Numerical Methods

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Teil I
Computing with Matrices

1 Computational Effort

We use *Asymptotic Computation Complexity* (O-bounds) to judge our algorithms. But note that in many cases this approach is imprecise because on today's computers, *memory* bandwidth and latency is a key bottleneck.

Lem. (Matrix Product) The matrix product of an $m \times n$ and $n \times k$ matrix has complexity $O(mnk)$

Tricks to reduce complexity:

- Exploit associativity of operations
- Exploit hidden summations

2 Machine Arithmetic

Machine Numbers have a finite precision, hence any involved computation suffers from **Roundoff**.

Def. (Cancellation) Subtraction of almost equal numbers and accordingly an extreme *amplification of relative errors.*

Tricks to avoid cancellation:

- Trigonometric Identities
- Case Distinction
- Taylor Approximations
- Computing Diff. Quot. through Approx.

Def. (Stability) An Algorithm F is **numerically stable** if for all $x \in X$ its result $F(x)$ (possibly affected by roundoff) is the exact result for 'slightly perturbed' data.

Teil II
Linear Systems

1 Theory

Def. The Condition Number of a matrix is

$$\text{cond}(\mathbf{A}) := \|\mathbf{A}\| \left\|\mathbf{A}^{-1}\right\|$$

and is a measure for stability (≈ 1 for stability). Note that it is infinite for singular matrices. Intuitively *cond*(A) catches the maximum stretching factor divided by the minimum stretching factor.

Lem. If $m = n$ and $\mathbf{A}$ is regular/invertible, then its 2-norm condition number is $\text{cond}_2(\mathbf{A}) = \sigma_1/\sigma_n$

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2 LU Decomposition

There are two methods to choose pivots in the elimination process with a Permutation Matrix:

- Partial Pivoting: Max of column
- Full Pivoting: Max of remaining matrix

Complexities:

- Gauss elim. $\mathcal{O}(n^3)$, back substitution $\mathcal{O}(n^2)$
- LU: decomp. $\mathcal{O}(n^3)$, solve $\mathcal{O}(n^2)$
- Inverse: compute $\mathcal{O}(n^3)$, solve $\mathcal{O}(n^2)$

3 Exploiting Structure when Solving Linear Systems

Thm. (Low Rank Modification) Use the *Sherman-Morrison-Woodbury* Formula to compute the inverse of slightly modified (rank-1-modification) matrix.

$$(\mathbf{A}+\mathbf{U}\mathbf{V}^H)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I}+\mathbf{V}^H\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^H\mathbf{A}^{-1}$$

4 Matrix Storage Formats

Dense Matrices are either stored in Column-Major Ordering or Row-Major Ordering. By default Eigen employs Column-Major Ordering.

Sparse Matrices can be stored in

- COO/Triplet: Vector of triplets of (row, col, value)
- CCS: Compressed Column Storage
- CRS: Compressed Row Storage

where CRS consists of the arrays:

- val: All non-zero values in Row-Major Order
- col_idx: Column index of each entry in val
- row_ptr: Pointers to first elements of row

Note that size$(row_ptr) = m + 1$ (Sentinel).

Teil III
Linear Least Squares

Goal: Generalisation of solving $\mathbf{Ax} = \mathbf{b}$, find

$$\mathbf{x} \in \underset{\mathbf{y}\in\mathbb{R}^n}{\text{argmin}} \|\mathbf{Ay} - \mathbf{b}\|_2^2$$

Critical comparison of Methods:

- Normal Equations: $\mathcal{O}(n^2m + n^3)$
  → blows up instability
- Extended Normal Equations:
  → same conditioning as $\mathbf{A}$
  → sparsity preserved
- Householder QR: $\mathcal{O}(n^2m)$
  → always stable
  → loss of sparsity
- SVD:
  → no full rank requirement for $\mathbf{A}$

1 Normal Equation Methods

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

Warning: Normal Equations are vulnerable to instability since $\text{cond}_2(\mathbf{A}^T\mathbf{A}) = \text{cond}_2(\mathbf{A})^2$

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2 Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^{\mathsf{H}}, \quad \mathbf{A} \in \mathbb{K}^{m\times n},$$

$p := \min\{m, n\}$, $r := \text{rank}(\mathbf{A})$,
$\mathbf{\Sigma} = \text{diag}(\sigma_1, \ldots, \sigma_p)$ $\sigma_1 > \sigma_2 > \ldots > \sigma_r > \sigma_{r+1} = \cdots = \sigma_p = 0$

