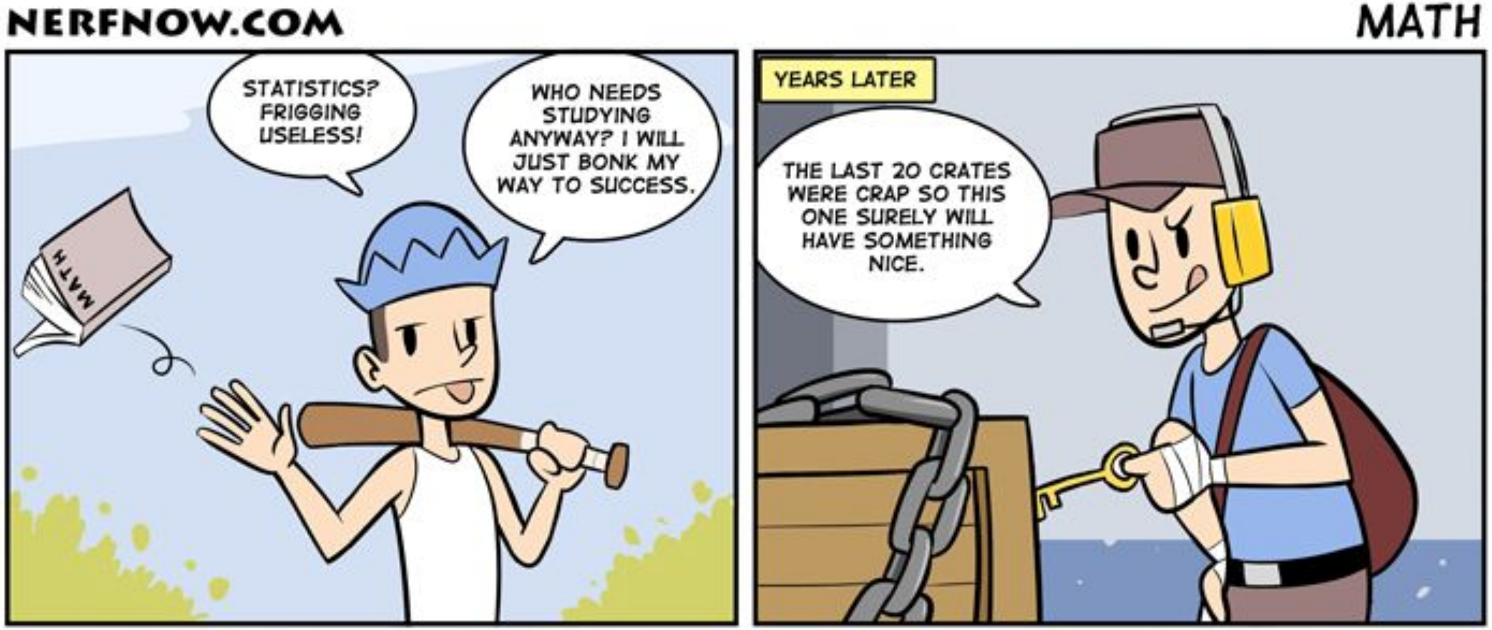
Lecture 8: More Discrete Random Variables and Their Distributions



Announcements and reminders

- HW 2 due next Thursday at 5 PM; HW 1 due **Today** at 5 PM.

  Quizlet 3 will be released tomorrow, due Sunday at 5 PM

  OH today 3 - 4 PM in ECOT 524

Previously, on CSCI 3022…

Definition: A **discrete random variable** (r.v.) X is a function that maps the elements of the sample space Ω to a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$

Definition: A discrete r.v. X ~ *Ber*(*p*), where $0 \leq p \leq 1$, if its probability mass function is given by

$$f(1) = p_X(1) = P(X=1) = p \quad \textbf{and} \quad f(0) = p_X(0) = P(X=0) = 1-p$$

Definition: A discrete r.v. X ~ *Bin*(*n*, *p*), where $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k \, (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

Binomial-like distributions

There are several discrete distributions that are similar in spirit to the Binomial distribution. We'll look at three of them today:

- Geometric distribution
- Negative Binomial distribution
- Poisson distribution

Binomial-like distributions

Example: You are doing an exit poll outside a voting station on Election Day. As people exit, you ask them questions about their political affiliation, who they voted for, etc.

In particular, you're interested in how registered Independents voted. Being well-prepared, you know that about 20% of registered voters are registered as Independents.

Goal: S'pose you interview 100 people. Let X be a random variable describing the number of actual Independents you encounter.

Distribution:

Binomial-like distributions

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Distribution: Binomial distribution (Bin(n=100, p=0.2))

Binomial-like distributions

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Goal: S'pose you talk to a lot of registered Republicans and Democrats, but haven't found an Independent yet. Let X be a random variable describing the number of people you have interviewed up to an including your first registered Independent voter.

Distribution:

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Distribution: Geometric distribution

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Goal: S'pose you're really interested in talking to a lot of Independents. Let X be the random variable describing the number of people you have to talk to in order to interview exactly 100 registered Independents.

Distribution:

Binomial-like distributions

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Distribution: Negative Binomial distribution

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Goal: You're concerned about being overwhelmed during peak voting times, so you track the number of people arriving in line at the voting station. Let X be a random variable describing the number of voters that arrive at the station over a 15-minute period.

