Analysis II

22. November 2020

Teil I
Ordinary Differential Equations

Def. A *differential equation* is an equation where the unknown is a function f, and the equation relates $f(x)$ with values of derivatives $f^{(i)}$ **at the same point** x.
Def. Ordinary $\iff$ One variable only

Def. A Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = D(y) = b$$

where $y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions.

Def. Homogenous $\iff$ $b(x) = 0$
Def. (Initial Condition) A set of equations

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

Thm. (2.2.3) $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.
(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.
$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$
(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution
(4) For any initial value problem, there is a unique solution $f \in S_b$

Remark: If $b \neq 0$, then S_b is not a vector space
Remark: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

1 Linear ODEs of order 1

Procedure: Consider $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$
$$f_0(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$
2. Find a solution of the inhomogeneous equation f_p, then $S_b = f_p + S_0$.
   - Guess: $b(x)$ should resemble f_p
   - Variation of Constants (Assume constants of S_0 are functions)
   - Formula: $f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$

Remark: The solutions are given by $f_0 + zf_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Remark: To solve the real value problem $f(x_0) = y_0$, one can solve $f_0(x_0) + zf_1(x_0) = y_0$
Remark: If $a \in \mathbb{R}$, then there exists $f_0, f_1 \in \mathbb{R}$

2 Lin. ODE with constant coefs.

The equation takes the form: Let $a_{k-1}, ..., a_0 \in \mathbb{C}$

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b(x)$$

Intuition: We look for solutions of the form $f(x) = e^{\lambda x}, \lambda \in \mathbb{C}$

$$\begin{aligned} 0 &= y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y \\ &= e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0) \\ &= e^{\lambda x} P(\lambda) \end{aligned}$$

$\implies$ f is a solution if and only if $P(\lambda) = 0$.
$\implies$ According to the Fundamental Theorem of Algebra, there are k roots for P in $\mathbb{C}$.
Remark: $P(\lambda)$ is the **characteristic polynomial** and the roots are called **eigenvalues**

Thm. Let $\lambda_1, ..., \lambda_r$ be the pairwise distinct roots of $P(\lambda)$ with corresponding multiplicity $m_1, ..., m_r$. Then the functions

$$x^l e^{\lambda_j x} \quad 1 \leq j \leq r, \quad 0 \leq l < m_j$$

form a basis of the space of solutions of the homogeneous equation.
E.g. for k distinct roots we get:

$$f(x) = z_1 e^{\lambda_1 x} + \cdots + z_k e^{\lambda_k x}, \text{ with } z_1, ..., z_2 \in \mathbb{C}$$

Remark: If we are only interested in real solutions, the solutions based on complex roots, the basis can be transformed. For $\lambda = a + bi$:

$$\text{span}(e^{\lambda x}, e^{\bar{\lambda} x}) = \text{span}(e^{ax}\cos(bx), e^{ax}\sin(bx))$$

2.1 Solving the inhomogenous eqn.

Procedure: (Ansatz)

$b(x)$	Ansatz $y_p(x)$
$P_n(x)$	$Q_n(x)$
$P_n(x)e^{\mu x}$	$Q_n(x)e^{\mu x}$
$P_n(x)\sin(\nu x)$ $+Q_n(x)\cos(\nu x)$	$R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x)$
$P_n(x)e^{\mu x}\sin(\nu x)$ $+Q_n(x)e^{\mu x}\cos(\nu x)$	$e^{\mu x}(R_n(x)\sin(\nu x)$ $+S_n(x)\cos(\nu x))$

Insert $y_p(x)$ in the inhomogeneous eqn. and solve for the constants. $P_n(x), Q_n(x), \ldots$ are polynomials of degree n.
Remark: If $y_p(x)$ is a root of $P(\lambda)$ of multiplicity m, multiply $y_p(x)$ by x^m.

Procedure: (Variation of constants)
This method can be derived from the matrix describing the problem. Assume $n = 2$.
Try $y_p = z_1(x)f_1 + z_2(x)f_2$ after solving the system:

$$\begin{cases} z_1'(x)f_1 + z_2'(x)f_2 = 0 \\ z_1'(x)f_1' + z_2'(x)f_2' = b \end{cases}$$

2.2 Other Methods

Separation -¿ See Assistance Summary

Teil II
Differential calculus in $\mathbf{R}^n$

If not specified, f is a function $f\colon X \subset \mathbb{R}^n \to \mathbb{R}^m$ and we denote

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

1 Continuity

Prop. (Sequences) The sequence $(x_k)_{k \in \mathbb{N}}, x_k \in \mathbb{R}^n$ converges to $y \in \mathbb{R}^n$ as $k \to +\infty$ iff the following two equivalent conditions hold:

1. $\forall i$, $\lim_{k \to +\infty}(x_k)_i = y_i$.
2. $\lim_{k \to +\infty} \|x_k - y\| = 0$.

