Numerical Methods

15. November 2020

Teil I
Computing with Matrices

1 Computational Effort

We use *Asymptotic Computation Complexity* (O-bounds) to judge our algorithms. But note that in many cases this approach is imprecise because on today's computers, *memory* bandwidth and latency is a key bottleneck.
Lem. (Matrix Product) The matrix product of an $m \times n$ and $n \times k$ matrix has complexity $O(mnk)$

Tricks to reduce complexity:

- Exploit associativity of operations
- Exploit hidden summations

2 Machine Arithmetic

Machine Numbers have a finite precision, hence any involved computation suffers from **Roundoff**.
Def. (Cancellation) Subtraction of almost equal numbers and accordingly an extreme *amplification of relative errors.*
Tricks to avoid cancellation:

- Trigonometric Identities
- Case Distinction
- Taylor Approximations
- Computing Diff. Quot. through Approx.

Def. (Stability) An Algorithm F is **numerically stable** if for all $x \in X$ its result $F(x)$ (possibly affected by roundoff) is the exact result for 'slightly perturbed' data.

Teil II
Linear Systems

1 Theory

Def. The Condition Number of a matrix is

$$\text{cond}(\mathbf{A}) := \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\|$$

and is a measure for stability (≈ 1 for stability). Note that it is infinite for singular matrices. Intuitively $cond(A)$ catches the maximum stretching factor divided by the minimum stretching factor.
Lem. If $m = n$ and $\mathbf{A}$ is regular/invertible, then its 2-norm condition number is $\text{cond}_2(\mathbf{A}) = \sigma_1/\sigma_n$

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2 LU Decomposition

There are two methods to choose pivots in the elimination process with a Permutation Matrix:

- Partial Pivoting: Max of column
- Full Pivoting: Max of remaining matrix

Complexities:

- Gauss elim. $\mathcal{O}(n^3)$, back substitution $\mathcal{O}(n^2)$
- LU: decomp. $\mathcal{O}(n^3)$, solve $\mathcal{O}(n^2)$
- Inverse: compute $\mathcal{O}(n^3)$, solve $\mathcal{O}(n^2)$

3 Exploiting Structure when Solving Linear Systems

Method: (Low Rank Modification) Use the *Sherman-Morrison-Woodbury* Formula to compute the inverse of slightly modified (rank-1-modification) matrix.

$$(\mathbf{A}+\mathbf{U}\mathbf{V}^H)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I}+\mathbf{V}^H\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^H\mathbf{A}^{-1}$$

4 Matrix Storage Formats

Dense Matrices are either stored in Column-Major Ordering or Row-Major Ordering. By default Eigen employs Column-Major Ordering.

Sparse Matrices can be stored in

- COO/Triplet: Vector of triplets of (row, col, value)
- CCS: Compressed Column Storage
- CRS: Compressed Row Storage

where CRS consists of the arrays:

- val: All non-zero values in Row-Major Order
- col_idx: Column index of each entry in val
- row_ptr: Pointers to first elements of row

Note that size(row_ptr) $= m + 1$ (Sentinel).

Teil III
Linear Least Squares

Critical comparison of Methods:

- Normal Equations: $\mathcal{O}(n^2 m + n^3)$
  → blows up instability
- Extended Normal Equations:
  → same conditioning as $\mathbf{A}$
  → sparsity preserved
- Householder QR: $\mathcal{O}(n^2 m)$
  → always stable
  → loss of sparsity
- SVD:
  → no full rank requirement for $\mathbf{A}$

1 Normal Equation Methods

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

Warning: Normal Equations are vulnerable to instability since $\text{cond}_2(\mathbf{A}^T\mathbf{A}) = \text{cond}_2(\mathbf{A})^2$

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2 Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^{\mathsf{H}}, \quad \mathbf{A} \in \mathbb{K}^{m\times n},$$

$p := \min\{m, n\}$, $r := \text{rank}(\mathbf{A})$,
$\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_p)$ $\sigma_1 > \sigma_2 > \dots > \sigma_r > \sigma_{r+1} = \dots = \sigma_p = 0$

Full:

- $\mathbf{U} \in \mathbb{K}^{m\times m}$ $[\mathcal{R}(\mathbf{A}) | \mathcal{N}(\mathbf{A})]$ (unitary)
- $\mathbf{\Sigma} \in \mathbb{K}^{m\times n}$ (generalized diagonal)
- $\mathbf{V} \in \mathbb{K}^{n\times n}$ $[\mathcal{R}(\mathbf{A}^{\mathsf{H}}) | \mathcal{N}(\mathbf{A}^{\mathsf{H}})]$ (unitary)

Economical:

- $\mathbf{U} \in \mathbb{K}^{m\times p}$ $[\mathcal{R}(\mathbf{A})]$ (orthogonal columns)
- $\mathbf{\Sigma} \in \mathbb{K}^{p\times p}$ (diagonal)
- $\mathbf{V} \in \mathbb{K}^{n\times p}$ $[\mathcal{R}(\mathbf{A}^{\mathsf{H}})]$ (orthogonal columns)

Numerical Rank $r := \max_{j\in\{1,\dots,p\}} \left(\frac{\sigma_j}{\sigma_1} \geq TOL\right)$
Cost of Eco SVD $\mathcal{O}(\min\{m,n\}^2 \max\{m,n\})$

Lem. (LSQ by SVD)

$$\mathbf{x}^* = \underset{\mathbf{x}\in\mathbb{R}^n}{\arg\min} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 = \mathbf{V}_1 \Sigma_r^{-1} \mathbf{U}_1^T \mathbf{b}$$

If the lsq has multiple solutions, $\mathbf{x}^*$ has minimal norm.
Thm. (Low Rank Approximation) Let $\mathbf{A}_k$ be the matrix resulting from only keeping the first k singular values of $\mathbf{A}$. $\mathbf{A}_k$ is the best rank-k approximation of $\mathbf{A}$.

Method: (Principal Component Analysis) Given a dataset in the form of a matrix $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}$, such that every column represents a time series.

- Columns of $\mathbf{U}$ capture dominant trends.
- Columns of $\mathbf{V}$ capture how strong the trends are in a particular column of $\mathbf{A}$
- Diagonal Entries of $\mathbf{\Sigma}$ tell us how dominant the trend is overall.

Method: (Proper orgthogonal decomp.) Closely related is the POD problem, where we seek k-dimensional subspace U_k of $\mathbb{R}^m$, for which the sum of squared distances of the data points $\mathbf{A} = [\mathbf{a}_1, \dots \mathbf{a}_n]$ to U_k is minimal.
Using the *Low Rank Approximation* Theorem we can prove that

$$U_k = \mathcal{R}(\mathbf{U}_{:,1:k})$$

3 QR Decomposition

$$\mathbf{A} = \mathbf{Q}\mathbf{R}, \quad \mathbf{A} \in \mathbb{K}^{m\times n} \quad \mathbf{Q} \in \mathbb{K}^{m\times m}$$

Idea: LLQ Problems are easier to solve for triangular matrices, hence we manage to decompose $\mathbf{A}$ into $\mathbf{QR}$ we have such a System since orthogonal matrices are norm-invariant.

Method: (Householder Reflections) The following Householder matrix performs a reflection

$$\mathbf{H}(\mathbf{v}) := \mathbf{I} - 2\mathbf{v}\mathbf{v}^\top$$

at the hyperplane with normal unit vector $\mathbf{v}$ (Intuitively $\text{H}(\mathbf{v})$ subtracts the projection of $\mathbf{x}$ on $\mathbf{v}$ twice).
We can reduce a matrix $\mathbf{A}$ to $\mathbf{R}$ using n successive transformations

$$\mathbf{H}(\mathbf{v}_n) \dots \mathbf{H}(\mathbf{v}_1)\mathbf{A} = \mathbf{R}$$

, where $\mathbf{H}(\mathbf{v}_i)$ reflects the lower part of the i'th column of the current $\mathbf{A}$ on $\mathbf{e}_i$. E.g

$$\mathbf{v}_1 = \mathbf{a}_1 \pm \|\mathbf{a}_1\| \, \mathbf{e}_1$$

Method: (Givens Rotations) We can selectively eliminate entries with Givens rotations. The following matrix rotates everything in the hyperplane defined by $\mathbf{e}_1, \mathbf{e}_k$:

$$\mathbf{G}(1,k,\theta) = \begin{bmatrix} c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{bmatrix}$$

Note that $c = \cos(\theta), s = \sin(\theta)$. Two eliminate a_k we solve the equations.

$$c^2 + s^2 = 1 \quad \text{and} \quad -sa_1 + ca_k = 0$$

As with householder reflections, there are always two possibilites, one of whom is cancellation free.

