Analysis II

24. Januar 2021

Teil I
Ordinary Differential Equations

Def. A *differential equation* is an equation where the unknown is a function f, and the equation relates $f(x)$ with values of derivatives $f^{(i)}$ **at the same point** x.
Def. Ordinary $\iff$ One (input) variable only

Def. A Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = D(y) = b$$

where $y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions.

Def. Homogenous $\iff$ $b(x) = 0$
Def. (Initial Condition) A set of equations

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

Thm. (2.2.3) $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.
(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.
$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$
(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution
(4) For any initial value problem, there is a unique solution $f \in S_b$

Remark: If $b \neq 0$, then S_b is not a vector space
Remark: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

1 Linear ODEs of order 1

Procedure: Consider $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$
$$f_0(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$
2. Find a solution of the inhomogeneous equation f_p, then $S_b = f_p + S_0$.
   - Guess: $b(x)$ should resemble f_p
   - Variation of Constants (Assume constants of S_0 are functions)
   - Formula: $f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$

Remark: The solutions are given by $f_0 + zf_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Remark: To solve the real value problem $f(x_0) = y_0$, one can solve $f_0(x_0) + zf_1(x_0) = y_0$
Remark: If $a \in \mathbb{R}$, then there exists $f_0, f_1 \in \mathbb{R}$

2 Lin. ODE with constant coefs.

The equation takes the form: Let $a_{k-1}, ..., a_0 \in \mathbb{C}$

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b(x)$$

Intuition: We look for solutions of the form $f(x) = e^{\lambda x}, \lambda \in \mathbb{C}$

$$\begin{aligned} 0 &= y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y \\ &= e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0) \\ &= e^{\lambda x} P(\lambda) \end{aligned}$$

$\implies$ f is a solution if and only if $P(\lambda) = 0$.
$\implies$ According to the Fundamental Theorem of Algebra, there are k roots for P in $\mathbb{C}$.
Remark: $P(\lambda)$ is the **characteristic polynomial** and the roots are called **eigenvalues**

Thm. Let $\lambda_1, ..., \lambda_r$ be the pairwise distinct roots of $P(\lambda)$ with corresponding multiplicity $m_1, ..., m_r$. Then the functions

$$x^l e^{\lambda_j x} \quad 1 \leq j \leq r, \quad 0 \leq l < m_j$$

form a basis of the space of solutions of the homogeneous equation.
E.g. for k distinct roots we get:

$$f(x) = z_1 e^{\lambda_1 x} + \cdots + z_k e^{\lambda_k x}, \text{ with } z_1, ..., z_2 \in \mathbb{C}$$

Remark: If we are only interested in real solutions, the solutions based on complex roots, the basis can be transformed. For $\lambda = a + bi$:

$$\text{span}(e^{\lambda x}, e^{\bar{\lambda} x}) = \text{span}(e^{ax}\cos(bx), e^{ax}\sin(bx))$$

2.1 Solving the inhomogenous eqn.

Procedure: (Ansatz)

$b(x)$	Ansatz $y_p(x)$
$P_n(x)$	$Q_n(x)$
$P_n(x)e^{\mu x}$	$Q_n(x)e^{\mu x}$
$P_n(x)\sin(\nu x)$ $+Q_n(x)\cos(\nu x)$	$R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x)$
$P_n(x)e^{\mu x}\sin(\nu x)$ $+Q_n(x)e^{\mu x}\cos(\nu x)$	$e^{\mu x}(R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x))$

Insert $y_p(x)$ in the inhomogeneous eqn. and solve for the constants. $P_n(x), Q_n(x), ...$ are polynomials of degree n.
Remark: If $y_p(x)$ is a root of $P(\lambda)$ of multiplicity m, multiply $y_p(x)$ by x^m.

Procedure: (Variation of constants)
This method can be derived from the matrix describing the problem. Assume $n = 2$.
Try $y_p = z_1(x)f_1 + z_2(x)f_2$ after solving the system:

$$\begin{cases} z_1'(x)f_1 + z_2'(x)f_2 = 0 \\ z_1'(x)f_1' + z_2'(x)f_2' = b \end{cases}$$

2.2 Separation of Variables

Recipe: One can try to separate the variables (e.g only y' on the left, only x's on the right) in order to solve a non-linear first order ODE. We get:

$$\int \frac{\mathrm{d}y}{g(y)} = \int \mathrm{d}x b(x)$$

If for any y_0 it is $g(y_0) = 0$, the constant function $y = y_0$ is a solution.

