Analysis II

26. September 2020

Teil I
Ordinary Differential Equations

Def. A *differential equation* is an equation where the unknown is a function f, and the equation relates $f(x)$ with values of derivatives $f^{(i)}$ **at the same point** x.
Def. Ordinary $\iff$ One variable only

Def. A Linear ODE is an equation of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = D(y) = b$$

where $y = f(x)$ is the unknown function
$a_{k-1}(x), ..., a_0(x), b(x)$ are continuous functions.

Def. Homogenous $\iff$ $b(x) = 0$
Def. (Initial Condition) A set of equations

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$

Thm. (2.2.3) $I \subset \mathbb{R}$, linear ODE of order $k \geq 1$

(1) Let S_0 be the set of solutions for $b = 0$. Then is S_0 a vector space of dimension k.
(2) For any initial conditions, there is a unique solution $f \in S_0$, s.t.
$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$$
(3) For an arbitrary b, the set of solutions is $S_b = \{f + f_p | f \in S_0\}$, where f_p is a particular solution
(4) For any initial value problem, there is a unique solution $f \in S_b$

Bem: If $b \neq 0$, then S_b is not a vector space
Bem: If f_1, f_2 are solutions for $b_1(x), b_2(x)$, $f_1 + f_2$ is a solution for $b_1(x) + b_2(x)$

1.1 Linear ODEs of order 1

Procedure: Consider $y' + ay = b$

1. Solve homogeneous equation $y' + ay = 0$
$$f_0(x) = z \cdot e^{-A(x)} \text{ for } z \in \mathbb{C}$$
2. Find a solution of the inhomogeneous equation f_p, then $S_b = f_p + S_0$.
   - Guess: $b(x)$ should resemble f_p
   - Variation of Constants (Assume constants of S_0 are functions)
   - Formula: $f_p(x) = \int b(x) \cdot e^{A(x)} dx \cdot e^{-A(x)}$

Bem: The solutions are given by $f_0 + zf_1$, where $z \in \mathbb{C}$ and f_1 is a basis of S
Bem: To solve the real value problem $f(x_0) = y_0$, one can solve $f_0(x_0) + zf_1(x_0) = y_0$
Bem: If $a \in \mathbb{R}$, then there exists $f_0, f_1 \in \mathbb{R}$

1.2 Linear ODE with constant coefs.

The equation takes the form: Let $a_{k-1}, ..., a_0 \in \mathbb{C}$

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b(x)$$

Procedure: Solving homogeneous equations
We look for solutions of the form $f(x) = e^{\lambda x}, \lambda \in \mathbb{C}$

$$\begin{aligned} 0 &= y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y \\ &= e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0) \\ &= e^{\lambda x}P(\lambda) \end{aligned}$$

Thm. f is a solution if and only if $P(\lambda) = 0$.
Bem: According to the Fundamental Theorem of Algebra, there are k roots for P in $\mathbb{C}$.
Bem: $P(\lambda)$ is the **characteristic polynomial** and the roots are called **eigenvalues**

Case 1: k distinct solutions for $P(\lambda) = 0$
$f_j(x) = e^{\lambda_j x}$ are linearly independent.

Every solution for the ODE is of the form:
$$f(x) = z_1 e^{\lambda_1 x} + \cdots + z_k e^{\lambda_k x}, \text{ with } z_1, ..., z_2 \in \mathbb{C}$$

Case 2: $\exists\lambda$, which is a root of order $2 \leq j \leq k$
$$f_{\lambda,0}(x) = x^0 e^{\lambda x}, \cdots, f_{\lambda,j-1}(x) = x^{j-1}e^{\lambda x}$$
Taking the union of the functions $f_{\lambda,j}$ for all roots of P, each with its multiplicity, gives a basis of the space of solutions.