Lem. (Modifications Techniques) Computing the QR decomposition of a slightly modified matrix (rank-1-modification, adding a row, adding a column) can be done efficiently in $\mathcal{O}(mn + n^2)$.

4 Constrained Least Squares

Problem: Find $x \in \mathbb{R}^n$ such that

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\| \to \min \quad \text{and} \quad \mathbf{C}\mathbf{x} = \mathbf{d}.$$

Method: (Lagrangian Multipliers) We intoduce the multiplier $\mathbf{m} \in \mathbb{R}^p$ to solve

$$\mathbf{x}^* = \operatorname*{arg\,min}_{\mathbf{x} \in \mathbb{R}^n} \sup_{\mathbf{m} \in \mathbb{R}^p} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \mathbf{m}^T(\mathbf{C}\mathbf{x} - \mathbf{d})$$

and we notice that for any finite solution $(\mathbf{C}\mathbf{x} - \mathbf{d}) = 0$. By realising that that fo the solution all partial derivatives must be zero we can obtain the *augmented normal equations*.

Method: (By SVD) We have

$$\mathbf{x} \in \mathbf{x}_0 + \mathcal{N}(\mathbf{C}) \quad \mathbf{x}_0 = \text{particular solution}$$

Hence since the SVD gives a basis of $\mathcal{N}(\mathbf{C})$ we write

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{V}_2\mathbf{y}$$

for some $\mathbf{y}$ which leads to the standard lsq:

$$\|\mathbf{A}\mathbf{V}_2\mathbf{y} - (\mathbf{b} - \mathbf{A}\mathbf{x}_0)\| \to \min$$

Teil IV
Filtering Algorithms

1 Filters & Convolution

Def. (Filter) A function $F : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ where $l^\infty(\mathbb{Z})$ is the space of bounded infinite sequences

$$l^\infty(\mathbb{Z}) = \left\{(x_j)_{j\in\mathbb{Z}} : \sup |x_j| < \infty\right\}$$

Def. (LT-FIR) A Filter that is:

- Linear
- Time-Invariant: Shifting the input in time leads to the same output shifted in time.
- Finite: $\exists M \forall j : x_j = 0$ if $|j| > M \implies \exists N \forall j : y_j = 0$ if $|j| > N$
- Causal: The output does not start before the input.

Such a filter is uniquely characterized by its **Impulse response**:

$$F(\delta_{j,0}) = \ldots, 0, h_0, h_1, \ldots, h_{n-1}, 0, \ldots$$

For inputs $\mathbf{x} \in \mathbb{R}^m$ we get outputs $\mathbf{y} \in \mathbb{R}^{m+n-1}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ h_3 & h_2 & h_1 & h_0 \\ 0 & h_3 & h_2 & h_1 \\ 0 & 0 & h_3 & h_2 \\ 0 & 0 & 0 & h_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_4 \end{bmatrix}$$

Def. (Discrete Convolution) A Filter where given $\mathbf{x} = [x_0, \ldots, x_{m-1}]^\top \in \mathbb{K}^m, \mathbf{h} = [h_0, \ldots, h_{n-1}]^\top \in \mathbb{K}^n$ their DCONV is the vector $\mathbf{y} \in \mathbb{K}^{m+n-1}$ with components.

$$y_k = \sum_{j=0}^{m-1} h_{k-j} x_j$$

Def. (Periodic Convolution) Given two n-periodic signals $(u_k)_{k\in\mathbb{Z}}, (x_k)_{k\in\mathbb{Z}}$ PCONV yields the n-periodic signal:

$$(y_k) := (u_k) *_n (x_k), y_k := \sum_{j=0}^{n-1} u_{k-j} x_j$$

PCONV can be represtend by a matrix

$$\mathbf{C} = [c_{ij}]_{i,j=1}^n, c_{ij} = u_{j-i}$$

. Such a matrix is called **circulant**.

