MATH2930: Differential Equations for Engineers

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1 First Order Ordinary Differential Equations

This course is about i) **finding** differential equations to model systems and ii) **solving** differential equations both **analytically** (closed form solution for $V(t)$) and **numerically** (plotting V at different times, making observations).

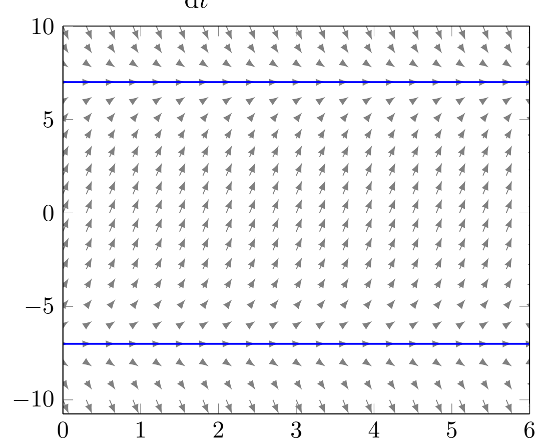
1.1 Direction Fields

Identify the direction field associated with a first order differential equation and sketch a particular solution or a representative family of solutions. Generate a direction field from a first order ODE.

Slope (direction) fields allow us to visualize *all* solutions of a system. To find a specific solutions, follow the path (through the initial condition) on the slope field.

We can find an **equlibrium solution** (a value of y for which $y' = 0, \forall t$) on a direction field if there exists a value of y where $y'(t) = 0$ (horizontal line).

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 0.2y^2$$



1.2 Separable ODEs

Determine whether a first order ordinary differential equation is separable. Solve a first order separable differential equation using integration.

First, a couple of definitions. A **solution** to a differential equation is a **function** that satisfies the equation. (Just like calc II, this is easy to verify correctness.) The **order** of a diffeq is the highest derivative.

Definition 1.1. A differential equation is **linear** if it follows the form:

$$a_0(t)y^n + a_1(t)y^{n-1} + a_2(t)y^{n-2} + ... + a_n(t)y = g(t)$$

and **nonlinear** otherwise (e.g., $y * y'$, etc)

Definition 1.2. A first order ODE is **separable** if it can be written in the form:

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

aka, if we can separate everything to do with y from everything to do with x.

**Note that linearity and separability are completely unrelated.

The solution methodology for a separable ODE:

1. separate: $...\mathrm{d}x = ...\mathrm{d}y$
2. integrate (to get implicit definition of general solution) $\rightarrow$ is also where the constant of integration C must appear
3. solve for y explicitly (if possible)
4. use initial condition to find C**
   **Note that this can also be done earlier

1.3 Integrating Factors

Solve a first order linear differential equation by using an integrating factor.

Definition 1.3. A **linear** first order ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + q(t)y = g(t)$$

can be solved with an **integrating factor** $\mu(t)$:

$$\mu = e^{\int p(t)dt}$$

Observe that this method relies on the **product rule** of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)g(t)] = \frac{\mathrm{d}f}{\mathrm{d}t}g + f\frac{\mathrm{d}g}{\mathrm{d}t} = f'g + fg'$$

Essentially, we manipulate the ODE into the form:

$$\int f'g + fg' dt = fg + C$$

by multiplying the integrating factor on both sides. The left hand side becomes

$$\int \mu(t)[\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y(t)] = \int \frac{\mathrm{d}}{\mathrm{d}t}[\mu y] = \mu y$$

while the right hand side evaluates to

$$\int \mu g dt$$

All in all, the differential equation becomes

$$\mu y = \int \mu g dt$$

**Take care to use the correct integrating factor (get the eqn in the correct form first) and don't forget to integrate the RHS

1.4 Existence and Uniqueness

Determine whether a first order ordinary differential equation has a solution and, if so, whether it is unique. Recognize additional solutions for differential equations with non-unique solutions. Understand and apply relevant theorems about existence and uniqueness of solutions to first order ODEs.

Sometimes there may not be a solution to an Initial Value Problem and/or a solution may not be unique. There are different theorems for linear and nonlinear first order ODEs.

Obverve that linear ODEs are a subset (more specific form of) nonlinear ODEs – so technically, the theorem for nonlinear ODEs would also apply. However, the interval specifically for linear ODEs provides more information.

1.4.1 Linear First Order ODEs

Theorem 1.1. *For a linear first order ODE:*

$$y' + p(t)y = g(t) \text{ with } y(t_0) = y_0$$

if $p(t)$ and $g(t)$ are ***continuous*** *on a interval containing t_0, then y' has a* ***unique solution***

1.4.2 Nonlinear First Order ODEs

Theorem 1.2. *For a nonlinear first order ODE:*

$$y' = f(t, y) \text{ with } y(t_0) = y_0$$

if f and $\frac{\partial f}{\partial y}$ are ***continuous*** *on an internval containing (t_0, y_0), there there exists a* ***subinterval*** *on which a unique solution exists – if only f is continuous and $\frac{\partial f}{\partial y}$ is not, then a solution exists but it* ***may not be unique***

1.5 Autonomous ODEs

Determine stable and unstable equilibria of autonomous ordinary differential equations. Sketch representative solutions with an emphasis on behaviors near equilibria.

Recall that a general first order ODE is written in the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$.