Teil II
Differential calculus in $\mathbb{R}^n$

If not specified, f is a function $f\colon X \subset \mathbb{R}^n \to \mathbb{R}^m$ and we denote

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

1 Continuity

Prop. (Sequences) The sequence $(x_k)_{k\in\mathbb{N}}, x_k \in \mathbb{R}^n$ converges to $y \in \mathbb{R}^n$ as $k \to +\infty$ iff the following two equivalent conditions hold:

1. $\forall i$, $\lim_{k\to+\infty}(x_k)_i = y_i$.
2. $\lim_{k\to+\infty} \|x_k - y\| = 0$.

Prop. (Limit) Let $x_0 \in X, y \in \mathbb{R}^m$.

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$$

iff for every sequence x_k which converges to x_0, $f(x_k)$ converges to y.
Def. (Continuity)

$$f \text{ is continuous at } x_0 \iff \lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = f(x_0)$$

Def. (Bounded, Closed, Open, Compact)
A subset $X \subset \mathbb{R}^n$ is called

- bounded $\iff$ $\exists M \forall x \in X \, \|x\| < M$.
- closed $\iff$ Every sequence in X converges in X
- open $\iff$ For any $x \in X$ there exists a ball around x in X
- compact $\iff$ closed and bounded.

Furthermore we have:

- $|X|$ is finite $\implies$ compact
- Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed (open) set $Y \subset \mathbb{R}^m$, the set
$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \subset \mathbb{R}^n$$
is closed (open).

Def. (C^k)

1. C^1 if f differentiable and all its partial derivatives are continuous
2. C^k if f differentiable and all its partial derivatives are of C^{k-1}

2 Derivatives

Def. (Partial derivatives) Let $X \subset \mathbb{R}^n$ be an open set, $f\colon X \to \mathbb{R}^m$ be a function. Then we decompose f into m functions f_j in order to write

$$\frac{\partial f}{\partial x_i}(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(x_0) \end{pmatrix}$$

where $\frac{\partial f_j}{\partial x_i}(x_0) = \lim_{h\to 0} \frac{f_j(x_0 + he_i) - f_j(x_0)}{h}$.

Def. (Directional derivative) Given $u \in \mathbb{R}^n$ with $\|u\| = 1$, the directional derivative at a is

$$D_u f(a) := \lim_{h\to 0} \frac{f(a + hu) - f(a)}{h} = \frac{d}{dh} f(a + hu)\bigg|_{h=0}$$

If f is differentiable in a

$$D_u f(a) = \vec{u} \cdot \nabla f(a)$$

Def. (Gradient) For $f : X \to \mathbb{R}$

$$\nabla f(x_0) := \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

The gradient is the direction of *steepest ascent*.

Def. (Jacobi Matrix)

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{i=1,\dots,m;\\ j=1,\dots,n}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Def. (Hessian Matrix) Let $f \in C^2(X; \mathbb{R}^n)$, then $\mathrm{Hess}_f(x_0)$ is given by

$$\left(\frac{\partial^2 f}{\partial x_i x_j}(x_0)\right)_{\substack{i=1,\dots,n;\\ j=1,\dots,n}} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\, \partial x_2} & \cdots \\ \frac{\partial^2 f}{\partial x_2\, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

3 The Differential

Def. (Differentiability) Let $u \colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $x_0 \in X$. We say that f is *differentiable at* x_0 if

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

The linear map u is then called the *differential of f at x_0* and is denoted by $df(x_0) = d_{x_0} f$.

Intuition: This means that we can approximate $f(x)$ by a linear map df such that $R(x, x_0)$ goes faster to zero than $\|x - x_0\|$.