Distribution:

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Distribution: Poisson distribution

The Geometric distribution

Example: S'pose you flip the same biased coin repeatedly. How many times do you flip the coin before you see your first Heads?

The Geometric distribution

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$P(H) = p$

1 flip: p

2 flips: $(1-p) \cdot p$

3 flips: $(1-p)^2 \cdot p$

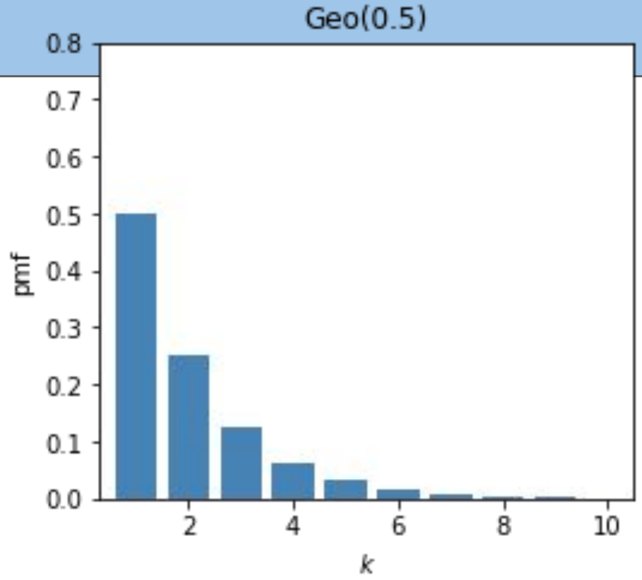
In general: $p_X(k) = (1-p)^{k-1} \cdot p$

The Geometric distribution

Definition: A discrete r.v. X has a **geometric distribution** with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(k) = P(X=k) = (1-p)^{k-1} \cdot p \quad \text{for } k = 1, 2, 3, \ldots$$

We say that $X \sim Geo(p)$.



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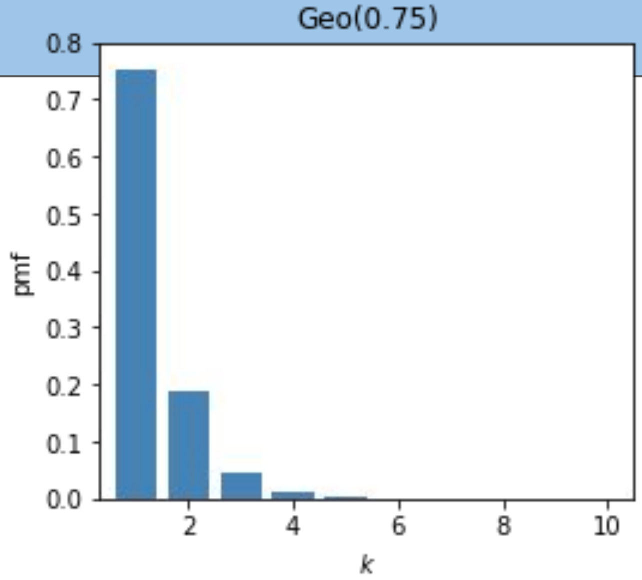

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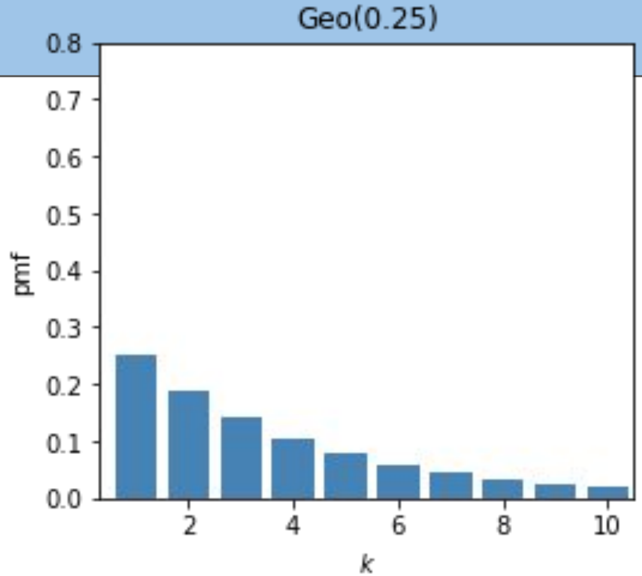
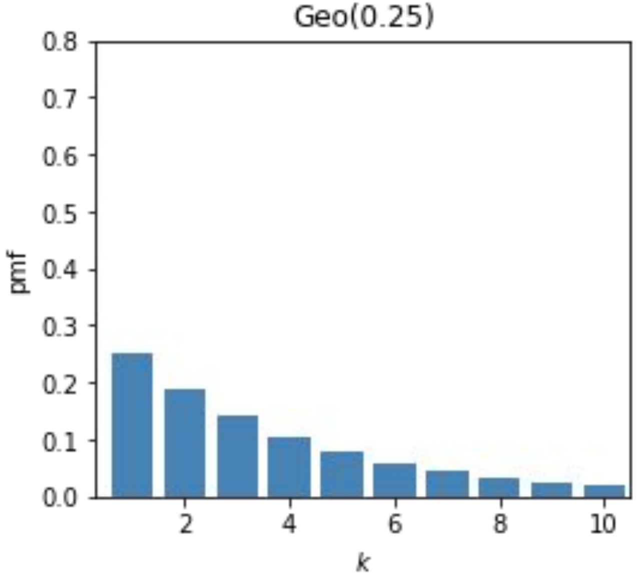