Definition 1.4. An ODE is **autonomous** if the independent variable does not show up in the derivative for function f:

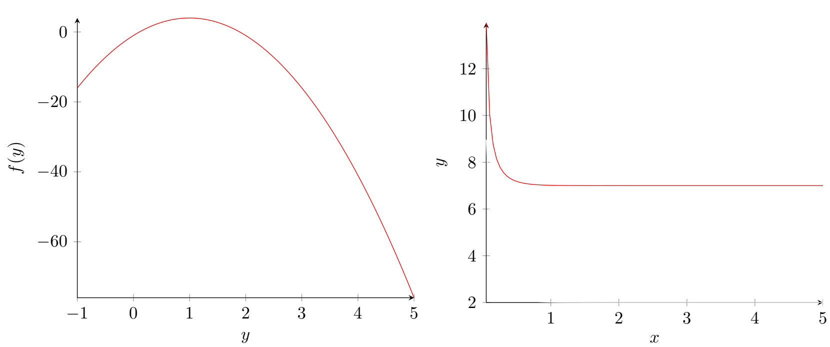
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

Like direction fields, the autonomous technique (?) is a *qualitative* approach: we can graph $f(y)$ vs y and note the characteristics of the graph

- zeros (**critical points**)
- areas where $f(y)$ is increasing or decreasing
- large or small values of f
- locations of equilibria (and if they're stable, unstable, semistable, etc)

and then use these characteristics to sketch a graph of y vs t.

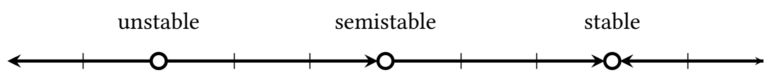
For example, we can take $f(y) = \frac{\mathrm{d}y}{\mathrm{d}t} = ry(1 - \frac{y}{k})$:



**One possible solution of y

Use **phase lines** to guide the sketch of y vs t, where we plot the equilibrium points of y and their stabilities:

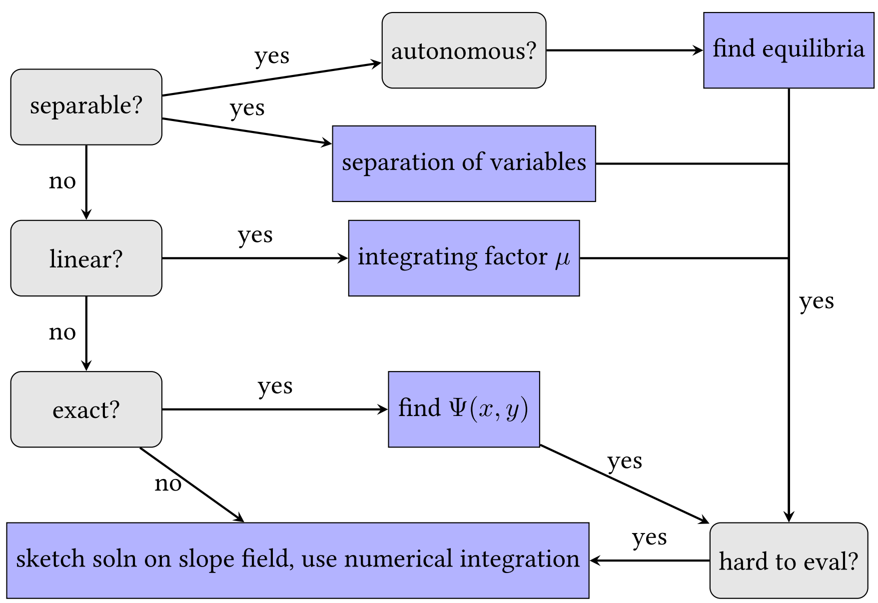
- **stable** equlibriums: neighboring values tend towards the equilibrium point as $t \to \infty$
- **unstable** equlibriums: neighboring values tend away from the equilibrium point as $t \to \infty$
- **semistable** equilibriums: solutions on one side of the eequation tend towards the equilibrium while solutions on a different side tend away as $t \to \infty$



1.6 Initial Value Problems

Given a first order differential equation, choose an appropriate solution method, determine the general solution, and determine particular solutions satisfying appropriate initial conditions.

For first order ODEs, a general strategy for approaching: (but really practice makes perfect)



This is a bit out of order as some of these strategies are discussed in the following pages... also for some reason exact ODEs isn't a LO, but we will discuss it briefly just in case.

Exact First Order ODEs

Definition 1.5. A differential equation of the form

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \text{is } \textbf{exact} \text{ if}$$

$$M_y = N_x$$

If an ODE is exact, there there must exist an **underlying function** $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x}=M, \frac{\partial \Psi}{\partial y}=N$$

Thus, the original $M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ becomes

$$\Psi_x+\Psi_y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

By 'un-chainruling', this can again be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\Psi(x,y(x)))=0$$

We know this because

$$\frac{\partial^2\Psi}{\partial x\partial y}=\frac{\partial}{\partial y}\frac{\partial\Psi}{\partial x}=\frac{\partial}{\partial y}M=\frac{\partial}{\partial x}\frac{\partial\Psi}{\partial y}=\frac{\partial}{\partial x}N$$

Therefore, we know that Ψ must be constant and

$$\Psi(x,y)=c$$

is an **implicit** solution to the diffeq. Solve for Ψ how you would in multi.

**Note exact equations aren't a learning outcome on their own – most likely to appear in Modelling or IVP 'if at all' (Ritz)

1.7 Modelling I

Recognize situations in which a first order differential equation is relevant. Develop an appropriate mathematical model of such systems, choose an appropriate technique for analyzing or solving the problem, and carry out the analysis.

