Policy Mix and the Sustainability of Public Debts in the Euro Area

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Abstract

This paper highlights the specifics of a monetary union, such as the Euro area, regarding the choices of monetary and fiscal policies. We show that only one policy can be active, while others fiscal and monetary policies should be passive. However, as the dynamics of public debts are specific to the choices made by each government, the stability requires coordination of fiscal policies, particularly in the case of a liquidity trap situation. Thus, a fiscal union, taking the form of a common debt as the one issued during the Covid19 crisis, guarantees the dynamic stability of the area, notwithstanding the monetary policy, chosen or constrained—thus improving institutional robustness of the European Union.

Keywords: Euro area, Public debt, Taylor rule, Fiscal rule, Fiscal theory of price level, NK model

JEL Classification: E32, E52, E62, F41, H63

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1 Introduction

When major crises occur asking for large stimulus, the choice of policy mix for ensuring the sustainability of public debts is at the center of macroeconomic analysis. The European Union (EU) requires specific analysis. Indeed, unlike a federal system like the United States (US), the decision-making centers of monetary and fiscal policies are not both centralized and, therefore, do not limit the process of political coordination between two actors. Therefore, the European policy mix also requires coordination of fiscal decisions, involving multiple decision-makers.

In this paper, we determine which set of rules, that is, which monetary-fiscal policies, make it possible to guarantee the stability of a monetary union such as the EU, and therefore the sustainability of public debts. We focus on equilibria where financial markets are complete (the "ideal" monetary union, where the financial integration would be achieved), or in incomplete.[1] Even these equilibria seem to be close to the closed economy equilibrium, it allows us to concentrate ourself only on the coordination between monetary and fiscal policies where the European specificity is restricted to the existence of several public debts.[2] Not surprisingly, but without our prior knowledge, we show that this set of rules is not unique, and it strongly depends on the effective margins of monetary policy. More specifically, we show that a fiscal union, that would take the form of a common debt, is necessary for European stability to overcome the limits of monetary policy when interest rates can no longer be lowered.

As this has been done since Leeper (1991), our paper considers that both monetary and fiscal authorities can only choose between two types of policies, active or passive. In a closed economy, the traditional policy mix associating an "active" monetary policy (application of the Taylor principle) and a "passive" fiscal policy (fiscal rule with a debt brake) ensures the dynamic stability of the New-Keynesian (hereafter, NK) model. However, another policy mix ensures equilibrium stability: it is based on a combination of a "passive" monetary policy (failure to respect the Taylor principle) and an "active" fiscal policy

[1]The integration of financial markets has long been considered to be a crucial condition to foster the adoption of a common currency (see Ingram (1969) and Mundell (1973)). Nevertheless, this view can be moderated by Auray and Eyquem (2014)'s results that shown that the welfare gains from complete markets induced by the possibility of risk sharing can be dominated by welfare costs induced by nominal rigidities when a unique central bank can not close all gaps and replicate the flexible prices equilibrium.

[2]We also focus on rules ensuring the existence of a single path around the steady state (a prerequisite for economic stabilization), leaving aside the analysis of multipliers that are specific to each of the policy mix regimes.

(fiscal rule without debt brake). On this last equilibrium, the Fiscal Theory of the Price Level (hereafter, FTPL) applies.[3] When considering the case of the EU, a cursory analysis could lead to the conclusion that European policy makers have the same arbitrations than those in the US because in open economies with a monetary union and complete financial markets the NK model has the same dynamic characteristics for the inflation and output gap than a closed economy. We show that this comfortable transposition of a policy mix analysis in a closed economy to the European case is not trivial. Indeed, given that each government of the Euro area maintains its fiscal independence, there are multiple public debt dynamics, a priori not controlled by the same debt brake.[4] By taking into account these differences in public debt dynamics across the EU members, we show that an active monetary policy is a coordination device because it reduces the political decision of each government to a binary choice: to fight or not the explosiveness of its public debt, the stability of inflation and output gap being insured by the ECB. When this active monetary policy is possible and thus implemented to apply European treaties, the governments of all countries of the area voluntary control the stability of their public debts. These passive fiscal policies are then in accordance with the European treaties that include the "1/20th rule", a fiscal rule which requires that 1/20th of "excess debt" is eliminated each year (see Fiscal Compact of 2012). A free rider behavior is excluded because it will lead to explosive dynamics of all the area. By contrast, when the ECB policy does not/can not respect the Taylor principle (passive monetary policy), we show that only one country of the Euro area must implement an active fiscal policy. This result is due to the integration of inflation rates as well as output gaps: with unique inflation rate dynamic in the area, only a single public debt is needed to anchor the price level. Indeed, as there is a common price level in a currency union, it only exists one degree of freedom to be pinned down by one monetary policy and two fiscal policies, thus implying that only one of these policies must be active and the others two have to be passive to implement a unique equilibrium.[5] But, this principle leaves unclear which country of the area will be free to escape the European treaty. This lack of coordination device advo-

[3] Leeper (1991), Sims (1994), Woodford (1998), and Cochrane (2001) developed the FTPL, where some intuitions are in Sargent and Wallace (1981) and Aiyagari and Gertler (1985). See Bassetto (2002) for an analysis of a game between market players and the government, where the equilibrium solution provides the foundations of the FTPL. See Bassetto (2008) for an overview of the FTPL or Christiano and Fitzgerald (2000), Leeper and Leith (2016), and Cochrane (2019) for surveys on FTPL. Bassetto and Sargent (2020) analyze the impact of several interactions between monetary and fiscal policies on the stability of equilibirum.

[4] This unique rule is lacking in the current treaties. Indeed, the European treaties provide some "general principles" for the fiscal policy, but not an explicit fiscal rule that is uniform for each state.

[5] This extension of the FTPL in a monetary union, has been already discussed in Bergin (2000).

cates in favor of a fiscal union for coordinating fiscal policies: this can be done by issuing bonds on behalf of the EU. This adds a new public debt in the Euro area that should be managed with a fiscal policy ensuring the dynamic stability of the Euro area whatever the monetary policy is.[6] We show that these results does not change qualitatively when financial markets are incomplete.

Therefore, we provide support to the way chosen to finance the "Next Generation EU" initiative.[7] Indeed, by allowing the European Commission to issue bonds on the financial markets on behalf of the EU, it introduces in the Euro area a new tool that ensures its stability when monetary policy can not react to inflation changes. With respect to design of new rules largely debated in the EU[8], we show that the "fiscal brakes" are crucial for the public debt sustainability of each country: they are necessary when the European Central Bank (ECB) controls inflation through nominal interest rate, but also when this last tool can not be used and is replaced by a common debt.[9]

The impractical implementation of the Taylor principle has been experimented when the nominal interest rate has been close or equal to zero in the past few years.[10] Policy choices

[6]This view goes in the same direction of Sims (2012) who wrote *"a solution would be to fill in the institutional gaps in the original euro framework. At a minimum, this would require a new institution with at least some taxing power, able to issue debt and to buy, or not buy, the debt of euro area governments. Such an institution would of course have to be subject to democratic control"* (p.217). We complement the arguments of Farhi and Werning (2017) and Berger, Dell'Ariccia and Obstfeld (2018) showing that a fiscal union has a greater ability to stabilize asymmetric shocks. In our paper, we focus on the ability of the fiscal union, that takes the form of a common debt, to guarantee the dynamic stability of the area, whatever may be the nature of the shocks. We also complement the arguments of Aguiar et al. (2015) that show why existence of a fiscal externality can rationalizes the imposition of debt ceilings in a monetary union, but above all, show that a high-debt country is less vulnerable to rollover crises in sovereign debt markets, and have higher welfare, when it belongs to a union with an intermediate mix of high- and low-debt members, than one where all other members are low-debt. We do not also discuss the issue on the use of the EU commission debt which can be devoted to green investments, as discussed e.g. by Garicano, (2022) or Darvas (2022).

[7]President von der Leyen announced on May 27, 2020 that the bulk of the recovery measures proposed by the EU will be powered by the "Next Generation EU" with financial firepower of €750 billion. The financing will be possible by borrowing up to €750 billion on behalf of the EU, through the issuance of bonds. See the speech of President von der Leyen, `https://ec.europa.eu/commission/presscorner/detail/en/speech_20_941`.

[8]See among other Blanchard (2023), Blanchard, Leandro and Zettelmeyer (2021), D'Amico, Giavazzi, Guerrieri, Lorenzoni and Weymuller (2022) or Martin, Pisani-Ferry and Ragot (2021) for discussions on the fiscal reforms for the EU.

[9]As it is shown in Bonam and Hobijn (2021), what matters is a credible commitment to apply fiscal breaks in the future, even if fiscal breaks are not strictly respected at each time.

[10]See Blanchard (2023) for a discussion of past and future interest rates.

for the management of the recent crises have been, therefore, implemented in a context where monetary policy is highly constrained. In a liquidity trap, or at the Zero-Lower-Bond (hereafter, ZLB), we show that a fiscal union, or a common debt, is a very useful tool to ensure the dynamic stability of the Euro area. Indeed, at the ZLB, dynamic stability must be ensured without using an active monetary policy. The situation is, therefore, close to an equilibrium where only an active fiscal policy, implemented by a unique European government (the European Council), would ensure the dynamic stability of the Euro area. Indeed, the multiplicity of public debts then raises the problem of the designation of the state which will need to have an active fiscal policy, thus allowing to anchor the price level in the Euro area. This suggests that a fiscal union, and thus a common debt, will solve this coordination problem. Of course, the liquidity trap (or ZLB) can only be perceived as transitory by agents. In this case, we show that what matters for the stability of the equilibrium are the agents' expectations on the policies implemented after this episode of liquidity trap. To this end, we follow Christiano, Eichenbaum, and Rebelo (2011)[11] by assuming a stochastic duration for this ZLB episode. We show that if agents expect that the monetary policy will become active after the ZLB period, then a stable equilibrium of the Euro area will be possible if each European country respects its commitment to the European Treaty by choosing a passive fiscal policy. Conversely, if agents expect the monetary policy to remain passive after the ZLB period, then a stable equilibrium of the Euro area will be possible only if one of the European countries does not respect its commitment to the European Treaty. This last result supports a fiscal union unifying the multiple debt dynamics of the Euro area, and thus making it possible to resolve the problem of the designation of the only state that can avoid the fiscal constraints of the European Treaty.

The remainder of the paper is structured as follows. Section 2 presents the arithmetics of public debt dynamics at partial equilibrium. In section 3, we present a two-country model of the Euro area with complete markets (CM) in which the countries are asymmetric because fiscal rules are country-specific and shocks have local realizations. We describe the dynamic properties of inflation, output gap, and public debts conditionally to the policy choices of the central bank and different governments. In Section 4, we show that our results regarding fiscal-monetary interactions apply to a more general framework where markets are incomplete (IM). Section 5 extends the analysis to episodes when the monetary policy is constrained by a zero lower bound. The link between the results of this paper and previous literature is discussed in section 6. Finally, section 7 concludes.

[11]These authors have based their analysis on the previous works of Eggertsson and Woodford (2003), Christiano (2004), and Eggertsson (2004)

2 The Public Debt Dynamics

Lets us denote $\mathcal{B}_t$ the nominal payoff during period t of the government bonds of the Home country bought during period $t-1$ and $1/R_t$ the nominal price at t of an asset that pays 1€ during period $t+1$. Using P_t as the consumer price index (CPI) in the Home country, the government's budget constraint is given by:

$$\frac{1}{R_t}\frac{\mathcal{B}_t}{P_t}+\frac{\mathcal{D}_t}{P_t}=\frac{\mathcal{B}_{t-1}}{P_t} \quad\Leftrightarrow\quad \frac{1}{R_t}\mathcal{B}^r_t+\mathcal{D}^r_t=\frac{1}{\Pi_t}\mathcal{B}^r_{t-1},$$

with $\Pi_t=\frac{P_t}{P_{t-1}}$ and where $\mathcal{D}^r_t$ are the real net transfers, namely, $\mathcal{D}^r_t = Tax_t - Tr_t$, with Tax_t a real lump-sum tax and Tr_t a real lump-sum transfer. We deflate the nominal debt ($\mathcal{B}_t$) by the CPI to define the real debt $\mathcal{B}^r_t$. Assume that it exists an exogenous growth characterized by the real growth rate g. If we focus on a balanced growth path, then the stationarized real variables are $B^r_t=\frac{\mathcal{B}^r_t}{(1+g)^t}$ and $D^r_t=\frac{\mathcal{D}^r_t}{(1+g)^t}$. This leads to:

$$\frac{1}{(1+r_t)}B^r_t+D^r_t=\frac{1}{(1+\pi_t)(1+g)}B^r_{t-1}, \quad \text{where} \begin{cases} R_t &= 1+r_t \\ \Pi_t &= 1+\pi_t. \end{cases}$$

Existence of a Steady State Public Debt. At the steady state, the government's budget constraint is given by:

$$D^r = \left(\frac{1}{(1+\pi)(1+g)}-\frac{1}{1+r}\right)B^r \approx \frac{r-\pi-g}{1+r+\pi+g}B^r \quad\Rightarrow\quad \frac{D^r}{B^r}=\Psi(r,\pi,g).$$

This relation defines a rule between D^r and B^r, given $\{r,\pi,g\}$, that must be satisfied. This rule doesn't say anything on the levels of D^r and B^r, only on the ratio $\frac{D^r}{B^r}$. Hence, if they are both expressed as a fraction of GDP, this rule is not able to set the Debt-to-GDP ratio and the Deficit-to-GDP ratio, but only the ratio of these two ratios.[12]

> *"The 60% debt ratio target was always arbitrary.... there is no such thing as a universal threshold over which debt becomes unsustainable,... the relevant debt level depends on many factors, in particular the real interest rate on debt."*
> Blanchard (2023), p. 75.

[12] This steady state restriction says that with a nominal interest rate on bonds at 3% (2%), an inflation rate at 2%, the real growth rate of the economy must be equal to 6.5% (5.5%) in order to satisfy $\frac{D^r}{B^r}=\frac{-3\% \text{ of the GDP}}{60\% \text{ of the GDP}}$, where values are those of the Maastricht treaty. With a nominal interest rate on bonds at 3%, an inflation rate at 2% and the real growth rate of the economy equal to 2%, the Deficit-to-Debt ratio must be closed to 0.01, thus compatible e.g. with the Deficit-to-GDP ratio of 2% and a Debt-to-GDP ratio of 200%.

In the following, we assume that this rule is satisfied, without any *a priori* on the level of B^r.

Public Debt Dynamics Around its Steady State. The linearized government's budget constraint around the steady state is given by

$$\widehat{b}_t^r \;=\; \beta^{-1}(\widehat{b}_{t-1}^r - \widehat{\pi}_t) + \widehat{r}_t + (1-\beta^{-1})\widehat{d}_t^r$$

where $\beta = \widetilde{\beta}(1+g)^\sigma = \frac{1+\pi}{1+r}$ from the Euler equation on consumption (see below), with σ the intertemporal elasticity of substitution for consumption. With $\beta < 1$ and $\{\widehat{\pi}_t, \widehat{r}_t, \widehat{d}_t^r\}$ some stationary processes independent from $\widehat{b}_t^r$, the debt explodes. In order to avoid debt explosion, the government can adopt a "fiscal rule" such that[13]:

$$D_t^r \;=\; D^r + \gamma\left(\frac{1}{R_{t-1}}\frac{B_{t-1}}{P_{t-1}} - \frac{B^r}{R}\right) + \varepsilon_t^F \quad \Rightarrow \quad \widehat{d}_t^r = \gamma\frac{1}{\beta^{-1}-1}(\widehat{b}_{t-1}^r - \widehat{r}_{t-1}) + \varepsilon_t^F$$

where ε_t^F is the discretionary component of the public deficit (stationary), D^r the structural (constant) deficit and $\gamma\left(\frac{1}{R_{t-1}}\frac{B_{t-1}}{P_{t-1}} - \frac{B^r}{R}\right)$ the deficit's component that brakes the debt's explosion. The linearized budget constraint of the government becomes:

$$\widehat{b}_t^r \;=\; (\beta^{-1}-\gamma)\widehat{b}_{t-1}^r - \beta^{-1}\widehat{\pi}_t + \widehat{r}_t + \gamma\widehat{r}_{t-1} + (1-\beta^{-1})\varepsilon_t^F.$$

The public debt does not explode iff $\beta^{-1} - \gamma < 1 \Leftrightarrow \gamma > \beta^{-1} - 1$, when $\{\widehat{\pi}_t, \widehat{r}_t\}$ are some stationary processes independent from $\widehat{b}_t^r$. This "fiscal rule" ($\gamma > 0$) can be viewed as an approximation of the "1/20th rule", which requires that 1/20th of "excess debt" is eliminated each year (see Fiscal Compact of 2012).[14] This basic analysis puts forward the idea that the debt brake, such the "1/20th rule", is a crucial tool to avoid public debts explosion.

If the stationarity (assumed previously) is a property that must share all processes of a non-explosive equilibrium, the independence of aggregates, such as $\{\widehat{\pi}_t, \widehat{r}_t\}$, with respect to public debt is not trivial. Hence, if the stability of the economy depends on the properties of the complete dynamic of the general equilibrium, one must go beyond this partial analysis based uniquely on the government's budget constraint in order to design policies insuring public debts sustainability.

[13] This type of fiscal rule was first discussed in Bohn (1998).

[14] With a gap between nominal interest rate (r) and inflation (π) of $\{-1; 0; 1; 2; 3\}$%, the Debt-to-GDP ratio initially equal to 120% comeback to 90% (a half of the gap to 60%) in $\{13; 15; 18; 24; 35\}$ years with $\gamma = 1/20$. Hence, the European Treaty ensures debt stability. But, with $\gamma = 1/40$, the debt explodes when $r - \pi = 3\%$ and the half of way to comeback to the targeted Debt-to-GDP ratio is done in $\{13; 15; 18; 24\}$ years for the other values for $r - \pi$.

3 Public Debt Sustainability in a Monetary Union

The Euro area is modeled as an economy comprising two "countries". The first one is "a real country", whereas the second one is "the rest of the Euro area", that is, aggregation of all the other Euro area countries.[15] The Euro area is populated by a continuum of infinitely lived households indexed by $j \in [0, 1]$. The population in segment $j \in [0, m)$ belongs to country H (Home country) and the population in segment $j \in [m, 1]$ belongs to country F (Foreign country). Therefore, m is the relative size of country H in the Euro area. The small Home country case can be derived by taking the limit of the two-country model as $m \to 0$. As in Gali's (2008) textbook, we assume that financial markets are complete, hence assuming that the financial integration promoted when monetary union was built has been realized. This simplify the analysis and allows us to concentrate ourself on the multiplicity of public debts dynamics in interaction with a unique monetary policy[16].

Home bias and country size. The home bias in Home (Foreign) households' preferences is crucial because it determines the gap between CPI and PPI inflation rates. Therefore, we assume that the share of imported goods denoted α (α^*) should decrease with the relative size of country H (F) and with the degree of home bias in Home (Foreign) households' preferences. A tractable way to formalize these ideas is to define $\alpha = \overline{\alpha}(1-m)$ ($\alpha^* = \overline{\alpha}^* m$), where the exogenous parameter $\overline{\alpha}$ ($\overline{\alpha}^*$) is inversely related to the degree of home bias in H (F) households' preferences. In the following, we assume symmetric preferences, namely $\overline{\alpha} = \overline{\alpha}^*$.

3.1 New Keynesian Equilibrium Dynamics

Lets us denoted $\widehat{\pi}_{H,t}$ and $\widehat{\pi}^*_{F,t}$ the inflation rates based on the Production Price Index (PPI) and $\widehat{y}_t$ and $\widehat{y}^*_t$ are the output gaps in the Home and Foreign countries, respectively.[17] The deviation of the nominal interest rate from its natural value is denoted by $\widehat{r}_t$. The log-deviation of the term-of-trade is denoted $\widehat{s}_t$. The complete set of equations describing the

[15] This presentation of a multiple countries economy in two blocks simplify the exposition, but results can be generalized to the N countries case. For example, if the model is extended to three countries, the result that only one of the two countries must have an active fiscal policy can be extended by saying that only one of the three must have an active fiscal policy.