The Geometric distribution

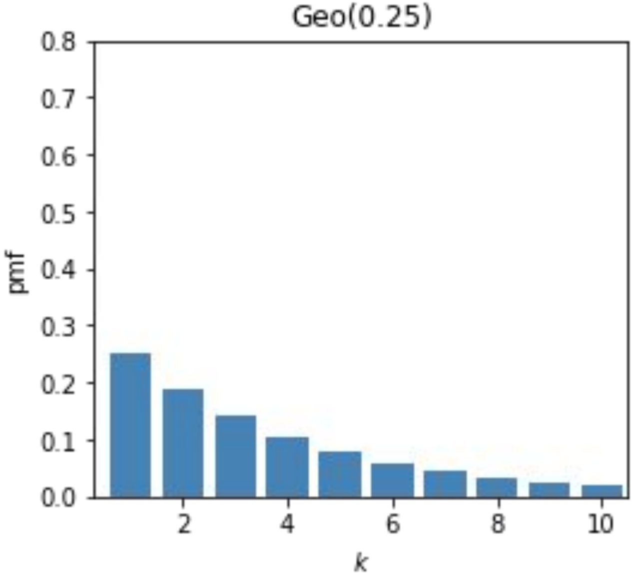
Question: What assumptions did we implicitly make in deriving the Geometric distribution?



The Geometric distribution

Question: What assumptions did we implicitly make in deriving the Geometric distribution?

- Each trial is **independent**
- Each trial is a Bernoulli r.v. with probability of success *p*



The Negative Binomial distribution

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X = random variable representing the number of flips total when we observe our 3rd Heads

$\rightarrow X \in \{3, 4, 5, \dots\}$

$p_X(k)$ = [probability of 2 Heads in the first k-1 flips] $\times$ [probability of Heads on k^{th} flip]

= [Binomial r.v. with n=k-1, and 2 successes] $\times$ p

$$= \binom{k-1}{2} p^2(1-p)^{(k-1)-2} \cdot p$$

$$= \binom{k-1}{2} p^3(1-p)^{k-3}$$ …. Can we generalize this?

The Negative Binomial distribution

Definition: A discrete r.v. X has a **negative binomial distribution** with parameters *r* and *p*, where $r > 1$ and $0 \le p \le 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k-1}{r-1} p^r \ (1-p)^{k-r}$$

We say that $X \sim NB(r, p)$

p = probability of success for each trial

r = number of successes we want to observe

X = number of trials needed before we observe *r* successes (r.v.)

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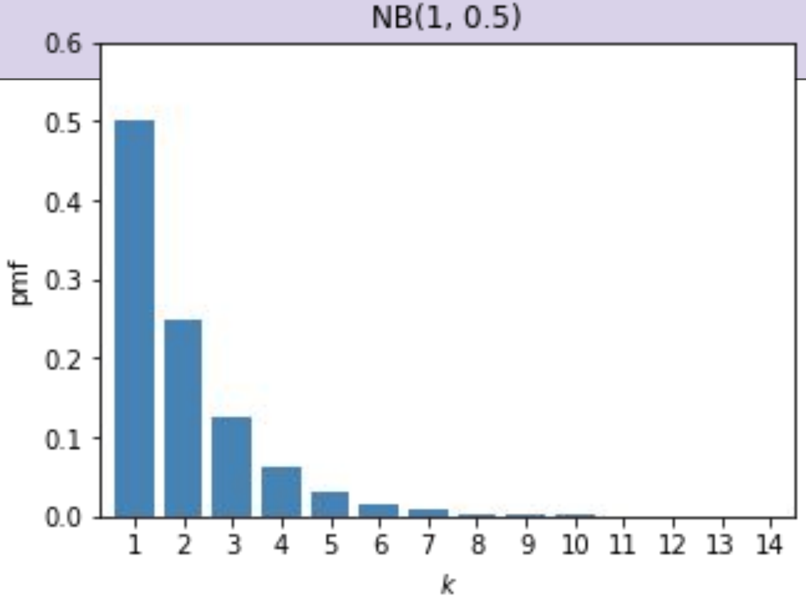

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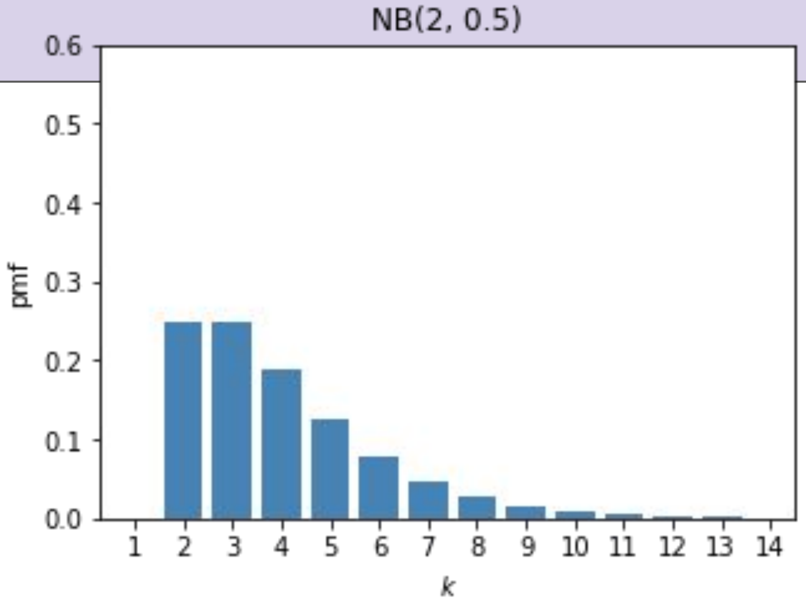