Prop. (Limit) Let $x_0 \in X, y \in \mathbb{R}^m$.

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y$$

iff for every sequence x_k which converges to x_0, $f(x_k)$ converges to y.
Def. (Continuity)

$$f \text{ is continuous at } x_0 \iff \lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = f(x_0)$$

Def. (Bounded, Closed, Open, Compact)
A subset $X \subset \mathbb{R}^n$ is called

- bounded $\iff$ $\exists M \forall x \in X \; \|x\| < M$.
- closed $\iff$ Every sequence in X converges in X
- open $\iff$ For any $x \in X$ there exists a ball around x in X
- compact $\iff$ closed and bounded.

2 Derivatives

Def. (Partial derivatives) Let $X \subset \mathbb{R}^n$ be an open set, $f\colon X \to \mathbb{R}^m$ be a function. Then we decompose f into m functions f_j in order to write

$$\frac{\partial f}{\partial x_i}(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(x_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(x_0) \end{pmatrix}$$

where $\frac{\partial f_j}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f_j(x_0 + he_i) - f_j(x_0)}{h}$.

Def. (Directional derivative) Given $u \in \mathbb{R}^n$ with $\|u\| = 1$, the directional derivative at a is

$$D_u f(a) := \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h} = \frac{d}{dh} f(a + hu)\Big|_{h=0}$$

If f is differentiable in a

$$D_u f(a) = \vec{u} \cdot \nabla f(a)$$

Def. (Gradient) For $f : X \to \mathbb{R}$

$$\nabla f(x_0) := \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

The gradient is the direction of *steepest ascent*.
Def. (Jacobi Matrix)

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{i=1,\ldots,m; \\ j=1,\ldots,n}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Def. (Hessian Matrix) Let $f \in C^2(X; \mathbb{R}^n)$, then $\mathrm{Hess}_f(x_0)$ is given by

$$\left(\frac{\partial^2 f}{\partial x_i x_j}(x_0)\right)_{\substack{i=1,\dots,n;\\ j=1,\dots,n}} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

3 The Differential

Def. (Differentiability) Let $u\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $x_0 \in X$. We say that f is *differentiable at* x_0 if

$$\lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

The linear map u is then called the *differential of* f *at* x_0 and is denoted by $df(x_0) = d_{x_0}f$.

Intuition: This means that we can approximate $f(x)$ by a linear map df such that $R(x, x_0)$ goes faster to zero than $\|x - x_0\|$.

Thm. If f is differentiable at x_0 then

1. f is continuous at x_0
2. All partial derivatives exist.
3. $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is given by $x \mapsto Ax$

$$A = J_f(x_0)$$

Thm. (Continuous Partials) If f has all partial derivatives and they are continuous on X, then f is differentiable on X

Def. (Tangent Space) Let f be differentiable then the tangent space of f at x_0 is

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

($\approx$ shifted image of the differential)

Thm. (Chain rule) Let $Y \subset \mathbb{R}^m$ be an open set and $f\colon X \to Y$ and $g\colon Y \to \mathbb{R}^p$ be differentiable functions on X and Y, respectively. Then $g \circ f\colon X \to \mathbb{R}^p$ is differentiable on X and the differential for $x_0 \in X$ is given by

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

In particular, the Jacobi matrix satisfies

$$J_{g\circ f}(x_0) = J_g(f(x_0))J_f(x_0)$$

Thm. (Change of Variables) If f is differentiable and $\det(J_f(x_0)) \neq 0$ then f is a change of variables around x_0.

Thm. (Order of Diff.) Let $f \in C^k$ for $k \geq 2$. Then the partial derivatives of order $\leq k$ are independent of the order of differentiation.

4 Taylor polynomials

Def. (Taylor Polynomial) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $f \in C^k(X; \mathbb{R})$. The *k-th Taylor polynomial of* f *at point* x_0 is a polynomial in n variables of degree $\leq k$ given by $T_k f(y; x_0) =$

$$\sum_{m_1+\ldots+m_n\leq k} \frac{1}{m_1!\cdots m_n!}\frac{\partial^k f}{\partial_1^{m_1}\cdots\partial_n^{m_n}} y_1^{m_1}\cdots y_n^{m_n}$$

For the 2nd degree case we have:

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0)\cdot\vec{y} + y^T \cdot \mathrm{Hess}_f(x_0)\cdot\vec{y}$$

5 Critical Points