[16] We show in section 4 that our results regarding the monetary-fiscal interactions are also valid when we consider a more general framework with incomplete markets

[17] These gaps are expressed in log-deviations from the steady state.

equilibrium dynamics are the following:[18]

The New Phillips curves (NP curves)

$$
\begin{aligned}
\widehat{\pi}_{H,t} &= \beta \mathbb{E}_t \left[\widehat{\pi}_{H,t+1}\right] + \kappa \widehat{y}_t - \lambda(\alpha m + \omega - 1)\widehat{s}_t + \lambda t_t && (1)\\
\widehat{\pi}^*_{F,t} &= \beta \mathbb{E}_t \left[\widehat{\pi}^*_{F,t+1}\right] + \kappa \widehat{y}^*_t + \lambda(\omega^* - \alpha m)\widehat{s}_t + \lambda t^*_t && (2)
\end{aligned}
$$

where κ and λ are the elasticities of the inflation to output gap and to marginal costs respectively (see appendix B), ω and ω^* are the weights of Home and Foreign countries respectively. The term-of-trade $\widehat{s}_t$ enters in the Phillips curves because households' labor supply is based on nominal wages deflated by the CPI while firms' labor demand is based on wages deflated by the PPI. Firms pay taxes based on their wage bill in each country, with the tax rates being t_t and t^*_t.

The Euler Equations (IS curves)

$$
\begin{aligned}
\widehat{y}_t &= \mathbb{E}_t[\widehat{y}_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}_{H,t+1}]\right) + \frac{1 - \overline{\alpha} m - \omega}{\sigma}\mathbb{E}_t \Delta \widehat{s}_{t+1} && (3)\\
\widehat{y}^*_t &= \mathbb{E}_t[\widehat{y}^*_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}^*_{F,t+1}]\right) + \frac{\omega^* - \overline{\alpha} m}{\sigma}\mathbb{E}_t \Delta \widehat{s}_{t+1} && (4)
\end{aligned}
$$

where σ is the intertemporal elasticity of consumption and $\Delta x_t \equiv x_t - x_{t-1}$, $\forall x$.

The Terms-Of-Trade (TOT)

$$
\begin{aligned}
\widehat{s}_t - \widehat{s}_{t-1} &= \widehat{\pi}^*_{F,t} - \widehat{\pi}_{H,t} && (5)\\
\widehat{s}_t &= \frac{\sigma}{\omega + \omega^*}(\widehat{y}_t - \widehat{y}^*_t) && (6)
\end{aligned}
$$

These relations are crucial to reduce the dynamics of inflation and output gap of each country to a single joint dynamic of inflation and output gap for the monetary union. This comes from the assumptions of preferences symmetry.

The Taylor Rule

$$
\begin{aligned}
\widehat{r}_t &= \alpha_\pi \left(m\widehat{\pi}_{H,t} + (1-m)\widehat{\pi}^*_{F,t}\right) + \alpha_y \left(m\widehat{y}_t + (1-m)\widehat{y}^*_t\right) + \varepsilon^M_t && (7)\\
\varepsilon^M_t &= \rho^M \varepsilon^M_{t-1} + e_t && (8)
\end{aligned}
$$

where $\{\alpha_y, \alpha_\pi\}$ are the sensitivities of the nominal interest rate to output and inflation gaps respectively. The discretionary component of the monetary policy is ε^M_t and we assume that $|\rho^M| < 1$, $e_t \sim iid$.

[18] The definitions of all the reduced form parameters as functions of the structural model's parameters are given in Appendix A. The complete description of the model is provided in Appendix B and a summary of the equations leading to the final reduced form is given in Appendix C.

Public Debt and the Fiscal Rules

$$\begin{aligned}
\widehat{b}_t^r &= \beta^{-1}(\widehat{b}_{t-1}^r - \widehat{\pi}_{H,t} - \bar{\alpha}(1-m)\Delta\widehat{s}_t) + \widehat{r}_t - (\beta^{-1}-1)\widehat{d}_t^r && (9)\\
\widehat{d}_t^r &= \gamma\frac{1}{\beta^{-1}-1}(\widehat{b}_{t-1}^r - \widehat{r}_{t-1}) + \varepsilon_t^F && (10)\\
\varepsilon_t^F &= \rho\varepsilon_{t-1}^F + \nu_t && (11)\\
\widehat{b}_t^{r*} &= \beta^{-1}(\widehat{b}_{t-1}^{r*} - \widehat{\pi}_{F,t}^* + \bar{\alpha}m\Delta\widehat{s}_t) + \widehat{r}_t - (\beta^{-1}-1)\widehat{d}_t^{r*} && (12)\\
\widehat{d}_t^{r*} &= \gamma^*\frac{1}{\beta^{-1}-1}(\widehat{b}_{t-1}^{r*} - \widehat{r}_{t-1}) + \varepsilon_t^{F*} && (13)\\
\varepsilon_t^{F*} &= \rho^*\varepsilon_{t-1}^{F*} + \nu_t^* && (14)
\end{aligned}$$

where the discretionary component of fiscal policies are $\{\varepsilon_t^F, \varepsilon_t^{F*}\}$ and we assume that $|\rho| < 1$, $|\rho^*| < 1$, $\nu_t \sim iid$, $\nu_t^* \sim iid$.

After integrating Equations (10) in (9) and (13) in (12) and the Taylor rule (7) in the IS curves (Equations (3) and (4)) and in the government budget constraints (Equations (9) and (12)), we obtain a system of eight equations, where the seven unknowns are $\{\widehat{\pi}_{H,t}, \widehat{\pi}_{F,t}^*, \widehat{y}_t, \widehat{y}_t^*, \widehat{s}_t, \widehat{b}_t^r, \widehat{b}_t^{r*}\}$ and five exogenous variables $\{\varepsilon_t^M, \varepsilon_t^F, \varepsilon_t^{F*}, t_t, t_t^*\}$.[19] In this system, there are four jump variables $\{\widehat{\pi}_{H,t}, \widehat{\pi}_{F,t}^*, \widehat{y}_t, \widehat{y}_t^*\}$, two predetermined variables $\{\widehat{b}_t^r, \widehat{b}_t^{r*}\}$, one static relation between $\widehat{s}_t$ and $\{\widehat{y}_t, \widehat{y}_t^*\}$ and a definition that links $\widehat{s}_t$ with $\{\widehat{\pi}_{H,t}, \widehat{\pi}_{F,t}^*\}$. The static relationship emphasizes that the two IS curves are not independent equations, whereas the definition of the TOT provides the link between the country specific inflation rates.

Note that with this linearized version of the model, the conditions under which a unique stable path exists can be determined in the deterministic version of the model. Therefore, for simplicity, we abstract in the following for the stochastic part of the model and consider that the exogenous variables $\{\varepsilon_t^M, \varepsilon_t^F, \varepsilon_t^{F*}, t_t, t_t^*\}$ are deterministic processes.

3.2 The Terms-Of-Trade Dynamic

In this section, we show that the dynamic of the terms-of-trade (TOT) can be solved independently of the rest of the model.

Proposition 1. *The TOT dynamic do not depend on the rules of monetary and/or fiscal policies (α_π, α_y, γ, and γ^*) as well as on their shocks (ε_t^M, ε_t^F, and ε^{F*}). The TOT dynamic only depends on fiscal gaps in labor costs ($t_t^* - t_t$).*

[19] The dynamics of these exogenous variables are described by Equations (8), (11), (14) and for any processes for $\{t_t, t_t^*\}$.

Proof. The difference between the two Phillips curves (Equations (1) and (2)) leads to:

$$\widehat{\pi}^*_{F,t} - \widehat{\pi}_{H,t} = \beta\left(\widehat{\pi}^*_{F,t+1} - \widehat{\pi}_{H,t+1}\right) + \kappa(\widehat{y}^*_t - \widehat{y}_t) + \lambda(\omega^* + \omega - 1)\widehat{s}_t + \lambda(t^*_t - t_t)$$

Given the definitions of ω and ω^*[20], we deduce, using Equation (6), the equilibrium dynamic of the TOT as follows:

$$0 = \widehat{s}_{t+1} - b\widehat{s}_t + \frac{1}{\beta}\widehat{s}_{t-1} + \frac{\lambda}{\beta}(t^*_t - t_t), \tag{15}$$

where $b = \left(1 + \beta + \frac{\kappa + (\kappa - \sigma\lambda)\overline{\alpha}(2-\overline{\alpha})(\sigma\eta - 1)}{\sigma}\right)/\beta$. The solution is such that:

$$(1 - r_1 L)(1 - r_2 L)\widehat{s}_{t+1} = \frac{\lambda}{\beta}(t^*_t - t_t) \Rightarrow \widehat{s}_{t+1} = r_1\widehat{s}_t - \lambda r_1 \sum_{j=0}^{\infty}\left(\frac{1}{r_2}\right)^j (t^*_{t+1+j} - t_{t+1+j}),$$

where $r_i = \frac{b \pm \sqrt{b^2 - 4/\beta}}{2}$, for $i = 1, 2$ and s.t. $0 < r_1 < 1 < r_2$ (See Appendix D). □

This result emphasizes the first property of an open economy dynamics: the TOT introduces backward and forward-looking components, independent of monetary and fiscal policies. The predetermined component of the TOT comes from the sticky prices, coupled with the fixed exchange rate assumption (Currency union). The TOT fluctuations are driven by tax rate shocks (t_t, t^*_t). This result is used by Farhi et al. (2014) to derive the optimal sequence of these taxes that would allow the equilibrium of a fixed exchange rate open economy to reach the equilibrium of an economy with flexible nominal exchange rates, better suited for damping business cycle shocks.

Corollary 1. *If $t_t, t^*_t = 0$, then $\widehat{s}_t = 0$, $\forall t$. We deduce that $\widehat{y}_t = \widehat{y}^*_t$ and $\widehat{\pi}_{H,t} = \widehat{\pi}^*_{F,t}$, $\forall t$.*

Proof. $\widehat{s}_t = 0\ \forall t$: obvious using equation (15) with $t_t, t^*_t = 0\ \forall t$ and if the initial condition for s_t is its equilibrium value. Using Equation (6), we deduce that $\widehat{y}_t = \widehat{y}^*_t$ and using Equation (5), we deduce $\widehat{\pi}_{H,t} = \widehat{\pi}^*_{F,t}$. □

Without any shocks on tax rates $(t_t, t^*_t = 0\ \forall t)$, the TOT dynamic is deterministic with a trivial solution $\widehat{s}_t = 0$, $\forall t$. Preferences symmetry imply that the equilibrium IS-Phillips equations of one of the two economies are completely redundant in the reduced form equilibrium. The redundancy of these conditions associated to the previous result on TOT dynamic lead to the fact that the model is isomorphic to the closed economy version of the model to which we add a foreign public debt dynamic.[21] Therefore, the study of the Euro area stability can be reduced to the analysis of the 4 dimensions system $\{\widehat{\pi}_{H,t}, \widehat{y}_t, \widehat{b}^r_t, \widehat{b}^{r*}_t\}$.

[20] See Appendix A for details on the parameters.

[21] Indeed, in a Monetary Union the nominal exchange rate is 1, $\forall t$, implying that $\widehat{\pi}_{F,t} = \widehat{\pi}^*_{F,t}$

3.3 A Limit Case: A Small Open Economy

The case of a small open economy (SOE) is obtained when $m \to 0$. Hence, the monetary policy, given by $\widehat{r}_t = \alpha_\pi \pi^*_{F,t} + \alpha_y \widehat{y}^*_t + \varepsilon^M_t$, does not depend on Home aggregates. The Home aggregates have then an autonomous dynamic, with the Foreign variables and the interest rate being exogenous. For simplicity, we thus assume that $\widehat{\pi}^*_{F,t} = \widehat{y}^*_t = t^*_t = 0$. This leads to $\widehat{s}_t = -\widehat{p}_{H,t}$. The equilibrium dynamic of a SOE is given by Equation (15), where $\widehat{s}_t$ is simply replaced by $-\widehat{p}_{H,t}$. Given the equilibrium path for $\widehat{p}_{H,t}$, we deduce the output gap using Equation (6), the equation being evaluated for $m \to 0$. This shows that the dynamic of the Home output gap is given by a Home price dynamic (here, the opposite of the TOT) and an inflation dynamic (the simple difference between the current and the past price levels). As for the TOT, the equilibrium price dynamic is deduced from Equation (15), where $\widehat{s}_t = -\widehat{p}_{H,t}$. Its solution is $\widehat{p}_{H,t+1} = r_1 \widehat{p}_{H,t} - \lambda r_1 \sum_{j=0}^{\infty} \left(\frac{1}{r_2}\right)^j t_{t+j}$, where the roots are $r_i = \frac{b \pm \sqrt{b^2 - 4/\beta}}{2}$, for $i = 1, 2$ and s.t. $0 < r_1 < 1 < r_2$. This is the result presented in Erceg and Linde (2012). It is also a corollary of Proposition 1: in a SOE, the equilibrium dynamic of Home price (opposite of the TOT in a SOE) does not depend on monetary and fiscal policies (rules and shocks).

Proposition 2. *In a small open economy of the Euro area, a debt brake, of a minimal size $\gamma > 1/\beta - 1$, is required to ensure the existence of a saddle path.*

Proof. See Appendix E. □

Given that the path of price gives both output and inflation dynamics, the equilibrium of a SOE is defined only by the price and public debt dynamics. The price and public debt are independent processes in a SOE because the price dynamic of the Home country does not depend on fiscal policy. The stability of the debt can only be ensured by the implementation of a fiscal brake.

From a policy perspective, this result implies that if each SOE of the Euro area adopts a fiscal rule that respect $1/\beta - 1 < \gamma$, then the aggregate public debt of the Euro area will have a sufficient debt break to ensure its stability. Therefore, the individual fiscal policies of each country imply the stability of the Euro aggregates. This is a credible strategy if all countries expect that the ECB respects its mandate which is to stabilize inflation via the implementation of the Taylor principle (active monetary policy). Indeed, given that inflation and output gaps abroad are taken as exogenous and stationary at the level of each SOE, the ECB must apply the European treaties by respecting the Taylor principle in order to stabilize the aggregate output-inflation dynamics of the Euro area.[22] If each SOE of the Euro area does not implement a debt brake, there is no chance for the ECB to correct for this instability.[23]

3.4 General Case: the Euro Area Stability

In the first subsection, we analyze a specific case where it is assumed that all governments of the Euro area adopt the same fiscal rule. This case corresponds to the one where all the governments behave as a representative one, obviously a hypothetical scenario in the current Euro area. We show that two equilibria are possible, as in a closed economy: first, the ECB has an active monetary policy (Taylor principle) and all governments have a passive fiscal policy (implementation of a fiscal brake); and second, the ECB has a passive monetary policy and all governments have an active fiscal policy (no debt brake). In the second subsection, we analyze the case where governments have different fiscal policies, implying that the equilibrium cannot be summarized by aggregates determined by a representative decision maker.

3.4.1 When All Governments Adopt the Same Fiscal Rule

Assume that fiscal brakes are identical across countries.[24] Note that only the components of fiscal policies that account for the debt brake (γ and γ^*) must be homogeneous across countries for the dynamic stability of the Euro aggregates.[25] The components of fiscal policies that account for discretionary decisions (ρ and ρ^*) can be heterogeneous across countries. Thus, we have $\gamma = \gamma^* \equiv \gamma_u$, but $\rho \neq \rho^*$. The Euro aggregates are defined as follows: inflation is $\widehat{\pi}_t^u = m\widehat{\pi}_{H,t} + (1-m)\widehat{\pi}_{F,t}$, output gap is $\widehat{y}_t^u = m\widehat{y}_t + (1-m)\widehat{y}_t^*$, and debt is $\widehat{b}_t^{ru} = m\widehat{b}_t^r + (1-m)\widehat{b}_t^{r*}$. By directly analyzing the aggregates of the Euro area,

[22]For analyzing the SOE, it is assumed that $\widehat{\pi}_{F,t}^* = \widehat{y}_t^* = 0$ for simplicity, but what matters is the stationarity of both $\widehat{\pi}_{F,t}^*$ and $\widehat{y}_t^*$. This stationarity is ensured if it exists a unique equilibrium path for these two variables representing inflation and the output gap of the Euro area.

[23]This is true because we consider here only one SOE for the Euro area, for which stability restrictions does not depend on the monetary policy.

[24]In this section, it is assumed for simplicity that $t_t = t_t^* = 0$, $\forall t$. These exogenous variables simply add shocks to the aggregate Phillips curve of the Euro area, which is not useful at this stage.

[25]Aggregation is also possible under the assumption that the structural parameters that govern nominal rigidities (Phillips curves) are the same across countries (assuming that all Europeans share the same preferences). This assumption is not rejected by the data (see Martin and Philippon (2017)).

the monetary policy described by the following Taylor rule (Equation (7))

$$\widehat{r}_t = \alpha_\pi \pi_t^u + \alpha_y \widehat{y}_t^u + \varepsilon_t^M \tag{16}$$

links the interest rate to inflation and output gap, without distinguishing countries by their weight on these aggregates. By aggregating the two Phillips curves (Equations (1) and (2))

$$\widehat{\pi}_t^u = \beta \widehat{\pi}_{t+1}^u + \kappa \widehat{y}_t^u,$$

the two IS curves (Equations (3) and (4))

$$\widehat{y}_t^u = \widehat{y}_{t+1}^u - \frac{1}{\sigma}\left(\widehat{r}_t - \widehat{\pi}_{t+1}^u\right),$$

the two debt dynamics (Equations 9) and (12) as well as the surpluses (Equations (10) and (13))

$$\begin{aligned}\frac{1}{\beta^{-1}-\gamma_u}\widehat{b}_{t+1}^{ru} &= \widehat{b}_t^{ru} - \widehat{\pi}_{t+1}^u - \frac{\beta^{-1}-1}{\beta^{-1}-\gamma^u}\left(m\varepsilon_{t+1}^F + (1-m)m\varepsilon_{t+1}^{F*}\right) \\ &+ \frac{1}{\beta^{-1}-\gamma_u}\left(\alpha_\pi \widehat{\pi}_{t+1}^u + \alpha_y \widehat{y}_{t+1}^u\right) + \frac{\gamma_u}{\beta^{-1}-\gamma_u}\left(\alpha_\pi \widehat{\pi}_t^u + \alpha_y \widehat{y}_t\right),\end{aligned}$$

and using the Taylor rule (Equation (16)), the equilibrium paths for $\{\widehat{\pi}_t^u, \widehat{y}_t^u, \widehat{b}_t^{ru}\}$ can be solved.[26]

Proposition 3. *When all governments adopt the same fiscal rules, (i) if the Taylor principle applies, then the stability of the Euro aggregates is ensured if a "passive" fiscal policy is implemented in each European state; (ii) if the Taylor principle does not apply, it can also exist a stable equilibrium if fiscal policies are "active" in each country.*

Proof. Using $\varepsilon_{t+1}^F = \rho \varepsilon_t^F$ and $\varepsilon_{t+1}^{F*} = \rho^* \varepsilon_t^{F*}$, the Philips and IS curves as well as the debt dynamics of the Euro area aggregates lead to:

$$\begin{bmatrix} \widehat{\pi}_t^u \\ \widehat{y}_t^u \\ \widehat{b}_t^u \end{bmatrix} = \begin{bmatrix} \begin{matrix} A \end{matrix} & 0 \\ \begin{matrix} a_1 & a_2 \end{matrix} & a_3 \end{bmatrix} \begin{bmatrix} \widehat{\pi}_{t+1}^u \\ \widehat{y}_{t+1}^u \\ \widehat{b}_{t+1}^u \end{bmatrix} + \Omega \begin{bmatrix} -\kappa & 0 & 0 \\ -1 & 0 & 0 \\ b_1 & m\rho b_2 & (1-m)\rho^* b_2 \end{bmatrix} \begin{bmatrix} \varepsilon_t^M \\ \varepsilon_t^F \\ \varepsilon_t^{F*} \end{bmatrix}$$

[26] Another way to obtain this dynamic system is to assume that Europeans have no home bias $\overline{\alpha} = 1$. In this case, the risk-sharing condition leads to $\widehat{c}_t = \widehat{c}_t^*$. Assuming for simplicity that $m = 1/2$, this implies that $\widehat{y}_t = \widehat{y}_t^* = \frac{\widehat{y}_t + \widehat{y}_t^*}{2} \equiv \widehat{y}_t^u$, $\widehat{\pi}_{H,t} = \widehat{\pi}_{F,t}^* = \frac{\widehat{\pi}_{H,t} + \widehat{\pi}_{F,t}^*}{2} \equiv \widehat{\pi}_t^u$, $\widehat{b}_t^{r,u} \equiv \frac{\widehat{b}_t^r + \widehat{b}_t^{r*}}{2}$ and $\widehat{d}_t^{r,u} \equiv \frac{\widehat{d}_t^r + \widehat{d}_t^{r*}}{2}$ are solutions of the same dynamic system.

where $\begin{cases} \Omega = \frac{1}{\sigma+\alpha_y+\kappa\alpha_\pi} & a_1 = -\frac{\beta}{1-\gamma_u\beta}\left(\alpha_y - \beta^{-1} + \gamma_u\sigma(\alpha_\pi\beta - 1)\Omega\right) \\ a_2 = -\frac{\beta}{1-\gamma_u\beta}\left(\alpha_y + \gamma_u\sigma(\alpha_y + \alpha_\pi\kappa)\Omega\right) & a_3 = \frac{\beta}{1-\gamma_u\beta} \\ b_1 = -\gamma_u\frac{\sigma\beta}{1-\gamma_u\beta} & b_2 = (1-\beta)\frac{\sigma+\alpha_\pi\kappa+\alpha_y}{1-\gamma_u\beta} \end{cases}$

This system is block-diagonal with an upper block defined by:

$$\begin{bmatrix} \widehat{y}_t^u \\ \widehat{\pi}_t^u \end{bmatrix} = A \begin{bmatrix} \widehat{y}_{t+1}^u \\ \widehat{\pi}_{t+1}^u \end{bmatrix} + B\varepsilon_t^M$$

$$\text{where} \quad A = \Omega \begin{bmatrix} \kappa + \beta(\sigma + \alpha_y) & \kappa\sigma \\ 1 - \beta\alpha_\pi & \sigma \end{bmatrix}, \quad B = -\Omega \begin{bmatrix} \kappa \\ 1 \end{bmatrix}$$

can be solved first. The two eigenvalues of A are positive, and the characteristic polynomial is such that $\mathbf{P}_A(1) = 1 - \mathrm{tr}(A) + \det(A) = \frac{\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y}{\sigma+\alpha_y+\kappa\alpha_\pi}$, which indicates that both roots are lower than 1 if $\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y > 0$. This is the Taylor principle.

