Homework - 1

3.1 $\{s\} \rightarrow$ fixed frame

(a) $\rightarrow$ given information about the frame $\{a\}$ w.r.t $\{s\}$

$\hat{x}_a = (0,0,1)$

$\hat{y}_a = (-1,0,0)$

$\hat{z}_a = \hat{x}_a \times \hat{y}_a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix}$

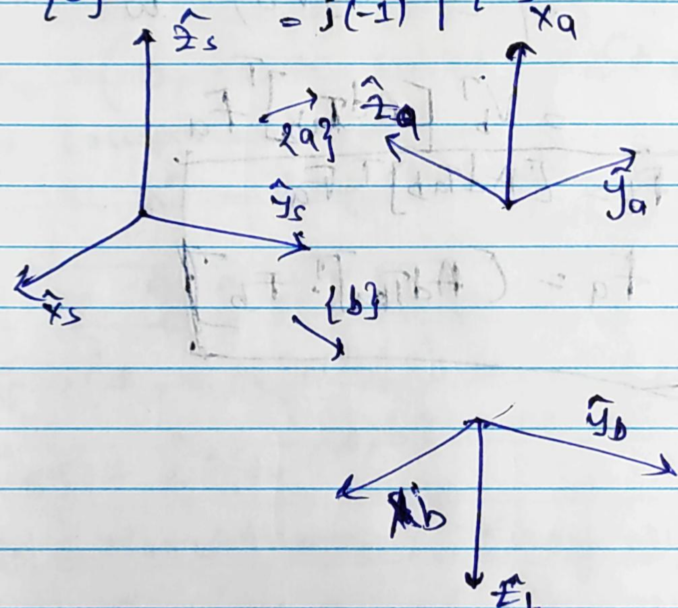
$= \hat{j}(-1)$

for frame $\{b\}$

$\hat{x}_b = (1,0,0)$

$\hat{y}_b = (0,0,-1)$

$\hat{z}_b = \hat{x}_b \times \hat{y}_b = (0,1,0)$



(b) Rotation matrices

The rotation matrices could be the collumn vectors of relative orientation

$R_{sa} = \left[\hat{x}_a^T \ \hat{y}_a^T \ \hat{z}_a^T \right]$

$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

$R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

ⓒ Given R_{sb} how do you calculate R_{sb}^{-1}

$\because \quad RR^T = I \;\&\; R^T = R^{-1}$

$$\therefore \quad R_{sb}^{-1} = R_{bs} = R_{sb}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

ⓓ Given R_{sa} & R_{sb} calculate R_{ab}

from $R_{as} = R_{sa}^T$

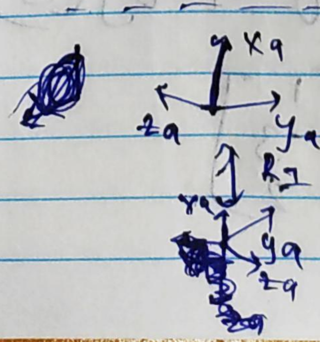
$$R_{ab} = R_{as} \cdot R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

ⓔ Given $R = R_{sb}$ find ① $R_1 = R_{sa} R$ find ② $R_2 = R R_{sa}$

Q: Does the new orientation R_1 correspond to a rotation of R_{sa} by $-90°$ in ~~any~~ both cases.

$$R_1 = R_{sa} \cdot R = R_{sa} \cdot R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$



$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Yes it corresponds to a $-90°$ rotation of R_{sa}.

$$R_2 = R \cdot R_{sa} = R_{sb} \cdot R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & +1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_2 does correspond to rotating R_{sa} by $+90°$ about $\hat{z}_s$-axis

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \{s\}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ if } \theta = -90$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} = R_{sa}$$

(f) Use R_{sb} to change representation of $p_b = (1,2,3)$ in $\{b\}$ to $\{s\}$

$$p_s = R_{sb} \cdot p_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(g) $P_s = (1,2,3)$ in $\{s\}$ calculate

$p' = R_{sb} \cdot P_s$ and $p'' = R_{sb}^T \cdot P_s'$

$$p' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(g) changing coordinates from $\{s\} \to \{b\}$ without changing or moving the point 'p' or as moving the location of the point 'p' without changing reference frame

$$p'' = R_{sb}^T \cdot P_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(h) Angular velocity in $\{s\}$ is given by $\omega = \{3,2,1\}$ what is ω in $\{a\}$

$$\omega_a = R_{as} \cdot \omega_s = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

3.5 find the exponential coordinates for

$T^{-1}\dot{T} = [V_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}$

aside:

Given $R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = Rot(\hat{\omega}, \theta)$

exp: $[\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$

$= I + \sin\theta [\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$

using $\theta = \cos^{-1}\left(\frac{1}{2}(tr R - 1)\right)$

$= \cos^{-1}\left(\frac{1}{2}(0-1)\right)$

$= \cos^{-1}(-1/2) = 2\pi/3$ — (1)

$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \frac{1}{2\sqrt{3}}$ — (2)

so $[\hat{\omega}]\theta = \begin{pmatrix} \frac{2\pi}{3\sqrt{3}} \\ \frac{-2\pi}{3\sqrt{3}} \\ \frac{2\pi}{3\sqrt{3}} \end{pmatrix}$

3.15

(a) Given fixed frame {0} & moving frame {1} initially aligned with {0}.

(1) Rotate {1} about the {0} frame $\hat{x}$-axis by by α. , call it {2}

ans:- Since both are initially aligned

$R_{01} = I$.

. To rotate by 'α' use $\boxed{R_{02} = Rot(\hat{x}_0, \alpha) R_{01}}$

(2) Rotate {2} about the frame {0} $\hat{y}$-axis call it {3} with 'β'. rad.

$\Rightarrow \boxed{R_{03} = Rot(\hat{y}, \beta) \cdot R_{02} = Rot(\hat{y}, \beta) \cdot Rot(\hat{x}, \alpha) \cdot R_{01}}$

(3) Rotate {3} about the {0} frame z-axis by γ call this frame {4}

$$R_{04} = Rot(\hat{z}, \gamma) R_{03} = Rot(\hat{z}, \gamma) \cdot Rot(\hat{y}, \beta) \cdot Rot(\hat{x}, \alpha) R_{01}$$

$$\boxed{R_{04} = Rot(\hat{z}, \gamma) \cdot Rot(\hat{y}, \beta) \cdot Rot(\hat{x}, \alpha) \cdot I}$$

$$\Rightarrow R_{04} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} *$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

(b) $R_{04} = R_{03} \cdot Rot(\hat{z}_3, \gamma)$

$= Rot(\hat{y}, \beta) \cdot Rot(\hat{x}_0, \alpha) \cdot Rot(\hat{z}_3, \gamma)$

$=$

(c) given

$$T_{ab} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{cb} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 1 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_{ca} = \cancel{T_{ab} \cdot T_{bc}} = T_{cb} \cdot T_{ba} = T_{cb} \cdot T_{ab}^{-1}$

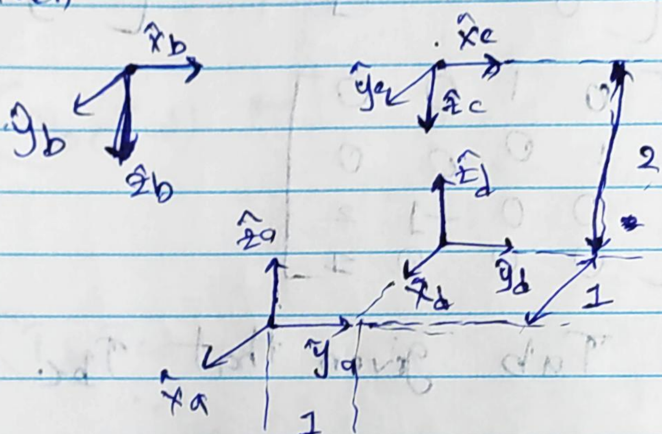
$= T_{cb} \cdot T_{ab}^{-1} = T_{cb} \cdot \begin{bmatrix} R_{ab}^T & -R_{ab}^T\, p_{ab} \\ 0 & 1 \end{bmatrix}$

$T_{ab}^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T\, p_{ab} \\ 0 & 1 \end{bmatrix}$

$$R_{ab}^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3-III

Given



(a) Find T_{ad} and T_{cd} in terms of dimension in the figure

$$T_{ad} = \begin{bmatrix} R_{ad} & P_{ad} \\ 0 & 1 \end{bmatrix}$$

position of the origin {d} w.r.t {a}

$$R_{ad} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{ad} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

They are identically oriented

$$T_{ad} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{cd} = \begin{bmatrix} R_{cd} & P_{cd} \\ 0 & 1 \end{bmatrix}$$

$$R_{cd} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad P_{cd} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$T_{cd} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find T_{ab} given that T_{bc}.