When studying a system, often want a mathematical model – in which we make assumptions about behavior, use experiments to determine numreical values of **model parameters**, and predict behavior. Not really sure what else – this is interpreting a word problem idk best of luck soldier

1.8 Numerical I

Solve a first order differential equation using Euler's method, graphically and algebraically. Identify error and stability. Describe differences between the implicit + explicit methods.

So far we've dealt with ODEs with closed form analytic solutions, but not all ODEs have them! In those cases, we can introduce a *numerical integration technique*: **Euler's Method** for first order ODEs.

1.8.1 Explicit Euler's Method

Here, we're approximating the value of a function near a point by taking the tangent line *at that point*. With the explicit Euler's method, we update the tangent line approximation as you go forward in time. The general algorithm is:

$$\begin{aligned} y_{n+1} &= y_n + (t_{n+1} - t_n) f(t_n, y_n) \\ &= y_n + \Delta t f_n \\ y_{n+1} &= y_n + h f_n \end{aligned}$$

To keep things easy when doing this shit by hand, use a table:

(x, y)	$\frac{\mathrm{d}x}{\mathrm{d}y} = x + y$	Δx	Δy	$(x + \Delta x, y + \Delta y)$

where in this case $\Delta x = \Delta t = h$.

This is all fine and dandy except it's not. As $h \downarrow$ error $\downarrow$. In fact, with the explicit Euler's method, $h \propto$ error. Besides possibly a larger error, the explicit method is also rather unstable. Thus, we have:

1.8.2 Implicit Euler's Method

With the implicit method, we take the tangent of the *point we are approximating* i.e., $n + 1$ instead of at n.

$$\begin{aligned} y_{n+1} &= y_n + \Delta t f_{n+1} \\ &= y_n + h f(t_{n+1}, y_{n+1}) \end{aligned}$$

However, in this case we still have error with order $O(h)$ aka $h \propto$ error. It is more stable, but it can be difficult to solve for y_{n+1}. Instead, we combine the best of both worlds for:

1.8.3 Improved Euler's Method

To make an analogy, Improved Euler's Method is the trapezoidal Reimann sum approximation of these tangent lines. We do two calculations of f at each step to decrease our error.

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}\right)$$

where we obtain the $f(t_{n+1}, y_{n+1})$ value from using explicit Euler's approximation.

2 Higher Order Ordinary Differential Equations

2.1 2^{nd} Order ODEs

Given a homogeneous second order linear differential equation with constant coefficients, find the general solution.

Definition 2.1. In general, a **linear 2^{nd} order ODE** can be written in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}x}{\mathrm{d}t} + q(t)x = g(t)$$

Note that we need two ICs, can be in the form of $x(t_0), x'(t_0)$ or $x(t_0), x(t_1)$, etc.

We're going to look at a simplified problem first:

Definition 2.2. A 2^{nd} order **homogeneous constant coefficient** ODE can be written as

$$ax'' + bx' + cx = 0$$

To solve this equation, we define the **characteristic equation** to be $ar^2 + br + c = 0$ where r are the roots of the equation such that this equals zero. From the quadratic equation, we have

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

based on the discriminant (not determinant lmfao), we take different approaches...

2.1.1 Two Distinct Real Roots

When we have two distinct real roots, aka $b^2 > 4ac$, the general solution takes the form

$$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

where we get the constants A_1 and A_2 from the initial conditions

2.1.2 Two Complex Roots

If $b^2 < 4ac$, we end up with complex roots (which are conjugates). We could plug into the same form as above, but we get complex exponentials which are ugly asf and make no physical sense. Thus, we take **Euler's formula**

$$e^{i\omega t} = \cos \omega t + i \sin \omega t, \qquad e^{i\pi} + 1 = 0$$

and do some algebra to manipulate our general solution into the form

$$y(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$$

where our roots $r_{1,2} = \lambda \pm \mu i$

Alternatively, we can see that $y(t) = A_1 e^{(\lambda + \mu i)t} + A_2 e^{(\lambda - \mu i)t} = e^{\lambda t}((A_1 + A_2)\cos \mu t + i(A_1 - A_2)\sin \mu t)$ – and just as how with vector equations the $\vec{i}, \vec{j}$ and $\vec{k}$ components each satisfy their own equations, the real and imaginary parts of the solution each need to satisfy the ODE.

2.1.3 Repeated Roots

The last case, when $b^2 = 4ac$, we need to find out how to get the second function of the general solution. In this case, we're given $ax'' + bx' + c = 0$ and $y_1(t) = e^{rt}$ (only one r...) and we need to find $y_2(t) = v(t)y_1 = v(t)e^{rt}$. If we

1. differentiate y_2: $y_2'' = v''e^{rt} + 2v're^{rt} + vr^2e^{rt}$
2. plug it into the problem statement
3. simplify

... we learn that $v'' = 0 \to$ if $v'' = 0$, then v is a straight line: $v = \alpha t + \beta$ Thus, the general solution takes the form

$$y(t) = B_1e^{rt} + B_2te^{rt}$$

To recap...

Distinct Real	Distinct Complex	Repeated
$b^2 > 4ac$	$b^2 < 4ac$	$b^2 = 4ac$
$y(t) = A_1e^{r_1t} + A_2e^{r_2t}$	$y(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$	$y(t) = B_1e^{rt} + B_2te^{rt}$

2.2 Reduction of Order

Given a linear ordinary differential equation of second or higher order, use the reduction of order technique to find the general solution.

