MATH2930: Differential Equations for Engineers

Instructor: Hadas Ritz, Cornell

Notes by Joyce Shen, FA24

1 First Order Ordinary Differential Equations

This course is about i) **finding** differential equations to model systems and ii) **solving** differential equations both **analytically** (closed form solution for $V(t)$) and **numerically** (plotting V at different times, making observations).

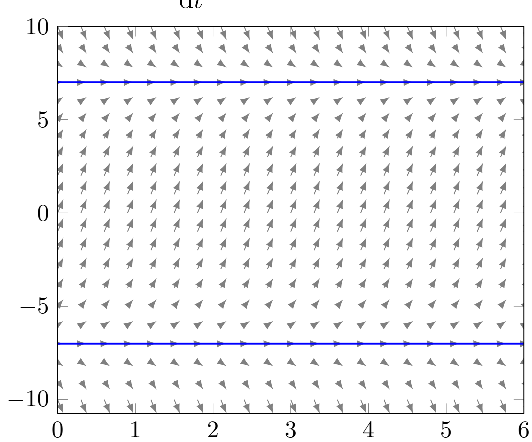
1.1 Direction Fields

Identify the direction field associated with a first order differential equation and sketch a particular solution or a representative family of solutions. Generate a direction field from a first order ODE.

Slope (direction) fields allow us to visualize *all* solutions of a system. To find a specific solutions, follow the path (through the initial condition) on the slope field.

We can find an **equlibrium solution** (a value of y for which $y' = 0, \forall t$) on a direction field if there exists a value of y where $y'(t) = 0$ (horizontal line).

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 0.2y^2$$



1.2 Separable ODEs

Determine whether a first order ordinary differential equation is separable. Solve a first order separable differential equation using integration.

First, a couple of definitions. A **solution** to a differential equation is a **function** that satisfies the equation. (Just like calc II, this is easy to verify correctness.) The **order** of a diffeq is the highest derivative.

Definition 1.1. A differential equation is **linear** if it follows the form:

$$a_0(t)y^n + a_1(t)y^{n-1} + a_2(t)y^{n-2} + \dots + a_n(t)y = g(t)$$

and **nonlinear** otherwise (e.g., $y * y'$, etc)

Definition 1.2. A first order ODE is **separable** if it can be written in the form:

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

aka, if we can separate everything to do with y from everything to do with x.

**Note that linearity and separability are completely unrelated.

The solution methodology for a separable ODE:

1. separate: $\dots\mathrm{d}x = \dots\mathrm{d}y$
2. integrate (to get implicit definition of general solution) $\rightarrow$ is also where the constant of integration C must appear
3. solve for y explicitly (if possible)
4. use initial condition to find C**
   **Note that this can also be done earlier

1.3 Integrating Factors

Solve a first order linear differential equation by using an integrating factor.

Definition 1.3. A **linear** first order ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + q(t)y = g(t)$$

can be solved with an **integrating factor** $\mu(t)$:

$$\mu = e^{\int p(t)dt}$$

Observe that this method relies on the **product rule** of differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)g(t)] = \frac{\mathrm{d}f}{\mathrm{d}t}g + f\frac{\mathrm{d}g}{\mathrm{d}t} = f'g + fg'$$

Essentially, we manipulate the ODE into the form:

$$\int f'g + fg'dt = fg + C$$

by multiplying the integrating factor on both sides. The left hand side becomes

$$\int \mu(t)[\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y(t)] = \int \frac{\mathrm{d}}{\mathrm{d}t}[\mu y] = \mu y$$

while the right hand side evaluates to

$$\int \mu g dt$$

All in all, the differential equation becomes

$$\mu y = \int \mu g dt$$

**Take care to use the correct integrating factor (get the eqn in the correct form first) and don't forget to integrate the RHS

1.4 Existence and Uniqueness

Determine whether a first order ordinary differential equation has a solution and, if so, whether it is unique. Recognize additional solutions for differential equations with non-unique solutions. Understand and apply relevant theorems about existence and uniqueness of solutions to first order ODEs.

Sometimes there may not be a solution to an Initial Value Problem and/or a solution may not be unique. There are different theorems for linear and nonlinear first order ODEs.