Full:

- $\mathbf{U} \in \mathbb{K}^{m\times m}$ $[\mathcal{R}(\mathbf{A})|\mathcal{N}(\mathbf{A})]$ (unitary)
- $\mathbf{\Sigma} \in \mathbb{K}^{m\times n}$ (generalized diagonal)
- $\mathbf{V} \in \mathbb{K}^{n\times n}$ $[\mathcal{R}(\mathbf{A}^{\mathsf{H}})|\mathcal{N}(\mathbf{A}^{\mathsf{H}})]$ (unitary)

Economical:

- $\mathbf{U} \in \mathbb{K}^{m\times p}$ $[\mathcal{R}(\mathbf{A})]$ (orthogonal columns)
- $\mathbf{\Sigma} \in \mathbb{K}^{p\times p}$ (diagonal)
- $\mathbf{V} \in \mathbb{K}^{n\times p}$ $[\mathcal{R}(\mathbf{A}^{\mathsf{H}})]$ (orthogonal columns)

Numerical Rank $r := \max_{j\in\{1,\ldots,p\}} \left(\frac{\sigma_j}{\sigma_1} \geq TOL\right)$
Cost of Eco SVD $\mathcal{O}(\min\{m,n\}^2 \max\{m,n\})$

Lem. (LSQ by SVD)

$$\mathbf{x}^* = \underset{\mathbf{x}\in\mathbb{R}^n}{\arg\min} \|\mathbf{Ax} - \mathbf{b}\|_2 = \mathbf{V}_1\Sigma_r^{-1}\mathbf{U}_1^T\mathbf{b}$$

If the lsq has multiple solutions, $\mathbf{x}^*$ has minimal norm.

Thm. (Low Rank Approximation) Let $\mathbf{A}_k$ be the matrix resulting from only keeping the first k singular values of $\mathbf{A}$. $\mathbf{A}_k$ is the best rank-k approximation of $\mathbf{A}$.

Method: (Principal Component Analysis)
Given a dataset in the form of a matrix $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}$, such that every column represents a time series.

- Columns of $\mathbf{U}$ capture dominant trends.
- Columns of $\mathbf{V}$ capture how strong the trends are in a particular column of $\mathbf{A}$
- Diagonal Entries of $\mathbf{\Sigma}$ tell us how dominant the trend is overall.

Method: (Proper orgthogonal decomp.)
Closely related is the POD problem, where we seek k-dimensional subspace U_k of $\mathbb{R}^m$, for which the sum of squared distances of the data points $\mathbf{A} = [\mathbf{a}_1, \ldots \mathbf{a}_n]$ to U_k is minimal.
Using the *Low Rank Approximation* Theorem we can prove that

$$U_k = \mathcal{R}(\mathbf{U}_{:,1:k})$$

3 QR Decomposition

$$\mathbf{A} = \mathbf{QR}, \quad \mathbf{A} \in \mathbb{K}^{m\times n} \quad \mathbf{Q} \in \mathbb{K}^{m\times m}$$

Idea: LLQ Problems are easier to solve for triangular matrices, hence we manage to decompose $\mathbf{A}$ into $\mathbf{QR}$ we have such a System since orthogonal matrices are norm-invariant.

Method: (Householder Reflections) The following Householder matrix performs a reflection

$$\mathbf{H}(\mathbf{v}) := \mathbf{I} - 2\mathbf{v}\mathbf{v}^\top$$

at the hyperplane with normal unit vector $\mathbf{v}$ (Intuitively H$(\mathbf{v})$ subtracts the projection of $\mathbf{x}$ on $\mathbf{v}$ twice).
We can reduce a matrix $\mathbf{A}$ to $\mathbf{R}$ using n successive transformations

$$\mathbf{H}(\mathbf{v}_n)\ldots\mathbf{H}(\mathbf{v}_1)\mathbf{A} = \mathbf{R}$$

, where $\mathbf{H}(\mathbf{v}_i)$ reflects the lower part of the i'th column of the current $\mathbf{A}$ on $\mathbf{e}_i$. E.g

$$\mathbf{v}_1 = \mathbf{a}_1 \pm \|\mathbf{a}_1\| \mathbf{e}_1$$

Method: (Givens Rotations) We can selectively eliminate entries with Givens rotations. The following matrix rotates everything in the hyperplane defined by $\mathbf{e}_1, \mathbf{e}_k$:

$$\mathbf{G}(1,k,\theta) = \begin{bmatrix} c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{bmatrix}$$

Note that $c = \cos(\theta), s = \sin(\theta)$. Two eliminate a_k we solve the equations.

$$c^2 + s^2 = 1 \quad \text{and} \quad -sa_1 + ca_k = 0$$

As with householder reflections, there are always two possibilites, one of whom is cancellation free.