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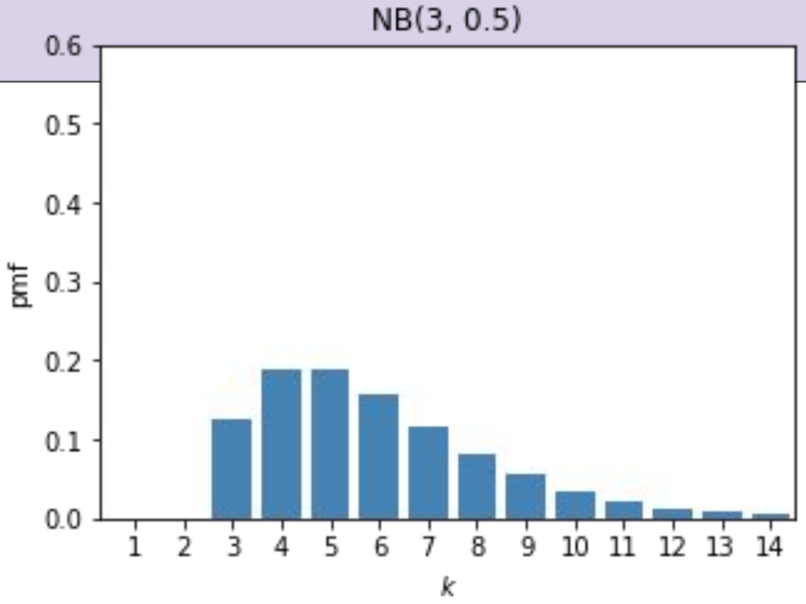

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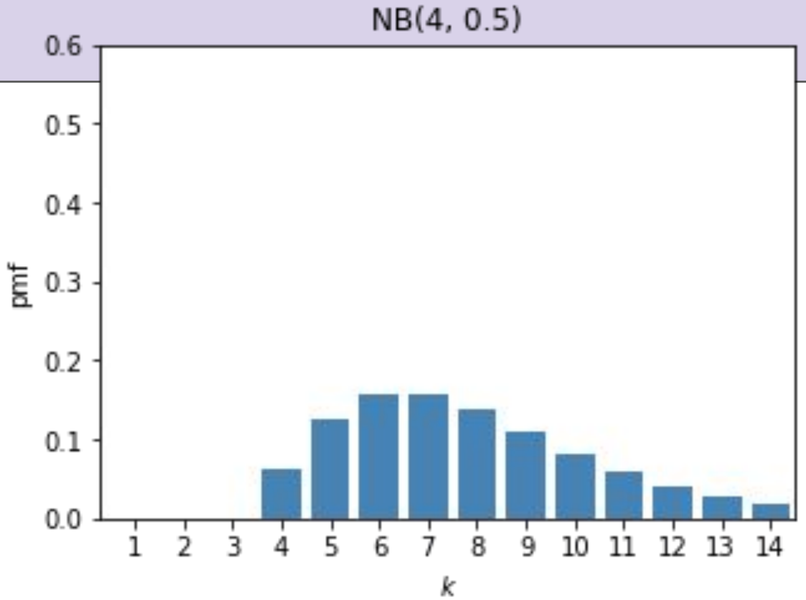

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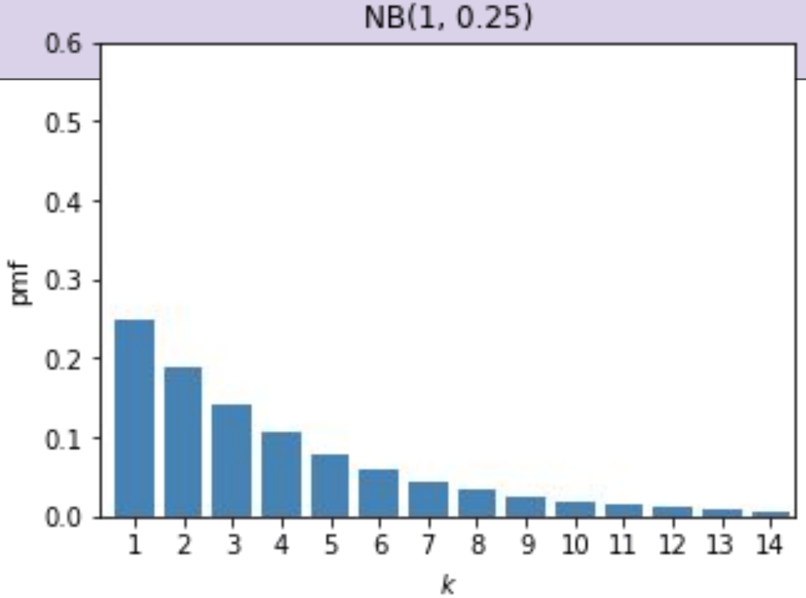

