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RESEARCH ARTICLE

A New Condition for Agglomeration in Bayesian Confirmation

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Abstract

Bayesian confirmation does not generally *agglomerate* over conjunction. That is, whenever a piece of evidence E confirms two hypotheses H_1 and H_2 *individually*, it does not follow that E also confirms them *conjunctively*. Here, I present a condition under which the latter *does* follow from the former. But this new condition reveals a surprising fact: Bayesian confirmation agglomerates over conjunction whenever the evidence in question also confirms that *both* target hypotheses are *false*.

1. Introduction

According to Bayesian confirmation theory, a piece of evidence E *confirms* two hypotheses H_1 and H_2 individually if and only if E makes each of them *more likely* to be true (see Fitelson 2001 or Strevens 2017). That is, the following two inequalities are satisfied:

$$P(H_1|E) > P(H_1) \quad \text{and} \quad P(H_2|E) > P(H_2) \tag{1}$$

This conception of confirmation is perhaps *the* most popular currently on the market. But it is a well-known fact, presumably first noted by Carnap (1950), that confirmation, thus understood, does not always *agglomerate* over conjunction (the label is from Leitgeb 2013). That is, condition (1) does not entail:

$$P(H_1 \wedge H_2|E) > P(H_1 \wedge H_2) \tag{2}$$

To see this more clearly, consider sampling a card from a standard deck. Let E be that the card is red, H_1 that it is a heart and H_2 that it is a diamond (see Roche 2012). Here, the agglomeration antecedent (1) is satisfied but the consequent (2) is not.[1]

Still, there are conditions under which Bayesian confirmation *does* agglomerate over conjunction, i.e. conditions under which (1) does entail (2). And in this paper, I would

[1]Conditional probability $P(H|E)$ is defined as usual by $P(H \wedge E)/P(E)$ provided $P(E) > 0$. To ensure well-definedness, I will tacitly assume that the relevant probabilities are non-extreme. Notice that the card example also shows that (1) neither entails that H_1 and H_2 are positively correlated *unconditionally* nor *conditional on* E. That is, it neither follows from (1) that $P(H_1 \wedge H_2) > P(H_1)P(H_2)$ nor that $P(H_1 \wedge H_2|E) > P(H_1|E)P(H_2|E)$. The example also helps us to see that (1) does not entail that E confirms the disjunction $H_1 \vee H_2$. That is, it does not follow from (1) that $P(H_1 \vee H_2|E) > P(H_1 \vee H_2)$. Simply let E be that the drawn card is black, H_1 that it is *not* a heart and H_2 that is *not* a diamond.

like to present a new one. This new condition will, however, turn out somewhat puzzling. I will introduce it in section 2 and point out in section 3 that a precursor can already be found in the work of Carnap and Salmon. I then discuss an objection in section 4 and examine how the new condition relates to previous agglomeration conditions in section 5. Finally, I conclude in section 6.

2. NOR-Confirmation

The new agglomeration condition I would like to present is the following:

$$P(\neg H_1 \wedge \neg H_2|E) > P(\neg H_1 \wedge \neg H_2) \tag{3}$$

Less formally, it states that the evidence in question E confirms that neither hypothesis H_1 nor H_2 is true, or equivalently, that *both* target hypotheses are *false*. Due to the obvious relationship to Peirce's (1933) NOR-connective $H_1 \downarrow H_2$, I will call condition (3) *NOR-confirmation*.

Intuitively, NOR-confirmation (3) is at odds with both the agglomeration antecedent (1) and the consequent (2). After all, the two latter conditions state that the evidence in question E confirms that *both* target hypotheses H_1 and H_2 are *true*, namely individually and conjunctively. It might therefore come as a surprise that NOR-confirmation (3) guarantees that agglomeration is *valid* for Bayesian confirmation. That is, whenever (3) holds, (1) entails (2). For a proof, first observe that (3) is equivalent to:[2]

$$P(H_1 \vee H_2|E) < P(H_1 \vee H_2)$$

Which by general additivity expands to:

$$P(H_1|E) + P(H_2|E) - P(H_1 \wedge H_2|E) < P(H_1) + P(H_2) - P(H_1 \wedge H_2)$$