Lem. (Modifications Techniques) Computing the QR decomposition of a slightly modified matrix (rank-1-modification, adding a row, adding a column) can be done efficiently in $\mathcal{O}(mn + n^2)$.

4 Constrained Least Squares

Problem: Find $x \in \mathbb{R}^n$ such that

$$\|\mathbf{Ax} - \mathbf{b}\| \to \min \quad \text{and} \quad \mathbf{Cx} = \mathbf{d}.$$

Method: (Lagrangian Multipliers) We intoduce the multiplier $\mathbf{m} \in \mathbb{R}^p$ to solve

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \sup_{\mathbf{m}\in\mathbb{R}^p} \|\mathbf{Ax} - \mathbf{b}\|_2 + \mathbf{m}^T(\mathbf{Cx} - \mathbf{d})$$

and we notice that for any finite solution $(\mathbf{Cx}-\mathbf{d}) = 0$. By realising that that fo the solution all partial derivatives must be zero we can obtain the *augmented normal equations*.

Method: (By SVD) We have

$$\mathbf{x} \in \mathbf{x}_0 + \mathcal{N}(\mathbf{C}) \quad \mathbf{x}_0 = \text{particular solution}$$

Hence since the SVD gives a basis of $\mathcal{N}(\mathbf{C})$ we write

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{V}_2\mathbf{y}$$

for some $\mathbf{y}$ which leads to the standard lsq:

$$\|\mathbf{AV}_2\mathbf{y} - (\mathbf{b} - \mathbf{Ax}_0)\| \to \min$$

Teil IV
Filtering Algorithms

1 Filters & Convolution

Def. (Filter) A function $F : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ where $l^\infty(\mathbb{Z})$ is the space of bounded infinite sequences

$$l^\infty(\mathbb{Z}) = \left\{(x_j)_{j\in\mathbb{Z}} : \sup|x_j| < \infty\right\}$$

Def. (LT-FIR) A Filter that is:

- Linear
- Time-Invariant: Shifting the input in time leads to the same output shifted in time.
- Finite: $\exists M \forall j : x_j = 0$ if $|j| > M \implies \exists N \forall j : y_j = 0$ if $|j| > N$
- Causal: The output does not start before the input.

Such a filter is uniquely characterized by its **Impulse response**:

$$F(\delta_{j,0}) = \ldots, 0, h_0, h_1, \ldots, h_{n-1}, 0, \ldots$$

For inputs $\mathbf{x} \in \mathbb{R}^m$ we get outputs $\mathbf{y} \in \mathbb{R}^{m+n-1}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ h_3 & h_2 & h_1 & h_0 \\ 0 & h_3 & h_2 & h_1 \\ 0 & 0 & h_3 & h_2 \\ 0 & 0 & 0 & h_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_4 \end{bmatrix}$$

Def. (Discrete Convolution) A Filter where given $\mathbf{x} = [x_0, \ldots, x_{m-1}]^\top \in \mathbb{K}^m, \mathbf{h} = [h_0, \ldots, h_{n-1}]^\top \in \mathbb{K}^n$ their DCONV is the vector $\mathbf{y} \in \mathbb{K}^{m+n-1}$ with components.

$$y_k = \sum_{j=0}^{m-1} h_{k-j} x_j$$

Def. (Periodic Convolution) Given two n-periodic signals $(u_k)_{k\in\mathbb{Z}}, (x_k)_{k\in\mathbb{Z}}$ PCONV yields the n-periodic signal:

$$(y_k) := (u_k) *_n (x_k), y_k := \sum_{j=0}^{n-1} u_{k-j} x_j$$

PCONV can be represtend by a matrix

$$\mathbf{C} = [c_{ij}]_{i,j=1}^n, c_{ij} = u_{j-i}$$

. Such a matrix is called **circulant**. The vector $\mathbf{u}$ s.t. $\mathbf{C} = \text{circul}(\mathbf{u})$ corresponds to $\mathbf{C}_{:,1}$.