This strategy for monetary policy enforces *all* governments of the Euro area to implement a strict fiscal rule, that is, to choose a parameter $\gamma = \gamma^* \equiv \gamma^u$ such that public debt does not explode. A simple restriction is $\gamma^u > \beta^{-1} - 1$, which represents the "strictness" of the fiscal rule. $\square$

This analysis reveals that the stability of the Euro aggregates, and in particular the public debt, does not depend on the nature of shocks, but on the coordination of the policy rules.[27] Moreover, because the TOT dynamic which captures the open relations can be solved independently, the reduced-form equilibrium conditions do not depend on parameters linked to the open-economy framework, namely the country size m, the home-bias preferences $\overline{\alpha}$ and the elasticity between Home and Foreign goods η.

This framework, where all the countries have exactly the same fiscal policy, can be interpreted as a **Federal state** where the central government coordinates fiscal policies which are thus unique. Therefore, we return to the same economy studied by Leeper, Traum, and Walker (2017) in the case of the US modeled as a closed economy. For the Euro area, this simplified model with a representative country is close to the one used by Woodford (1996) in discussing the impact of the Maastricht treaty on maintaining price stability in Europe. **If** the Taylor principle is satisfied, namely $\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y > 0$, **then** a passive fiscal rule **in each country** is necessary to ensure that public debt remains non-explosive. Indeed, with $\{\widehat{\pi}_t^u, \widehat{y}_t^u\}$ (the stationary solutions of the dynamic system for the aggregate economy when the Taylor principle is respected), debt is sustainable if

[27]Note that this analysis, based on the aggregates of the Euro area, does not imply that the output gap and inflation of each country are the same and equal to the ones of the Euro area.

$\gamma > \beta^{-1} - 1$. This fiscal brake ($\gamma > \beta^{-1} - 1$) represents the "strictness" of the fiscal rule. The commitment of each government of the Euro area is crucial to ensure economic stability. **By contrast**, if the ECB does not respect the Taylor principle, then all countries of the Euro area (in fact, the representative government of the Euro area) must abandon fiscal rules incorporating a fiscal brake. This is an easy strategy for a federal state but less easy for the Euro area where European treaties maintain fiscal independence. Note that this problem is not a theoretical curiosity: it can be concrete when an economy enters a liquidity trap, such as the period when the ZLB constraint is perceived as very persistent (see section 5).

3.4.2 When Fiscal Rules Are Different Across Countries

We now break the simplistic view of unique IS and Phillips curves in Europe. Each country has a specific fiscal policy (rule and shock). The complete dynamics of the two-country model are the solution of the following system (using Equations (1), (3), (9), and (12)):

$$\begin{aligned}
\widehat{\pi}_{H,t} &= \beta\widehat{\pi}_{H,t+1} + \kappa\widehat{y}_t - \lambda(\bar{\alpha}m + \omega - 1)\widehat{s}_t + \lambda t_t \\
\widehat{y}_t &= \widehat{y}_{t+1} - \frac{1}{\sigma}\left(\widehat{r}_t - \widehat{\pi}_{H,t+1}\right) + \frac{1 - \bar{\alpha}m - \omega}{\sigma}(\widehat{s}_{t+1} - \widehat{s}_t) \\
\frac{\beta}{1-\beta\gamma}\widehat{b}^r_{t+1} &= \widehat{b}^r_t - \frac{\bar{\alpha}(1-m)}{1-\beta\gamma}\Delta\widehat{s}_{t+1} - \frac{1}{1-\beta\gamma}\widehat{\pi}_{H,t+1} + \frac{\beta}{1-\beta\gamma}\widehat{r}_{t+1} + \frac{\gamma\beta}{1-\beta\gamma}\widehat{r}_t - \frac{1-\beta}{1-\beta\gamma}\varepsilon^F_{t+1} \\
\frac{\beta}{1-\gamma^*\beta}\widehat{b}^{r*}_{t+1} &= \widehat{b}^{r*}_t + \frac{\bar{\alpha}m - 1}{1-\gamma^*\beta}\Delta\widehat{s}_{t+1} - \frac{1}{1-\gamma^*\beta}\widehat{\pi}_{H,t+1} + \frac{\beta}{1-\beta\gamma^*}\widehat{r}_{t+1} + \frac{\gamma^*\beta}{1-\beta\gamma^*}\widehat{r}_t - \frac{1-\beta}{1-\beta\gamma^*}\varepsilon^{F*}_{t+1}
\end{aligned}$$

where the Taylor rule, given by:

$$\widehat{r}_t = \alpha_\pi\widehat{\pi}_{H,t} + \alpha_y\widehat{y}_t + (1-m)\left(\alpha_\pi - \alpha_y\frac{\omega + \omega^*}{\sigma}\right)\widehat{s}_t - \alpha_\pi(1-m)\widehat{s}_{t-1} + \varepsilon^M_t \quad (17)$$

shows that despite the asymmetric weight of each country, the sensitivity of the Taylor rule to inflation and output gap does not depend on the weight of each country. This result is a direct implication of the TOT equation (6).

After integrating Equation (17) in the IS and public debt equations, the dynamics evolves according to $Z_t = \widehat{M}Z_{t+1} + \widehat{D}S_t + \widehat{P}\varepsilon_t$, where $Z_t = [\widehat{\pi}^H_t, \widehat{y}_t, \widehat{b}^r_t, \widehat{b}^{r*}_t]'$, $S_{t+1} = [\widehat{s}_{t+1}, \widehat{s}_t, \widehat{s}_{t-1}]'$, and $\varepsilon_t = [\varepsilon^M_t, \varepsilon^F_t, \varepsilon^{F*}_t]'$. For a given sequence of $\{\widehat{s}_t\}_{t=0}^\infty$, this system has two jump variables $\{\widehat{\pi}^H_t, \widehat{y}_t\}$ and two predetermined variables $\{\widehat{b}^r_t, \widehat{b}^{r*}_t\}$. Its stability is then ensured if $\widehat{M}$ has two eigenvalues inside and two outside the unit circle. Using the solution $\widehat{s}_t = 0$ if $t_t = t^*_t = 0$ $\forall t$ (see Proposition 1), $\widehat{y}^*_t$ is deduced from $\widehat{y}^*_t = \widehat{y}_t - \frac{\omega+\omega^*}{\sigma}\widehat{s}_t$, and $\widehat{\pi}^*_{F,t}$ from $\widehat{\pi}^*_{F,t} \equiv \widehat{\pi}_{H,t}$.[28]

[28]See appendix F for the complete description of this system.

Proposition 4. *If the Taylor principle is respected, the stability of the Euro area is ensured if all the countries have a passive fiscal policy, that is, if all the countries implement a fiscal brake satisfying* $\gamma, \gamma^* > \beta^{-1} - 1$.

If the Taylor principle is not respected, the stability of the Euro area is ensured if only one country implement an active fiscal policy, all others must respect their commitment to implement a fiscal brake: $\gamma > \beta^{-1} - 1$ *&* $\gamma^* < \beta^{-1} - 1$*, or* $\gamma < \beta^{-1} - 1$ *&* $\gamma^* > \beta^{-1} - 1$.

Proof. Given that $\widehat{M}$ is triangular (See appendix F), its eigenvalues are those of the matrix A and $\left\{\frac{1}{\beta^{-1}-\gamma}, \frac{1}{\beta^{-1}-\gamma^*}\right\}$. In Proposition 3, it is shown that the two eigenvalues of A are inside the unit circle if $\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y > 0$ (Taylor principle).

If $\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y > 0$, then the two eigenvalues of A are inside the unit circle. Therefore, the two last eigenvalues of $\widehat{M}$ must be outside the unit circle: $\frac{1}{\beta^{-1}-\gamma} > 1$ and $\frac{1}{\beta^{-1}-\gamma^*} > 1$, that is, $\gamma, \gamma^* > \beta^{-1} - 1$.

If $\kappa(\alpha_\pi - 1) + (1-\beta)\alpha_y < 0$, then one eigenvalue of A is inside the unit circle, the other being outside the unit circle. Therefore, only one of the two last eigenvalues of $\widehat{M}$ must be outside the unit circle. This implies that one set of the following restrictions must be satisfied: *i)* $\frac{1}{\beta^{-1}-\gamma} > 1$ and $\frac{1}{\beta^{-1}-\gamma^*} < 1$, that is, $\gamma > \beta^{-1} - 1$ and $\gamma^* < \beta^{-1} - 1$, or *ii)* $\frac{1}{\beta^{-1}-\gamma} < 1$ and $\frac{1}{\beta^{-1}-\gamma^*} > 1$, that is, $\gamma < \beta^{-1} - 1$ and $\gamma^* > \beta^{-1} - 1$. □

If the Taylor principle is respected, that is, when the ECB implements an "active" monetary policy rule, **all** countries of the Euro area must implement a fiscal brake. This shows that these two policies are complementary to ensure the stability of the area, and thus the sustainability of public debts. The active monetary policy implemented by the ECB can be viewed as a coordination device guiding all governments to respect the European treaties.

If the Taylor principle is not respected, only one country is constrained to adopt a passive fiscal policy. The other country must have an active fiscal policy. Therefore, when the monetary policy of the ECB does not respect the Taylor principle, **one** of the two countries of the Euro area must adopt an active fiscal policy, that is, it must not implement the fiscal brake of the Fiscal Stability Treaty. In this case, the fiscal theory of the price level applies, showing that a non-unified fiscal policy provides the freedom to choose which country can renounce to its European treaty commitments. This degree of freedom can also lead to a coordination problem, unforeseen by the European treaties.

Instability of the Euro area: a common debt as a stabilizing device. The two regimes of Proposition 4 suggest that a *fiscal union*, that takes the form of a common debt, would be the best solution to avoid any destabilization of the Euro area linked to the non-compliance of one of the member states to the Fiscal Stability Treaty[29].

The issue of common debt leads “simply” to an additional public debt dynamic, managed by the European Council. Therefore, the dynamic system of the Euro area would be composed by five equations: the NP and the IS curves, the public debts of each of the two countries and the common public debt of the EU. In this context, even if the monetary policy is not “active”, the stability of the Euro area can be ensured by an “active” fiscal policy for the common EU budget, the stability of the two other public debts being ensured by the implementation of debt brakes (“passive” fiscal policy inside each country). By contrast, if the monetary policy can be “active”, then the three fiscal policies must be “passive”. However, the risk of instability for the Euro area seems lower when the monetary policy is active: if one country deviates, it knows that it could lead to global instability in the area, its gains then being able to exist only in the very short term. Moreover, if this country is small, Proposition 2 suggests that it is not rational for a SOE to find grounds for non-compliance with treaties. Obviously, this reasoning also applies to the governance of the European common debt. Hence, when the monetary policy is active, nobody seems to be tempted to give up its commitment to European treaties. An active monetary policy also acts as a coordination device.

This optimism about the Euro system is undermined in situations where the ECB would choose not to have an active monetary policy. In this case, one of the members must renounce to these European commitments, but no one knows which among the states. This indeterminacy of equilibrium presents a clear problem of stability in the Euro area. The critical situation in the Euro area would then be the moment when the ECB would be forced not to have an active monetary policy at a time where the issue of bonds on behalf of the EU would not be possible.

4 Incomplete markets

In this section, we show that our results are robust even when financial markets are incomplete. For the sack of simplicity, we will consider a framework where incomplete markets take the form of financial autarky where households can only hold national public

[29]However, this solution will require some changes in the European legal framework as article 310(1) of TFEU imposes that Union’s annual budget should be balanced, i.e excluding active fiscal policy at the level of the union.

debt[30]. However, the conclusions regarding the monetary-fiscal interactions that ensure a unique saddle-path of the dynamics also generalize to a more realistic framework where financial trade is allowed[31].

The previous framework with CM made the hypothesis that perfect international risk-sharing through state contingent assets was available for households in each country. This conditions materialized in the model through the IRS (or Backus-Smith (1993)) condition:

$$\widehat{c}_t(s) = \frac{1}{\sigma}\widehat{q}_t(s) + \widehat{c}^*_t(s) \quad \forall t,\ \forall s \tag{18}$$

However, when markets are incomplete and financial trade forbidden, there is no possibility for a country to be net debtor or net creditor internationally. It implies that even if households can trade goods, a country cannot accumulate a trade balance surplus or deficit and imports should be equal to exports at each period no matter the shocks, i.e $\widehat{tb}_t = 0$ and $\widehat{tb}^*_t = 0$, $\forall t$.

4.1 New Keynesian Equilibrium Dynamics

The equilibrium conditions of the two-country model with incomplete markets and financial autarky can be summarized by the following conditions:

The New Phillips curves (NP curves)

$$\widehat{\pi}_{H,t} = \beta\mathbb{E}_t\left[\widehat{\pi}_{H,t+1}\right] + \kappa\widehat{y}_t + \lambda\bar{\alpha}(1-m)(1-\sigma)\widehat{s}_t + \lambda t_t \tag{19}$$

$$\widehat{\pi}^*_{F,t} = \beta\mathbb{E}_t\left[\widehat{\pi}^*_{F,t+1}\right] + \kappa\widehat{y}^*_t - \lambda\bar{\alpha}m(2\eta\sigma+\sigma+1)\widehat{s}_t + \lambda t^*_t \tag{20}$$

The Euler Equations (IS curves)

$$\widehat{y}_t = \mathbb{E}_t[\widehat{y}_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}_{H,t+1}]\right) + \bar{\alpha}(1-m)(\sigma^{-1}-1)\mathbb{E}_t\Delta\widehat{s}_{t+1} \tag{21}$$

$$\widehat{y}^*_t = \mathbb{E}_t[\widehat{y}^*_{t+1}] - \frac{1}{\sigma}\left(\widehat{rz}_t - \mathbb{E}_t[\widehat{\pi}^*_{F,t+1}]\right) - \bar{\alpha}m(2\eta-1+\sigma^{-1})\mathbb{E}_t\Delta\widehat{s}_{t+1} \tag{22}$$

where $\widehat{rz}_t$ is the interest rate in the foreign economy.

The Terms-Of-Trade (TOT)

$$\widehat{s}_t - \widehat{s}_{t-1} = \widehat{\pi}^*_{F,t} - \widehat{\pi}_{H,t} \tag{23}$$

$$\widehat{s}_t = \frac{(\widehat{y}_t - \widehat{y}^*_t)}{(2\eta - 1 - \bar{\alpha}\eta + \bar{\alpha} - 2\bar{\alpha}m\eta)} \tag{24}$$

[30] The new budget constraints with regards to the IM framework with financial autarky can be found in Appendix G.1

[31] The description of the model and numerical simulations can be found in Appendix G.2.

Interest rate differential

$$\widehat{r}_t - \widehat{rz}_t = (2\eta\sigma - \sigma - \bar{\alpha}\eta\sigma + \bar{\alpha} - 1)(\widehat{s}_{t+1} - \widehat{s}_t) \quad (25)$$

In the two-country model with IM and financial autarky, there will be a gap between interest rates on public debt in the two countries $\widehat{r}_t - \widehat{rz}_t$. Indeed, by combining the two IS curves (21 and 22) with the equilibrium condition that links output gap in the two regions 24, we get $\widehat{\pi}^*_{F,t+1} - \widehat{\pi}_{H,t+1} = (\widehat{s}_{t+1} - \widehat{s}_t)(2\eta\sigma - \sigma - \bar{\alpha}\eta\sigma + \bar{\alpha}) + (\widehat{rz}_t - \widehat{r}_t)$, and by using the definition of the TOT (Equation 23) that should hold $\forall t$, we deduce (25). This relation highlights that $\widehat{r}_t \neq \widehat{rz}_t$ if $\widehat{s}_t \neq 0$. Interest rate differentials between the two countries is a *sine qua non* condition in a model with a monetary union and incomplete markets, even when there are no risk-premia. This is due to the incompatibility of having at the same time a fixed nominal exchange rate $e_t = 1$ and a peg interest rate for one of the two countries $\widehat{rz}_t = \widehat{r}_t$, $\forall t$.[32] As long as there exists shocks affecting the TOT, interest rate on public debt in the two countries may follow different paths. This result was absent in the specific context of CM where the presence of Arrow-Debreu securities implied an equalization of rate of return no matter the shocks. This condition imply that until financial integration would have not been achieved in the EU, there must be an interest rate differential between EZ countries interest rates.

Public Debt

$$\widehat{b}^r_t = \widehat{r}_t + \widehat{b}^r_{t-1}(\beta^{-1} - \gamma) + \gamma\widehat{r}_{t-1} - \frac{1}{\beta}\widehat{\pi}_{H,t} - \frac{\bar{\alpha}(1-m)}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon^F_t \quad (26)$$

$$\widehat{b}^{r*}_t = \widehat{rz}_t + \widehat{b}^{r*}_{t-1}(\beta^{-1} - \gamma^*) + \gamma^*\widehat{rz}_{t-1} - \frac{1}{\beta}\widehat{\pi}^*_{F,t} + \frac{\bar{\alpha}m}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon^{F*}_t \quad (27)$$

where the dynamics of ε^F_t and ε^{F*}_t are given by (11) (14).

With a monetary policy given by (7) and (8), these equilibrium conditions are equivalent to the ones defined in the equilibrium with CM, except that the parameters that are linked to the TOT are different. The new specification regarding financial market will materialize through different adjustments in TOT. As a consequence, following a shock, the different adjustments in TOT will imply different responses for output and inflation gaps in the two countries. Finally, incomplete markets will imply a different rate of return on public debt in the two countries.

[32] Note that having an interest peg in one of the two countries *de facto* imply that the Taylor principle is not respected in that country.

