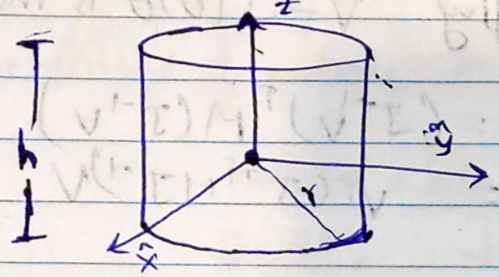
Problem set. 3.

8.1 Derive moment of Inertia.

(b) circular cylinder.



$V = \pi r^2 h$

→ using $I_b = \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix}$

where I_{xx}, I_{yy}, I_{zz} are the moment of Inertia along x, y, z axes respectively

converting to cylindrical coordinates

$x = r\cdot\cos\phi \quad y = r\sin\phi \quad z = z.$

In the cylindrical coordinates infinitesimal volume $dV = r\, dr\, d\phi\, dz$

$$\therefore I_b = \begin{bmatrix} z^2+r^2\sin^2\phi & -r^2\cos\phi\sin\phi & rz\cos\phi \\ -r^2\cos\phi\sin\phi & z^2+r^2\cos^2\phi & -rz\sin\phi \\ -rz\cos\phi & -rz\sin\phi & r^2 \end{bmatrix}$$

where the limits of integration in the equation

$$I_{ij} = \iiint_D \bar{I}_{ij}\, dx\,dy\,dz\,\rho$$

$$= \iiint_D I_{ij}\, r\,dr\,d\phi\,dz\cdot\rho$$

$\phi \to \{0, 2\pi\}$, $z = \{-h/2, h/2\}$, $r = \{\ , R\}$, $\rho = 1$

By observation of symmetry (axis, alignment) $I_{xx}, I_{yy}, I_{zz} \neq 0$ and rest of them are 0

$$I_{xx} = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^R (r^3 \sin^2\phi + r z^2)\,dr\,d\phi\,dz\cdot\rho$$

$$I_{xx} = \rho \underbrace{\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^R (r^2 \sin^2\phi)\,dr\,r\,d\phi\,dz}_{Int_1} + \underbrace{\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^R r z^2\,dr\,r\,d\phi\,dz}_{Int_2}$$

solving Int_2

$$Int_2 = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^R r z^2\,dr\,r\,d\phi\,dz$$

$$= \frac{1}{12} h^3 \cdot \pi R^2 \rho$$

solving Int_1 note that $\sin^2\phi = 1 - \cos^2\phi$

and $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$

$\therefore$ the Int_1 becomes

$$Int_1 = \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^R r^3 \cdot \frac{1}{2}(1+\cos 2\phi)\, dr\, d\phi\, dz$$

$$Int_1 = \rho \cdot \frac{1}{4} h \pi R^4$$

$$\therefore I_{xx} = \rho \left(\frac{\pi}{4} R^4 h + \frac{\pi R^2 h^3}{12} \right)$$

using $V = \pi R^2 h$. & $\rho = \frac{M}{V}$.

$$\rho = \frac{M}{\pi R^2 h}$$

$$\therefore I_{xx} = \left(\frac{1}{4} MR^2 + \frac{1}{12} MH^2 \right)$$

By symmetry $I_{xx} = I_{yy}$

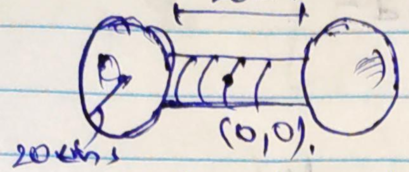
and I_{zz}

$$I_{zz} = \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^R r^2 r\, dr\, d\phi\, dz$$

$$= \frac{1}{2} \pi h R^4 = MR^2/2$$

$$\therefore I = \begin{bmatrix} \frac{1}{4}MR^2 + \frac{1}{12}MH^2 & 0 & 0 \\ 0 & \frac{1}{4}MR^2 + \frac{1}{12}MH^2 & 0 \\ 0 & 0 & MR^2 \end{bmatrix}$$

(8-2)

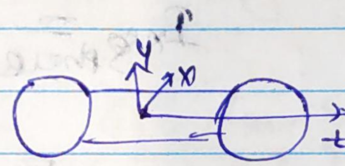


$\rho_{durnbell} = 7500 kg/m^3$

cylinder diameter $= 4cm$

length $= 20cm$

center of mass $= (0, 0, 0)$ aligning z along height



$V.\rho = \frac{4}{3}\pi r^2 h.\rho$

calculating mass of sphere $= 31.41592653$ kg

mass of cylinder $= 47.12388980$ kg

using results from (8-1).

