AMATH 301
Midterm 3: Autumn 2021

**DUE: Friday, 8pm PST, December 10, 2021.**

**NOTE:** You get five attempts, you are not allowed to talk to others, it is open book and open notes, you must submit MATLAB or Python code that is executed by gradescope to produce the answers.

1. Consider Fourier mode expansions on the domain $x \in [0, 2\pi]$ (divide the domain into 200 points, include the endpoints!) which take the following form:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=0}^{\infty} b_n \cos(nx)$$

Thus both sines and cosines are used to represent a function. First we need to normalize the Fourier modes to make an orthonormal basis. Thus do the following.

(a) Construct the sine basis $\psi_n = c_n \sin(nx)$ where $c_n$ is a normalization constant in order to make an orthonormal basis. Do the same for the cosine basis $\phi_n = c_n \cos(nx)$ where $c_n$ is a normalization constant in order to make an orthonormal basis. Construct a matrix A1 with 10 columns (and 200 rows) of orthonormal vectors with the first five columns corresponding to $n = 1, 2, \cdots 5$ of the sine modes and the second five columns corresponding to $n = 0, 1, \cdots 4$ of the cosine modes. Compute the correlation matrix A2=A1$^T$A1.

**ANSWER**: Should be written out as matrices A1 and A2 respectively.

(b) Using the sine and cosine basis A1, approximate the function $f(x) = \exp(-2(x-2)^2)$ using a 10 mode expansion so that

$$f(x) = \exp(-2(x-2)^2) = \sum_{n=1}^{5} a_n \psi_n + \sum_{n=0}^{4} b_n \phi_n.$$

Save this approximation as A3. Then compute the norm of the error between the function and your approximation

$$E = \text{norm}\left(\exp(-2(x-2)^2) - \left[\sum_{n=1}^{5} a_n \psi_n + \sum_{n=0}^{4} b_n \phi_n\right]\right).$$

Compute the projection of each mode onto the function $f(x)$, i.e. the $a_n$ and $b_n$ is the dot product of $f(x)$ with the orthonormal modes $\psi_n$ and $\phi_n$ respectively. Save the error as A4.

**ANSWER**: Save A3 as a 200 by 1 vector and A4 as the scalar number **E**.

2. Consider the temperature data (from homework 7) taken over a 24-hour (military time) cycle:

75 at 1, 77 at 2, 76 at 3, 73 at 4, 69 at 5, 68 at 6, 63 at 7, 59 at 8, 57 at 9, 55 at 10, 54 at 11, 52 at 12, 50 at 13, 50 at 14, 49 at 15, 49 at 16, 49 at 17, 50 at 18, 54 at 19, 56 at 20, 59 at 21, 63 at 22, 67 at 23, 72 at 24.

Develop a Least-Squares algorithm and calculate $E_2$ for:

$$y = A \sin(Bx + C) + D \tag{1}$$

(Hint: use the FMINSEARCH (matlab) or FMIN (python) command to help. YOUR INITIAL GUESS IS CRITICAL! So be sure to plot your results to see if they look right!) Evaluate the curve for $x = 1 : 0.01 : 24$ and save this in a column vector.

**ANSWER**: The error (which is a scalar, only calculated at the data points!) and curve (computed at every 0.01, it is a column vector 2301 by 1) should be written out as A5 and A6 respectively.

3. Consider the data set where the first column is $t$ and the second column is $y$.

```
X=[-3.0  -0.2
-2.2   0.1
-1.7   0.05
-1.5   0.2
-1.3   0.4
-1.0   1.0
-0.7   1.2
-0.4   1.4
-0.25  1.8
-0.05  2.2
 0.07  2.1
 0.15  1.6
 0.3   1.5
 0.65  1.1
 1.1   0.8
 1.25  0.3
 1.8   -0.1
 2.5    0.2]
```

Develop a Least-Squares algorithm and calculate $E_2$ for:

$$y = A\sin(Bt) + C\exp(-Dt^2) \qquad (2)$$

(Hint: use the FMINSEARCH (matlab) or FMIN (python) command to help. YOUR INITIAL GUESS IS CRITICAL! So be sure to plot your results and tune the guess to get the best result.) Evaluate the curve for $x = -3 : 0.1 : 3$ and save this in a column vector.

**ANSWER**: The error (which is a scalar, only calculated at the data points!) and curve (computed at every 0.1, it is a column vector 61 by 1) should be written out as A7 and A8 respectively.