With 2$^{\text{nd}}$ Homo, we assumed $y_2(t) = v(t)y_1$ to solve for the repeated roots. Here, we'll generalize that strategy to all **linear** 2$^{\text{nd}}$ Order ODEs. We'll be working with

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t), q(t)$ are functions of t and y_1 is a solution. The goal is to find y_2 such that

$$y = C_1y_1 + C_2y_2$$

The plan is to

1. assume $y_2 = v(t)y_1$ and differentiate twice **
2. plut into ODE + gather terms of v'', v', v ***
3. simplify the resulting equation
4. define $w = v'$ (order has been reduced!), solve the first order ODE
5. $\int w$ to find v
6. sub back into $y_2 = v(t)y_1$

** When differentiating, **do not sub in** y_1 – STAY GENERIC

*** The v terms should *always* cancel

General process:

$$\begin{aligned}
y'' + p(t)y' + q(t)y &= 0 \\
(vy_1)'' + p(t)(vy_1)' + q(t)(vy_1) &= 0 \\
v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)(vy_1) &= 0 \\
v''y_1 + v'(2y_1' + p(t)y_1) + v(y_1'' + p(t)y_1' + q(t)y_1) &= 0 \\
v''y_1 + v'(2y_1' + p(t)y_1) &= 0
\end{aligned}$$

Define $w = v'$:

$$\begin{aligned}
w'y_1 + w(2y_1' + p(t)y_1) &= 0 \\
\frac{\mathrm{d}w}{\mathrm{d}t}y_1 + w(2y_1' + p(t)y_1) &= 0 \\
\frac{\mathrm{d}w}{\mathrm{d}t} + w(\frac{2y_1'}{y_1} + p(t)) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= -w(\frac{2y_1'}{y_1} + p(t)) \\
\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} &= -\frac{2y_1'}{y_1} - p(t) \\
\ln w &= -2\ln(y_1) - \int p(t)dt \\
w &= Ae^{\ln(1/{y_1}^2) - \int p(t)dt} \\
w &= A\frac{e^{-\int p(t)dt}}{y_1^2}
\end{aligned}$$

Integrating w to get v,

$$v = \int w dt = \int A\frac{e^{-\int p(t)dt}}{y_1^2}dt$$

From here, we have

$$y_2 = vy_1$$

and we can get the general solution

$$y = C_1y_1 + C_2y_2$$

Realistically, you wouldn't memorize the formula and go this general – instead, start plugging in explicit values/derivatives at around the 'define $w = v$ step'.

Note than when solving more explicitly, ignore constants of integration when appropriate. Also note that the first order ODE will always be separable.

2.3 Undetermined Coefficients

Determine a particular solution to a nonhomogeneous second order linear differential equation with constant coefficients using the technique of undetermined coefficients.

Thus far, only looked at **homogeneous** 2^{nd} Order ODEs. A **nonhomogeneous** 2^{nd} order ODE has the form:

$$y'' + p(t)y' + q(t) = g(t)$$

We'll define $L[\alpha] = \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}\alpha}{\mathrm{d}t} + q(t)\alpha$. Note that when $L[y] = 0$, the system is homogeneous, and when $L[y] = g(t) \neq 0$ it is nonhomogeneous.

Note that $L[\alpha]$ is a **linear operator**, i.e.,

$$L[af + bg] = aL[f] + bL[g]$$

where a, b are constants and f, g are functions. Then assuming $L[y_p] = g(t)$ and $L[y_1] = 0, L[y_2] = 0$, by linearity we have

$$L[C_1y_1 + C_2y_2 + y_p] = C_1L[y_1] + C_2L[y_2] + L[y_p] = g(t)$$

Therefore, the general solution to $L[y] = g(t)$ is

$$y = y_c + y_p$$

- $y_c = C_1y_1 + C_2y_2$ is the solution to the homogeneous equation and depends *only on the system* – **complementary solution**
- y_p depends on the excitation – **particular solution**

For constant coefficient ODEs, *guess* the form of the paricular solution based on the *form of the RHS*

**and the form of the complementary solution

In this class, there will be three possible forms: polynomial, exponential, or sines/cosines.

- **Polynomial:** $P_n(t) = a_0t^n + a_1t^{n-1} + ... + a_n$– for a **polynomial** RHS of order n (even if some terms are missing), we will assume the y_p takes the form

  $$t^s[A_0t^n + A_1t^{n-1} + ... + A_n]$$

  where s is the number of times $r = 0$ appears in the characteristic equation.
- **Exponential:** $P_n(t)e^{\alpha t}$– where the RHS is a polynomial of order n times an exponential, we expect y_p to follow:

  $$t^s[A_0t^n + A_1t^{n-1} + ... + A_n]e^{\alpha t}$$

  where s is the number of times $r = \alpha$ appears in the characteristic equation.

- **Sines and Cosines:** $P_n(t)e^{\alpha t}\begin{cases}\sin(\beta t)\\ \cos(\beta t)\end{cases}$ – where the RHS is a polynomial of order n times an exponential times a sine and/or cosine, we expect y_p to follow:

$$t^s[(A_0t^n + A_1t^{n-1} + ... + A_n)e^{\alpha t}\sin(\beta t) +$$
$$(B_0t^n + B_1t^{n-1} + ... + B_n)e^{\alpha t}\cos(\beta t)]$$

  where s is the number of times $\alpha \pm \beta i$ are roots of the char equation. Note that we match the exponential and always guess both sine and cosine terms, regardless of if both appear in the original RHS. Also note that the sines and cosines get their own separate polynomial terms.