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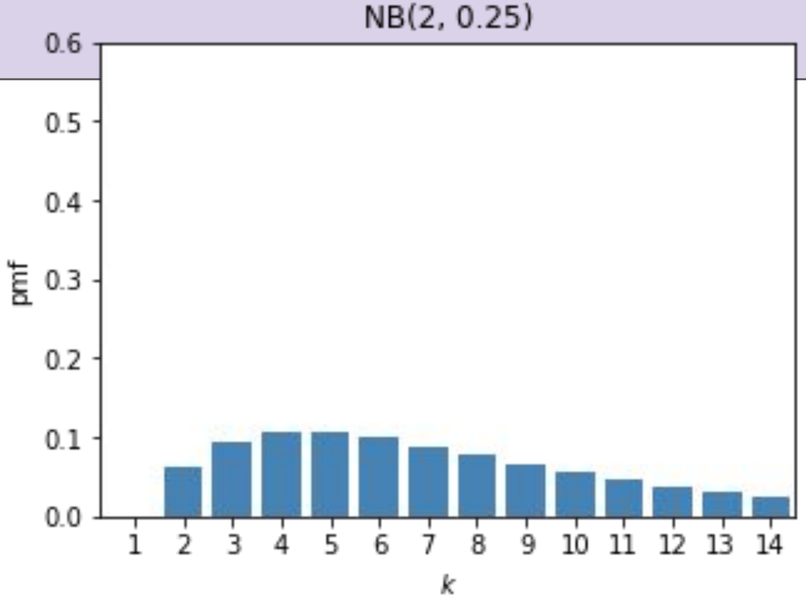

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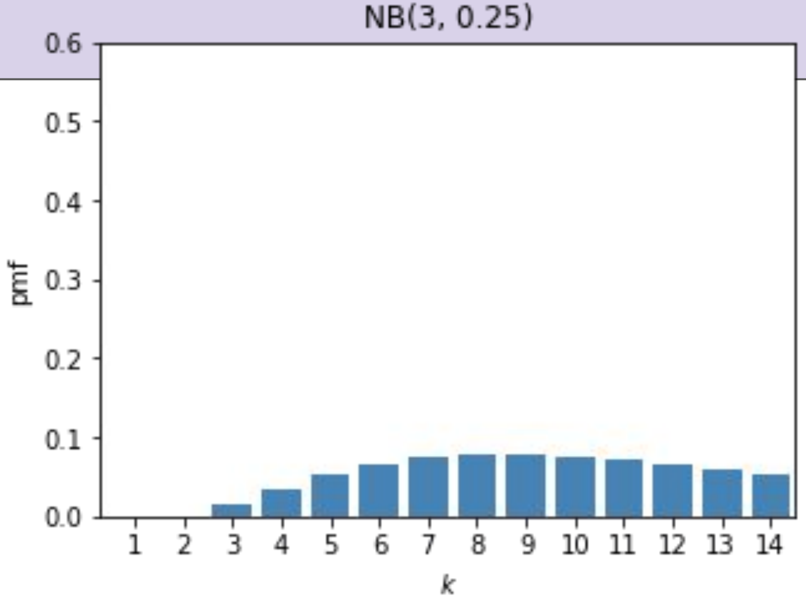

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Question: What assumptions did we implicitly make in deriving the Negative Binomial distribution?

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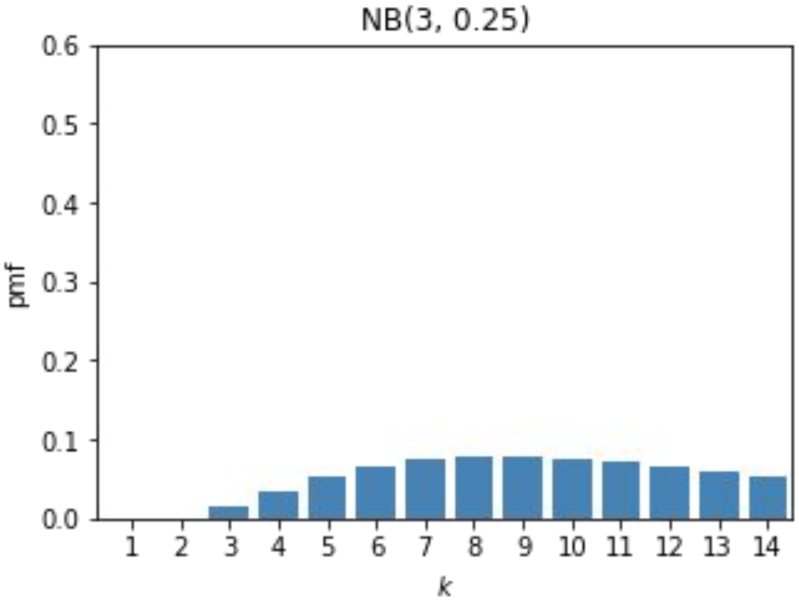
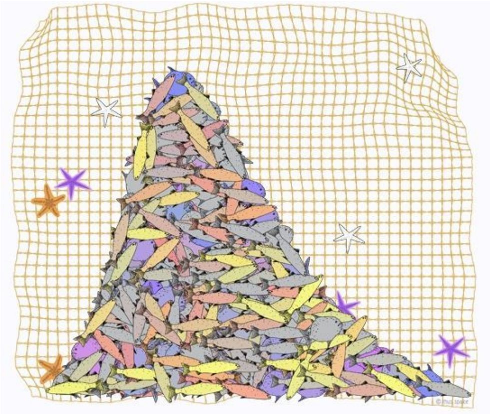


The Poisson distribution

Example: A grocery store check-out line moves people through at an average rate of 1 customer per 3 minutes. What is the probability that they check-out:
(i) 1 customer in 5 minutes? (ii) 3 customers in 5 minutes? (iii) 10 customers in 5 minutes?