By simple algebra, this is equivalent to:

$$P(H_1|E) - P(H_1) + P(H_2|E) - P(H_2) < P(H_1 \wedge H_2|E) - P(H_1 \wedge H_2)$$

And by condition (1), the following equivalent of (2) follows:

$$0 < P(H_1 \wedge H_2|E) - P(H_1 \wedge H_2)$$

To see that NOR-confirmation (3) is a *non-trivial* condition for agglomeration, i.e. (3) is *consistent* with (1), consider an urn containing 10 balls with 3 binary attributes, distributed as shown in Table 1. Let the evidence E be that a randomly drawn ball is blue, H_1 that it is small and H_2 that it is clean. Then, NOR-confirmation (3) is satisfied, i.e. the evidence confirms that the drawn ball is *not* small and *not* clean:

$$P(\neg H_1 \wedge \neg H_2|E) = 4/10 > P(\neg H_1 \wedge \neg H_2) = 3/10$$

[2] Notice that I am not arguing that the sufficiency of NOR-confirmation (3) for agglomeration is *mathematically* surprising. It is surprising from a confirmation-theoretic perspective. Thanks to an anonymous referee for pushing me to be more explicit here. Also notice that NOR-confirmation (3) and the agglomeration antecedent (1) entail more than just (2). For instance, they also entail that E confirms the two material conditionals $H_1 \supset H_2$, $H_2 \supset H_1$ and their conjunction $H_1 \leftrightarrow H_2$. And it also follows that E confirms each hypothesis H_1 and H_2 conditional on the other and that E confirms each negated hypothesis $\neg H_1$ and $\neg H_2$ conditional on the other. See also section 5.

Table 1. Urn model under which NOR-confirmation (3), (1) and thus (2) are jointly satisfied.

	Dirty		Clean		
	Big	Small	Big	Small	Total
Red	1	2	2	0	5
Blue	2	0	0	3	5
Total	3	2	2	3	10

The agglomeration antecedent (1) is satisfied, i.e. the evidence confirms that the drawn ball is small and clean *individually*:

$$\forall i \in 1,2 : P(H_i|E) = 6/10 > P(H_i) = 5/10$$

And hence, the agglomeration consequent (2) is satisfied, i.e. the evidence confirms that the drawn ball is small and clean *conjunctively*:

$$P(H_1 \wedge H_2|E) = 6/10 > P(H_1 \wedge H_2) = 3/10$$

This shows that NOR-confirmation (3) is a non-trivial condition for agglomeration.

3. Carnap and Salmon

The observation that NOR-confirmation (3) and the agglomeration antecedent (1) are consistent is not entirely new: this fact was already noted *implicitly* by Carnap (1950) and Salmon (1983). The two authors discussed examples in which a piece of evidence E confirms two hypotheses H_1 and H_2 individually while *disconfirming* their disjunction $H_1 \vee H_2$, the latter being equivalent to NOR-confirmation (3). What *is* new, however, is that this makes Carnap's and Salmon's examples rather peculiar instances of agglomeration. To see this more clearly, consider Salmon's example:

> a medical researcher finds evidence confirming the hypothesis that Jones is suffering from viral pneumonia and also confirming the hypothesis that Jones is suffering from bacterial pneumonia—yet this very same evidence disconfirms the hypothesis that Jones has pneumonia! It is difficult to entertain such a state of affairs, even as an abstract possibility. (Salmon, 1983, Section 3)[3]

Salmon found the fact that such situations can arise "shocking and counterintuitive" (Salmon, 1983, Section 3). But he overlooked that being an instance of agglomeration, it follows that the evidence also confirms the hypothesis that Jones has viral *and* bacterial pneumonia. Just imagine the following dialogue:

> Researcher: Mr. Jones, good to see you! I just received your lab results. I have some good and some bad news for you. The bad news is that the results confirm that you have viral pneumonia; and they also confirm that you have bacterial pneumonia.
>
> Jones: Oh dear! So I have both viral *and* bacterial pneumonia?! *That* explains why I feel so miserable!