With the form of y_p, can differentiate and plug into problem statement and solve for the unknown coefficients. Think partial fractions decomposition → combine like terms and solve the resulting system of equations.

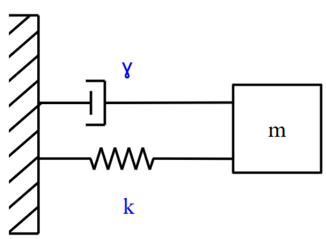
2.4 Vibrations

Determine whether a second order differential equation represents a free or forced system with or without damping. Identify whether resonance will occur. Find the solution and analyze the results in context.

2.4.1 Free Response System

For a free response system with no forcing function, the entire system is determined by initial conditions. For a string-mass-damper system with mass m, force per unit (length per unit time) γ, and force per length k, we can model the system as:

$$m\dot{x} + \gamma\ddot{x} + kx = 0$$



Alernatively, if we define

$$\omega_0 \equiv \sqrt{\frac{k}{m}} \text{ as the } \textbf{natural frequency}$$
$$\zeta \equiv \frac{\gamma}{2\sqrt{mk}} \text{ as the } \textbf{damping ratio}$$

we can rewrite our model as a second order *homogeneous* constant coefficient ODE:

$$m\dot{x} + \gamma\ddot{x} + kx = 0$$
$$\ddot{x} + \frac{\gamma}{m}k + \frac{k}{m}x = 0$$
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

Solving, we get roots

$$r_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Clearly, the roots depend on ζ (or γ) and we can analyze four distinct cases:

1. **Undamped** $\zeta = 0$ **(**$\gamma = 0$**)**:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

2. **Underdamped** $0 < \zeta < 1(\gamma^2 < 4mk)$:

$$x(t) = e^{-\zeta\omega_0 t}(A\cos\omega_0\sqrt{1-\zeta^2}t + B\sin\omega_0\sqrt{1-\zeta^2}t)$$

   note that $\omega_o\sqrt{1-\zeta^2} \equiv \omega_d$, the **damped natural frequency

3. **Critically Damped** $\zeta = 1(\gamma^2 = 4mk)$:

$$x(t) = Ae^{-\omega_0 t} + Be^{-\omega_0 t}$$

   This case is the fastest return to equilibrium: fastest settling time w/ no overshoot

4. **Overdamped** $\zeta > 1(\gamma^2 > 4mk)$:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

   note ω doesn't appear here: expected bc no oscillations in overdamped case

2.4.2 Forced Response

In this case, there is a forcing function and the system looks like:

$$m\dot{x} + \gamma\ddot{x} + kx = F_0\cos\omega_e t \text{ or}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{F_0}{m}cos\omega_e t$$

where F_0 is the magnitude of the input force and ω_e is the **excitation frequency**.

The general solution is of the form $x(t) = x_c + x_p$

- x_c is the **transient** solution – as $t \to \infty$, $x_c \to 0$ and 'dies away'
- x_p is the **steady state** solution – once $x_c \to 0$, x_p will dominate

Using the method of undetermined coefficients, we get $x_p = A\cos\omega_e t + B\sin\omega_e t$ or equivalently,

$$x_p = R\cos(\omega_e t + \phi)$$

where $R = \sqrt{A^2+B^2}$ and $\phi = \tan^{-1}(\frac{B}{A})$***

$$\frac{R}{F_0/k} = \frac{1}{\sqrt{(1-(\frac{w_e}{w_0})^2)^2 + 4\zeta^2(\frac{\omega_e}{\omega_0})^2}}$$

*** assuming $\zeta \neq 0$. If $\zeta = 0$, then $x_p = tR\cos(\omega_0 t + \phi)$

For some intuition, notice that R is the **amplitude** of the steady state response and F_0/k is how far the spring would stretch due to a **constant** F_0 – thus, this ratio compares the response of the oscillations to that of a constant force.

With this amplitude ratio, we have:

$$\begin{aligned}
\omega_e << \omega_0 &\rightarrow \frac{R}{F_0/k} = 1 \\
\omega_e = \omega_0 &\rightarrow \frac{R}{F_0/k} \rightarrow 0 \\
\omega_e >> \omega_0 &\rightarrow \frac{R}{F_0/k} = \frac{1}{2\zeta}
\end{aligned}$$

Note sometimes Γ is defined as $\gamma^2/(mk)$.

2.5 Higher Order ODEs

Given a higher order linear differential equation with constant coefficients, find the general solution.

For **homogeneous** (constant coefficient) higher order ODEs, solve by first finding the roots of the characteristic equation:

For example, if examining third order ODE of the form

$$y''' + ay'' + by' + c = 0$$

- **complex conjugate pair + real**: $C_1e^{r_1t} + e^{\lambda t}(C_2\sin(\mu t) + C_3\cos(\mu t))$
- **3 distinct real roots**: $C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t}$
- **2 real roots, 1 repeated**: $C_1e^{r_1t} + C_2e^{r_2t} + C_3te^{r_3t}$
- **1 real root, thrice repeated**: $C_1e^{r_1t} + C_2te^{r_1t} + C_3t^2e^{r_1t}$

For **non-homogeneous** higher order ODEs, use the method of undetermined coefficients. Solve for the complementary solution as above, and guess the form of the particular solution, making sure there are no terms of the same form.