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

~~T_{abc}~~ =

* we have matrices of T_{ad} T_{cd}

$$T_{ab} \cdot T_{bc} \cdot T_{cd} = T_{ad}$$

$$T_{ab} = T_{ad}(T_{bc} T_{cd})^{-1}$$

and

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad T_{ab} = T_{ad}\,(T_{bc}T_{cd})^{-1}$$

$$(T_{bc}\cdot T_{cd}) = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_{bc}\cdot T_{cd}) = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_{bc}T_{cd})^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_{ad}

$$T_{ab} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8.24 Given information

screw joint of pitch $h=2$

$L_1 = 10$

$L_2 = L_3 = 5$

$L_4 = 3$

$\omega_1 = \pi/4 \text{ rad s}^{-1}$

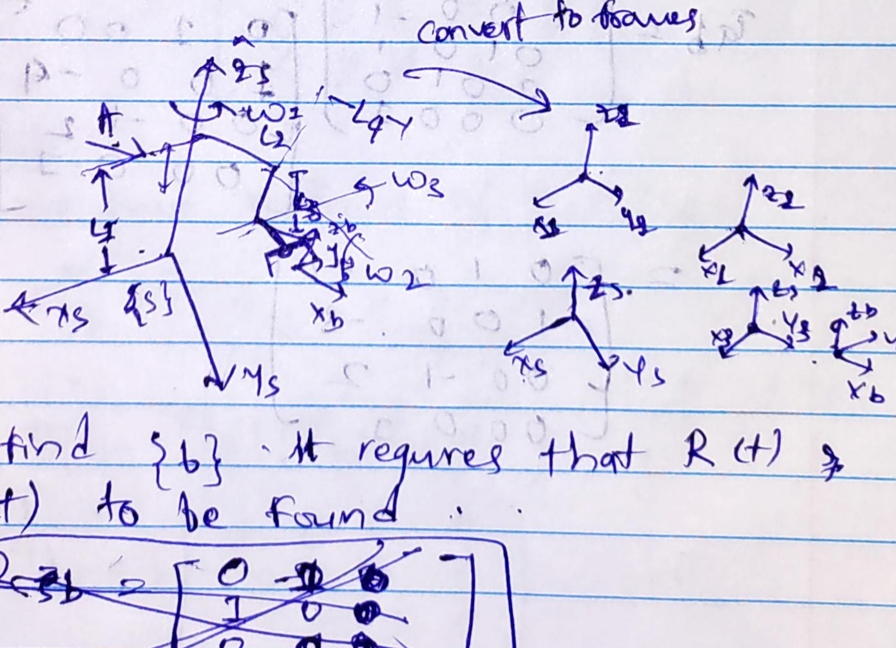
$\omega_2 = \pi/8 \text{ rad s}^{-1}$

$\omega_3 = -\pi/4 \text{ rad s}^{-1}$

find $T_{sb}(4)$

h = pitch = linear speed / angular speed.

$$T = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$



To find $\{b\}$ it requires that $R(t)$ & $p(t)$ to be found.

$$\boxed{\cancel{R_{sb} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

rough

the Rotation matrix at "t = 4"

$(\theta_1, \theta_2, \theta_3) = (\omega_1 \cdot t, \omega_2 \cdot t, \omega_3 t)$

$= (\pi/4 * 4, \pi/8 * 4, -\pi/4 * 4)$

$= (\pi, \pi/2, -\pi)$

$h = 2 = (\text{linear speed})/(\text{angular vel}) = \text{mts/rad}$

$\cancel{2 * (\omega_1) = v}$

$\cancel{2 * \pi/4 = v}$ rough

$\cancel{v = (\pi/2)}$

∴ ~~distance~~ moved = $v * t = \pi/2 * 4 = 2\pi$

→ since Pitch = 2. ⇒ it moves 2 units every 2π rads.

∴ $2 \rightarrow 2\pi$

$? \leftarrow \pi$ ⇒ $x = \frac{2 * \pi}{2\pi} = 1$ mts.

∴ $L_1 = 10 + 1 = 11$.

The transforms between frames $\{(s,1)\ (1,2)\ (2,3)\ (3,b)\}$ are

$$T_{s1} = \begin{bmatrix} R_z(\pi) & \begin{matrix} 0 \\ 0 \\ 11 \end{matrix} \\ 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} R_y(\pi/2) & \begin{matrix} 0 \\ 5 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \quad T_{23} = \begin{bmatrix} R_x(-\pi) & \begin{matrix} 0 \\ 0 \\ -5 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$T_{3b} = \begin{bmatrix} R_z(\pi/2) & \begin{matrix} 0 \\ 3 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$T_{sb} = T_{s1} T_{12} T_{23} T_{3b}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.26

since $\hat{s} = (0,0,1)$. $q = (3,0,0)$ $h = 2$ $\dot{\theta} = 1$

$$V\dot{\theta} = S = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s} \times q\dot{\theta} + h\hat{s}\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$