2 Discrete Fourier Transform

The Fourier Matrix for inputs $\mathbf{y} \in \mathbb{C}^n$ is given by

$$\mathbf{F}_n = \left[\omega_n^{lj}\right]_{l,j=0}^{n-1} \in \mathbb{C}^{n,n} \quad \omega_n = \exp\left(\frac{-2\pi i}{n}\right)$$

and $\mathrm{DFT}_n(\mathbf{y}) := \mathbf{F}_n\mathbf{y} = \mathbf{c}$. Properties include

$$\mathbf{F}_n^{-1} = \frac{1}{n}\mathbf{F}_n^{\mathsf{H}} = \frac{1}{n}\overline{\mathbf{F}}_n$$

Method: (Frequency Filtering) The columns of $\mathbf{F}_n$ are trigonometric basis vectors, where

$$\mathbf{v}_k = \left[\cos\left(j\frac{2\pi}{n} \cdot k\right) + \imath \sin\left(\frac{2\pi jk}{n}\right)\right]_{j=0}^{n-1}$$

k is the 'frequency' and $j2\pi/n$ is the sample point. Hence the DFT_n is a basis transformation

$$B_E \leftrightarrow B_{trig}$$

Denoising and low/high filters can be implemented by manipulating the signal in the frequency domain.

Lem. (Diagonalizing circulant matrices) For any circulant matrix $\mathbf{C} \in \mathbb{C}^{n,n}$ we have

$$\mathbf{C} = \mathbf{F}_n^{-1} \operatorname{diag}(d_1, \ldots, d_n) \mathbf{F}_n,$$

$$[d_0, \ldots, d_{n-1}]^\top = \mathbf{F}_n [u_0, \ldots, u_{n-1}]^\top$$

In other words, the columns of $\mathbf{F}_n$ are eigenvectors of $\mathbf{C}$.

Thm. (Convolution Theorem) Periodic convolution in the time-domain equals to multiplication in the frequency-domain.

$$(\mathbf{u}) *_n (\mathbf{x}) = \mathbf{F}_n^{-1} \left[(\mathbf{F}_n\mathbf{u})_j (\mathbf{F}_n\mathbf{x})_j\right]_{j=1}^n$$

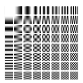
Implementation Speedup: $\mathcal{O}(n^2) \to \mathcal{O}(n \log_2(n))$

Thm. (2D DFT) Is given by

$$\mathbf{C} = \mathbf{F}_m \left(\mathbf{F}_n\mathbf{Y}^\top\right)^\top = \mathbf{F}_m\mathbf{Y}\mathbf{F}_n$$

The real basis vectors can be visualized. Analogously there is a 2D-Convolution-Theorem:

$$\mathbf{X} *_{m,n} \mathbf{Y} = \mathrm{DFT}_{m,n}^{-1}(\mathrm{DFT}_{m,n}(\mathbf{X}) \odot \mathrm{DFT}_{m,n}(\mathbf{Y}))$$



Method: (Deblurring an image) A blurred image can be modeled as the 2D-Convolution of the actual image with a small filter. In the 2D-frequency-domain deblurring equals component wise division.

3 Fast Fourier Transform

The DFT of a vector of length pq can be divided in p DFT's of vectors of length q. The asymptotic runtime for this special 'divide and conquer' is

$$\mathrm{FFT}(n) \in \mathcal{O}(n \log n)$$

Special Case $p = 2$:

$$\mathrm{DFT}(\mathbf{v}) = \mathrm{DFT}(\mathbf{v}_{even}) + s\,\mathrm{DFT}(\mathbf{v}_{odd})$$

Def. (Toeplitz Matrix) $\mathbf{T} = (t_{ij})_{i,j=1}^n \in \mathbb{K}^{mn}$ is a Toeplitz matrix if it has constant diagonals given by a vector $\mathbf{u} \in \mathbb{K}^{m+n-1}$.

Lem. (Fast Arithmetic)

Vector Multiplication: Construct Circulant Matrix $\mathbf{C} \in \mathbb{K}^{m+n,m+n}$ s.t.

$$\begin{bmatrix} \mathbf{T}\mathbf{x} \\ \star \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$$

which can be computed in the frequency domain by Convolution Theorem in $\mathcal{O}((n+m)\log(n+m))$.

LSE Solution: The Levinson Algorithm solves the LSE in $\mathcal{O}(n^2)$

Teil V
Data Interpolation in 1D