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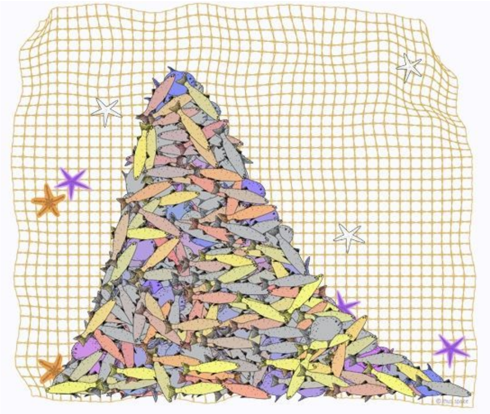
Derivation:

Think of this process as the limit of a Binomial r.v., as we pack more and more trials into a fixed slice of time.

$\mu = np$ n = time slices; p = prob. of a customer in that time slice

→ What is the probability of seeing *k* successes in that slice of time?



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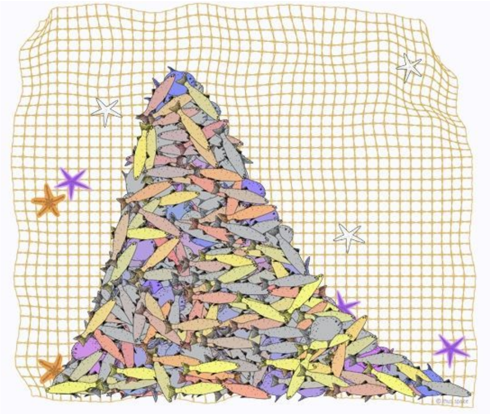
$$= \lim_{n\to\infty} \binom{n}{k} p^k \, (1-p)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} p^k \, (1-p)^{n-k}$$



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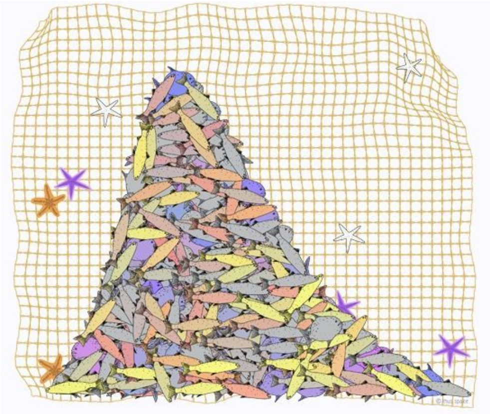

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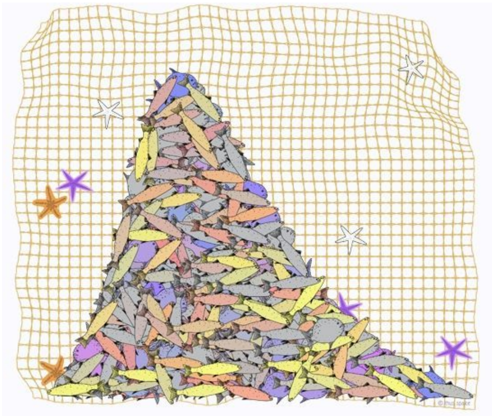
$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\mu}{n}\right)^k \left(1-\left(\frac{\mu}{n}\right)\right)^{n-k}$$

$$= \frac{\mu^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\, n^k} \frac{\left(1-\frac{\mu}{n}\right)^n}{\left(1-\frac{\mu}{n}\right)^k}$$

$$= \frac{\mu^k}{k!} \cdot 1 \cdot \frac{e^{-\mu}}{1}$$

$$f(k) = \frac{\mu^k\, e^{-\mu}}{k!}$$



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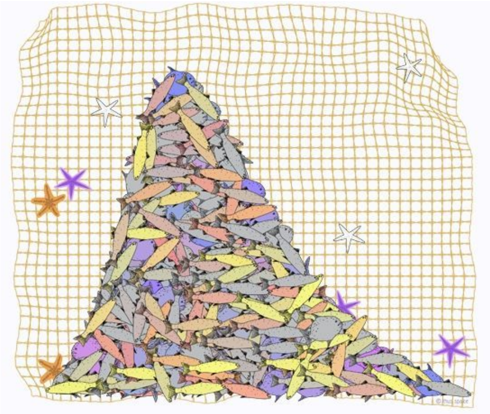
We say that $X \sim Pois(\mu)$



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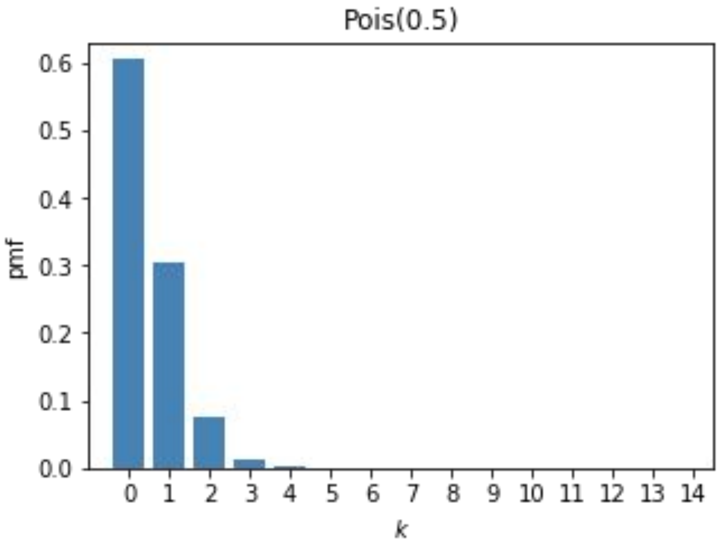