4.2 The Terms-Of-Trade Dynamic

Proposition 5. *As in the CM framework, the TOT dynamic do not depend on the rules of monetary and/or fiscal policies* (α_π, α_y, γ, *and* γ^*) *as well as on their shocks* (ε_t^M, ε_t^F, *and* ε^{F*})*. The TOT dynamic only depends on fiscal gaps in labor costs and can be solved independently of the rest of the model.*

Proof. The difference between the two Phillips curves (Equations (19) and (20)) leads to:

$$\widehat{\pi}_{F,t}^* - \widehat{\pi}_{H,t} \;=\; \beta\left(\widehat{\pi}_{F,t+1}^* - \widehat{\pi}_{H,t+1}\right) + \kappa(\widehat{y}_t^* - \widehat{y}_t) - \lambda\Upsilon\widehat{s}_t + \lambda(t_t^* - t_t)$$

where $\Upsilon = (\bar{\alpha}m(2\eta + \sigma + 1) + \bar{\alpha}(1-m)(1-\sigma))$. By using the the definition of the TOT 23 and the equilibrium condition that links home and foreign output 24 we deduce the equilibrium dynamic of the TOT:

$$0 \;=\; \widehat{s}_{t-1} - \chi\widehat{s}_t + \beta\widehat{s}_{t+1} + \lambda(t_t^* - t_t) \tag{28}$$

where $\chi = [(1+\beta) + \bar{\alpha}(1-m)(1-\sigma)\lambda + \kappa(2\eta - 1 - \bar{\alpha}\eta + \bar{\alpha} - 2\bar{\alpha}m\eta) + \lambda\bar{\alpha}m(2\eta\sigma + \sigma + 1)]$. The way to solve this recurrence equation is the same than in Proposition 1. □

Corollary 2. *As in the CM framework, If* $t_t, t_t^* = 0$*, then* $\widehat{s}_t = 0$*,* $\forall t$*. We deduce that* $\widehat{y}_t = \widehat{y}_t^*$ *and* $\widehat{\pi}_{H,t} = \widehat{\pi}_{F,t}^*$*,* $\forall t$*.*

Proof. $\widehat{s}_t = 0$ $\forall t$: obvious using equation (28) with $t_t, t_t^* = 0$ $\forall t$ and if the initial condition for s_t is its equilibrium value. Using Equation (24), we deduce that $\widehat{y}_t = \widehat{y}_t^*$ and using Equation (23), we deduce $\widehat{\pi}_{H,t} = \widehat{\pi}_{F,t}^*$. □

Without any shocks on tax rates ($t_t, t_t^* = 0$, $\forall t$) , the TOT dynamic is deterministic with a trivial solution $\widehat{s}_t = 0$, $\forall t$. As with CM, preferences symmetry imply that the equilibrium IS-Phillips curves of one of the two economies are completely redundant. Therefore, the study of the Euro area stability with IM can be reduced to the analysis of the 4 dimensions system $\{\widehat{\pi}_{H,t}, \widehat{y}_t, \widehat{b}_t^r, \widehat{b}_t^{r*}\}$.

From Proposition 5, we deduce that $\widehat{r}_t \neq \widehat{rz}_t$ if $\widehat{s}_t \neq 0$ and it will be the case only when country-specific tax rate shocks (t_t, t_t^*) are different. On the other hand, interest rate in the two region will follow the same path ($\widehat{r}_t = \widehat{rz}_t$, $\forall t$) for shocks that do not affect the TOT, namely, fiscal and monetary shocks ($\varepsilon_t^F, \varepsilon^{F*}, \varepsilon_t^M$).

4.3 The Euro Area Stability

As in the CM framework, the size of the previous equilibrium system can be reduced in order to study its stability. The complete dynamics of the two-country model with incomplete markets and financial autarky can be summarized by the following equations:

$$
\begin{aligned}
\widehat{y}_t &= \mathbb{E}_t[\widehat{y}_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}_{H,t+1}]\right) + \bar{\alpha}(1-m)(\sigma^{-1}-1)\mathbb{E}_t\Delta\widehat{s}_{t+1} \\
\widehat{\pi}_{H,t} &= \beta\mathbb{E}_t\left[\widehat{\pi}_{H,t+1}\right] + \kappa\widehat{y}_t + \lambda\bar{\alpha}(1-m)(1-\sigma)\widehat{s}_t + \lambda t_t \\
\widehat{b}^r_t &= \widehat{r}_t + \widehat{b}^r_{t-1}(\beta^{-1}-\gamma) + \gamma\widehat{r}_{t-1} - \frac{1}{\beta}\widehat{\pi}_{H,t} - \frac{\bar{\alpha}(1-m)}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon^F_t \\
\widehat{r}_t - \widehat{rz}_t &= (2\eta\sigma - \sigma - \bar{\alpha}\eta\sigma + \bar{\alpha} - 1)\Delta\widehat{s}_{t+1} \\
\widehat{b}^{r*}_t &= \widehat{rz}_t + \widehat{b}^{r*}_{t-1}(\beta^{-1}-\gamma^*) + \gamma^*\widehat{rz}_{t-1} - \frac{1}{\beta}\widehat{\pi}_{H,t} + \frac{\bar{\alpha}m-1}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon^{F*}_t \\
\widehat{r}_t &= \alpha_\pi\widehat{\pi}_{H,t} + \alpha_y\widehat{y}_t + (1-m)\Big[\alpha_\pi - (2\eta - 1 - \bar{\alpha}\eta + \bar{\alpha} - 2\bar{\alpha}m\eta)\Big]\widehat{s}_t - \alpha_\pi(1-m)\widehat{s}_{t-1} + \varepsilon^M_t
\end{aligned}
$$

This system is almost equivalent to the system with CM. However, we have an additional condition due to interest rate differentials that as mentioned in the previous section, will exist each time the economies face shocks affecting the TOT. Under the hypothesis that tax rate shocks are equal to zero ($t_t = t^*_t = 0$)[33], the dynamic stability of the two-country model with incomplete markets and financial autarky is exactly the same than with CM. It resumes to the four equations system that is composed of the Philipps curve and the IS curve of the home economy and the two debt dynamics of home and foreign. As a consequence, the conditions regarding the monetary-fiscal interactions that ensure the uniqueness of the saddle path are equivalent in the two framework and are not affected by the hypotheses regarding financial markets.

Proposition 6. *In a two-country model of the EZ with incomplete markets and financial autarky, the policy interactions that ensure the uniqueness of the saddle-path are:*
If the Taylor principle is respected, the stability of the Euro area is ensured if all the countries have a passive fiscal policy, that is, if all the countries implement a fiscal brake satisfying $\gamma, \gamma^* > \beta^{-1} - 1$.
If the Taylor principle is not respected, the stability of the Euro area is ensured if only one country implement an active fiscal policy, all others must respect their commitment to implement a fiscal brake: $\gamma > \beta^{-1} - 1$ & $\gamma^* < \beta^{-1} - 1$, or $\gamma < \beta^{-1} - 1$ & $\gamma^* > \beta^{-1} - 1$.

These conditions are equivalent to the ones with CM (see Proposition 4). Thus, this underlines their robustness.

[33] As in the previous framework with CM, this hypothesis is useful to simplify the exposition and relaxing it do not affect the conditions that ensure the uniqueness of the saddle-path.

5 Zero-Lower-Bound: A Risk Arguing in Favor of a Fiscal Union

When the Euro area is caught in a liquidity trap, it could be not possible for an active monetary policy to coordinate the governments of the Euro area on an equilibrium where they all choose a passive fiscal policy (Proposition 4). This section presents the options for a policy mix in this context by highlighting that these strategies mainly depend on agents' expectations.

5.1 A highly persistent Zero-Lower-Bound

If the ZLB is expected to be highly persistent and can be approximated by a permanent state, a passive monetary policy will be favored in the agents' expectations. Therefore, Proposition 4 shows that one country of the Euro area must leave the Fiscal Stability Treaty in order to stabilize the Euro area. However, which country will do this? If each country considers itself as one among the many SOEs in the Euro area, then none of them will be interested in choosing an active fiscal policy (abandonment of the fiscal brake of the European treaties) because this strategy destabilizes a SOE (see proposition 2). However, in the case of larger economies of the Euro area, such as Germany, France, Italy or Spain, it is possible that these governments do not consider their countries as a SOE, and therefore rely on their partners to ensure the stability of the Euro area, while "enjoying" an active fiscal policy. However, without a coordination system, each country can consider itself as one that can have an active fiscal policy, thereby leading to instability in the Euro area.

Proposition 7. *When the ZLB is perceived to be permanent, a fiscal union stabilizes the Euro area by selecting a unique active fiscal policy at the federal state level.*

Proof. Using proposition 4, we know that if the monetary policy is passive (the Taylor principle cannot be respected), an active fiscal policy must be implemented because one eigenvalue of A is outside the unit circle.

Given that the monetary policy is constrained to be passive at the ZLB, a fiscal union that would take the form of a unification of all public debts, can ensure that $\gamma = \gamma^* \equiv \gamma^u$ with $\gamma^u < \beta^{-1} - 1$. With only one real-debt dynamic (fiscal union), the dynamics would have two eigenvalues inside the unit circle and one outside.

Another way, consist of issuing bonds on behalf of the EU, that will manage them by implementing an active fiscal policy. At the same time, all the states must respect their European commitments by implementing a passive fiscal policy, as in the case where monetary policy is active. $\square$

Proposition 7 shows that a fiscal union is a solution to ensure the stability of the Euro area when deprived of an active monetary policy. The policy of this fiscal union would then be to select the equilibrium where the Fiscal Theory of Price Level (FTPL) would apply. The non-Ricardian properties of this equilibrium would, therefore, allow fiscal transfer policies to revive the Euro area economy. The budgetary tools for regulating economic activities would, therefore, be greatly appreciated during periods when the interest rate is constant, that is, locked in at zero. It seems easier to issue new public debt on behalf of the EU than to unify the debts of the member states. This is the Franco-German proposal (Macron-Merkel), presented by the President of the European Commission on May 27, 2020.

Note that even in an extreme context of permanence of the ZLB constraint, policies other than those leading to an equilibrium where the FTPL applies are possible. Specifically, it is possible to envisage rules inducing fiscal devaluations in response to the business cycle and then allowing to stabilize the Euro area around a "Ricardian" equilibrium. On this equilibrium, the fiscal policy of each state must remain passive (a fiscal brake exists) as each of them is committed to it by signing the European treaties. In this equilibrium, employment subsidies in each state depend on economic activities (inflation and output gaps), and then induce fluctuations in the real exchange rate. This policy rule replaces a stabilization by the nominal interest rate.[34] This selection of a stable equilibrium via adjustments to the TOT is a specificity of open economies that opted for a monetary union.

5.2 Transitory Zero-Lower-Bound

The assumption of permanence of the ZLB constraint is certainly too extreme. Nonetheless, it allows us to see how essential anticipation is for the design of a policy mix: by considering that the ZLB constraint will be permanent, then the stability of the economy

[34] Appendix H creates restrictions on the rules such that $t_t = \nu_y \widehat{y}_t + \nu_\pi \widehat{\pi}_{H,t}$, allowing to have stable dynamics of inflation and the output gap in the absence of the Taylor rule (passive monetary policy or ZLB). Given this stability in the inflation-output gap dynamics, the fiscal policy must then be passive to ensure the dynamic stability of the Euro area.

can only be assessed around this stationary state where the monetary policy is doomed to be silent. In a more realistic case where the ZLB period is finite in expectation, the dynamics of the Euro area depend on agents' expectations of the monetary policy that the ECB will likely lead, once the economy has left the ZLB. To model a ZLB constraint that is only transitory, a stochastic ending of this regime à la Christiano, Eichenbaum, and Rebelo (2011) is considered.

Proposition 8. *If the ZLB is only a transitory period, what matters is the expected monetary policy at the end of the liquidity trap:*

- *If the monetary policy is expected to be active after the period when the ZLB is binding, then* ***all*** *governments must choose a passive fiscal policy. The fiscal union is not necessary to coordinate fiscal policies to achieve stability.*
- *If the monetary policy is expected to be passive after the period when the ZLB is binding, then one of the government must choose an active fiscal policy, or a common public debt must ensure this stabilizing task.*

Proof. See Appendix I. □

A fiscal union that takes the form of bond issues on behalf of the EU makes it possible to guarantee the stability of the Euro area whatever maybe the expectations of agents. This will be necessary only if the agents expect that an active monetary policy will no longer be possible in the future, even when the economy will leave the ZLB regime. When the agents expect that an active monetary policy will revive, a fiscal union is not necessary; however, cannot destabilize the dynamics if the European council adopts a passive fiscal policy, as all the member states. Therefore, a fiscal union can be viewed as a guarantee, ensuring, whatever maybe agents' expectations and the constraints on monetary policy, the non-explosiveness of the Euro area dynamics.

6 Discussion

Our paper relates to several strands of the literature.

First, it offers complementary arguments in favor of a fiscal union to those presented in Farhi and Werning (2017) and Berger, Dell'Ariccia, and Obstfeld (2018). These authors show that the loss of adjustment margin induced by the fixed exchange rate of a currency union, can be compensated by transfers between countries, or organized by a fiscal union. They also show that a fiscal union can discipline moral hazard problems induced by this more efficient risk sharing. Hence, such studies focus on the higher ability of a fiscal union to stabilize asymmetric shocks. We complement these arguments by highlighting the ability of a fiscal union to guarantee the dynamic stability of the area, irrespective of the nature of shocks. This is shown by considering explicitly the public debt dynamics of each country of the area, a dimension too often overlooked in business cycle analysis.

Second, our analysis provides answers to the stability problems of the Euro area raised by Woodford (1996) and Sims (1999):[35]

> "*Even if one government is fiscally responsible and keeps its real primary deficit at some sustainable constant level, variations in the budget deficit of the other government will result in price level instability for that government as well. Thus, there is a clear reason for a government concerned to maintain stable prices to care about fiscal policies of the other governments with which it shares a common currency.*" Woodford (1996), p. 30

> "*What is the weakness of the EMU system in other circumstances—the fiscal free rider problem—would work toward resolving the difficulty. Even one country that is sufficiently fiscally expansive, despite the Maastricht rules, could undo the liquidity trap, with the resultant reversal of deflation benefiting all members of the EMU. On the other hand, if the logical foundations of the need for fiscal coordination are not understood, the need to break the Maastricht rules in this situation could undermine adherence to them more generally.*" Sims, (1999), p. 425

Following these remarks, we show that the construction of European institutions cannot be founded on a single vision of price stability, based on the effectiveness of Taylor's principle. If, the policy mix of the Euro area is open to a wider range of practices than the conventional ones, exposed in Gali's (2008) textbook, the FTPL, presented by Leeper (1991), Sims (1994), Woodford (1998), and Cochrane (2001), can help to design new and more robust rules.[36] In this paper, we show that a fiscal union is not necessary when the monetary policy can be active. In this case, no country has the incentive to

[35]See also Dornbusch (1996) and Feldstein (1997) for criticisms of the European institutions.

[36]As emphasized by Woodford (1998), the study by D'Autume and Michel (1987) is important in the literature on the fiscal theory of the price level: "*It thus appears to be usually an admissible policy for the government to distribute to the public subsidies financed by an ever-increasing monetary debt. The*

deviate from the implementation of the fiscal brake suggested by the Treaties. But a fiscal union can be a necessity when monetary policy cannot be active. In this case where the FTPL applies, the Euro area's inflation is determined by the nominal public debt dynamic of one country of the area that must implement an active fiscal policy, whereas the others must respect their fiscal commitments (passive fiscal policy by implementing a fiscal brake). The degree of freedom concerning the choice of the country that must have an active fiscal policy, advocates in favor of a fiscal union that trivially solves this coordination problem. Therefore, our paper complements Bergin (2000) who analyzed the implications of the FTPL in an endowment open economy. Bergin (2000) shows that if an unifying central bank (the ECB) controls the issue of new money through open market bond purchases, and transfers its interest income back to the national governments via lump sum transfers, then a unique equilibrium can exist when monetary policy follows an interest rate peg (a passive monetary policy) only if one country's fiscal policy is passive while the other country's fiscal policy is active. Our results generalize Bergin's ones by showing that this policy mix can be one solution for the Euro area stability, among all the other combinations presented in our paper, in particular to issue bonds on behalf of the EU.[37]

Third, since the 2008 global financial crisis and for a period that will last certainly for another few years after the COVID-19 crisis, we also contribute to the analysis of the Euro area's policy mix under the ZLB constraint, as it has already been investigated by Farhi and Werning (2016). These authors focus on the fiscal multiplier size during a liquidity trap period and highlight the limits of conventional stabilizers in this context.[38] Our study suggests that other fiscal multipliers can be evaluated in the context where individuals expect that monetary policy will never be active in the future. But, beyond the size of the multipliers, our results suggest that issuing bonds on the financial markets on behalf of the EU could also reinforce the dynamic stability of the Euro area, and thus ensure the robustness of the recovery measures.

Fourth, the sequence of the Ukrainian crisis after that of Covid-19 has pushed up the debts

reason, of course, is that the government does not pay any interest on this special kind of debt" (p. 1351). "*If the government freely chooses the paths of its instruments without caring about its intertemporal budget constraint, the choice of the initial price level must ensure that [...] the real value of initial debt are such that the government budget constraint is satisfied ex post*" (p. 1363).

[37]Dupor (2000) also analyzes the implications of the FTPL in the framework of an open economy, but in flexible exchange rates system, which does not apply directly to the current European experience.

[38]Indeed, exchange rate adjustments greatly reduce the impact of conventional stimuli, as the fiscal multipliers are less than one in an open economy in a currency union and greater than one in the context of a closed economy with a liquidity trap.

of European countries well beyond what the treaties authorize, leading to the temporary suspension of EU budget rules. This new context has carried out to reform treaties in 2023.

> *"For each Member State with a government deficit above 3% of GDP or public debt above 60% of GDP, the Commission will issue a country-specific "technical trajectory". This trajectory will seek to ensure that debt is put on a plausibly downward path or stays at prudent levels, and that the deficit remains or is brought and maintained below 3% of GDP in the medium term.... The ratio of public debt to GDP will have to be lower at the end of the period covered by the plan than at the start of that period; and a minimum fiscal adjustment of 0.5% of GDP per year as a benchmark will have to be implemented so long as the deficit remains above 3% of GDP. "* European Commission (2023)

Therefore, new rules keep the targets of 3% deficit, 60% debt, but replace the "1/20th rule" by the "0.5% rule" accompanied by the commitment of "lower at the end of the plan's period" Debt-to-GDP ratio. This could be interpreted as the non-uniqueness of γ in the future, but a γ country-specific (the case analyzed in section 3.4.2). This flexibility of γ could also manage its state-dependency with respect to its threshold insuring public debts sustainability.[39]

> *"The green and digital transitions, the strengthening of economic and social resilience and the need to bolster Europe's security capacity will require large and sustained public investment in the years to come.... The positive interaction between reforms (of fiscal rules) and investment is already showing its benefits under NextGenerationEU's **Recovery and Resilience Facility**."* European Commission (2023)

where "Recovery and Resilience Facility" plan allows the Commission to raise funds for investments in order to address European challenges, thus validating the principle of a common debt managed by the European Commission, also advocated in this paper.

But all these analyzes are limited to studying the stability of equilibria in the context of models with representative agents. Kaplan et al (2023) have analyzed how the interactions between monetary and fiscal policies can be affected by the presence of heterogeneous

[39] As explained in section 2, the γ-threshold depends on $\{r, \pi, g\}$ which are stochastic variables. See Blanchard, Leandro and Zettelmeyer (2021) for a detailed discussion on the "stochastic" nature of public debt management.

agents in a HANK model. They shed light on the fact that some policy-mix equilibria, unstable in a representative agent model, would be consistent with the uniqueness of the saddle-path when agents are heterogeneous. In that perspective, it seems paramount for us to extend our paper's analyses to a case where agents are heterogeneous as in Kaplan et al (2023) but with the specificities of the Euro Area, in particular the multiplicity of the fiscal policies.

