AMATH 301
Midterm 2: Autumn 2021

DUE: Friday, 8pm PST, November 19, 2021.

NOTE: You get five attempts, you are not allowed to talk to others, it is open book and open notes, you must submit MATLAB or Python code that is executed by gradescope to produce the answers.

Consider the damped nonlinear pendulum equation

$$\Theta'' + \gamma\Theta' + \sin(\Theta) = 0$$

where $\gamma = 0.1$ is the damping coefficient. In what follows, assume $\Delta t = 0.01$ is your time-step and you are solving on the interval $t \in [0, 10]$. Write the dynamics as a system of two first order equations for Θ and Θ', thus

$$\mathbf{y}(t) = \begin{bmatrix} \Theta \\ \Theta' \end{bmatrix} \quad \text{where at } t = 0\text{:} \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$

(a) Use the following Euler scheme to produce the solution on the time interval.

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t \; \mathbf{f}(t_n, \mathbf{y}_n)$$

ANSWER: Should be 2 rows (Θ and Θ' respectively) and 1001 columns written as A1.

(b) Using the computed solution in (a), calculate the accumulated phase $\int_0^{10} \Theta dt$ using a trapezoidal rule and a Simpson's rule.
ANSWERS: Trapezoidal rule vector (1×1001) is A2 and Simpsons rule vector (1×501) is A3.

(c) Using the computed solution in (a), calculate the angular acceleration by computing the derivative of $\Theta'(t)$ using a second-order accurate and fourth-order accurate formula for the derivative with 2nd order accurate forward- and backward-differentiation schemes at the endpoints. (For the fourth-order scheme, you need to used the forward- and backward- schemes for two endpoints on each side).
ANSWERS: 2nd-order scheme (1×1001) is A4 and 4th-order scheme (1×1001) is A5.

(d) Using the computed solution in (a), calculate the angular acceleration by computing the second derivative of $\Theta(t)$ using a fourth-order accurate formula for the second derivative with 2nd order accurate forward- and backward-second derivative schemes at the endpoints. (For the fourth-order scheme, you need to used the forward- and backward- schemes for two endpoints on each side).

ANSWERS: Computation (1×1001) is A6.

(e) Use the following Adams-Bashforth scheme to produce the solution on the time interval. (NOTE: use the Euler method of part (a) to generate $\mathbf{y}_1 = \mathbf{y}(t = \Delta t)$ to go along with the $\mathbf{y}_0$.)

$$\mathbf{y}_{n+1} = \mathbf{y}_n + (\Delta t/2) \;\; [3\mathbf{f}(t_n, \mathbf{y}_n) - \mathbf{f}(t_{n-1}, \mathbf{y}_{n-1})]$$

ANSWER: Should be 2 rows (Θ and Θ' respectively) and 1001 columns written as A7.

(f) Use the following 2nd-order Runge-Kutta scheme to produce the solution on the time interval.

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t \; \mathbf{f}\left(t_n + \Delta t/2, \mathbf{y}_n + (\Delta t/2) f(t_n, \mathbf{y}_n)\right)$$

ANSWER: Should be 2 rows (Θ and Θ' respectively) and 1001 columns written as A8.