[3] Atkinson et al.'s (2009) so-called *Alan Author Effect* is structurally equivalent to the phenomenon described by Salmon. The effect occurs when a piece of evidence E confirms a conjunction $H_1 \wedge H_2$ while disconfirming its conjuncts H_1 and H_2 individually. This is equivalent to confirming the negated hypotheses individually while disconfirming their disjunction.

> Researcher: Well, *that* is not quite what I said, Mr. Jones! In any case, the good news is that the results also confirm that you have *neither* viral *nor* bacterial pneumonia.
>
> Jones: Wait, didn't you just tell me the opposite? Do the results confirm that I *have* viral and bacterial pneumonia or do they confirm that I *don't*?!
>
> Researcher: Well, they confirm both, Mr. Jones, albeit in different ways.
>
> Jones: How can this be? Is there something wrong with the lab results?
>
> Researcher: No, I can assure you that our lab results are flawless and absolutely reliable. In fact, it follows that they also confirm that you have viral *and* bacterial pneumonia at the same time.

I suspect that most readers will find the researcher's utterances confusing and unhelpful. Perhaps, some will even question the validity of her inference, arguing that the lab results should *disconfirm* the hypothesis that Jones has viral *and* bacterial pneumonia. But the researcher's inference is valid and everything she says is consistent.[4]

4. The Rarity Objection

One might try to relativize the phenomenon above by arguing that it is probably very *rare*. My response to this objection is two-fold: I admit that the phenomenon is not very *prevalent*, but this does not make it less *unsettling*. More precisely, the conjunctive prevalence of cases where NOR-confirmation (3) *and* (1) are jointly satisfied is around 2.5%. And the conditional prevalence of cases where NOR-confirmation (3) is satisfied *if* (1) is satisfied is around 10%. This can be shown using Monte Carlo integration based on 10 million regular probability functions over an algebra generated by 3 variables (see Metropolis and Ulam, 1949). The left-hand graph in Figure 1 shows how the prevalence stabilizes with increasing number of probability functions.

To put these values into context, compare them with Simpson's (1951) paradox, a different but similarly puzzling probabilistic phenomenon where a piece of evidence E confirms a hypothesis H conditional on some assumption X and conditional on $\neg X$, but E fails to confirm H *unconditionally* (see Sprenger and Weinberger 2021). The conjunctive prevalence of such cases is only around 0.83%, and their conditional prevalence is around 3.33%, as shown in the right-hand side of Figure 1. But the low prevalence of Simpson's paradox has not kept researchers from finding the phenomenon unsettling. So, even if cases where NOR-confirmation (3) and the agglomeration antecedent (1) are jointly satisfied are rare, they are prevalent *enough* to care about.

5. Previous Agglomeration Conditions

NOR-confirmation (3) is not the *only* agglomeration condition for Bayesian confirmation. As Reichenbach (1956) showed in his analysis of common-cause structures,

[4] Taking inspiration from Hempel (1960), we might call cases where a single piece of evidence consistently confirms a number of jointly inconsistent hypotheses *evidential inconsistencies*. See also the phenomenon of *floating conclusions* where two contradicting lines of reasoning confirm the same conclusion (see Makinson and Schlechta 1991 or Horty 2002).