2 Discrete Fourier Transform

The Fourier Matrix for inputs $\mathbf{y} \in \mathbb{C}^n$ is given by

$$\mathbf{F}_n = \left[\omega_n^{lj}\right]_{l,j=0}^{n-1} \in \mathbb{C}^{n,n} \quad \omega_n = \exp\left(\frac{-2\pi i}{n}\right)$$

and $\text{DFT}_n(\mathbf{y}) := \mathbf{F}_n\mathbf{y} = \left[\sum_{j=0}^{n-1} y_j \omega_n^{kj}\right]_{k=0}^{n-1}$.
Properties include

$$\mathbf{F}_n^{-1} = \frac{1}{n}\mathbf{F}_n^{\mathsf{H}} = \frac{1}{n}\overline{\mathbf{F}}_n$$

Method: (Frequency Filtering) The columns of $\mathbf{F}_n$ are trigonometric basis vectors, where

$$\mathbf{v}_k = \left[\cos\left(j\frac{2\pi}{n}\cdot k\right) + \imath \sin\left(\frac{2\pi jk}{n}\right)\right]_{j=0}^{n-1}$$

k is the 'frequency' and $j2\pi/n$ is the sample point. Hence the DFT_n is a basis transformation

$$B_E \leftrightarrow B_{trig}$$

Denoising and low/high filters can be implemented by manipulating the signal in the frequency domain.

Lem. (Diagonalizing circulant matrices) For any circulant matrix $\mathbf{C} \in \mathbb{C}^{n,n}$ we have

$$\mathbf{C} = \mathbf{F}_n^{-1} \operatorname{diag}(d_1, \ldots, d_n) \mathbf{F}_n,$$
$$[d_0, \ldots, d_{n-1}]^\top = \mathbf{F}_n [u_0, \ldots, u_{n-1}]^\top$$

In other words, the columns of $\mathbf{F}_n$ are eigenvectors of $\mathbf{C}$.

Thm. (Convolution Theorem) Periodic convolution in the time-domain equals to multiplication in the frequency-domain.

$$(\mathbf{u}) *_n (\mathbf{x}) = \mathbf{F}_n^{-1}\left[(\mathbf{F}_n\mathbf{u})_j (\mathbf{F}_n\mathbf{x})_j\right]_{j=1}^n$$

Implementation Speedup: $\mathcal{O}(n^2) \to \mathcal{O}(n\log_2(n))$

Thm. (2D DFT) Is given by

$$\mathbf{C} = \mathbf{F}_m\left(\mathbf{F}_n\mathbf{Y}^\top\right)^\top = \mathbf{F}_m\mathbf{Y}\mathbf{F}_n$$

The real basis vectors can be visualized. Analogously there is a 2D-Convolution-Theorem:

$$\mathbf{X} *_{m,n} \mathbf{Y} = \text{DFT}_{m,n}^{-1}(\text{DFT}_{m,n}(\mathbf{X}) \odot \text{DFT}_{m,n}(\mathbf{Y}))$$

Method: (Deblurring an image) A blurred image can be modeled as the 2D-Convolution of the actual image with a small filter. In the 2D-frequency-domain deblurring equals component wise division.

3 Fast Fourier Transform

The DFT of a vector of length pq can be divided in p DFT's of vectors of length q. The asymptotic runtime for this special 'divide and conquer' is

$$\text{FFT}(n) \in \mathcal{O}(n\log n)$$

Special Case $p = 2$:

$$\text{DFT}(\mathbf{v}) = \text{DFT}(\mathbf{v}_{even}) + s\,\text{DFT}(\mathbf{v}_{odd})$$

Def. (Toeplitz Matrix) $\mathbf{T} = (t_{ij})_{i,j=1}^n \in \mathbb{K}^{mn}$ is a Toeplitz matrix if it has constant diagonals given by a vector $\mathbf{u} \in \mathbb{K}^{m+n-1}$. Note that every circulant matrix is also a Toeplitz matrix.