Obverve that linear ODEs are a subset (more specific form of) nonlinear ODEs – so technically, the theorem for nonlinear ODEs would also apply. However, the interval specifically for linear ODEs provides more information.

1.4.1 Linear First Order ODEs

Theorem 1.1. *For a linear first order ODE:*

$$y' = p(t)y = g(t) \ \text{ with } \ y(t_0) = y_0$$

*if $p(t)$ and $g(t)$ are **continuous** on a interval containing t_0, then y' has a **unique solution***

1.4.2 Nonlinear First Order ODEs

Theorem 1.2. *For a nonlinear first order ODE:*

$$y' = f(t, y) \text{ with } y(t_0) = y_0$$

if f and $\frac{\partial f}{\partial t}$ are ***continuous*** *on an internval containing (t_0, y_0), there there exists a* ***subinterval*** *on which a unique solution exists – if only f is continuous and $\frac{\partial f}{\partial t}$ is not, then a solution exists but it* ***may not be unique***

1.5 Autonomous ODEs

Determine stable and unstable equilibria of autonomous ordinary differential equations. Sketch representative solutions with an emphasis on behaviors near equilibria.

Recall that a general first order ODE is written in the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$.

Definition 1.4. An ODE is **autonomous** if the independent variable does not show up in the derivative for function f:

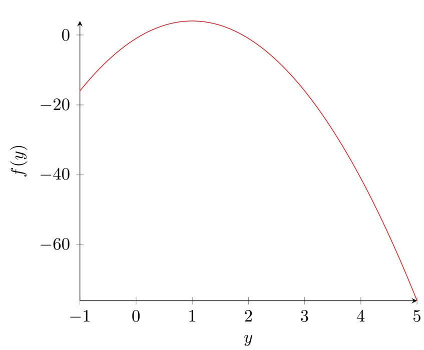
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

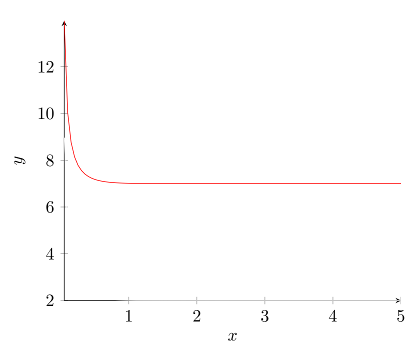
Like direction fields, the autonomous technique (?) is a *qualitative* approach: we can graph $f(y)$ vs y and note the characteristics of the graph

- zeros (**critical points**)
- areas where $f(y)$ is increasing or decreasing
- large or small values of f
- locations of equilibria (and if they're stable, unstable, semistable, etc)

and then use these characteristics to sketch a graph of y vs t.

For example, we can take $f(y) = \frac{\mathrm{d}y}{\mathrm{d}t} = ry(1 - \frac{y}{k})$:

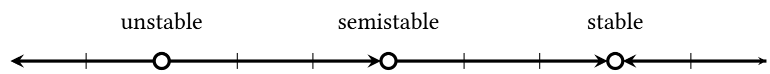




**One possible solution of y

Use **phase lines** to guide the sketch of y vs t, where we plot the equilibrium points of y and their stabilities:

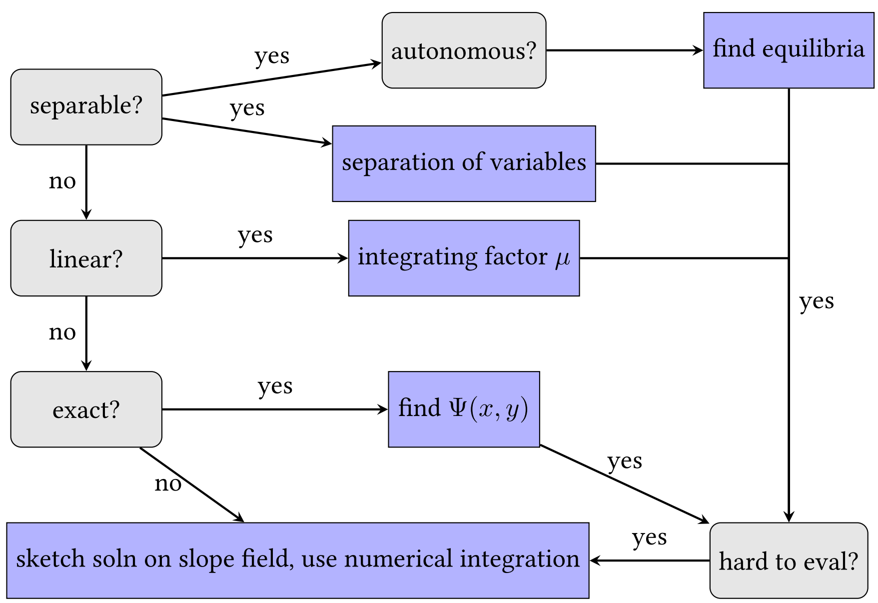
- **stable** equlibriums: neighboring values tend towards the equilibrium point as $t \to \infty$
- **unstable** equlibriums: neighboring values tend away from the equilibrium point as $t \to \infty$
- **semistable** equilibriums: solutions on one side of the eequation tend towards the equilibrium while solutions on a different side tend away as $t \to \infty$



1.6 Initial Value Problems

Given a first order differential equation, choose an appropriate solution method, determine the general solution, and determine particular solutions satisfying appropriate initial conditions.

For first order ODEs, a general strategy for approaching: (but really practice makes perfect)



This is a bit out of order as some of these strategies are discussed in the following pages... also for some reason exact ODEs isn't a LO, but we will discuss it briefly just in case.

Exact First Order ODEs

Definition 1.5. A differential equation of the form

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \text{is } \textbf{exact} \text{ if}$$

$$M_y = N_x$$

If an ODE is exact, there there must exist an **underlying function** $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M, \frac{\partial \Psi}{\partial y} = N$$

Thus, the original $M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ becomes

$$\Psi_x + \Psi_y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

By 'un-chainruling', this can again be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\Psi(x, y(x))) = 0$$

We know this because

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial y}M = \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial x}N$$

Therefore, we know that Ψ must be constant and

$$\Psi(x,y) = c$$

is an **implicit** solution to the diffeq. Solve for Ψ how you would in multi.

**Note exact equations aren't a learning outcome on their own – most likely to appear in Modelling or IVP 'if at all' (Ritz)

1.7 Modelling I

Recognize situations in which a first order differential equation is relevant. Develop an appropriate mathematical model of such systems, choose an appropriate technique for analyzing or solving the problem, and carry out the analysis.

When studying a system, often want a mathematical model – in which we make assumptions about behavior, use experiments to determine numreical values of **model parameters**, and predict behavior. Not really sure what else – this is interpreting a word problem idk best of luck soldier

1.8 Numerical I

Solve a first order differential equation using Euler's method, graphically and algebraically. Identify error and stability. Describe differences between the implicit + explicit methods.

So far we've dealt with ODEs with closed form analytic solutions, but not all ODEs have them! In those cases, we can introduce a *numerical integration technique*: **Euler's Method** for first order ODEs.

1.8.1 Explicit Euler's Method

Here, we're approximating the value of a function near a point by taking the tangent line *at that point.* With the explicit Euler's method, we update the tangent line approximation as you go forward in time. The general algorithm is:

$$\begin{aligned} y_{n+1} &= y_n + (t_{n+1} - t_n) f(t_n, y_n) \\ &= y_n + \Delta t f_n \\ y_{n+1} &= y_n + h f_n \end{aligned}$$

To keep things easy when doing this shit by hand, use a table:

$$(x, y) \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = x + y \qquad \Delta x \qquad \Delta y \qquad (x + \Delta x, y + \Delta y)$$

where in this case $\Delta x = \Delta t = h$.

This is all fine and dandy except it's not. As $h \downarrow$ error $\downarrow$. In fact, with the explicit Euler's method, $h \propto$ error. Besides possibly a larger error, the explicit method is also rather unstable. Thus, we have:

1.8.2 Implicit Euler's Method

With the implicit method, we take the tangent of the *point we are approximating* i.e., $n + 1$ instead of at n.

$$\begin{aligned} y_{n+1} &= y_n + \Delta t f_{n+1} \\ &= y_n + h f(t_{n+1}, y_{n+1}) \end{aligned}$$

However, in this case we still have error with order $O(h)$ aka $h \propto$ error. It is more stable, but it can be difficult to solve for y_{n+1}. Instead, we combine the best of both worlds for:

1.8.3 Improved Euler's Method

To make an analogy, Improved Euler's Method is the trapezoidal Reimann sum approximation of these tangent lines. We do two calculations of f at each step to decrease our error.