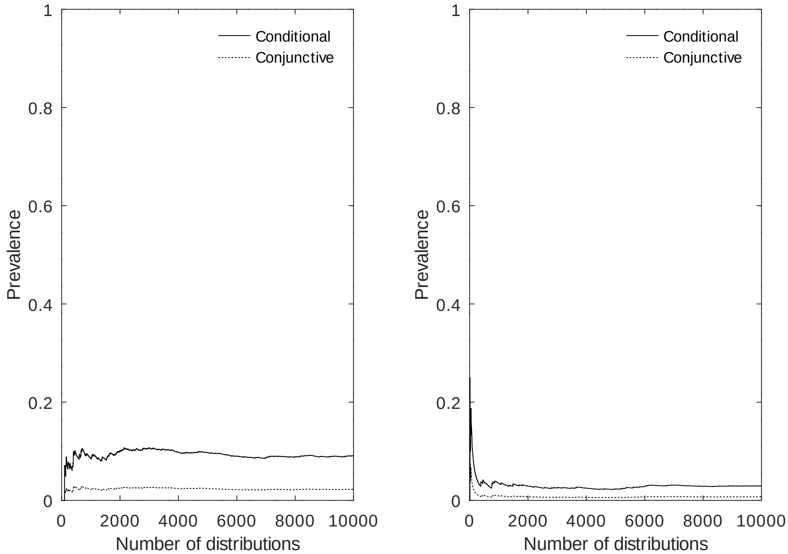

Figure 1. Prevalence of the NOR-effect left and the Simpson-effect right.

agglomeration is also valid if the evidence *screens-off* both hypotheses from each other:

$$P(H_1|E \wedge H_2) = P(H_1|E) \quad \text{and} \quad P(H_1|\neg E \wedge H_2) = P(H_1|\neg E) \tag{4}$$

And as Falk (1986) pointed out in his discussion of Cohen's (1977) corroboration theorem, agglomeration remains valid even if screening-off is relaxed as follows:[5]

$$P(H_1|E \wedge H_2) \geq P(H_1|E) \quad \text{and} \quad P(H_1|\neg E \wedge H_2) \leq P(H_1|\neg E) \tag{5}$$

Salmon (1983) uncovered another interesting condition. While agglomeration can also fail for independent hypotheses, it cannot if additionally, the two target hypotheses are independent *conditional on the evidence*:

$$P(H_1|H_2 \wedge E) = P(H_1|E) \quad \text{and} \quad P(H_1|H_2) = P(H_1) \tag{6}$$

Finally, there are two more recent conditions from the literature on the problem of irrelevant conjunction (see Schurz 2022). The first is part of Fitelson's (2002) *confirmational irrelevance condition*:

$$P(H_2|H_1 \wedge E) = P(H_2) \quad \text{and} \quad P(H_2|H_1) = P(H_2) \tag{7}$$

And the second is Hawthorne and Fitelson's (2004) conditional irrelevance condition which states that the evidence is irrelevant for one hypothesis conditional on the other:

$$P(H_2|H_1 \wedge E) = P(H_2|H_1) \tag{8}$$

[5]Cohen's condition (5) should not be confused with *weak screening-off* $P(H_1|E \wedge H_2) \geq P(H_1|E)$ and $P(H_1|\neg E \wedge H_2) \geq P(H_1|\neg E)$ (see Atkinson and Peijnenburg, 2021). The two conditions only differ in the second conjunct. But (5) guarantees agglomeration while weak screening-off does not. And weak screening-off guarantees transitivity while (5) does not (see Suppes 1986 or Roche 2012).

Now, interestingly, NOR-confirmation (3) is *logically independent* of each of the aforementioned conditions (4) to (8). That is, NOR-confirmation (3) is consistent with each of them but neither entails nor is entailed by any of them. A proof of this statement is provided in Appendix I.

Notice, however, that most of these logical independence relationships *break down* once the agglomeration antecedent (1) is satisfied. More precisely, if (1) holds, then NOR-confirmation (3) is inconsistent with screening-off (4), full independence (6), confirmational irrelevance (7) and conditional irrelevance (8). A proof of this is provided in Appendix II. With these remarks, I close my discussion of NOR-confirmation (3).

6. Conclusion

In this short paper, I presented a new condition under which Bayesian confirmation agglomerates over conjunction. One might think that such a condition is helpful because it allows us to establish claims about Bayesian confirmation without tedious case-by-case examination (see Shogenji 2003 or Roche 2012). But the condition presented here is more puzzling than helpful: it is difficult to see why Bayesian confirmation should agglomerate over conjunction whenever the new condition is satisfied. I hope that Bayesian confirmation theorists can help with an explanation.

