambda_1} \cancel{yz}\,dx + xy\,dy + \cancel{xz}\,dz = \int_0^1 t^3\cdot 1\,dt = \frac{t^4}{4}\Big|_0^1 = 1/4$

(ii) $\int_{\lambda_2} yz\,\cancel{dx} + xy\,\cancel{dy} + xz\,dz = \int_0^1 t\cdot 1\,dt = \frac{t^2}{2}\Big|_0^1 = 1/2$

$\Rightarrow \int_{\lambda=\lambda_1\cup\lambda_2} yz\,dx + xy\,dy + xz\,dz = \frac{3}{4}$