$= I_{yy\,cylinder} = I_{xx} = \frac{1}{4}(MR^2) + \frac{1}{12}MH^2 = 0.0064$

$I_{zz} = \frac{1}{2}MR^2 = 0.00377$

and using

$$I_{sphere} = \begin{bmatrix} 2/5\,MR^2 & 0 & 0 \\ 0 & 2/2\,MR^2 & 0 \\ 0 & 0 & 2/5\,MR^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1256 & 0 & 0 \\ 0 & 0.1256 & 0 \\ 0 & 0 & 0.1256 \end{bmatrix}$$

using the parallel axis theorem.

$$I_d = I_{cm} + md^2$$

$$I_{xx_c} = I_{xx_a} + md^2$$

	I_{xxc}	r
Sph	0.126	0.2
cydn	0.0064	0

$I_{xx\,sphere} = 0.126 + 0.2^2 \times M_{sp}$

$= 1.3891$

$\therefore I_{xx} = (I_{xx\,sphere}) \times 2 + I_{cyl}$

$= 2 * (1.3891) + 0.0064$

$= 2.77174$

	M	I_{zz}	r
sp	31.4	0.126	[illegible]
cy	68	0.0003	0

$I_{xx\,sphere} = 0.126 + ([illegible])^2 * M$

$= (0.126)$

$I_{zz} = (I_{zz\,sphere} \times 2) + I_{cyl_{zz}} = 0.252$

$$\therefore I = \{\underset{I_{xx}}{2.771}, \underset{I_{yy}}{2.771}, \underset{I_{zz}}{0.252}\}$$

(b) The spatial matrix G_b

$$G_b = \begin{bmatrix} I_b & 0 \\ 0 & mI \end{bmatrix}$$

$$G_b = \text{diag}\{I_b, mI\}$$

$$m = M_{dumbbell} = M_{sphere \times 2} + M_{cyl}$$
$$= 64.2\,kg$$

$$\therefore \quad G_b = \text{diag}\{2.47, 2.47, 0.25, 64.72, 64.72, 64.72\}.$$

(8.3)

The spatial inertia matrix in the adjoint representation is given by

$$G_a = [Ad_{T_{ba}}]\, G_b\, [Ad_{T_{ba}}] \quad -(1)$$

The steiner's theorem equation is.

$$I_a = I_b + m(q^T q I - q q^T) \quad -(2)$$

need to prove (1) = (2)

expanding (1)

$$\begin{bmatrix} I_a & 0 \\ 0 & mI \end{bmatrix} = \begin{bmatrix} R_{ba}^T & R_{ba}^T [P_{ba}]^T \\ 0 & R_{ba}^T \end{bmatrix} \begin{bmatrix} I_b & 0 \\ 0 & mI \end{bmatrix} * \begin{bmatrix} R_{ba} & 0 \\ [P_{ba}] R_{ba} & R_{ba} \end{bmatrix}$$

$$\Rightarrow I_a = R_{ba}^T \cdot I_b \cdot R_{ba} + m R_{ba}^T [P_{ba}]^T [P_{ba}] R_{ba}$$

If frame $\{a\}$ is aligned with $\{b\}$ then $R_{ba} = I$

$$\Rightarrow I_a = I_b + m [P_{ba}]^T [P_{ba}]$$

using $P_{ba} = q = (q_x, q_y, q_z)$

$$\Rightarrow I_a = I_b + m [q]^T [q]$$

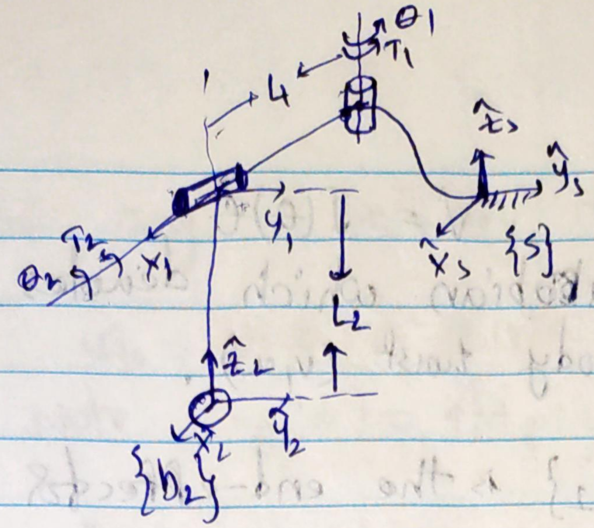
$$I_a = I_b + m \begin{bmatrix} q_y^2 + q_z^2 & -q_x q_y & -q_x q_z \\ -q_y q_z & q_x^2 + q_z^2 & -q_y q_z \\ -q_z q_x & -q_z q_y & q_x^2 + q_y^2 \end{bmatrix}$$

$$I_a = I_b + m (q^T q I - q q^T) \text{ --- } (2)$$

$\therefore$ (1) = (2)