Appendix I: Logical Independence

The probability distributions provided in Table 2 show that (3) is logically independent of (4) to (8). Under distribution 1, all conditions (3) to (8) are satisfied and thus, none of them entails the negation of the other. Under distribution 2, NOR-confirmation (3) is satisfied while none of the other conditions is. This shows that the former condition does not entail any of the latter. And under distribution 3, NOR-confirmation (3) is violated while the other conditions are satisfied.

Table 2. Probability distributions showing that NOR-confirmation (3) is logically independent of (4) to (8).

E	H_1	H_2	Distribution 1	Distribution 2	Distribution 3
0	0	0	1/16	1/16	2/16
0	0	1	1/16	1/16	2/16
0	1	0	2/16	2/16	1/16
0	1	1	2/16	2/16	1/16
1	0	0	2/16	2/16	2/16
1	0	1	2/16	3/16	2/16
1	1	0	3/16	3/16	3/16
1	1	1	3/16	2/16	3/16

Distribution 1

Under this distribution, NOR-confirmation (3) is satisfied:

$$P(H_1 \vee H_2|E) = 8/10 < P(H_1 \vee H_2) = 13/16$$

Screening-off (4) and thus relaxed screening-off (5) are satisfied:

$$P(H_1|H_2 \wedge E) = P(H_1|E) = 3/5 \quad \text{and} \quad P(H_1|H_2 \wedge \neg E) = P(H_1|\neg E) = 1/2$$

We also have:

$$P(H_2|H_1 \wedge E) = P(H_2) = 1/2 \quad \text{and} \quad P(H_1|H_2) = P(H_1) = 10/16$$

Thus, full independence (6), confirmational irrelevance (7) and also conditional irrelevance (8) are satisfied.

Distribution 2

NOR-confirmation (3) is satisfied:

$$P(H_1 \vee H_2|E) = 8/10 < P(H_1 \vee H_2) = 13/16$$

Screening-off (4), relaxed screening-off (5) and full independence (6) are violated:

$$P(H_1|H_2 \wedge E) = 2/5 < P(H_1|E) = 1/2$$

Conditional irrelevance (8) and thus confirmational irrelevance (7) are violated:

$$P(H_2|H_1 \wedge E) = 2/5 < P(H_2|H_1) = 5/9$$

Distribution 3

NOR-confirmation (3) is violated:

$$P(H_1 \vee H_2|E) = 16/20 > P(H_1 \vee H_2) = 12/16$$

Screening-off (4) and thus relaxed screening-off (5) are satisfied:

$$P(H_1|H_2 \wedge E) = P(H_1|E) = 3/5 \quad \text{and} \quad P(H_1|H_2 \wedge \neg E) = P(H_1|\neg E) = 1/3$$

We also have:

$$P(H_2|H_1 \wedge E) = P(H_2) = 1/2 \quad \text{and} \quad P(H_1|H_2) = P(H_1) = 1/2$$

Thus, full independence (6), confirmational irrelevance (7) and also conditional irrelevance (8) are satisfied.

Appendix II: Breakdown of Logical Independence

If the agglomeration antecedent (1) is satisfied, then (3) is no longer logically independent of (4) to (8). More precisely, if (1) holds, then NOR-confirmation (3) NOR-confirmation (3) and the agglomeration antecedent (1) jointly entail that the evidence E confirms H_1 conditional on H_2 and that E confirms H_2 conditional on H_1:

$$P(H_1|H_2 \wedge E) > P(H_1|H_2) \quad \text{and} \quad P(H_2|H_1 \wedge E) > P(H_2|H_1)$$

And the two conditions also entail that the evidence E disconfirms H_1 conditional on $\neg H_2$ and that E confirms H_2 conditional on $\neg H_1$:

$$P(H_1|\neg H_2 \wedge E) < P(H_1|\neg H_2) \quad \text{and} \quad P(H_2|\neg H_1 \wedge E) < P(H_2|\neg H_1)$$

Together with screening-off (4) or full-independence (6), the second condition yields a contradiction. And together with confirmational irrelevance (7) and thus with conditional irrelevance (8), the first condition yields a contradiction.

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