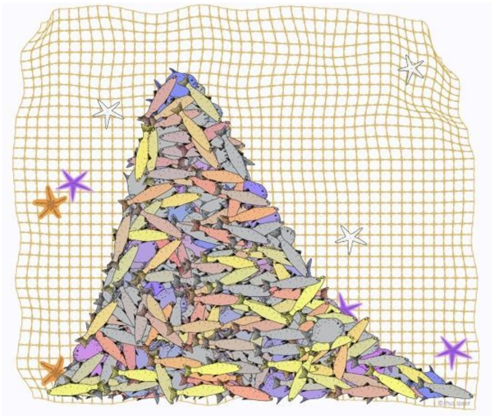

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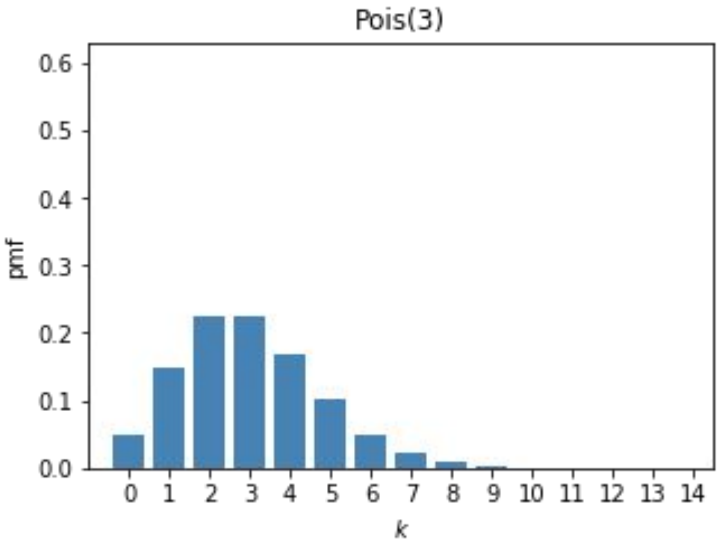

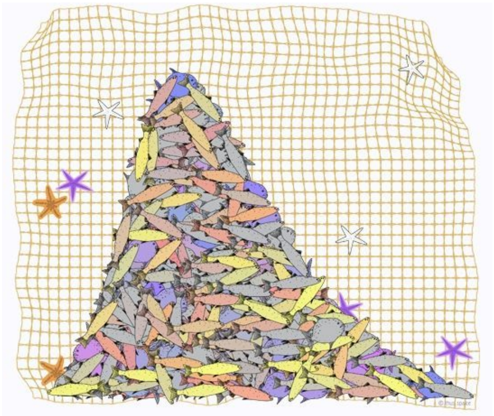

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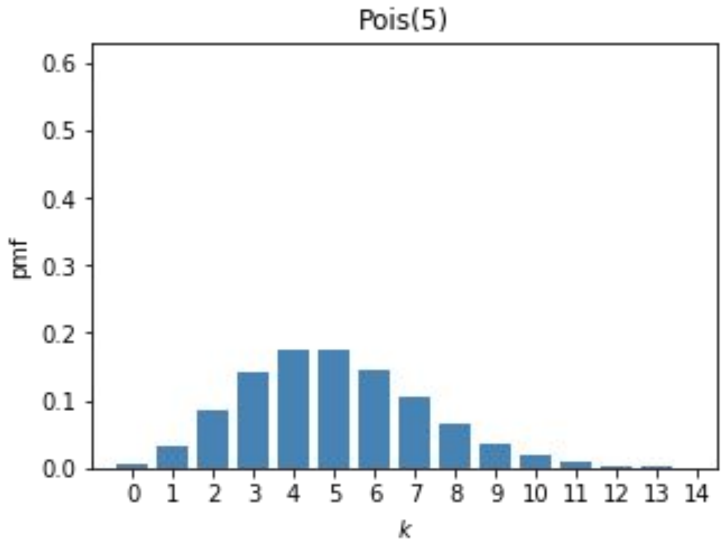

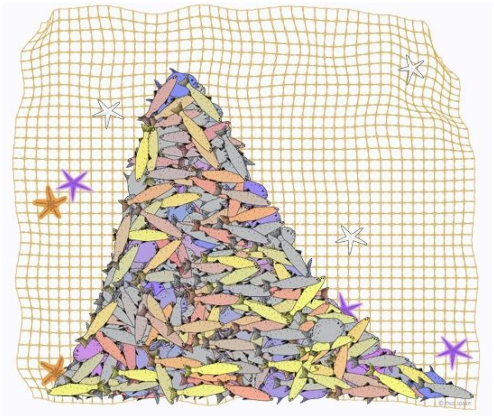

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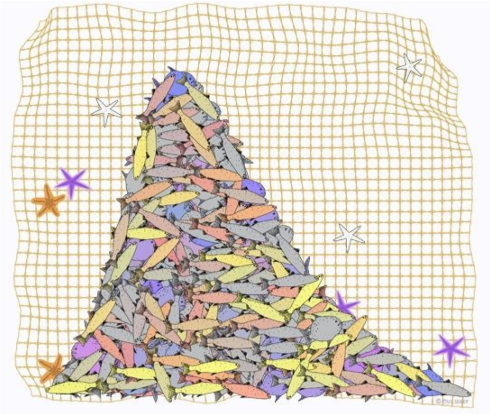
$\mu = \frac{1}{3}$ customer/min × time interval