Thm. If f is differentiable at x_0 then

1. f is continuous at x_0
2. All partial derivatives exist.
3. $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is given by $x \mapsto Ax$

$$A = J_f(x_0)$$

Thm. (Continuous Partials) If f has all partial derivatives and they are continuous on X, then f is differentiable on X

Def. (Tangent Space) Let f be differentiable then the tangent space of f at x_0 is

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

($\approx$ shifted image of the differential)

Thm. (Chain rule) Let $Y \subset \mathbb{R}^m$ be an open set and $f \colon X \to Y$ and $g \colon Y \to \mathbb{R}^p$ be differentiable functions on X and Y, respectively. Then $g \circ f \colon X \to \mathbb{R}^p$ is differentiable on X and the differential for $x_0 \in X$ is given by

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

In particular, the Jacobi matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$$

Thm. (Order of Diff.) Let $f \in C^k$ for $k \geq 2$. Then the partial derivatives of order $\leq k$ are independent of the order of differentiation.

4 Taylor polynomials

Def. (Taylor Polynomial) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ and $f \in C^k(X; \mathbb{R})$. The *k-th Taylor polynomial of f at point x_0* is a polynomial in n variables of degree $\leq k$ given by $T_k f(y; x_0) =$

$$\sum_{m_1 + \ldots + m_n \leq k} \frac{1}{m_1! \cdots m_n!} \frac{\partial^k f}{\partial_1^{m_1} \cdots \partial_n^{m_n}} \tilde{y}_1^{m_1} \cdots \tilde{y}_n^{m_n}$$

where $\tilde{y} = (y - x_0)$.
For the 2nd degree case we have:

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot \tilde{y} + 0.5 \tilde{y}^T \cdot \mathrm{Hess}_f(x_0) \cdot \vec{y}$$

5 Critical Points

Def. (Critical Point) For $f \colon X \to \mathbb{R}$, a point $x_0 \in X$ is critical if $\nabla f(x_0) = 0$.

- x_0 local max., if $f(x) \leq f(x_0) \quad \forall x \in B_r(x_0)$
- A critical point which is neither a local maximum nor a local minimum is a *saddle point*.

Thm. (2nd Derivative Test) Let $X \subset \mathbb{R}^n$ be open and $f \in C^2(X; \mathbb{R})$. Let $x_0 \in X$ be a non-degenerate critical point of f ($\nabla f(x_0) = 0, \det(Hess_f(x_0)) \neq 0$). Then

1. x_0 is a local min. if $Hess_f(x_0)$ pos. definite,
2. x_0 is a local max. if $Hess_f(x_0)$ neg. definite,
3. x_0 is a saddle point if the Hessian matrix is indefinite.

Recipe: (Degenerate Critical Point) If there exists $g \colon \mathbb{R} \to X$ such that $f \circ g$ has no local maximum (minimum, saddle) with $f(x_0)$ than neither does f.

Teil III

Integrals in $\mathbb{R}^n$

1 Line Integrals

Def. (Line Integral) Let $I = [a, b]$ be compact.

(1) Let $f(t) = (f_1(t), \ldots, f_n(t))$ be continuous.

$$\int_a^b f(t) dt = \left(\int_a^b f_1(t), \ldots, \int_a^b f_n(t) dt\right) \in \mathbb{R}^n$$

(2) A parameterized curve in $\mathbb{R}^n$ is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$ that is piecewise C^1, i.e., there exists $k \geqslant 1$ and a partition

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$$

such that the restriction of f to $]t_{j-1}, t_j[$ is C^1 for $1 \leqslant j \leqslant k$. We say that γ is a parameterized curve between $\gamma(a)$ and $\gamma(b)$

(3) Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterized curve. Let $X \subset \mathbb{R}^n$ be a subset containing the image of γ, and let $f : X \to \mathbb{R}^n$ be continuous. The line integral of f along γ is:

$$\int_\gamma f(s) \cdot ds := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

Lem. (Invariance under orientation)
Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterized curve. An oriented reparameterization of γ is a $\sigma : [c, d] \to \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$, where $\varphi : [c, d] \to [a, b]$ is a continuous map, differentiable on $]a, b[$, that is strictly increasing and satisfies $\varphi(a) = c$ and $\varphi(b) = d$.
Let X be a set containing the image of γ, $f : X \to \mathbb{R}^n$ continuous. Then

$$\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$$

Def. (Conservative Vector Field)
Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous vector field. X is called conservative if, for any x_1, x_2 in X, the line integral

$$\int_\gamma f(s) \cdot d\vec{s}$$

is independent of the choice of a parametrized curve γ in X from x_1 to x_2,
Equivalently f is conservative if and only if $\int_\gamma f(s) \cdot d\vec{s} = 0$ for any closed parametrized curve in X.