2.6 Modelling II

Recognize situations in which a higher order ordinary differential equation or a system of first order differential equations is relevant. Develop a mathematical model of such situations, choose an appropriate technique for analyzing or solving the problem, and carry out the analysis.

Good luck soldier.

2.7 Numerical II

Numerically solve a higher order differential equation or system of first order differential equations using explicit Euler integration.

Recall **Euler's Method**:

$$y_{n+1} = y_n + \Delta t f_n$$

When applying to higher order ODEs, this course will examine two flavors of questions:

1. **2^{nd} (Higher) Order ODEs**

   Approach these questions by assuming

   $$\frac{\Delta y}{\Delta t} = y', \frac{\Delta y'}{\Delta t} = y'', \ldots \frac{\Delta y^{n-1}}{\Delta t} = y^n$$

   and writing the higher order derivatives in terms of a function

   $$f(t, \text{all lower derivatives})$$

   Then, use the Euler Formation on each 'level' of derivative. For a two order system, we'd see something akin to

   $$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$
   $$y'_{n+1} = y'_n + \Delta t f(t_n, y_n, y'_n)$$

   Where f is *problem dependent.*

2. **2^{nd} (Higher) Order System** – a system of multiple related dependent variables

   Approach these questions be defining **state variables**and then writing the derivative for each state variable in terms of t, z_i for the state variables z_i.

   Also, could just think about it: if a reaction between two chemical species $x(t)$ and $y(t)$ can be modeled to consume the two species according to

   $$\dot{x} = -\frac{xy}{2} \text{ and } \dot{y} = -\frac{xy}{2}$$

   we can write the formula for x_{n+1} and y_{n+1} with time step h as:

   $$x_{n+1} = x_n + h * \dot{x_n} = x_n - \frac{hx_n y_n}{2} = x_n(1 - \frac{hy_n}{2})$$
   $$y_{n+1} = y_n + h * \dot{y_n} = y_n - \frac{hx_n y_n}{2} = y_n(1 - \frac{hx_n}{2})$$

2.8 Boundary Value Problems

Determine the eigenvalues and eigenfunctions associated with an ordinary differential equation with boundary conditions. Identify all solutions to a given boundary value problem, if any exist.

A **boundary value problem (BVP)** consists of

1. **governing equation** e.g., $y'' + y = 0$
2. **domain** e.g., $x \in [0, L]$
3. **boundary conditions** e.g., $y(0) = 0, y(L) = 1$

A BVP of the form

$$y'' + p(x)y' + q(x)y = g(x)$$

with $y(x_0) = y_0$ and $y(x_1) = y_1$ is **homogeneous** iff $g(x) = 0$ and $y_0 = y_1 = 0$ aka, we have a **homogeneous ODE** *and* **homogeneous boundary conditions**.

2.8.1 Fully Defined BVP

To find all solutions to a fully defined BVP, need the governing equation and boundary conditions. Show that a given BVP (e.g., $y'' + y = 0$) has one of the following:

- no solution $\qquad y(0) = 0, y(\pi) = 1$
- only trivial solutions $\qquad y(0) = 0, y(1) = 0$
- one unique solution $\qquad y(0) = 0, y(1) = 1$
- infinite solutions $\qquad y(0) = 0, y(\pi) = 0$

2.8.2 Eigvenvalue & Eigenfunction Problem

Given a paramatrized BVP, find the values of λ (the parameter) that result in nontrivial solutions (or show that none exist).

- **Eigenvalues:** values of λ that give nontrivial solutions
- **Eigenfunctions:** corresponding solutions to the eigenvalues

Methodology is to first identify ranges** of λ that may give different results. Then, for each case find the general solution and use the boundary conditions to search for nontrivial solutions.

**note that we see $\lambda >, <, = 0$ frequently, but these are not the only possible ranges: for example, $y'' + \lambda y' + y = 0$ has characteristic equation $r_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}$ with $\lambda^2 = 4$ and ranges of $\lambda^2 >, <, = 4$ to consider.

3 Partial Differential Equations

Recall **ordinary differential equations (ODEs)** have one independent variable. **Partial differential equations (PDEs)** have multiple independent variables.

3.1 Fourier Series

Calculate the Fourier series associated with a function. Determine convergence of the series.

3.1.1 Intuition

Consider a function $f(x)$. If we want to work with an easier version of $f(x)$ – aka an approximation – our current option is a **Taylor series** expansion about a certain point:

$$T(x) = a_0 + a_1 x + a_2 x^2 + ...a_n x^n$$

However, we need a recipe/formula to find smart values of a_i. A Taylor series uses a recipe of matching higher and higher order derivaties – however, *polynomial approximations are bad for periodic functions.*

A **Fourier Series** uses *periodic building blocks* to approximate a function. If we want to represent a function $f(x)$ as a sum of other functions, we need to find out how much of each function to add (aka the a_i coefficients).