Lem. (Fast Arithmetic for Toeplitz)
Vector Multiplication in $\mathcal{O}((n+m)\log(n+m))$: Construct Circulant Matrix $\mathbf{C} \in \mathbb{K}^{m+n,m+n}$ s.t.

$$\begin{bmatrix} \mathbf{Tx} \\ \star \end{bmatrix} = \mathbf{C}\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$$

which can be solved in the frequency domain (Convolution Theorem).
LSE Solution in $\mathcal{O}(n^2)$: The Levinson Algorithm.

Teil V

Data Interpolation in 1D

Goal: Given data points (t_i, y_i) and a finite-dimensional vector-space of functions V, reconstruct $f \in V$ such that $\forall i \quad f(t_i) = y_i$.

Def. (Interpol. as linear mapping)
For data points $(t_i, y_i) \quad i = 0, \dots, n$ and V spanned by basis functions $b_0, \dots b_m$ we have

$$\mathbf{Ac} := \begin{bmatrix} b_0(t_0) & \dots & b_m(t_0) \\ \vdots & & \vdots \\ b_0(t_n) & \dots & b_m(t_n) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_m \end{bmatrix} = \mathbf{y}$$

→ The interpolant is uniquely solv. iff $m = n$
→ The basis is **cardinal** if $b_j(t_i) = \delta_{ij}$

1 Global Polynomial Interpol.

→ In general not suitable due to potentially very high sensitivity!

Def. (Monomial Basis)
→ Evaluation in $\mathcal{O}(n)$ by exploiting associativity.
Def. (Lagrange Basis)

$$L_i(t) := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{t - t_j}{t_i - t_j} = I_{\mathcal{T}}(\mathbf{e}_{i+1})$$

→ Cardinal Basis.
Def. (Newton Basis)

$$N_i(t) := \prod_{j=0}^{i-1} t - t_j$$

→ Evaluation in $\mathcal{O}(n)$ by exploiting associativity.

Method: (Multiple Evaluations)
→ Barycentric Interpolation Formula.
Using precomputed weights in $\mathcal{O}(n^2)$ it achieves $\mathcal{O}(Nn)$ to evaluate N points.
Method: (Single Point Evaluation)
→ Aitken-Neville scheme.
Evaluation in $\mathcal{O}(n^2)$ but addPoint in $\mathcal{O}(n)$.
Method: (Multipoint+Updates)
→ Divided Differences Method.
Evaluation in $\mathcal{O}(Nn)$ and addPoint in $\mathcal{O}(n)$.

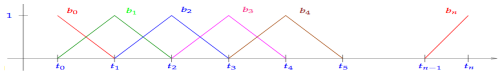
Method: (Extrapolation)

Method: (Sensitivity)
→ We can give an upper bound on the relative absolute error independent of data points! Lebesgue constant catches the maximal error.

2 Shape-Preserving Interpol.

Def. (Shape preserving)
positive data ⟶ positive interpolant.
monotonic data ⟶ monotonic interpolant.
convex data ⟶ convex interpolant.

Method: (Piecewise linear Interpol.)
→ locally shape preserving
Tent functions make a cardinal basis:



Method: (Cubic Hermite Interpol.)
→ f $\in C^1([t_0, t_n])$
Idea: Interpolate each interval as cubic polynomial but edge points have to match, and derivatives on edges have to match. But we have to decide what we want derivates to be at edge:

- Linear: (weighted mean of adjacent derivatives) -¿ No Preservation of monotonicity
- Pchip: Since slope must be zero at boundary in some cases -¿ Loss of linearity

3 Splines

Instead of fixing the boundary slopes, we add additional continuity constraints. The interpolating function f is then in the Spline Space.

Def. (Spline Space)

$$\mathcal{S}_{d,\mathcal{M}} := \left\{ s \in C^{d-1}(I) : s_j := s_{|[t_{j-1}, t_j]} \in \mathcal{P}_d \forall j = 1, \dots, n \right\}$$

$\mathcal{S}_{d,\mathcal{M}}$ is the space of piecewise polynomial functions in C^{d-1}. One can easily see that

$$\dim \mathcal{S}_{d,\mathcal{M}} = n + d$$

Method: (Cubic Spline Interpolation) The cubic spline interpolant is a function $s \in \mathcal{S}_{3,\mathcal{M}}$ that complies with the $n+1$ interpolation conditions. Note that we still have 2 degrees of freedom, this leads to the following flavours:

- natural: $s'(t_0) = 0, s'(t_n) = 0$
- periodic: $s'(t_0) = s'(t_n), s''(t_0) = s''(t_n)$
- complete: $s'(t_0) = c_0, s'(t_n) = c_n$

Note that: cubic splines are not shape preserving but weakly local.