Thm. (Potential) Let $f : X \to \mathbb{R}^n$ be a conservative vector field. Then there exists a C^1 function g on X such that $f = \nabla g$. If X is a path-connected set, then g is unique up to addition of a constant.

Prop. (4.1.13) Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ a vector field of class C^1. If f is conservative, then we have for any integers with $1 \leqslant i \neq j \leqslant n$.

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Def. (Star-shaped) $X \subset \mathbb{R}^n$ is star-shaped if there exists x_0 such that $\forall x \in X$, the line segment joining x_0 to x is contained in X.

Thm. (4.1.17) Let $X \subset \mathbb{R}^n$ be star-shaped and open. Let f be a C^1 vector field such that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

on X for all $i \neq j$ between 1 and n. Then the vector field f is conservative.

Def. (Curl) Let $X \subset \mathbb{R}^3$ be an open set and $f : X \to \mathbb{R}^3$ a C^1 vector field. Then $\mathrm{curl}(f)$ is the continuous vector field on X defined by

$$\mathrm{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

2 Riemann Integral in $\mathbb{R}^n$

Thm. (Fubini) Let $f : X \to \mathbb{R}^n$ be a function on a compact subset X. If n_1 and n_2 are integers $\geqslant 1$ such that $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^{n_1}$, let

$$Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$$

Let X_1 be the set of $x_1 \in \mathbb{R}^n$ such that Y_{x_1} is not empty. Then X_1 is compact in $\mathbb{R}^{n_1}$ and Y_{x_1} is compact in $\mathbb{R}^{n_2}$ for all $x_1 \in X_1$. If the function

$$g(x_1) = \int_{Y_{x_1}} f(x_1, x_2) \, dx_2$$

on X_1 is continuous, then

$$\int_X f(x_1, x_2) \, dx = \int_{X_1} g(x_1) \, dx_1 = \int_{X_1} \left(\int_{Y_{x_1}} f(x_1, x_2) \, dx_2\right) dx_1$$

Def. (Negligible set) Let $1 \leqslant m \leqslant n$

(1) A parameterized m-set is a continuous map

$$f : [a_1, b_1] \times \cdots \times [a_m, b_m] \to \mathbb{R}^n$$

which is differentiable on

$$]a_1, b_1[\times \cdots \times]a_m, b_m[$$

(2) A subset $B \subset \mathbb{R}^n$ is negligible if there exist an integer $k \geqslant 0$ and parameterized m_i-sets $f_i : X_i \to \mathbb{R}^n$, with $1 \leqslant i \leqslant k$ and $m_i < n$, such that

$$X \subset f_1(X_1) \cup \cdots \cup f_k(X_k)$$

Lem. (Integral of neglibible set) Let $X \subset \mathbb{R}^n$ be compact and X be negligible. Then for any continuous function, we have

$$\int_X f(x)dx = 0$$

2.1 Improper integrals

Def. (4.3.1) Let $X \in \mathbb{R}^n$ be non-compact and $f : X \to \mathbb{R}^n$ continuous and positive. Let X_k be a sequence, such that $X_k \subset X_{k+1}$ and $\bigcup_{k=1}^{\infty} X_k = X$ We say that $\int_X f(x)dx$ converges if

$$\int_X f(x)dx = \lim_{k\to\infty} \int_{X_k} f(x)dx \text{ and exists}$$

2.2 Change of variable

Def. (Change of variables) If f is differentiable and $\det(J_f(x_0)) \neq 0$ then f is a change of variables around x_0.

Thm. (Change of variables) Let $\overline{X}, \overline{Y} \subset \mathbb{R}^n$ be compact and $\varphi\colon \overline{X} \to \overline{Y}$ a continuous function that is of class C^1 and bijective on the interiors X_0 and Y_0 of $\overline{X}$ and $\overline{Y}$ where $\overline{X} = X_0 \cup B$ and $\overline{Y} = Y_0 \cup C$ with X_0, Y_0 open and C, D negligible. Then

$$\int_{\overline{Y}} f(y)dy = \int_{\overline{X}} f(\varphi(x)) \cdot |\det J_\varphi(x)|\, dx$$

Examples:

1. polar coordinates
$$dxdy = rdrd\varphi$$
2. cylindrical coordinates
$$dxdydz = rd\theta drdz$$
3. spherical coordinates
$$dxdydz = r^2 \sin\varphi drd\theta d\varphi$$

$$f\colon (0,\infty) \times [0, 2\pi) \times (0,\pi) \to \mathbb{R}^3$$
$$(r,\theta,\phi) \mapsto (x,y,z) = (r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi))$$

2.3 Geometric applications

(1) [Center of mass] Let $X \subset \mathbb{R}^n$ be compact, such that the volume of X is positive. The center of mass (or barycenter) of X is the point $\bar{x} \in R^n$ such that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ with

$$\bar{x}_i = \frac{1}{\mathrm{Vol}(X)} \int_X x_i dx$$

(2) [Surface area] Consider a continuous function

$$f : [a,b] \times [c,d] \to \mathbb{R}$$

which is C^1 on $]a,b[\times]c,d[$. Let

$$\Gamma = \{(x,y) \in [a,b] \times [c,d], z = f(x,y)\} \subset \mathbb{R}^3$$

be the graph of f. Intuitively, this is a surface, and it should have an area. This is in fact given by

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x,y))^2 + (\partial_y f(x,y))^2} dxdy$$

Such a result also holds for the graphs of functions defined on other sets, such as discs, provided they are C^1 in the ïnterioröf the domain.

There is an analogue formula for the length of the graph of a function $f : [a,b] \to \mathbf{R}$, namely it is equal to

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

3 Green's Formula

Thm. (Green) . Let $X \subset \mathbb{R}^2$ be compact with a boundary ∂X that is the union of finitely many simple closed parameterized curves $\gamma_1, \ldots, \gamma_k$. Assume that

$$\gamma_i : [a_i, b_i] \to \mathbb{R}^2$$

has the property that X lies always to the **left** of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Let $f = (f_1, f_2)$ be a vector field of class C^1 defined on some open set containing X. Then we have

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}$$

Recipe: (Enclosed Area) : Let $X \subset \mathbb{R}^2$ and $F = \int f dx$ then this implies for example

$$\int_X f(x) = \int_\gamma (0, F)\vec{s}$$

Kor. (4.6.5) Let $X \subset \mathbb{R}^2$ be compact with a boundary ∂X that is the union of finitely many simple closed parameterized curves $\gamma_1, \ldots, \gamma_k$. Assume that

$$\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) : [a_i, b_i] \to \mathbf{R}^2$$

has the property that X lies always to the left of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Then we have

$$\mathrm{Vol}(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t)\gamma_{i,2}'(t)dt$$

Def. (Simple closed parameterized curve) $\gamma : [a,b] \to \mathbb{R}^2$ is a closed parameterized curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or $\{s,t\} = \{a,b\}$, and such that $\gamma'(t) \neq 0$ for $a < t < b$. (If γ is only piecewise C^1 inside $]a,b[$, this condition only applies where $\gamma'(t)$ exists).

Teil IV
Miscelaneous

1 Linalg

Def. (Positive/Negative definite) A symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ with $\det(A) \neq 0$ (meaning that its eigenvectors are non-zero) is

1. *positive definite* $(A > 0)$ if
$$x^T Ax > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$
meaning that its eigenvalues are strictly positive
2. *negative definite* $(A < 0)$ if
$$x^T Ax < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$
meaning that its eigenvalues are strictly negative
3. *indefinite* otherwise, meaning that its eigenvalues are both positive and negative.

Lem. (Eigenvalues) The eigenvalues are the zeros of the characteristic Polynomial:

$$\det(A - \lambda Id)$$

Furthermore for eigenvalues $\lambda_1, \ldots \lambda_n$:

$$\sum \lambda_i = \mathrm{trace}(A) \qquad \prod \lambda_i = \det(A)$$

Hence:

$$\det(A) < 0 \wedge \mathrm{trace}(A) \geq 0 \implies \text{indefinite}$$

Thm. (Sylvester) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if and only if for every $1 \leq j \leq n$ the determinant of

$$A_j = (a_{kl})_{\substack{1\leq k\leq j \\ 1\leq l\leq j}}$$

is greater than zero $\det(A_j) > 0$. (Doesn't work for negative definitiveness)

Lem. (Determinant)

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - gec - hfa - idg$$

2 Exam Tipps

- Determine Order of ODE: Sometimes case distinction needed!
- When using Green, make sure curve is counter clockwise!