Can also think of this as an analogy to vectors – given $\vec{v} = v_x \hat{i} + v_y \hat{j}$, we can find v_r, v_s such that $\vec{v} = v_r \hat{r} + v_s \hat{s}$ by taking the **dot (inner) product** to project $\vec{v}$ onto $\hat{r}, \hat{s}$. However, for Fourier Series, instead of the 'dot product' we will define our inner product operation as

$$\frac{1}{L}\int_{-L}^{L} f(x)g(x)dx$$

Evaluating for integers m, n:

$$\begin{aligned}
\cos(mx)\cos(nx) &\text{ on } [-\pi,\pi] \text{ for } m \neq \text{ and } = n \\
\sin(mx)\sin(nx) &\text{ on } [-\pi,\pi] \text{ for } m \neq \text{ and } = n \\
\sin(mx)\cos(nx) &\text{ on } [-\pi,\pi] \text{ for } m \neq \text{ and } = n
\end{aligned}$$

we see that

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \begin{cases} 0, m \neq n \\ 1, m = n \end{cases} \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= \begin{cases} 0, m \neq n \\ 1, m = n \end{cases} \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= 0 \text{ for all } m, n
\end{aligned}$$

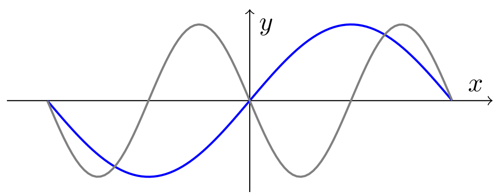
Our goal is to represent a **periodic function** $f(x)$ with a **fundamental period** $2L$ as:

$$\begin{aligned} f(x) \approx h(x) = c_0 + c_1 \cos(\omega_1 x) + b_1 \sin(\omega_1 x) \\ + c_2 \cos(\omega_2 x) + b_2 \sin(\omega_2 x) \\ ... + c_n \cos(\omega_n x) + b_n \sin(\omega_n x) \end{aligned}$$

We're now faced with choosing ω_i, c_i, b_i.

3.1.2 Choosing Fourier Frequencies – ω_i

Want to make sure that our function – every term in $h(x)$ – repeats an *integer* number of times between $-L$ and L.



Thus, we have $\omega_n = (n\pi)/L$ and we can write:

$$h(x) = c_0 + \sum_{n=1}^{\infty} (c_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$$

3.1.3 Determining Fourier Series Coefficients – c_i, b_i

First, we'll modify the integrals in the form of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx$$

with a change of variable: let's define

$$u = \frac{\pi x}{L}, \text{ implying } x = \frac{Lu}{\pi} \text{ and } dx = \frac{L}{\pi} du$$

we see that when $x = \pm L$, $u = \pm \pi$. Thus, we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(mu) \cos(nu) du = \frac{1}{L} \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, m = n \\ 1, m \neq n \end{cases}$$

If we want our approximation of $f(x) \approx h(x)$ to be accurate, then we want their integrals to also match:

$$\int_{-L}^{L} f(x) = \int_{-L}^{L} h(x) = \int_{-L}^{L} \left[c_o + \sum_{n=1}^{\infty} (c_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \right]$$

The integrals of the $\cos$ and $\sin$ terms inside the sum go to 0 and we're left with

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

aka, the average value of $f(x)$.

Next, we'll use the **orthogonality** of the $\cos$ and $\sin$ terms to determine c_n, b_n. With some cool math and properties we saw (on pg 17), we have

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Putting everything together, we see that a function $f(x)$ with a fundamental period of $2L$ (i.e., $f(x) = f(x+2L)$) can be represented as

$$f(x) \approx h(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(c_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\text{with } c_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \text{ and } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

3.1.4 Convergence of a Fourier Series

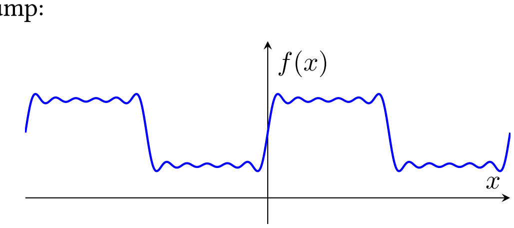
Theorem 3.1 (Fourier Series Convergence Theorem). *A Fourier Series for $f(x)$ on $a \le x \le b$ always converges* ***unless****:*

- *$f(x)$ has an infinite number of discontinuities OR*
- *$f(x)$ or $f'(x)$ approaches $\pm\infty$ on $a \le x \le b$*

Formally, if $f(x), f'(x)$ are **piecewise continuous** on $-L \le x \le L$ and **periodic** outside that interval, then

- $\widetilde{f}(x) \to f(x)$ where it is continuous
- $\widetilde{f}(x) = (f(x^+) + f(x^-))/2$ where it is not

At a discontinuity, observe the **Gibbs Phenomenon**: at discontinuities, the partial sums do not converge smoothly to the mean value and instead overshoot the mark at the end of each jump:

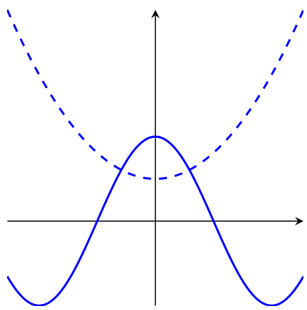


3.2 Even & Odd Fourier Series

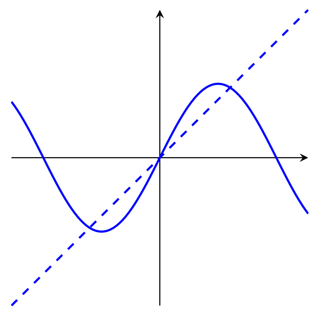
Determine whether a Fourier sine or cosine series is appropriate for a model and calculate it. Create an even or odd periodic extension of a function defined on a limited domain and find the associated Fourier series coefficients.

