Analysis II

15. Oktober 2020

Teil I

Ordinary Differential Equations

Def. A *differential equation* is an equation where the unknown is a function f, and the equation relates $f(x)$ with values of derivatives $f^{(i)}$ **at the same point** x.

Def. Ordinary $\iff$ One variable only

Def. A Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = D(y) = b$$

where $y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions.

Def. Homogenous $\iff$ $b(x) = 0$

Def. (Initial Condition) A set of equations

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

Thm. (2.2.3) $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.

(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution

(4) For any initial value problem, there is a unique solution $f \in S_b$

Remark: If $b \neq 0$, then S_b is not a vector space
Remark: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

1 Linear ODEs of order 1

Procedure: Consider $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$
$$f_0(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$
2. Find a solution of the inhomogeneous equation f_p, then $S_b = f_p + S_0$.
   - Guess: $b(x)$ should resemble f_p
   - Variation of Constants (Assume constants of S_0 are functions)
   - Formula: $f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$

Remark: The solutions are given by $f_0 + zf_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Remark: To solve the real value problem $f(x_0) = y_0$, one can solve $f_0(x_0) + zf_1(x_0) = y_0$
Remark: If $a \in \mathbb{R}$, then there exists $f_0, f_1 \in \mathbb{R}$

2 Lin. ODE with constant coefs.

The equation takes the form: Let $a_{k-1}, ..., a_0 \in \mathbb{C}$

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b(x)$$

Intuition: We look for solutions of the form $f(x) = e^{\lambda x}, \lambda \in \mathbb{C}$

$$\begin{aligned} 0 &= y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y \\ &= e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0) \\ &= e^{\lambda x} P(\lambda) \end{aligned}$$

$\implies$ f is a solution if and only if $P(\lambda) = 0$.
$\implies$ According to the Fundamental Theorem of Algebra, there are k roots for P in $\mathbb{C}$.
Remark: $P(\lambda)$ is the **characteristic polynomial** and the roots are called **eigenvalues**

Thm. Let $\lambda_1, ..., \lambda_r$ be the pairwise distinct roots of $P(\lambda)$ with corresponding multiplicity $m_1, ..., m_r$. Then the functions

$$x^l e^{\lambda_j x} \quad 1 \leq j \leq r, \quad 0 \leq l < m_j$$

form a basis of the space of solutions of the homogeneous equation.
E.g. for k distinct roots we get:

$$f(x) = z_1 e^{\lambda_1 x} + \cdots + z_k e^{\lambda_k x}, \text{ with } z_1, ..., z_2 \in \mathbb{C}$$

Remark: If we are only interested in real solutions, the solutions based on complex roots, the basis can be transformed. For $\lambda = \beta + i\gamma$:

$$\text{span}(e^{\lambda x}, e^{\bar{\lambda} x}) = \text{span}(e^{\beta x} \cos x, e^{\beta x} \sin x)$$

2.1 Solving the inhomogenous eqn.

Procedure: (Ansatz)

$b(x)$	Ansatz $y_p(x)$
$P_n(x)$	$Q_n(x)$
$P_n(x)e^{\mu x}$	$Q_n(x)e^{\mu x}$
$P_n(x)\sin(\nu x)$ $+Q_n(x)\cos(\nu x)$	$R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x)$
$P_n(x)e^{\mu x}\sin(\nu x)$ $+Q_n(x)e^{\mu x}\cos(\nu x)$	$e^{\mu x}(R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x))$

Insert $y_p(x)$ in the inhomogeneous eqn. and solve for the constants. $P_n(x), Q_n(x), \ldots$ are polynomials of degree n.
Remark: If d is a root of $P(\lambda)$ of multiplicity m, multiply $y_p(x)$ by x^m.

Procedure: (Variation of constants)
This method can be derived from the matrix describing the problem. Assume $n = 2$.
Try $y_p = z_1(x)f_1 + z_2(x)f_2$ after solving the system:

$$\begin{cases} z_1'(x)f_1 + z_2'(x)f_2 = 0 \\ z_1'(x)f_1' + z_2'(x)f_2' = b \end{cases}$$