4 Trigonometric Interpol.

bla

5 Least Squares Data Fitting

Goal: Given m data points (t_i, y_i) and a set of functions V, find a continuous function $f \in V$ such that

$$f \in \underset{\mathbf{g} \in S}{\operatorname{argmin}} \|\mathbf{g}(t_i) - y_i\|_2^2$$

We focus on **linear** data fitting, i.o.w. let V be an n-dimensional vector space spanned by functions $b_1(t), \dots, b_n(t)$. We look for coefficients

$$[x_1, \dots, x_n]^T = \underset{\mathbf{z} \in \mathbb{R}^n}{\operatorname{argmin}} \sum_{i=1}^{m} \left| \sum_{j=1}^{n} (\mathbf{z})_j b_j(t_i) - y_i \right|^2$$

Thm. (Linear Data Fitting Solution)
The solution to the linear least squares fitting problem is the least squares solution of the system

$$\begin{bmatrix} b_1(t_1) & \dots & b_n(t_1) \\ \vdots & & \vdots \\ b_1(t_m) & \dots & b_n(t_m) \end{bmatrix} \mathbf{x} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Method: (Regularization) We can punish large oscillations by introducing a regularization term:

$$f \in \underset{g \in V}{\operatorname{argmin}} \left\{ \sum_{i=0}^{n} |g(t_i) - y_i|^2 + \alpha \int_a^b |g''(t)|^2 \, \mathrm{d}t \right\}$$

Teil VI

Approximation in 1D

Goal: For a given function f on $I \subset \mathbb{R}$ find a simpler function $\tilde{f}$ (called *surrogate*) such that the approximation error $\left\| f - \tilde{f} \right\|$ is small for some norm.

⟶ Focus on **Approximation by interpolation**
⟶ Hence, good interpolation nodes are key.

1 Global Polynomial Approx.

Def. (Polynomial Interp. Approx. scheme)
Given a node set $\mathcal{T} = \{t_0, \dots, t_n\} \subset I$, define

$$\tilde{f} = p \in \mathcal{P}_n \quad \text{s.t } p(t_j) = f(t_j)$$

as the interpolating polynomial.

Thm. (Weierstrass) Every continuous function defined on a close interval can be uniformly approximated by a polynomial (of increasing degree).

Thm. (Best approximation, Jackson) For $f \in C^r([-1, 1])$ we have

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_\infty \leq \left(1 + \pi^2/2\right)^r \frac{(n-r)!}{n!} \left\| f^{(r)} \right\|_\infty$$
$$\in \quad \mathcal{O}(n^{-r})$$

Def. (Error convergence) We distinguish

- Algebraic convergence: $\left\| f - \tilde{f} \right\| = \mathcal{O}(n^{-p})$ with a rate $p > 0$
- Exponential convergence: $\left\| f - \tilde{f} \right\| = \mathcal{O}(q^n)$ for $0 < q < 1$

Def. (Chebychev interpolation) Lagrange interpolation on $I = [-1, 1]$ using nodes

$$t_k = \cos\left(\frac{2k+1}{2n} \pi \right), \quad k = 0, \dots, n-1$$

. These nodes are a priori optimal for the approximation error, note that they gather around the borders
Error Estimate: Almost algebraic

2 Trigonometric Poly. Approx.

x
x
x

3 Piecewise Polynomial Approx.

Method: (Piecewise Lagrange)

Given an interval $[a, b]$ endowed with a mesh $\mathcal{M}$, choose local degree and local interpolation nodes for each mesh cell to perform Lagrange Interpolation.
Error Estimate: For constant polynomial degree n and mesh width $h_{\mathcal{M}}$ we have

$$\|f - s\|_\infty \leqslant \frac{h_{\mathcal{M}}^{n+1}}{(n+1)!} \left\|f^{(n+1)}\right\|_\infty$$

Method: (Piecewise Cubic Hermite)

Given $f, \mathcal{M}$, the piecewise cubic hermite interpolant is defined as (Note that we use exact slopes!)

$$s_{|[x_{j-1}, x_j]} \in \mathcal{P}_3, \quad s(x_j) = f(x_j), \quad s'(x_j) = f'(x_j)$$