By observing characteristics of period functions, we can determine ahead of time what FS terms must be 0. Recall the definitions of **even** and **odd** functions:

Even Function: $f(-x) = f(x)$



Odd Function: $f(-x) = -f(x)$



If we integrate over $[-L, L]$ we get:

$2\int_0^L f(x)dx$ for an **even** function

0 for an **odd** function

If we take a function $f(x)$ and multiply by sin (odd) or cos (even) and integrate over a symmetric domain, we see:

	f(x) even	f(x) odd
$b_n : \left(\sin \frac{n\pi x}{L}\right) \cdot f(x)$	0	$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$
$c_n : \left(\cos \frac{n\pi x}{L}\right) \cdot f(x)$	$c_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx$	0

Therefore, we know that for

- **even** $f(x)$, sin terms are 0
- **odd** $f(x)$, cos terms are 0

3.3 Heat Equation

Determine the general solution to the homogeneous heat equation with homogeneous boundary conditions and analyze the results in context.

A **heat equation** looks at the temperature variation of a 1-dimensional spatial domain. We'll consider either **well insulated** situations or where one dimension is much smaller than the others.

Problem Statement: governing equation of

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

- **Domain**: $0 \leq x \leq L, t \geq 0$
- **Boundary Conditions**: $u(0,t) = 0, u(L,t) = 0$ (homogeneous BCs, also ok if $\partial u/\partial x(L,t) = 0$ and/or $\partial u/\partial x(0,t) = 0$, etc)
- **Initial Conditions**: $u(x,0) = h(x)$ – totally arbitrary function

Find: $u(x,t)$ satisfying the governing equation, BCs, ICs

Method: guess *separation of variables* and try

$$u(x,t) = X(x) \cdot T(t)$$

assuming the temperature function can be written as the product of a function of x and a function of t. In this case, the governing equation becomes

$$XT' = \alpha X''T$$

Separating the variables and setting them equal to the same constant λ, we can turn the PDE into two ODEs.

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda \text{ becomes } X'' = -\lambda X \text{ and } T' = -\lambda\alpha^2 T$$

Solving $X'' = -\lambda X$ as a BVP: finding $X(x)$ such that $X'' + \lambda X = 0$ with boundary conditions $X(0) = 0$ and $X(L) = 0$ (from $u(0,t) = 0$ and $u(L,t) = 0$), we get nontrivial solutions for

$$\lambda = \frac{n^2\pi^2}{L^2} \qquad X(x) = A_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, 3...$$

Plugging this λ value into $T' = -\lambda\alpha^2 T$ and solving the ODE, we get

$$T(t) = B_n e^{\left(-\frac{\alpha n \pi}{L}\right)^2 t} \text{ for } n = 1, 2, 3...$$

Putting the two functions together, we get a general solution of

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{\left(-\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

Where for initial conditions of $u(x,0) = f(x)$, then we have

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}$$

which is the **half range Fourier sine series** with

$$C_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})$$

After using governing equation and boundary conditions, apply initial conditions.

3.3.1 Other Boundary Conditions

Thus far only considered $u(0,t) = 0$ and $u(L,t) = 0$, but many possibilities exist:

- **Dirichlet** *(essential – value of fcn itself)*: fixed temperature
  - *Homogeneous Dirichlet*: $u(0,t) = 0, u(L,t) = 0$
  - *Nonhomogeneous Dirichlet*: $u(0,t) = T_a$
- **Neumann** *(natural – value of fcn slope)*: fixed $\partial u / \partial x$ @ specific x locations
  - *Homogeneous Neumann*: insulated $u_x(0,t) = 0$
- **Robin** *(mixed – relationship btwn temp + slope)*: convection BC $u + Au_x = B$

This learning outcome specifically 'targets' homogeneous boundary conditions. However, we'll look at non-homogeneous cases more in depth in the Non-Homogeneous PDE learning outcome.

3.4 Homogeneous PDEs

Given a homogeneous differential equation, determine the solution satisfying appropriate initial conditions and homogeneous boundary conditions. Note that this can encompass the heat equation, the wave equation, the Laplace equation, and other similar PDEs.

3.5 Non-Homogeneous PDEs

Given a nonhomogeneous differential equation, determine the solution to the differential equation satisfying appropriate initial and/or boundary conditions. Note that this can encompass the heat equation, the wave equation, the Laplace equation, and other similar PDEs. Both the equation itself and the boundary conditions may be nonhomogeneous.

3.5.1 The Non-Homogeneous Heat Equation

Recall the types of multiple types of boundary conditions possible:

- **Dirichlet** *(essential – value of fcn itself)*: fixed temperature
  - *Homogeneous Dirichlet*: $u(0,t) = 0, u(L,t) = 0$
  - *Nonhomogeneous Dirichlet*: $u(0,t) = T_a$
- **Neumann** *(natural – value of fcn slope)*: fixed $\partial u/\partial x$ @ specific x locations
  - *Homogeneous Neumann*: insulated $u_x(0,t) = 0$
- **Robin** *(mixed – relationship btwn temp + slope)*: convection BC $u + Au_x = B$

For the non-homogeneous heat equation, we change the *governing equation (e.g., heat generation)* and/or *boundary conditions* to non-zero values.

We'll solve the non-homo heat equation by separating the problem into two parts:

1. Homogeneous problem: transient solution, $t \to \infty, u \to 0$
2. Non-homogeneous problem: steady state solution to nonhomo, note this is $w(x)$, not $w(x,t)$