7 Conclusion

In this paper, we show that, in a monetary union, the dynamic stability of open economies, which are all in charge of their public debt, depends on the combination of the choice of fiscal and monetary policy rules. If an active monetary policy can be implemented, or if agents expect its implementation after experiencing a period of a liquidity trap situation, then we show that each country will choose to have a passive fiscal policy (debt brake). Indeed, if the central bank is able to ensure the dynamic stability of the inflation and output gap by applying the Taylor principle, then each government will voluntarily choose to implement a fiscal brake so that its debt does not explode. The fiscal union is not rationally necessary in this environment, with all governments choosing the same option, which is to implement a fiscal brake already present in the European treaties.

Conversely, when monetary policy is constrained, as it has been the case for the past few years, it becomes necessary to adapt the current EU institutions. Thus, we show that a fiscal union, that would take the form of bond issues on behalf of the EU, and thus define a public debt common to the countries of the Euro area, makes it possible to overcome the coordination problem induced by the need to manage differently a single public debt among those of the area. At the ZLB, it becomes necessary for an actor to have an active fiscal policy to guarantee the stability of inflation on a balance where the fiscal theory of the price level applies. This actor could be the European Council, as it will be the case for the management of the "Next Generation EU" initiative financed by a common debt.

Consequently, these results provide the basis for the new European budgetary rules (European Commission (2023)): a deficit reduction plan must be put in place when the public debt becomes excessive, and the issuance of a common debts can allow better coordination.

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Appendix

A From Structural to Reduced Form Parameters

Parameter		Interpretation
β		Subjective discount factor
σ		Intertemporal elasticity of substitution for consumption
φ		Intertemporal elasticity of substitution for employment
$\overline{\alpha}$		Home Bias
m		Size of the Home country
α	$\overline{\alpha}(1-m)$	Trade openness in Home country
α^*	$\overline{\alpha}m$	Trade openness in the Foreign country
η		Elasticity of substitution between Home and Foreign baskets
ε		Elasticity of substitution between Home (Foreign) goods
ρ_a		Persistence of technological shocks
θ		Probability to not reset price for a firm
γ, γ^*		Fiscal brake in Home (Foreign) country
ρ, ρ^*		Persistence of fiscal transfer in Home (Foreign) country
α_π		Elasticity of the interest rate to inflation (Taylor rule)
α_y		Elasticity of the interest rate to output gap (Taylor rule)
Parameters for the reduced form		
λ	$\frac{1-\theta}{\theta}(1-\theta\beta)$	Elasticity of the inflation to marginal costs
κ	$\lambda(\sigma+\varphi)$	Elasticity of the inflation to output gap in a closed economy
ω	$1-\alpha^*$ $+\alpha(2-\overline{\alpha})(\sigma\eta-1)$	Weight of the Home country
ω^*	α^* $+\alpha^*(2-\overline{\alpha})(\sigma\eta-1)$	Weight of the Foreign country

Table 1: Model parameters

B Individual Behaviors and Equilibrium

B.1 Households

The quantities and prices in country F are denoted by an asterisk and those in country H are without asterisks. The representative households' preferences are given by

$$\mathcal{U}^j = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} (C_t^j)^{1-\sigma} - \frac{1}{1+\varphi} (N_t^j)^{1+\varphi} \right) \right] \tag{29}$$

$$\mathcal{U}^{j*} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} (C_t^{j*})^{1-\sigma} - \frac{1}{1+\varphi} (N_t^{j*})^{1+\varphi} \right) \right], \tag{30}$$

where $\beta \in (0,1)$ is the subjective discount factor. $N_t = \int_0^m N_t^j dj = mN_t^j$ and $N_t^* = \int_m^1 N_t^{j*}(i)di = (1-m)N_t^{j*}$ are the total hours worked in countries H and F, respectively. C_t^j and C_t^{j*} denote the consumption index of a household in countries H and F, and are defined as follows:

$$C_t^j = \left((1-\alpha)^{-\frac{1}{\eta}} (C_{H,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{-\frac{1}{\eta}} (C_{F,t}^j)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \tag{31}$$

$$C_t^{j*} = \left((1-\alpha^*)^{-\frac{1}{\eta}} (C_{F,t}^{j*})^{\frac{\eta-1}{\eta}} + (\alpha^*)^{-\frac{1}{\eta}} (C_{H,t}^{j*})^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}, \tag{32}$$

where the index of the consumption of goods produced in country H, and the index of the consumption of goods produced in country F are

$$C_{H,t}^j = \left(\left(\frac{1}{m} \right)^{\frac{1}{\varepsilon}} \int_0^m C_t^j(h)^{\frac{\varepsilon-1}{\varepsilon}} dh \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{33}$$

$$C_{F,t}^{j*} = \left(\left(\frac{1}{1-m} \right)^{\frac{1}{\varepsilon}} \int_m^1 C_t^{j*}(f)^{\frac{\varepsilon-1}{\varepsilon}} df \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{34}$$

For simplicity, assume that the number of goods produced in each "country" is equal to the number of people living in each "country" (one skill-one good). The elasticity of substitution between the differentiated goods produced in each country satisfies $\varepsilon > 1$. The trade openness of each "country" is measured by $\alpha, \alpha^* \in [0,1]$. The elasticity of substitution between domestic and foreign goods from the viewpoint of the domestic consumer is given by $\eta \in (0,1)$.

The maximizations of (29) and (30) are subject to sequences of budget constraints of the following form:

$$
\begin{aligned}
& \int_0^m P_t(h)C_t^j(h)dh + \int_0^m P_t(f)C_t^j(f)df + \mathbb{E}_t[Q_{t,t+1}A_t^j] + \frac{1}{R_t}B_t^j \\
= \; & A_{t-1}^j + B_{t-1}^j + W_t N_t^j + T_t^j \qquad (35) \\
& \int_m^1 P_t^*(h)C_t^{j*}(h)dh + \int_m^1 P_t^*(f)C_t^{j*}(f)df + \mathbb{E}_t[Q_{t,t+1}A_t^{j*}] + \frac{1}{R_t}B_t^{j*} \\
= \; & A_{t-1}^{j*} + B_{t-1}^{j*} + W_t^* N_t^* + T_t^{j*}, \qquad (36)
\end{aligned}
$$

where $P_t(h)$ and $P_t^*(f)$ are the prices of goods produced in country H and country F, respectively. A_t denotes the nominal payoff during period $t+1$ of the portfolio (Arrow securities) bought during period t and $Q_{t,t+1}$ the prices of these contingent assets (i.e., the price in t to have 1 € in period $t+1$). B_t denotes the nominal payoff during period t of the government bonds bought during period $t-1$ and $1/R_t$ the nominal price at t of an asset that pays 1 € during period $t+1$. The asset price ($Q_{t,t+1}$) and bond price $1/R_t$ are the same in both countries because all exchanges take place in the Euro area. W_t and T_t are the nominal wage and lump-sum transfers (taxes)[40], respectively. The transfers include the sum of dividends redistributed by the firm to households and the lump-sum tax net of government transfers.

Within each category of goods, the optimal choices are given by

$$
C_t^j(h) = \frac{1}{m}\left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t}^j \;, \qquad C_t^j(f) = \frac{1}{1-m}\left(\frac{P_t(f)}{P_{F,t}}\right)^{-\varepsilon} C_{F,t}^j \qquad (37)
$$

$$
C_t^{j*}(h) = \frac{1}{m}\left(\frac{P_t^*(h)}{P_{H,t}^*}\right)^{-\varepsilon} C_{H,t}^{j*} \;, \qquad C_t^{j*}(f) = \frac{1}{1-m}\left(\frac{P_t^*(f)}{P_{F,t}^*}\right)^{-\varepsilon} C_{F,t}^{j*}, \qquad (38)
$$

where $P_{H,t} = \left(\frac{1}{m}\int_0^m P_t(h)^{1-\varepsilon}dh\right)^{\frac{1}{1-\varepsilon}}$ and $P_{F,t}^* = \left(\frac{1}{1-m}\int_m^1 P_t(f)^{1-\varepsilon}df\right)^{\frac{1}{1-\varepsilon}}$ denote the PPI. The price indices of imported goods, $P_{H,t}^*$ and $P_{F,t}$, are defined analogously to $P_{H,t}$ and $P_{F,t}^*$. The optimal allocations of expenditure between domestic and imported goods are

$$
C_{H,t}^j = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t^j \quad , \qquad C_{F,t}^j = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t^j \qquad (39)
$$

$$
C_{H,t}^{j*} = \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*} \quad , \qquad C_{F,t}^{j*} = (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^{j*}, \qquad (40)
$$

where the CPIs are given by

$$
\begin{aligned}
P_t &= \left((1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}\right)^{\frac{1}{1-\eta}} \qquad (41) \\
P_t^* &= \left((1-\alpha^*)(P_{F,t}^*)^{1-\eta} + \alpha^*(P_{H,t}^*)^{1-\eta}\right)^{\frac{1}{1-\eta}} \qquad (42)
\end{aligned}
$$

[40] T_t comprises lump-sum transfers and taxes from the government and profits from the firms.

Equations (37) and (38) imply that the total expenditure in country H is the sum of $\int_0^m P_t(h)C_t^j(h)dh = P_{H,t}C_{H,t}^j$ and $\int_0^m P_t(f)C_t^j(f)df = P_{F,t}C_{F,t}^j$. Similarly, the total expenditure in country F is the sum of $\int_m^1 P_t^*(h)C_t^{j*}(h)dh = P_{H,t}^*C_{H,t}^{j*}$ and $\int_m^1 P_t^*(f)C_t^{j*}(f)df = P_{F,t}^*C_{F,t}^{j*}$. Then, from Equations (39) and (40), the following relations hold: $P_{H,t}C_{H,t}^j + P_{F,t}C_{F,t}^j = P_tC_t^j$ and $P_{H,t}^*C_{H,t}^{j*} + P_{F,t}^*C_{F,t}^{j*} = P_t^*C_t^{j*}$. These expressions can be used to rewrite Equations (35) and (36) as:

$$\begin{aligned} P_tC_t^j + \mathbb{E}_t[Q_{t,t+1}A_t^j] + \frac{1}{R_t}B_t^j &= A_{t-1}^j + B_{t-1}^j + W_tN_t^j + T_t && (43) \\ P_t^*C_t^{j*} + \mathbb{E}_t[Q_{t,t+1}A_t^{j*}] + \frac{1}{R_t}B_t^{j*} &= A_{t-1}^{j*} + B_{t-1}^{j*} + W_t^*N_t^{j*} + T_t^{j*} && (44) \end{aligned}$$

Using the definition of the aggregates $C_t = mC_t^j$, $C_t^* = (1-m)C_t^{j*}$, $N_t = mN_t^j$, and $N_t^* = (1-m)N_t^{j*}$, the first-order conditions of the maximizations of (29) subject to (43), and the maximizations of (30) subject to (94) are:

$$\begin{aligned} \beta\frac{C_{t+1}^{-\sigma}P_t}{C_t^{-\sigma}P_{t+1}} &= Q(z_{t+1}|z_t) \quad \forall z && (45) \\ \beta\mathbb{E}_t\left[\frac{C_{t+1}^{-\sigma}P_t}{C_t^{-\sigma}P_{t+1}}\right] &= \frac{1}{R_t} && (46) \\ m^{-(\sigma+\varphi)}C_t^\sigma N_t^\varphi &= \frac{W_t}{P_t} && (47) \\ \beta\frac{(C_{t+1}^*)^{-\sigma}P_t^*}{(C_t^*)^{-\sigma}P_{t+1}^*} &= Q(z_{t+1}|z_t) \quad \forall z && (48) \\ \beta\mathbb{E}_t\left[\frac{(C_{t+1}^*)^{-\sigma}P_t^*}{(C_t^*)^{-\sigma}P_{t+1}^*}\right] &= \frac{1}{R_t} && (49) \\ (1-m)^{-(\sigma+\varphi)}(C_t^*)^\sigma(N_t^*)^\varphi &= \frac{W_t^*}{P_t^*}, && (50) \end{aligned}$$

where $Q_{t,t+1}$ is the vector of $Q(z_{t+1}|z_t)$. Equations (45) and (48) imply that:

$$\begin{aligned} \beta\mathbb{E}_t\left[\frac{C_{t+1}^{-\sigma}P_t}{C_t^{-\sigma}P_{t+1}}\right] &= \mathbb{E}_t\left[Q(z_{t+1}|z_t)\right] \equiv Q_t && (51) \\ \beta\mathbb{E}_t\left[\frac{(C_{t+1}^*)^{-\sigma}P_t^*}{(C_t^*)^{-\sigma}P_{t+1}^*}\right] &= \mathbb{E}_t\left[Q(z_{t+1}|z_t)\right] \equiv Q_t. && (52) \end{aligned}$$

The short-term nominal interest rate, R_t, which is also the ECB's policy instrument, is linked to the nominal bond price, namely $1/R_t$. The no-arbitrage condition leads to $1/R_t = Q_t$.

Log-linearizing around the steady state for the symmetric case (i.e., where $\overline{\alpha} = \overline{\alpha}^*$), Equations (51) and (52), and using the expression for R_t, we find that Home and Foreign aggregate consumptions are:

$$\widehat{c}_t = \mathbb{E}_t[\widehat{c}_{t+1}] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\pi_{t+1}] + \log(\beta)\right) \quad (53)$$

$$\widehat{c}^*_t = \mathbb{E}_t[\widehat{c}^*_{t+1}] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\pi^*_{t+1}] + \log(\beta)\right), \quad (54)$$

where $\widehat{x}_t = \log(X_t/X)$, $\forall x_t$ with X the steady-state value of X_t and $x_t = \log(X_t)$. Therefore, $\pi_t = p_t - p_{t-1}$ ($\pi^*_t = p^*_t - p^*_{t-1}$) is the Home (Foreign) CPI inflation.

B.2 PPI and CPI inflation rates, real exchange rate, TOT, and international risk sharing

To determine the link between CPI and PPI inflation rates, we consider the price of goods produced in country F in terms of the price of goods produced in country H. The TOT are defined by $\mathcal{S}_t = \frac{P_{F,t}}{P_{H,t}}$. The log-linearized CPI inflation rate is

$$\widehat{\pi}_t = \widehat{\pi}_{H,t} + \overline{\alpha}(1-m)(\widehat{s}_t - \widehat{s}_{t-1}) \quad ; \quad \widehat{\pi}^*_t = \widehat{\pi}^*_{F,t} - \overline{\alpha}m(\widehat{s}_t - \widehat{s}_{t-1}), \quad (55)$$

where $\widehat{\pi}_{H,t} = \widehat{p}_{H,t} - \widehat{p}_{H,t-1}$ and $\widehat{\pi}^*_{F,t} = \widehat{p}^*_{F,t} - \widehat{p}^*_{F,t-1}$ denote the PPI inflation in countries H and F, respectively. Equation (55) shows that CPI inflation depends on PPI inflation and changes in the TOT. The effect of a change in the Home country's TOT on the gap between the Home CPI and PPI increases with the weight of the imported (Foreign) goods in Home households' preferences, given by $\overline{\alpha}(1-m) = \alpha$, which decreases with the relative size (m) of country H and with the degree of home bias, inversely related to $\overline{\alpha}$. Therefore, in the specific case of a small country H, when m is close to 0, $\widehat{p}^*_t = \widehat{p}^*_{F,t}$ ($\widehat{\pi}^*_t = \widehat{\pi}^*_{F,t}$); **however**, we maintain a gap between the CPI and PPI in country H ($\widehat{\pi}_t = \widehat{\pi}_{H,t} + \overline{\alpha}(\widehat{s}_t - \widehat{s}_{t-1})$ because $\alpha = \overline{\alpha}(1-m)$ and $m \to 0$).

We assume that the law of one price (LOP) always holds for individual goods, both for import and export prices. This implies that $P_t(f) = \mathcal{E}_t P^*_t(f)$, $\forall f$ and $P_t(h) = \mathcal{E}_t P^*_t(h)$, $\forall h$, where $\mathcal{E}_t$ denotes the price of country F's currency in terms of country H's currency. Substituting these conditions into the price indices for the two countries yields $P_{F,t} = \mathcal{E}_t P^*_{F,t}$ and $P_{H,t} = \mathcal{E}_t P^*_{H,t}$. In the Euro area, we have $\mathcal{E}_t = 1$, therefore, $P_{F,t} = P^*_{F,t}$ and $P_{H,t} = P^*_{H,t}$. However, due to the home bias introduced in the preferences over consumption bundles ($\alpha \neq \frac{1}{2}$), purchasing power parity (PPP) does not hold in general. Hence, $P_t \neq \mathcal{E}_t P^*_t$.

The real exchange rate, denoted by $\mathcal{Q}_t = \frac{\mathcal{E}_t P^*_t}{P_t}$, reduces to $\mathcal{Q}_t = \frac{P^*_t}{P_t}$ in the Euro area, where $\mathcal{E}_t = 1$. Hence, we have[41]:

$$\mathcal{Q}_t = \frac{\left((1-\alpha^*)\mathcal{S}_t^{1-\eta} + \alpha^*\right)^{\frac{1}{1-\eta}}}{\left((1-\alpha) + \alpha\mathcal{S}_t^{1-\eta}\right)^{\frac{1}{1-\eta}}} \;\Rightarrow\; \widehat{q}_t = \widehat{s}_t(1-\overline{\alpha}), \tag{56}$$

where we use the properties that for any m, the steady-state solution is such that $\mathcal{S} = 1$, when preferences are identical across countries $\overline{\alpha} = \overline{\alpha}^*$. Hence, home bias is the only source of the violation of PPP.[42] The real exchange rate's volatility increases with the degree of Home bias and volatility in the TOT. Although the LOP holds for all goods individually, the real exchange rate $\widehat{q}_t$ is directly related to the TOT $\widehat{s}_t$, which fluctuates over time in response to shocks in both countries, because the Foreign (Home) preference places a higher weight on Foreign (Home) goods than the Home (Foreign) preference does.

Turning to the financial market and uncovered interest parity (UIP) conditions, we have in a monetary union (the nominal exchange rate is equal to one, $\mathcal{E}_t = 1$):

$$\left.\begin{array}{rcl} \beta\frac{(C_{t+1}^j)^{-\sigma}}{(C_t^j)^{-\sigma}} & = & \frac{P_{t+1}}{P_t}Q(z_{t+1}|z_t) \\ \beta\frac{(C_{t+1}^{j*})^{-\sigma}}{(C_t^{j*}(i))^{-\sigma}} & = & \frac{\mathcal{Q}_{t+1}}{\mathcal{Q}_t}\frac{P_{t+1}}{P_t}Q(z_{t+1}|z_t) \end{array}\right\} \Rightarrow\; C_t = \frac{m}{1-m}\vartheta\mathcal{Q}_t^{\frac{1}{\sigma}}C_t^*, \tag{57}$$

where $\vartheta = \mathcal{Q}_t^{-\frac{1}{\sigma}}\frac{C_0}{C_0^*}$ and with $C_t = \int_0^m C_t^j dj = mC_t$ and $C_t^{j*} = \int_m^1 C_t^{j*}dj = (1-m)C_t^*$. Hence, home bias allows for a variable gap between Home and Foreign households' consumption growth rates, even if the international financial market structure is complete. The ratio between the Home and Foreign aggregate consumption levels collapses to zero as the Home country becomes a small economy.

Following a general procedure in the literature, we assume the same initial conditions for Home and Foreign households, so that $\vartheta = 1$. Log-linearizing (57) around the steady state for the symmetric case ($\overline{\alpha} = \overline{\alpha}^*$), we obtain:

$$\widehat{c}_t = \frac{1}{\sigma}\widehat{q}_t + \widehat{c}_t^* \tag{58}$$

[41] We use the definitions of price indexes:

$$\begin{array}{rcl} P_t & = & \left((1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}\right)^{\frac{1}{1-\eta}} \;\Rightarrow\; \frac{P_t}{P_{H,t}} = \left((1-\alpha) + \alpha\mathcal{S}_t^{1-\eta}\right)^{\frac{1}{1-\eta}} \\ P_t^* & = & \left((1-\alpha^*)P_{F,t}^{1-\eta} + \alpha^* P_{H,t}^{1-\eta}\right)^{\frac{1}{1-\eta}} \;\Rightarrow\; \frac{P_t^*}{P_{H,t}} = \left((1-\alpha^*)\mathcal{S}_t^{1-\eta} + \alpha^*\right)^{\frac{1}{1-\eta}} \end{array}$$

With $\mathcal{Q}_t = \frac{P_t^*}{P_t}$, this leads to

$$\widehat{q}_t = (1-\alpha^*-\alpha)\,\widehat{s}_t = (1-\overline{\alpha}m - \overline{\alpha}(1-m))\,\widehat{s}_t = (1-\overline{\alpha})\,\widehat{s}_t$$

[42] $\widehat{q}_t = 0$ in every period when there is no home bias (i.e., when $\overline{\alpha} = 1$).

showing that risk sharing implies a perfect international correlation between consumption levels.