(8.9)

The Furuta Pendulum model is shown as



Assumption 1: mass of each link is concentrated at tip

$M_1 = M_2 = 2$, $L_1 = L_2 = 1$, $g = 10$

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad I_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q derive dynamic equations & determine i/p torque τ_1, τ_2 when $\theta_1 = \theta_2 = \pi/4$

To find torques we need to solve the equation (8.8).

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \left(\frac{\partial L}{\partial \theta_i}\right)$$

$L = K_i - P_i$ (8.7).

from.

$K(\theta, \dot{\theta}) = \frac{1}{2} V_b^T \mathcal{G}_b V_b$ (eq. 8.20)

$P = $ P.E.

Note that $V = J(\theta)\dot{\theta}$

$J(\theta) =$ Jacobian which denotes the body twist (v, ω).

Assume $\{b_1\}$ is the end-effector frame

Then the body jacobian in b_1 w.r.t $\{s\}$

ω_1	q_1	$v_1 = -\omega_1 \times q_1$
$(0,0,1)$	$(-L_1, 0, 0)$	$(0, L_1, 0)$

$$J_{b_1} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_1 \\ 0 \end{bmatrix}$$

$$\therefore V_{b_1} = J_{b_1} \dot{\theta}_1$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_1 \\ 0 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \\ 0 \\ L_1\dot{\theta}_1 \\ 0 \end{bmatrix}$$

Now assume frame b_2 is the end-effector's frame then

$$\omega_1 = Rot(\hat{x}_2, -\theta_2) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (0, \sin\theta_2, \cos\theta_2)$$

$$q_1 = (-L_1, 0, L_2)$$

$$v_1 = -\omega_1 \times q_1 = (-L_2 \sin\theta_2, L_1 \cos\theta_2, -L_1 \sin\theta_2)$$

$$\omega_2 = [1, 0, 0]$$
$$q_2 = [0, 0, L_2]$$

$$v_2 = -\omega_2 \times q_2 = [0, L_2, 0]$$

$$\therefore J_{b_2} = \begin{bmatrix} 0 & 1 \\ \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ -L_2 \sin\theta_2 & 0 \\ L_1 \cos\theta_2 & L_2 \\ -L_1 \sin\theta_2 & 0 \end{bmatrix}$$

$$\therefore V_{b_2} = J_{b_2} \dot{\theta} = \begin{bmatrix} 0 & 1 \\ \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ -L_2 \sin\theta_2 & 0 \\ L_1 \cos\theta_2 & L_2 \\ -L_1 \sin\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$V_{b_2} = \begin{bmatrix} \dot\theta_2 \\ \dot\theta_1 \sin\theta_2 \\ \dot\theta_1 \cos\theta_2 \\ -L_2\dot\theta_1 \sin\theta_2 \\ L_1\dot\theta_1 \cos\theta_2 + L_2\dot\theta_2 \\ -L_1\dot\theta_1 \sin\theta_2 \end{bmatrix}$$

$$K(\theta, \dot\theta) = \frac{1}{2} V_{b_1}^T G_{b_1} V_{b_1} + 1/2\, V_{b_2}^T G_{b_2} V_2$$

$$= \frac{1}{2} V_{b_1}^T \begin{bmatrix} I_1 & 0 \\ 0 & m_1 I \end{bmatrix} V_{b_1} + 1/2\, V_{b_2}^T \begin{bmatrix} I_2 & 0 \\ 0 & m_2 I \end{bmatrix} V_{b_2}$$

using $L_1 = L_2 = 1$, and $m_1 = m_2 = 2$

$$K(\theta, \dot\theta) = 3\dot\theta_1^2 + (\dot\theta_1^2 + 3\dot\theta_2^2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2)$$

$$= 4\dot\theta_1^2 + 3\dot\theta_2^2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2$$

$P(\theta)$ = potential energy.

assuming the P.E at zero position to be zero.