→ 5/3 cust./5 min block



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μ = ⅓ customer/min × time interval

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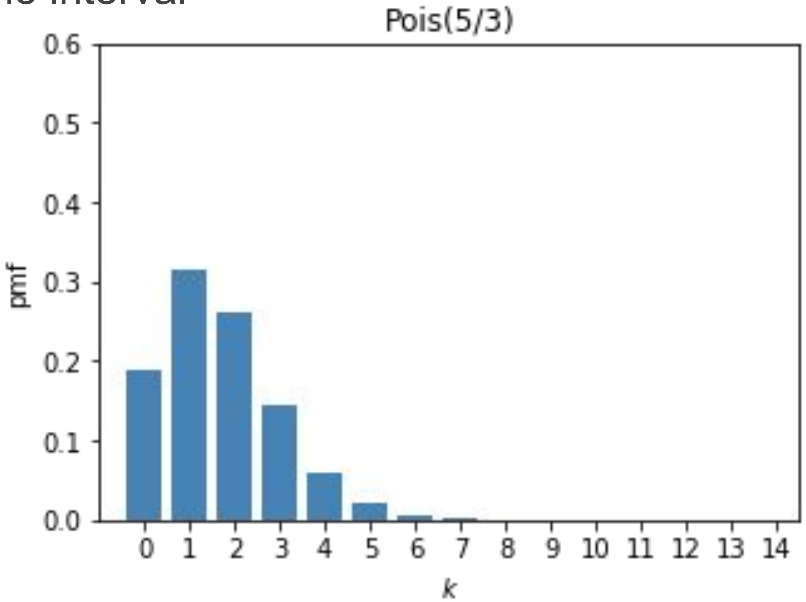
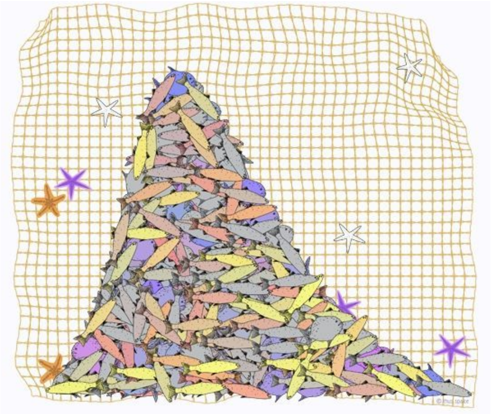
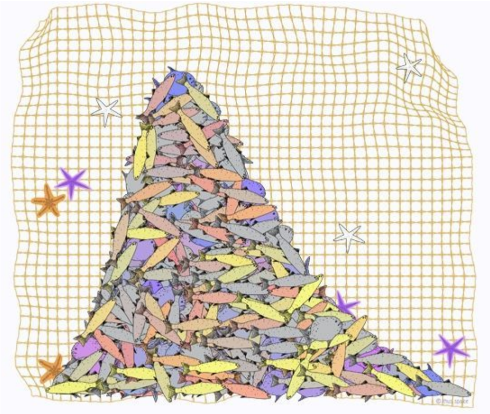




The Poisson distribution

Question: What assumptions did we implicitly make in deriving the Poisson distribution?

- Probability of observing a single event over a small interval is **proportional to** the size of the interval
- Each event/arrival is **independent**

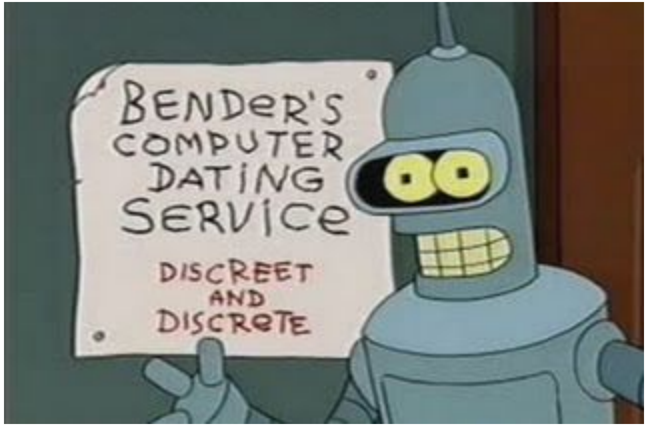


What just happened?

- We learned about some *more* important **discrete** distributions!
  - **Geometric** distribution: ________________________________
  - **Negative binomial** distribution: ________________________________
  - **Poisson** distribution: ________________________________

Next time: More **distributions**!

But, we won't be *discrete* about it.



Okay! Let's get to work!

Get in groups, get out laptops, and open **nb08** notebook

Let's…

- Practice identifying applications for the distributions we've learned
- Confirm our theoretical distributions with some simulations
- Look at the *Challenger* disaster