B.3 Firms

There is a continuum of firms indexed by $i \in [0,1]$. Firms on the interval $[0,m)$ are located in country H, while firms on the interval $[m,1]$ are located in country F. Each monopolistic competitive firm sets the relative price of its differentiated goods, faced an isoelastic and downward-sloping demand curve and is subject to a technological constraint. Firms use only a homogeneous type of labor for production and there is no investment. The labor market is competitive.

B.3.1 Cost minimization

All Home firms operate with an identical constant returns-to-scale technology:

$$Y_t(i) = A_t L_t(i),$$

where $Y_t(i)$ is the Home firm i's output and $L_t(i)$ is the Home firm i's labor demand. A_t is the Home's total factor productivity shifter. Its log-linearized counterpart follows an $AR(1)$ process:

$$\widehat{a}_t = \log(A_t/A) = \rho \widehat{a}_{t-1} + \epsilon_{a,t},$$

where $0 < \rho < 1$ and $\epsilon_{a,t}$ are i.i.d Gaussian shocks. All Foreign firms operate with a similar technology, but these processes may face different persistence ($\rho \neq \rho^*$). The shocks $\epsilon_{a,t}$ and $\epsilon^*_{a,t}$ may be correlated. Technology constraints imply that the Home and Foreign ith firms' labor demands are given respectively by:

$$\begin{aligned} L_t(i) &= \frac{Y_t(i)}{A_t} \\ L^*_t(i) &= \frac{Y^*_t(i)}{A^*_t} \end{aligned}$$

so that the Home and Foreign nominal marginal costs are given by $MC^n_t = (1+t_t)W_t/A_t$ and $MC^{n*}_t = (1+t^*_t)W^*_t/A^*_t$, while the Home and Foreign real marginal costs are defined as:

$$MC_t = \frac{MC^n_t}{P_{H,t}} = \frac{(1+t_t)W_t}{P_{H,t}A_t} \tag{59}$$

$$MC^*_t = \frac{MC^{n*}_t}{P^*_{F,t}} = \frac{(1+t^*_t)W^*_t}{P^*_{F,t}A^*_t}, \tag{60}$$

where t_t and t^*_t are payroll tax rates in Home and Foreign economies.

B.3.2 Price setting

Firms set prices in a staggered way, as in Calvo (1983): every period, a measure of $1-\theta$ randomly selected firms set a new price, with an individual firm's probability of readjusting at each period being independent of the time elapsed since it last reset its price. A Home firm i adjusting its price in period t sets a new price $\overline{P}_{H,t}$ to maximize the present value of its stream of expected future profits:

$$\max_{\overline{P}_{H,t}(i)} \sum_{\tau=0}^{\infty} \theta^{\tau} \mathbb{E}_t \left[Q_{t,t+\tau} (\overline{P}_{H,t}(i) - MC^n_{t+s}) Y_{t+s}(i) \right]$$

subject to the demand constraint:

$$Y_{t+s}(i) = \left(\frac{\overline{P}_{H,t}}{P_{H,t+s}} \right)^{-\varepsilon} C_{t+s}.$$

A firm cannot decide on its optimal price in each period. To simplify the notation, we omit the index i for the firm. Following Calvo (1983), on each date, the firm receives a signal advising it whether it can revise its price $P_{H,t}$ in an optimal manner. There is a probability θ that the firm cannot revise its price in a given period. When a firm receives a positive signal (with probability $1-\theta$), it chooses price $P_{H,t}$ that maximizes its profit discerning that during the next periods it may be unable to choose its price optimally. Let $\widetilde{\mathcal{V}}_t$ be the value of a firm that receives a positive signal in period t and $\mathcal{V}_t(P_{H,t-1})$ the value of a firm that receives a negative signal. As a firm that receives a negative signal simply follows the *ad hoc* pricing rule $P_{H,t} = P_{H,t-1}$, its value at time t depends only on $P_{H,t-1}$. Denoting $\Pi(\overline{P}_{H,t}) = (\overline{P}_{H,t} - MC^n_t) Y_t$, the value of a firm that receives a positive signal in period t is:

$$\widetilde{\mathcal{V}}_t = \max_{\overline{P}_{H,t}} \left\{ \Pi(\overline{P}_{H,t}) + \beta \mathbb{E}_t \left[Q_{t+1} \left((1-\theta) \widetilde{\mathcal{V}}_{t+1} + \theta \mathcal{V}_{t+1}(\overline{P}_{H,t}) \right) \right] \right\} \tag{61}$$

The value of a firm that cannot re-optimize is:

$$\mathcal{V}_t(P_{H,t-1}) = \Pi(P_{H,t-1}) + \beta \mathbb{E}_t \left[Q_{t+1} \left((1-\theta) \widetilde{\mathcal{V}}_{t+1} + \theta \mathcal{V}_{t+1}(P_{H,t-1}) \right) \right] \tag{62}$$

The first-order and envelope conditions associated with (61)–(62) determine the dynamics of inflation. These are given by:

$$\Pi'_t(\overline{P}_{H,t}) + \beta\theta \mathbb{E}_t \left[Q_{t+1} \mathcal{V}'_{t+1}(\overline{P}_{H,t}) \right] = 0 \tag{63a}$$

$$\mathcal{V}'_t(P_{H,t-1}) = \Pi'_t(P_{H,t-1}) + \beta\theta \mathbb{E}_t \left[Q_{t+1} \mathcal{V}'_{t+1}(P_{H,t-1}) \right] \tag{63b}$$

Iterating forward on these first-order conditions, we can obtain:

$$
\begin{aligned}
\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j Q_{t+j}Y_{t+j}\left(\overline{P}_{H,t}-\frac{\varepsilon}{\varepsilon-1}MC^n_{t+j}\right)\right] &= 0 \\
\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta^j Q_{t+j}Y_{t+j}\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}-\frac{\varepsilon}{\varepsilon-1}MC_{t+j}\frac{P_{H,t+j}}{P_{H,t-1}}\right)\right] &= 0 \qquad (64)
\end{aligned}
$$

Log-linearizing this equation around the zero-inflation steady state and given that $Q_{t+j} = \beta^j$ must hold at the steady state (see (45) and (48)), we obtain:

$$
\widehat{\overline{p}}_{H,t}-\widehat{p}_{t-1} = (1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j\mathbb{E}_t\left[\widetilde{mc}_{t+j}+(\widehat{p}_{t+j}-\widehat{p}_{t-1})\right],
$$

where $\widetilde{mc}_t$ is the deviation of the real marginal cost from its steady-state level. It is equal to $\widetilde{mc}_t = t_t + \widehat{w}_t - \widehat{p}_{H,t} - \widehat{a}_t$.

$$
\widehat{\pi}_{H,t} = \beta\mathbb{E}_t\left[\widehat{\pi}_{H,t+1}\right] + \lambda\widetilde{mc}_t,
$$

where $\lambda = \frac{1-\theta}{\theta}(1-\theta\beta)$.

Proceeding in the same way with the Foreign country, we obtain:

$$
\widehat{\pi}^*_{F,t} = \beta\mathbb{E}_t\left[\widehat{\pi}^*_{F,t+1}\right] + \lambda\widetilde{mc}^*_t,
$$

where $\widetilde{mc}^*_t$ is the deviation of the real marginal cost from its steady-state level. The Home and Foreign real marginal costs under flexible prices are given by $mc_t = mc^*_t = -\log\left(\frac{\varepsilon}{\varepsilon-1}\right)$, where $\log\left(\frac{\varepsilon}{\varepsilon-1}\right)$ is the gross markup (in log).

B.4 Government budget constraints

Each agent j in country H (or F) can only invest in its own country public debts. We assume for simplicity that a lump sum transfer (Tr_t and Tr^*_t) allows each government to redistribute the revenues of the payroll taxes paid only in sticky price economies: this implies that $Tr_t = t_t W_t N_t$ and $Tr^*_t = t^*_t W^*_t N^*_t$.

The budget constraint (after receiving and redistributing the payroll taxes) of the state in the Home country is given by:

$$
\frac{1}{R_t}\frac{B_t}{P_t} + D^r_t = \frac{B_{t-1}}{P_t} \quad\Leftrightarrow\quad \frac{1}{R_t}B^r_t + D^r_t = \frac{1}{\pi_t}B^r_{t-1},
$$

where D^r_t are the real net transfers from the government to households, namely, $D^r_t = Tax_t - Tr_t$, with Tax_t a real lump-sum tax and Tr_t a real lump-sum transfer. We deflate the nominal debt (B_t) by the CPI to define the real debt B_t^r. By the forward iteration of this budgetary constraint, the price level P_t depends on the sequence of the choice of the state ($\{D_{t+i}^r\}_{i=0}^{\infty}$), via the valuation of the debt:

$$\frac{B_{t-1}}{P_t} = \mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \frac{\pi_{t+j}}{R_{t+j-1}} \right) D_{t+i}^r$$

Given an initial condition for the nominal debt B_{t-1}, this equation shows that the general price level depends on the sequence of net spending by the state of the Home country ($\{D_{t+i}^r\}_{i=0}^{\infty}$): this is one of the basis of the price tax theory.

The linearized versions of the budget constraints of each state (Home and Foreign) are given by:

$$\widehat{b}_t^r = \beta^{-1}(\widehat{b}_{t-1}^r - \widehat{\pi}_t) + \widehat{r}_t + (1-\beta^{-1})\widehat{d}_t^r \quad (65)$$

$$\widehat{b}_t^{r*} = \beta^{-1}(\widehat{b}_{t-1}^{r*} - \widehat{\pi}_t^*) + \widehat{r}_t + (1-\beta^{-1})\widehat{d}_t^{r*} \quad (66)$$

As for the Taylor rule, one can model the government's choices using a fiscal rule. For example, we can assume that the surplus of the Home government $\widehat{d}_t$ follows the following rule:

$$D_t^r = D^r + \gamma\left(\frac{1}{R_{t-1}}\frac{B_{t-1}}{P_{t-1}} - \frac{B^r}{R}\right) + \varepsilon_t^F$$

$$\Rightarrow \quad \widehat{d}_t^r = \gamma\frac{1}{\beta^{-1}-1}(\widehat{b}_{t-1}^r - \widehat{r}_{t-1}) + \varepsilon_t^F \quad (67)$$

This last equation represents the fiscal rule. The same rule exists in the Foreign country. Introducing (67) into the budget constraints of the government in the Home (Foreign) country (Equations (65) and (66)), and using the CPI definition 55 we obtain:

$$\widehat{b}_t^r = (\beta^{-1}-\gamma)\widehat{b}_{t-1}^r - \beta^{-1}\widehat{\pi}_{H,t} + \widehat{r}_t + \gamma\widehat{r}_{t-1} - \frac{\bar{\alpha}(1-m)}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon_t^F \quad (68)$$

$$\widehat{b}_t^{r*} = (\beta^{-1}-\gamma^*)\widehat{b}_{t-1}^{r*} - \beta^{-1}\widehat{\pi}_{F,t}^* + \widehat{r}_t + \gamma\widehat{r}_{t-1} + \frac{\bar{\alpha}m}{\beta}\Delta\widehat{s}_t + (1-\beta^{-1})\varepsilon_t^{F*} \quad (69)$$

These budgetary constraints depend on the ECB's policy, that is, directly through the interest rate and indirectly through inflation and TOT.

B.5 Equilibrium

The log-linearization around the steady state of the equilibrium equations of the goods market around the steady state symmetric ($\overline{\alpha} = \overline{\alpha}^*$), using the UIP condition (Equation

58), leads to

$$\widehat{y}_t = \widehat{c}_t + \frac{\omega + \overline{\alpha} - 1}{\sigma}\widehat{s}_t \tag{70}$$

$$\widehat{y}_t^* = \widehat{c}_t^* - \frac{\omega^*}{\sigma}\widehat{s}_t, \tag{71}$$

where $\omega = 1 - \overline{\alpha}m + (1-m)\overline{\alpha}(2-\overline{\alpha})(\sigma\eta - 1) > 0$ and $\omega^* = \overline{\alpha}m + m\overline{\alpha}(2-\overline{\alpha})(\sigma\eta - 1) > 0$. Using Equations (55), (70), and (71), Equations (53) and (54) become:

$$\begin{aligned}
\widehat{y}_t &= \mathbb{E}_t[\widehat{y}_{t+1}] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\widehat{\pi}_{H,t+1}] + \log(\beta)\right) + \frac{1 - \overline{\alpha}m - \omega}{\sigma}\Delta\mathbb{E}_t\widehat{s}_{t+1} \\
\widehat{y}_t^* &= \mathbb{E}_t[\widehat{y}_{t+1}^*] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\widehat{\pi}_{F,t+1}^*] + \log(\beta)\right) - \frac{\overline{\alpha}m - \omega^*}{\sigma}\mathbb{E}_t\Delta\widehat{s}_{t+1}
\end{aligned}$$

Using Equations (58) and (56), we obtain:

$$\widehat{s}_t = \frac{\sigma}{\omega + \omega^*}\left(\widehat{y}_t - \widehat{y}_t^*\right) \tag{72}$$

B.5.1 Flexible price equilibrium

In the flexible price economy, it is assumed that $t_t = t_t^* = 0$. Home and Foreign firms optimize their prices in each period. Hence, these prices are equal to a mark-up over the marginal cost. We find that the Home and Foreign real marginal costs are given by:

$$mc_t = mc_t^* = -\log\left(\frac{\varepsilon}{\varepsilon - 1}\right),$$

where $mc_t = mc_t^n - p_{H,t}$ and $mc_t^* = mc_t^n - p_{F,t}^*$. Hence, we deduce the TOT

$$\overline{s}_t = \frac{1 + \varphi}{1 + \frac{\varphi}{\sigma}(\omega + \omega^*)}(\widehat{a}_t - \widehat{a}_t^*),$$

and the natural level of the Home and Foreign outputs:

$$\begin{aligned}
\overline{y}_t &= \log m - \frac{1}{\varphi + \sigma}\log\left(\frac{\varepsilon}{\varepsilon - 1}\right) + (1 - \Theta)\frac{1 + \varphi}{\varphi + \sigma}\widehat{a}_t + \Theta\frac{1 + \varphi}{\varphi + \sigma}\widehat{a}_t^* \\
\overline{y}_t^* &= \log(1 - m) - \frac{1}{\varphi + \sigma}\log\left(\frac{\varepsilon}{\varepsilon - 1}\right) + \Theta^*\frac{1 + \varphi}{\varphi + \sigma}\widehat{a}_t + (1 - \Theta^*)\frac{1 + \varphi}{\varphi + \sigma}\widehat{a}_t^*,
\end{aligned}$$

with $\Theta = \frac{\sigma(1 - \overline{\alpha}m - \omega)}{\sigma + \varphi(\omega + \omega^*)}$ and $\Theta^* = \frac{\sigma(\overline{\alpha}m - \omega^*)}{\sigma + \varphi(\omega + \omega^*)}$. Using the $AR(1)$ processes for $\widehat{a}_t$ and $\widehat{a}_t{}^*$, we obtain:

$$\mathbb{E}_t\Delta\overline{s}_{t+1} = \frac{1 + \varphi}{1 + \frac{\varphi}{\sigma}(\omega + \omega^*)}[(\rho - 1)\widehat{a}_t - (\rho^* - 1)\widehat{a}_t^*]$$

This leads us to deduce the natural expected interest rate ($r_t^e \equiv \overline{r}_t - \mathbb{E}_t\overline{\pi}_{H,t}$ and $r_t^{e*} \equiv \overline{r}_t - \mathbb{E}_t\overline{\pi}^*_{F,t}$):

$$\begin{aligned} r_t^e &= \frac{(1-\Gamma)\sigma(1+\varphi)(\rho-1)}{\varphi+\sigma}\widehat{a}_t + \frac{\Gamma\sigma(1+\varphi)(\rho^*-1)}{\varphi+\sigma}\widehat{a}_t^* - \log\beta && (73)\\ r_t^{e*} &= \frac{\Gamma^*\sigma(1+\varphi)(\rho-1)}{\varphi+\sigma}\widehat{a}_t + \frac{(1-\Gamma^*)\sigma(1+\varphi)(\rho^*-1)}{\varphi+\sigma}\widehat{a}_t^* - \log\beta, && (74)\end{aligned}$$

where $\Gamma = \frac{\varphi(\overline{\alpha}m+\omega-1)}{\sigma+\varphi(\omega+\omega^*)}$ and $\Gamma^* = \frac{\varphi(\omega-\overline{\alpha}m)}{\sigma+\varphi(\omega+\omega^*)}$.

B.5.2 Equilibrium with nominal rigidities

As in Gali and Monacelli (2005), we assume that the deviations of the dispersion of the Home and Foreign firms' output induced by price rigidities around the steady state symmetric are of second order, so that up to a first-order approximation, we can set them to zero. Therefore, using the labor market equilibrium, we have:

$$\begin{aligned} mc_t &= (\sigma+\varphi)(y_t - \log m) - (\overline{\alpha}m+\omega-1)s_t - (1+\varphi)a_t + t_t\\ mc_t^* &= (\sigma+\varphi)(y_t^* - \log(1-m)) - (\overline{\alpha}m-\omega^*)s_t - (1+\varphi)a_t^* + t_t^*\end{aligned}$$

Assuming that t_t and t_t^* are different from zero only in the sticky price economies, we obtain:

$$\begin{aligned} \widehat{\pi}_{H,t} &= \beta\mathbb{E}_t\left[\widehat{\pi}_{H,t+1}\right] + \kappa\widehat{y}_t - \lambda(\alpha m+\omega-1)\widehat{s}_t + \lambda t_t && (75)\\ \widehat{\pi}^*_{F,t} &= \beta\mathbb{E}_t\left[\widehat{\pi}^*_{F,t+1}\right] + \kappa\widehat{y}_t^* + \lambda(\omega^*-\alpha m)\widehat{s}_t + \lambda t_t^*, && (76)\end{aligned}$$

which are the Home and Foreign New Phillips curves, where $\kappa = \lambda(\varphi+\sigma)$. The IS curves are given by:

$$\begin{aligned} \widehat{y}_t &= \mathbb{E}_t[\widehat{y}_{t+1}] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\widehat{\pi}_{H,t+1}] - \overline{r}_t^e\right) + \frac{1-\overline{\alpha}m-\omega}{\sigma}\mathbb{E}_t\Delta\widehat{s}_{t+1} && (77)\\ \widehat{y}_t^* &= \mathbb{E}_t[\widehat{y}^*_{t+1}] - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t[\widehat{\pi}^*_{F,t+1}] - \overline{r}_t^{e*}\right) + \frac{\omega^*-\overline{\alpha}m}{\sigma}\mathbb{E}_t\Delta\widehat{s}_{t+1} && (78)\end{aligned}$$

Therefore, the equilibrium dynamics of this two-country NK model with asymmetric countries is defined by:

- the New Phillips curves (Equations (75) and (76)),
- the IS curves (Equations (77) and (78)), and
- the natural expected interest rates (Equations (73) and (74)),

- The Taylor rule of the ECB:

$$\widehat{r}_t = \alpha_\pi \widehat{\pi}_t^u + \alpha_y \widehat{y}_t^u + \epsilon_{r,t}, \tag{79}$$

  where $\widehat{\pi}_t^u = m\widehat{\pi}_{H,t} + (1-m)\widehat{\pi}_{F,t}^*$ and $\widehat{y}_t^u = m\widehat{y}_t + (1-m)\widehat{y}_t^*$. The distinction between CPI inflation and PPI inflation, while meaningful at the level of each country, disappears for the ECB. Formally, given that $p_t = p_{H,t} + \overline{\alpha}(1-n)s_t$ and $p_t^* = p_{F,t}^* - \overline{\alpha}ns_t$, we have $p_t^u = np_t + (1-n)p_t^* = np_{H,t} + (1-n)p_{F,t}^*$ (see Equation (55)).