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}\right)$$

where we obtain the $f(t_{n+1}, y_{n+1})$ value from using explicit Euler's approximation.

2 Higher Order Ordinary Differential Equations

2.1 2^{nd} Order ODEs

2^{nd} order ODEs can appear in many physics + engineering contexts – acceleration and position, RLC circuits, interdependent 1^{st} order interdependent phenomena, etc

Definition 2.1. In general, a **linear 2^{nd} order ODE** can be written in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}x}{\mathrm{d}t} + q(t)x = g(t)$$

Note that we need two ICs, can be in the form of $x(t_0), x'(t_0)$ or $x(t_0), x(t_1)$, etc.

We're going to look at a simplified problem first:

Definition 2.2. A 2^{nd} order **homogeneous constant coefficient** ODE can be written as

$$ax'' + bx' + cx = 0$$

To solve this equation, we define the **characteristic equation** to be $ar^2 + br + c = 0$ where r are the roots of the equation such that this equals zero. From the quadratic equation, we have

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

based on the discriminant (not determinant lmfao), we take different approaches…

2.1.1 Two Distinct Real Roots

When we have two distinct real roots, aka $b^2 > 4ac$, the general solution takes the form

$$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

where we get the constants A_1 and A_2 from the initial conditions

2.1.2 Two Complex Roots

If $b^2 < 4ac$, we end up with complex roots (which are conjugates). We could plug into the same form as above, but we get complex exponentials which are ugly asf and make no physical sense. Thus, we take **Euler's formula**

$$e^{i\omega t} = \cos\omega t + i\sin\omega t, \qquad e^{i\pi} + 1 = 0$$

and do some algebra to manipulate our general solution into the form

$$y(t) = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$$

where our roots $r_{1,2} = \lambda \pm \mu i$

Alternatively, we can see that $y(t) = A_1 e^{(\lambda+\mu i)t} + A_2 e^{(\lambda-\mu i)t} = e^{\lambda t}((A_1+A_2)\cos\mu t + i(A_1 - A_2)\sin\mu t)$ – and just as how with vector equations the $\vec{i}, \vec{j}$ and $\vec{k}$ components each satisfy their own equations, the real and imaginary parts of the solution each need to satisfy the ODE.

2.1.3 Repeated Roots

The last case, when $b^2 = 4ac$, we need to find out how to get the second function of the general solution. In this case, we're given $ax'' + bx' + c = 0$ and $y_1(t) = e^{rt}$ (only one r...) and we need to find $y_2(t) = v(t)y_1 = v(t)e^{rt}$. If we

1. differentiate y_2: $y_2'' = v''e^{rt} + 2v're^{rt} + vr^2e^{rt}$
2. plug it into the problem statement
3. simplify

... we learn that $v'' = 0 \rightarrow$ if $v'' = 0$, then v is a straight line: $v = \alpha t + \beta$ Thus, the general solution takes the form

$$y(t) = B_1 e^{rt} + B_2 t e^{rt}$$

To recap...

Distinct Real	Distinct Complex	Repeated
$b^2 > 4ac$	$b^2 < 4ac$	$b^2 = 4ac$
$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$	$y(t) = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$	$y(t) = B_1 e^{rt} + B_2 t e^{rt}$

2.2 Reduction of Order

2.3 Undetermined Coefficients

2.4 Vibrations

2.5 Higher Order ODEs

2.6 Modelling II

2.7 Numerical II

2.8 Boundary Value Problems

Summary

3 Partial Differential Equations

Recall **ordinary differential equations (ODEs)** have one independent variable. **Partial differential equations (PDEs)** have multiple independent variables.

3.1 Fourier Series

3.2 Even & Odd Fourier Series

3.3 Heat Equation

3.4 Homogeneous PDEs

3.5 Non-Homogeneous PDEs

Summary