$$P(\theta) = m_2 g L_2 (1 - \cos\theta_2)$$

using $m_2 = 2$, $g = 10$, $L_2 = 1$

$$P(\theta) = 20 - 20\cos\theta_2$$

$$L(\theta, \dot\theta) = K(\theta, \dot\theta) - P(\theta)$$

$$= 4\dot\theta_1^2 + 3\dot\theta_2^2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2 + 20\cos\theta_2 - 20$$

$$\therefore \tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$$

$$= \frac{d}{dt}\begin{bmatrix} \partial L/\partial \dot{\theta}_1 \\ \partial L/\partial \dot{\theta}_2 \end{bmatrix} - \begin{bmatrix} \partial L/\partial \theta_1 \\ \partial L/\partial \theta_2 \end{bmatrix}$$

calculating the above terms in sympy

$$= \begin{bmatrix} 8\ddot{\theta}_1 + 6\ddot{\theta}_1 \sin^2\theta_2 + 12\dot{\theta}_1\dot{\theta}_2 \sin\theta_2 \cos\theta_2 + 2\ddot{\theta}_2 \cos\theta_2 - 2\dot{\theta}_2^2 \sin\theta_2 \\ 6\ddot{\theta}_2 + 2\ddot{\theta}_1 \cos\theta_2 - 6\dot{\theta}_1^2 \sin\theta_2 \cos\theta_2 + 20\sin\theta_2 \end{bmatrix} \quad -(1)$$

Substituting $\theta_1 = \theta_2 = \pi/4$ and $\dot{\theta}_1 = \dot{\theta}_2 = \ddot{\theta}_1 = \ddot{\theta}_2 = 0$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10\sqrt{2} \end{bmatrix}$$

(b) The torque ellipsoid when $\theta_1 = \theta_2 = \pi/4$ is obtained by substituting in (1) the values $\theta_1 = \theta_2 = \pi/4$ which gives

$$\tau = \begin{bmatrix} 11\ddot{\theta}_1 + 6\dot{\theta}_1\dot{\theta}_2 + \sqrt{2}\ddot{\theta}_2 - \sqrt{2}\dot{\theta}_2^2 \\ 6\ddot{\theta}_2 + \sqrt{2}\ddot{\theta}_1 - 3\dot{\theta}_1^2 + 10\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 11 & \sqrt{2} \\ \sqrt{2} & 6 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\sqrt{2} \\ -3 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \cdots$$

∴ Mass matrix is

$$\underline{M(\theta)}\,\ddot{\theta} + h(\theta,\dot{\theta}) = \tau.$$

$$\Rightarrow M(\theta) = \begin{bmatrix} 11 & \sqrt{2} \\ \sqrt{2} & 6 \end{bmatrix}$$

eigen values are

$$|(M(\theta) - \lambda I)| = 0$$

$$\rightarrow \begin{vmatrix} 11-\lambda & \sqrt{2} \\ \sqrt{2} & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow [(11-\lambda)(6-\lambda)] = 2\} = 64 - 17\lambda + \lambda^2$$

⇒ solving the quadratic equation

$$\lambda_1 = 5.62,\ \lambda_2 = 11.378.$$

corresponding eigen vectors are:

$$M\lambda_1 = \lambda_1 V \Rightarrow (M - \lambda_1 I)\,v = 0$$

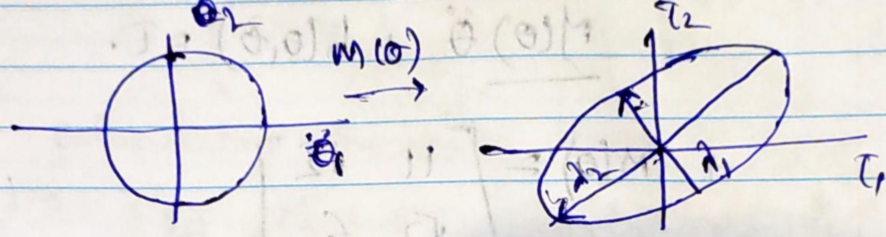
$$\begin{bmatrix} 11-\lambda_1 & \sqrt{2} \\ \sqrt{2} & 6-\lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \qquad \lambda_1 = 5.62$$

gives $v_1 = (0.25, -0.94)$

and similarly $v_2 = (-0.96, -0.25)$

The ellipsoid is drawn by mapping

$\{\ddot{\theta} \mid \ddot{\theta}^T \ddot{\theta} = 1\}$ when $(\theta_1 = \theta_2 = \pi/4)$



$\tau_1 = 11\ddot{\theta}_1 + \sqrt{2}\,\ddot{\theta}_2$

$\tau_2 = \sqrt{2}\,\ddot{\theta}_1 + 6\ddot{\theta}_2$

s.t. $\ddot{\theta}_1^2 + \ddot{\theta}_2^2 = 1$

direction of principle axes = $\lambda_1 v_1$, $\lambda_2 v_2$

lengths = λ_1, λ_2