- Gap in the TOT is given by:

$$\widehat{s}_t = \frac{\sigma}{\omega + \omega^*}(\widehat{y}_t - \widehat{y}_t^*) \tag{80}$$

- Public debt of each country given by Equations (65) and (66) and the fiscal rules in Equation (67).

For simplicity, we assume in the following that $\widehat{a}_t = \widehat{a}_t^* = 0$, $\forall t$, leading to $r^e = r^{e*} = -\log(\beta)$. This does not change the properties of the dynamic system. The dynamic system is

$$Z_t = A^{-1}B \ \mathbb{E}_t Z_{t+1} + A^{-1}C \ e_t$$

C Summary of the Model's Equations

This appendix provides the equations used to define the model equilibrium. The individual behaviors allowing us to derive these reduced forms are explained in Appendix B. Appendix A provides a complete description of the model parameters.

C.1 Households

Consumption demand

$$\widehat{c}_t = \mathbb{E}_t[\widehat{c}_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}_{t+1}]\right) \quad (81)$$

$$\widehat{c}^*_t = \mathbb{E}_t[\widehat{c}^*_{t+1}] - \frac{1}{\sigma}\left(\widehat{r}_t - \mathbb{E}_t[\widehat{\pi}^*_{t+1}]\right) \quad (82)$$

Labor supply

$$\sigma\widehat{c}_t + \varphi\widehat{n}_t = \widehat{w}_t - \widehat{p}_t \quad (83)$$

$$\sigma\widehat{c}^*_t + \varphi\widehat{n}^*_t = \widehat{w}^*_t - \widehat{p}^*_t \quad (84)$$

C.2 Prices, exchange rates, and international risk sharing

Terms-of-Trade

$$\mathcal{S}_t = \frac{P_{F,t}}{P_{H,t}} \quad \Rightarrow \quad \widehat{s}_t = \widehat{p}_{F,t} - \widehat{p}_{H,t}$$

Price indexes

$$P_t = \left((1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}\right)^{\frac{1}{1-\eta}} \quad \Rightarrow \quad \frac{P_t}{P_{H,t}} = \left((1-\alpha) + \alpha\mathcal{S}_t^{1-\eta}\right)^{\frac{1}{1-\eta}}$$

$$P^*_t = \left((1-\alpha^*)(P^*_{F,t})^{1-\eta} + \alpha^*(P^*_{H,t})^{1-\eta}\right)^{\frac{1}{1-\eta}} \quad \Rightarrow \quad \frac{P^*_t}{P^*_{H,t}} = \left((1-\alpha^*)\mathcal{S}_t^{1-\eta} + \alpha^*\right)^{\frac{1}{1-\eta}}$$

$$\Rightarrow \begin{cases} \widehat{p}_t - \widehat{p}_{H,t} = \alpha\widehat{s}_t = \overline{\alpha}(1-m)\widehat{s}_t \\ \widehat{p}^*_t - \widehat{p}^*_{H,t} = (1-\alpha^*)\widehat{s}_t = (1-\overline{\alpha}m)\widehat{s}_t \end{cases}$$

Nominal exchange rate

$$P_{F,t} = \mathcal{E}_t P^*_{F,t} \quad \Rightarrow \quad \widehat{p}_{F,t} = \widehat{e}_t + \widehat{p}^*_{F,t}$$

Real exchange rate

$$\mathcal{Q}_t = \frac{\mathcal{E}_t P_t^*}{P_t} \quad \Rightarrow \quad \begin{aligned} \widehat{q}_t &= \widehat{e}_t + \widehat{p}_t^* - \widehat{p}_t \\ \widehat{q}_t &= \widehat{e}_t + \widehat{p}_{H,t}^* + (1-\alpha^*)\widehat{s}_t - \widehat{p}_{H,t} - \alpha\widehat{s}_t \\ \widehat{q}_t &= \widehat{e}_t + (1-\alpha^*-\alpha)\widehat{s}_t + \widehat{p}_{H,t}^* - \widehat{p}_{H,t} \end{aligned}$$

In the Euro Area, we have $\widehat{e}_t = 0$, $\forall t$ and thus:

$$\widehat{q}_t = (1-\alpha^*-\alpha)\,\widehat{s}_t = (1-\overline{\alpha}m - \overline{\alpha}(1-m))\,\widehat{s}_t = (1-\overline{\alpha})\,\widehat{s}_t$$

CPI Inflation

$$\begin{aligned} \widehat{\pi}_t &= \widehat{\pi}_{H,t} + \overline{\alpha}(1-m)(\widehat{s}_t - \widehat{s}_{t-1}) \\ \widehat{\pi}_t^* &= \widehat{\pi}_{F,t}^* - \overline{\alpha}m(\widehat{s}_t - \widehat{s}_{t-1}) \end{aligned}$$

Financial markets

$$\left.\begin{aligned} \beta\frac{(C_{t+1}^j)^\sigma}{(C_t^j)^\sigma} &= \frac{P_{t+1}}{P_t}Q(z_{t+1}|z_t) \\ \beta\frac{(C_{t+1}^{j*})^\sigma}{(C_t^{j*}(i))^\sigma} &= \frac{\mathcal{Q}_{t+1}}{\mathcal{Q}_t}\frac{P_{t+1}}{P_t}Q(z_{t+1}|z_t) \end{aligned}\right\} \Rightarrow \quad C_t = \frac{m}{1-m}\vartheta\mathcal{Q}_t^{\frac{1}{\sigma}}C_t^*, \tag{85}$$

where $\vartheta = \mathcal{Q}_t^{-\frac{1}{\sigma}}\frac{C_0}{C_0^*}$ and with $C_t = \int_0^m C_t^j dj = mC_t^j$ and $C_t^* = \int_m^1 C_t^{j*} dj = (1-m)C_t^{j*}$. This leads to:

$$\widehat{c}_t = \frac{1}{\sigma}\widehat{q}_t + \widehat{c}_t^* \tag{86}$$

C.3 Price-setting rule

The Phillips curves are:

$$\begin{aligned} \widehat{\pi}_{H,t} &= \beta\mathbb{E}_t\left[\widehat{\pi}_{H,t+1}\right] + \kappa\widehat{y}_t - \lambda(\alpha m + \omega - 1)\widehat{s}_t & (87)\\ \widehat{\pi}_{F,t}^* &= \beta\mathbb{E}_t\left[\widehat{\pi}_{F,t+1}^*\right] + \kappa\widehat{y}_t^* + \lambda(\omega^* - \alpha m)\widehat{s}_t, & (88) \end{aligned}$$

which are the Home and Foreign New Phillips curves, where $\kappa = \lambda(\varphi + \sigma)$.

C.4 Resource constraints

$$\begin{aligned} Y_t &= C_{H,t} + C_{H,t}^* = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta} C_t^* \\ Y_t^* &= C_{F,t} + C_{F,t}^* = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t + (1-\alpha^*)\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta} C_t^* \end{aligned}$$

Using (85) $\Leftrightarrow \frac{1-m}{m}\frac{1}{\vartheta}\mathcal{Q}_t^{-\frac{1}{\sigma}}C_t = C_t^*$, we have:

$$Y_t = \left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} + \alpha^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta}\frac{1-m}{m}\frac{1}{\vartheta}\mathcal{Q}_t^{-\frac{1}{\sigma}}\right]C_t,$$

where

$$\begin{aligned}
\mathcal{Q}_t &= \frac{\mathcal{E}_t P_t^*}{P_t} = \frac{\left((1-\alpha^*)\mathcal{S}_t^{1-\eta}+\alpha^*\right)^{\frac{1}{1-\eta}}}{\left((1-\alpha)+\alpha\mathcal{S}_t^{1-\eta}\right)^{\frac{1}{1-\eta}}} \\
\frac{P_{H,t}}{P_t} &= \left((1-\alpha)+\alpha\mathcal{S}_t^{1-\eta}\right)^{-\frac{1}{1-\eta}} \\
\frac{P_{H,t}^*}{P_t^*} &= \left((1-\alpha^*)\mathcal{S}_t^{1-\eta}+\alpha^*\right)^{-\frac{1}{1-\eta}}
\end{aligned}$$

This shows that $Y_t = \mathcal{Y}(\mathcal{S}_t, C_t)$. Log-linearizing these equations lead to:

$$\widehat{y}_t = \widehat{c}_t + \frac{\omega+\overline{\alpha}-1}{\sigma}\widehat{s}_t \tag{89}$$

$$\widehat{y}_t^* = \widehat{c}_t^* - \frac{\omega^*}{\sigma}\widehat{s}_t, \tag{90}$$

where $\omega = 1-\overline{\alpha}m+(1-m)\overline{\alpha}(2-\overline{\alpha})(\sigma\eta-1) > 0$ and $\omega^* = \overline{\alpha}m+m\overline{\alpha}(2-\overline{\alpha})(\sigma\eta-1) > 0$. Remark that (89) and (90) can be rewritten as follows:

$$\widehat{y}_t = \widehat{c}_t + \frac{\omega+\overline{\alpha}-1}{\sigma}(\widehat{p}_{F,t}-\widehat{p}_{H,t}) \tag{91}$$

$$\widehat{y}_t^* = \widehat{c}_t^* - \frac{\omega^*}{\sigma}(\widehat{p}_{F,t}-\widehat{p}_{H,t}) \tag{92}$$

Therefore, the IRS condition (86) can be rewritten as follows:

$$\widehat{y}_t - \widehat{y}_t^* = \frac{\omega+\omega^*}{\sigma}(\widehat{p}_{F,t}-\widehat{p}_{H,t}) = \frac{\omega+\omega^*}{\sigma}\widehat{s}_t$$

D Roots of the Terms-Of Trade Dynamic Equation

The following equation gives the deterministic part of the TOT dynamic:[43]

$$0 = \widehat{s}_{t+1} - \underbrace{\left(\frac{1+\beta+\frac{\kappa+(\kappa-\sigma\lambda)\overline{\alpha}(2-\overline{\alpha})(\sigma\eta-1)}{\sigma}}{\beta} \right)}_{=b} \widehat{s}_t + \frac{1}{\beta}\widehat{s}_{t-1}$$

The solution is such that $(1-r_1L)(1-r_2L)\widehat{s}_t = 0$, where the roots are:

$$r_i = \frac{b \pm \sqrt{b^2 - 4/\beta}}{2} \quad i = 1,2 \quad \text{where} \quad 0 < r_1 < 1 < r_2$$

First, remark that:

$$b = \frac{1+\beta+\frac{\kappa+(\kappa-\sigma\lambda)\alpha(2-\alpha)(\sigma\eta-1)}{\sigma}}{\beta} = \frac{1+\beta}{\beta} + \frac{\lambda(\varphi+\sigma)+\lambda\varphi\alpha(2-\alpha)(\sigma\eta-1)}{\beta\sigma} \equiv \frac{1+\beta}{\beta} + \Gamma,$$

where $\Gamma > 0$ when $\eta\sigma > 1$. Therefore, assuming $\eta\sigma > 1$, we have $|r_1| < 1$ and $|r_2| > 1$ iff:

$$\begin{array}{cc}
|r_1| < 1 & |r_2| > 1 \\
\frac{b - \sqrt{b^2 - 4/\beta}}{2} < 1 & \frac{b - \sqrt{b^2 + 4/\beta}}{2} > 1 \\
\frac{1+\beta}{\beta} < \frac{1+\beta}{\beta} + \Gamma & \frac{1+\beta}{\beta} < \frac{1+\beta}{\beta} + \Gamma
\end{array}$$

These two inequalities lead to the same restriction, which is always satisfied. Thus, given that $0 < r_1 < 1 < r_2$ when $\eta\sigma > 1$, the TOT equation has a unique saddle path.

[43]See Appendix A for a description of the model parameters.

E Equilibrium Dynamics of a SOE

The system describing the equilibrium is:[44]

$$
\begin{aligned}
\widehat{p}_{H,t+1} &= b\widehat{p}_{H,t} - \frac{1}{\beta}\widehat{p}_{H,t-1} \\
\widehat{b}^r_{t-1} &= \frac{\beta}{1-\beta\gamma}\widehat{b}^r_t + \frac{(1-\bar{\alpha}(1-m))}{1-\beta\gamma}(\widehat{p}_{H,t} - \widehat{p}_{H,t-1}) + \frac{\beta^{-1}-1}{\beta^{-1}-\gamma}\varepsilon^F_t
\end{aligned}
$$

This system can be rewritten as follows:

$$
\begin{aligned}
\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\beta} & b & 0 \\ \frac{(1-\bar{\alpha}(1-m))}{1-\beta\gamma} & \frac{(\bar{\alpha}(1-m)-1)}{1-\beta\gamma} & 1 \end{bmatrix} \begin{bmatrix} \widehat{p}_{H,t-1} \\ \widehat{p}_{H,t} \\ \widehat{b}_{t-1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\beta}{1-\beta\gamma} \end{bmatrix} \begin{bmatrix} \widehat{p}_{H,t} \\ \widehat{p}_{H,t+1} \\ \widehat{b}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\beta^{-1}-1}{\beta^{-1}-\gamma} \end{bmatrix} \varepsilon^F_t \\
\Rightarrow \begin{bmatrix} \widehat{p}_{H,t-1} \\ \widehat{p}_{H,t} \\ \widehat{b}_{t-1} \end{bmatrix} &= \begin{bmatrix} b\beta & -\beta & 0 \\ 1 & 0 & 0 \\ \varpi & \delta & \frac{\beta}{1-\beta\gamma} \end{bmatrix} \begin{bmatrix} \widehat{p}_{H,t} \\ \widehat{p}_{H,t+1} \\ \widehat{b}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\beta^{-1}-1}{\beta^{-1}-\gamma} \end{bmatrix} \varepsilon^F_t
\end{aligned}
$$

where:

$$
\varpi = \frac{(1-\bar{\alpha}-b\beta+\bar{\alpha}m+\bar{\alpha}b\beta-\bar{\alpha}b\beta m)}{1-\beta\gamma},\ \delta = \frac{(\beta-\bar{\alpha}\beta+\bar{\alpha}\beta m)}{1-\beta\gamma}
$$

The eigenvalues of its deterministic dynamics are:

$$
\lambda_{1,2} = \beta\frac{b\pm\sqrt{b^2-4/\beta}}{2} < 1, \quad \lambda_3 = \frac{\beta}{1-\beta\gamma} \lesseqgtr 1
$$

We have $\lambda_3 > 1$ if $1/\beta - 1 < \gamma$. This restriction gives the minimum size of the debt brake.

[44] See Appendix A for a description of the model parameters.

F Dynamic System for the Two-Country Model

The dynamic system for the two-country model when each country has a specific fiscal policy is:[45]

$$
\begin{aligned}
\widehat{\pi}_{H,t} - \kappa\widehat{y}_t &= \beta\widehat{\pi}_{H,t+1} - \lambda(\bar{\alpha}m + \omega - 1)\widehat{s}_t \\
\widehat{y}_t\left(1 + \frac{\alpha_y}{\sigma}\right) + \frac{\alpha_\pi}{\sigma}\pi_{H,t} &= \widehat{y}_{t+1} + \frac{1}{\sigma}\widehat{\pi}_{H,t+1} \\
&\quad + \frac{1 - \bar{\alpha}m - \omega}{\sigma}\widehat{s}_{t+1} - \frac{\Lambda}{\sigma}\widehat{s}_t + \frac{\alpha_\pi(1-m)}{\sigma}\widehat{s}_{t-1} - \frac{1}{\sigma}\varepsilon^M_t \\
\frac{\beta}{1-\beta\gamma}\widehat{b}^r_{t+1} &= \widehat{b}^r_t - \frac{1}{1-\beta\gamma}\widehat{\pi}_{H,t+1} - \frac{1-\beta}{1-\beta\gamma}\varepsilon^F_{t+1} + \frac{\beta}{1-\beta\gamma}\varepsilon^M_{t+1} + \frac{\beta\gamma}{1-\beta\gamma}\varepsilon^M_t \\
&\quad + \frac{\beta}{1-\beta\gamma}\left(\alpha_\pi\widehat{\pi}_{H,t+1} + \alpha_y\widehat{y}_{t+1}\right) + \frac{\beta\gamma}{1-\beta\gamma}\left(\alpha_\pi\widehat{\pi}_{H,t} + \alpha_y\widehat{y}_t\right) \\
&\quad + \frac{(1-m)\zeta}{1-\beta\gamma}\widehat{s}_{t+1} + \frac{\Upsilon}{1-\beta\gamma}\widehat{s}_t - \frac{\beta\gamma}{1-\beta\gamma}\alpha_\pi(1-m)\widehat{s}_{t-1} \\
\frac{\beta}{1-\beta\gamma^*}\widehat{b}^{r*}_{t+1} &= \widehat{b}^{r*}_t - \frac{1}{1-\beta\gamma^*}\widehat{\pi}_{H,t+1} - \frac{1-\beta}{1-\beta\gamma}\varepsilon^{F*}_{t+1} + \frac{\beta}{1-\beta\gamma^*}\varepsilon^M_{t+1} + \frac{\beta\gamma^*}{1-\beta\gamma^*}\varepsilon^M_t \\
&\quad + \frac{\beta}{1-\beta\gamma^*}\left(\alpha_\pi\widehat{\pi}_{H,t+1} + \alpha_y\widehat{y}_{t+1}\right) + \frac{\beta\gamma^*}{1-\beta\gamma^*}\left(\alpha_\pi\widehat{\pi}_{H,t} + \alpha_y\widehat{y}_t\right) \\
&\quad + \frac{\Phi}{1-\beta\gamma^*}\widehat{s}_{t+1} + \frac{\Xi}{1-\beta\gamma}\widehat{s}_t - \frac{\beta\gamma^*}{1-\beta\gamma^*}\alpha_\pi(1-m)\widehat{s}_{t-1}
\end{aligned}
$$

where:

$$
\begin{aligned}
\Lambda &= \left[(1-m)\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right) + 1 - \bar{\alpha}m - \omega\right] \\
\zeta &= \left[\beta\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right) - \bar{\alpha}\right] \\
\Upsilon &= \left[\bar{\alpha} - \beta\alpha_\pi(1-m) + \beta\gamma\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right)\right] \\
\Phi &= \left[(\bar{\alpha}m - 1) + \beta(1-m)\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right)\right] \\
\Xi &= \left[(1-\bar{\alpha}m) - \beta\alpha_\pi(1-m) + \beta\gamma^*(1-m)\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right)\right]
\end{aligned}
$$

Given that:

$$
\begin{aligned}
\widehat{\pi}^*_{F,t+1} &= \widehat{s}_{t+1} - \widehat{s}_t + \widehat{\pi}_{H,t+1} \\
\widehat{r}_t &= \alpha_\pi\widehat{\pi}_{H,t} + \alpha_y\widehat{y}_t + (1-m)\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right)\widehat{s}_t - \alpha_\pi(1-m)\widehat{s}_{t-1} + \varepsilon^M_t,
\end{aligned}
$$

[45]See Appendix A for a description of the model parameters.

we deduce:

$$\begin{bmatrix} \widehat{\pi}_{H,t} \\ \widehat{y}_t \\ \widehat{b}_t^r \\ \widehat{b}_t^{r*} \end{bmatrix} = \begin{bmatrix} & A & 0 & 0 \\ m_1 & m_2 & m_3 & 0 \\ m_1^* & m_2^* & 0 & m_3^* \end{bmatrix} \begin{bmatrix} \widehat{\pi}_{H,t+1} \\ \widehat{y}_{t+1} \\ \widehat{b}_{t+1}^r \\ \widehat{b}_{t+1}^{r*} \end{bmatrix} + N \begin{bmatrix} \widehat{s}_{t+1} \\ \widehat{s}_t \\ \widehat{s}_{t-1} \end{bmatrix} + P \begin{bmatrix} \varepsilon_{t+1}^M \\ \varepsilon_t^M \\ \varepsilon_{t+1}^F \\ \varepsilon_t^F \\ \varepsilon_{t+1}^{F*} \\ \varepsilon_t^{F*} \end{bmatrix}$$

where $m_3 = \frac{1}{\beta^{-1}-\gamma}$, $m_3^* = \frac{1}{\beta^{-1}-\gamma^*}$ and

$$\begin{aligned}
m_1 &= [\tfrac{\alpha_y\alpha_\pi\beta-\sigma-\alpha_\pi\kappa-\alpha_y+\alpha_y\beta\gamma+\alpha_\pi\beta\sigma+\alpha_\pi^2\beta\kappa+\alpha_\pi\beta^2\gamma\sigma+\alpha_\pi\beta\gamma\kappa}{(\beta\gamma-1)(\alpha_y+\sigma+\alpha_\pi\kappa)}] \\
m_2 &= [\tfrac{\beta(\alpha_y\sigma+\alpha_y^2+\alpha_y\alpha_\pi\kappa+\alpha_y\gamma\sigma+\alpha_\pi\gamma\kappa\sigma)}{(\beta\gamma-1)(\alpha_y+\sigma+\alpha_\pi\kappa)}] \\
m_1^* &= [\tfrac{\alpha_y\alpha_\pi\beta-\sigma-\alpha_\pi\kappa-\alpha_y+\alpha_y\beta\gamma^*+\alpha_\pi\beta\sigma+\alpha_\pi^2\beta\kappa+\alpha_\pi\beta^2\gamma^*\sigma+\alpha_\pi\beta\gamma^*\kappa}{(\beta\gamma^*-1)(\alpha_y+\sigma+\alpha_\pi\kappa)}] \\
m_2^* &= [\tfrac{\beta(\alpha_y\sigma+\alpha_y^2+\alpha_y\alpha_\pi\kappa+\alpha_y\gamma^*\sigma+\alpha_\pi\gamma^*\kappa\sigma)}{(\beta\gamma^*-1)(\alpha_y+\sigma+\alpha_\pi\kappa)}]
\end{aligned}$$

G Incomplete markets

G.1 Incomplete markets with financial autarky

In the two-country model with incomplete markets and financial autarky, the households new budget constraints where state-contingent assets are not available:

$$P_t C_t^j + \frac{1}{R_t} B_t^j = B_{t-1}^j + W_t N_t^j + T_t \tag{93}$$

$$P_t^* C_t^{j*} + \frac{1}{R_{zt}} B_t^{j*} = B_{t-1}^{j*} + W_t^* N_t^{j*} + T_t^{j*} \tag{94}$$

The only financial assets available are their national public debt.

G.2 Incomplete markets with financial trade

In this section, we will provide numerical simulations to show that our results apply also when we consider a more general framework when financial trade is allowed across countries. When trade in assets is available and markets incomplete, Schmitt-Grohé and Uribe (2003) have demonstrated that for a small open-economy (SOE), the model is non-stationary (i.e the steady-state depends on initial conditions). Therefore, there is a need to introduce stationarity inducing devices. Without such mechanisms, the model will exhibit a unit-root. This feature is obviously also present in a two-country model of the EZ and thus, we need to introduce stationarity inducing mechanisms in order to assess the stability properties of the model. In that perspective, we will introduce portfoflio adjustment costs (PAC) to foreign asset.

G.2.1 The model

With foreign debt and PAC, the representative households in each region j and j^* want to maximize the following flows of utilities:

$$\mathcal{U}^j = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} (C_t^j)^{1-\sigma} - \frac{1}{1+\varphi} (N_t^j)^{1+\varphi} \right) \right] \tag{95}$$

$$\mathcal{U}^{j*} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} (C_t^{j*})^{1-\sigma} - \frac{1}{1+\varphi} (N_t^{j*})^{1+\varphi} \right) \right] \tag{96}$$

under the following budget constraints:

$$P_t C_t^j + \frac{B_t^j}{R_t} + \frac{B_{h,t}^j}{R_t^h} + \frac{B_{f,t}^j}{R_t^f} = B_{t-1}^j + B_{h,t-1}^j + B_{f,t-1}^j + W_t N_t^j + T_t^j - \frac{\kappa}{2}\Big(B_{f,t}^j - \bar{B_f^j}\Big)^2 \quad (97)$$

$$P_t^* C_t^{j*} + \frac{B_t^{j*}}{R_{zt}} + \frac{B_{h,t}^{j*}}{R_t^h} + \frac{B_{f,t}^{j*}}{R_t^f} = B_{t-1}^{j*} + B_{h,t-1}^{j*} + B_{f,t-1}^{j*} + W_t^* N_t^{j*} + T_t^{j*} - \frac{\kappa^*}{2}\Big(B_{h,t}^{j*} - \bar{B_h^{j*}}\Big)^2 \quad (98)$$

In addition to their national public debt, households can now trade private assets internationally. Households in country H can hold their private domestic bonds $B_{h,t}^j$ that pay interest rate R_t^h but they can also hold foreign private bonds $B_{f,t}^j$ that pay interest rate R_t^f. Similarly, households in country F can hold their domestic private debt $B_{f,t}^{j*}$ remunerated at rate R_t^f but also foreign private debt $B_{h,t}^{j*}$ that pays R_t^h. Both households face quadratic adjustment costs each time they change their holding of foreign debt $\{B_{f,t}^j, B_{h,t}^{j*}\}$ relative to their steady-state target $\{\bar{B_f^j}, \bar{B_h^{j*}}\}$ where κ and κ^* measure the strength of these costs. Without this hypothesis, the model exhibits a unit-root similar to the one in Schmitt-Grohé and Uribe (2003) in the case of a SOE. The FOCs associated to this new problem are given by:

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j,\sigma}}{C_{t+1}^{j,\sigma}} \frac{1}{\pi_{t+1}} \Big] = \frac{1}{R_t} \quad (99)$$

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j,\sigma}}{C_{j,t+1}^{j,\sigma}} \frac{1}{\pi_{t+1}} \Big] = \frac{1}{R_t^h} \quad (100)$$

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j,\sigma}}{C_{j,t+1}^{j,\sigma}} \frac{rer_{t+1}}{rer_t} \frac{1}{\pi_{t+1}^*} \Big] = \frac{1}{R_t^f} + \kappa(b_{f,t} - \bar{b_f}) \quad (101)$$

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j^*,\sigma}}{C_{t+1}^{j^*,\sigma}} \frac{1}{\pi_{t+1}^*} \Big] = \frac{1}{R_{zt}} \quad (102)$$

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j^*,\sigma}}{C_{t+1}^{j^*,\sigma}} \frac{1}{\pi_{t+1}^*} \Big] = \frac{1}{R_t^f} \quad (103)$$

$$\beta \mathbb{E}_t \Big[\frac{C_t^{j^*,\sigma}}{C_{t+1}^{j^*,\sigma}} \frac{rer_t}{rer_{t+1}} \frac{1}{\pi_{t+1}} \Big] = \frac{1}{R_t^h} + \kappa^*(b_{h,t}^* - \bar{b_h^*}) \quad (104)$$

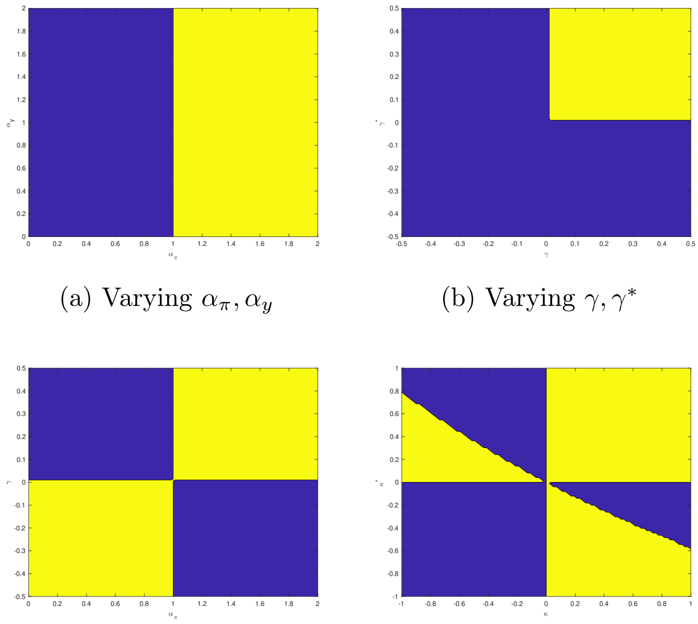
where $\{b_{f,t}, b_{h,t}^*\}$ are real value of private debt assets.

The introduction of private debt adds two new market clearing conditions where in equilibrium, net interhouseholds lending should be zero.

$$m b_{h,t} + (1-m) b_{h,t}^* = 0 \quad (105)$$

$$m b_{f,t} + (1-m) b_{f,t}^* = 0 \quad (106)$$

G.2.2 Numerical analysis



(a) Varying α_π, α_y

(b) Varying γ, γ^*

(c) Varying α_π, γ

(d) Varying κ, κ^*

Figure 1: **Stability of the Euro Area with IM**. Each panel ((a)-(d)) represent the regions where eigenvalues ensure the uniqueness of the saddle-path (yellow) or not (purple).

Panels (a), (b) and (c) of the Figure 1 highlights that the policy-mix regimes that ensures the uniqueness of the saddle-path (yellow surfaces) when financial trade is allowed are similar to the cases with CM and with IM and financial autarky.[46] Finally, under the constraint that adjustment costs parameters $\{\kappa, \kappa^*\} \geq 0$, the model is determinate for strictly positive values of these parameters (see Panel (d) of the Figure 1).

[46]The same graphic of the one of panel (c) is obtained for varying values for $\{\alpha_\pi, \gamma^*\}$.

H Fiscal Devaluation and Stability of the Euro Area

When the monetary policy is constrained to be passive ($\widehat{r}_t = 0$), we show in this appendix that it is possible to stabilize inflation and output via the introduction of policy rules on payroll taxes. This policy allows governments to respect their European commitments by implementing a budgetary brake.

We assume the rates of the payroll tax change as follows:

$$\begin{aligned} t_t &= \nu_y \widehat{y}_t + \nu_\pi \widehat{\pi}_{H,t} \\ t_t^* &= \nu_y^* \widehat{y}_t^* + \nu_\pi^* \widehat{\pi}_{F,t} \end{aligned}$$

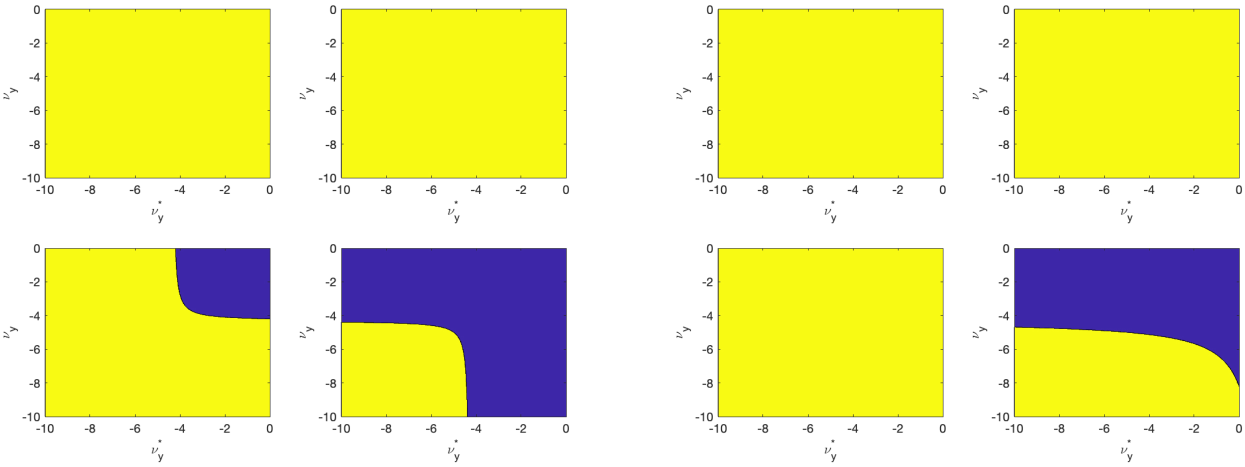
Therefore, the dynamics of the inflation and output gap are given by, knowing the TOT dynamic:

$$\begin{aligned} (1-\lambda\nu_\pi)\widehat{\pi}_{H,t} &= \beta\widehat{\pi}_{H,t+1} + (\kappa + \lambda\nu_y)\widehat{y}_t - \lambda(\alpha m + \omega - 1)\widehat{s}_t \\ \left(1+\frac{\alpha_y}{\sigma}\right)\widehat{y}_t &= \widehat{y}_{t+1} + \frac{1}{\sigma}\widehat{\pi}_{H,t+1} - \frac{\alpha_\pi}{\sigma}\widehat{\pi}_{H,t} + \frac{1-\overline{\alpha}m-\omega}{\sigma}\widehat{s}_{t+1} + \frac{\alpha_\pi(1-m)}{\sigma}\widehat{s}_{t-1} \\ & \quad -\frac{1}{\sigma}\left[1-\overline{\alpha}m-\omega+(1-m)\left(\alpha_\pi - \alpha_y\frac{\omega+\omega^*}{\sigma}\right)\right]\widehat{s}_t - \frac{1}{\sigma}\varepsilon_t^M \\ \widehat{s}_{t+1} &= \left(b - \frac{\lambda}{\beta}\left(\nu_\pi^* - \nu_y^*\frac{\omega+\omega^*}{\sigma}\right)\right)\widehat{s}_t - \frac{1}{\beta}\left(1-\nu_\pi^*\right)\widehat{s}_{t-1} \\ & \quad -\frac{\lambda}{\beta}\left[(\nu_\pi^* - \nu_\pi)\widehat{\pi}_{H,t} + (\nu_y^* - \nu_y)\widehat{y}_t\right] \end{aligned}$$

This system can be rewritten as $Z_t = MZ_{t+1} + R\varepsilon_t^M$, where $Z_t = [\widehat{\pi}_{H,t}, \widehat{y}_t, \widehat{s}_t, \widehat{s}_{t-1}]$. This dynamic system will be stable if and only if three of the four eigenvalues of M are inside the unit circle. These three dimensions correspond to the two jump variables that are inflation and the output gap and to the "forward" component of the real exchange rate. The last eigenvalue must be outside the unit circle because it corresponds to the "backward" component of the real exchange rate.

The eigenvalues of the matrix M, for different values of the parameters $\{\nu_y, \nu_\pi, \nu_y^*, \nu_\pi^*\}$, are presented in Figure 1.[47] This figure only reports cases where the tax rates are countercyclical, that is, cases where governments implement a fiscal devaluation when the country's economic activity is poor. This figure shows two results. The first is that it is more likely to obtain a stable dynamic system when the tax rates depend simultaneously on inflation and the output gap. Indeed, panel (a) of Figure 1 shows that two eigenvalues can be simultaneously greater than one, if $\nu_\pi = \nu_\pi^* = 0$, while in panel (b), where

[47] For this numerical illustration, the model parameters are the following: $\beta = 0.98$, $m = 0.5$, $\sigma = 2$, $\alpha = 0.5$, $\eta = 2$, $\theta = 0.8$, $\varphi = 3$.



(a) Varying ν_y, ν_y^* and $\nu_\pi = \nu_\pi^* = 0$

(b) Varying ν_y, ν_y^* and $\nu_\pi = \nu_\pi^* = -0.5$

Figure 2: **Stability of the Euro Area with Fiscal Devaluation Policies**. The four graphics on each panel ((a) and (b)) represent the regions, where an eigenvalue is larger (purple) or lower (yellow) than one. In panel (a), $\nu_\pi = \nu_\pi^* = 0$, whereas in panel (b) $\nu_\pi = \nu_\pi^* = 1.5$. In these two cases, we have $\nu_y \in [-10, 0]$ and $\nu_y^* \in [-10, 0]$, respectively.

both $\nu_\pi < 0$ and $\nu_\pi^* < 0$, only one eigenvalue can be greater than unity. However, even in this case, the uniqueness of the dynamic path is not always guaranteed. The stable path is unique only when ν_y is not less than a threshold value, here around -4 (see panel (b) of the Figure 1). For all the parameters allowing this "automatic" fiscal devaluation policy to guarantee dynamic stability, it would then be interesting to assess its impact on welfare, compared to the case where stabilization is obtained via an active fiscal policy implemented on one of the public debt of one of the two countries. This study is left to future research.

I Model solution at the ZLB

As in Christiano, Eichenbaum and Rebelo (2011)[48], we assume that the economy hits the ZLB after a "shock" on the discount factor. The discount factor follows a two-state Markov process, where the low value is an absorbing state. At the time of shock, the discount factor is at its high value and is sufficient to ensure that the zero bound binds. Thereafter, the discount factor can return to its low value with a probability of $1-p$.

A) If the monetary policy is expected to be active after the ZLB, the model is solved by the agents by taking $\{\widehat{\pi}_{H,t}, \widehat{y}_t\}$ as the two jump variables of the economy. This is the solution provided by Christiano, Eichenbaum and Rebelo (2011). Given the paths of $\{\widehat{\pi}_{H,t}, \widehat{y}_t\}$ that solve this system with stochastic regime switches, it is then necessary to impose stability conditions on the real debt dynamics.

B) If the monetary policy is expected to be passive after the ZLB, the model is solved by the agents by taking $\{\widehat{\pi}_{H,t}, \widehat{b}_t^r\}$ as the two jump variables of the economy. After solving for $\{\widehat{\pi}_{H,t}, \widehat{b}_t^r\}$, we deduce the dynamics of the Home output gap followed by the Foreign variables.

First, we show how to solve the model in this case when the ZLB is not binding.[49] Introducing the Taylor rule in the debt dynamics, we obtain:

$$\begin{aligned} \widehat{b}_{t-1}^r &= \beta\widehat{b}_t^r + (1-\beta\alpha_\pi)\widehat{\pi}_{H,t} + (1-\beta)\widehat{d}_t^r - \beta\varepsilon_t^M \\ \sum_{i=0}^{\infty}\beta^i\widehat{\pi}_{H,t+i} &= \frac{1}{1-\beta\alpha_\pi}\left[\widehat{b}_{t-1}^r - (1-\beta)\sum_{i=0}^{\infty}\beta^i\widehat{d}_{t+i}^r + \beta\sum_{i=0}^{\infty}\beta^i\varepsilon_{t+i}^M\right], \end{aligned} \tag{107}$$

where the right-hand side provides the expected value of government revenues, while the left-hand side is the expected value of inflation. The expected value of inflation (left-hand side) is the solution of the inflation dynamic, which is given by:

$$\widehat{\pi}_{H,t+1} = \lambda_1\widehat{\pi}_{H,t} + \frac{\sigma^{-1}\kappa}{\beta\lambda_2}\varepsilon_t^M\sum_{j=0}^{\infty}\left(\frac{\rho_M}{\lambda_2}\right)^j = \lambda_1\widehat{\pi}_{H,t} + \frac{\sigma^{-1}\kappa}{\beta(\lambda_2-\rho_M)}\varepsilon_t^M,$$

where $|\lambda_1| < 1 < |\lambda_2|$ are the eigenvalues of A and using $\varepsilon_{t+1} = \rho_M\varepsilon_t$. The present value of the expected inflation is then given by:

$$L^{-j}\widehat{\pi}_{H,t} = \lambda_1^j\widehat{\pi}_{H,t} + \left(\lambda_1^{j-1} + \lambda_1^{j-2}L^{-1} + ... + \lambda_1 L^{-(j-2)} + L^{-(j-1)}\right)\frac{\sigma^{-1}\kappa}{\beta(\lambda_2-\rho_M)}\varepsilon_t^M$$

[48]These authors have based their analysis on the previous works of Eggertsson and Woodford (2003), Christiano (2004), and Eggertsson (2004)

[49]In the following, we always assume that $\alpha_y = 0$ without loss of generality, but with significant gains for the exposal simplicity.

Given that $L^{-i}\varepsilon_t = \rho_M \varepsilon_t$, the expected value of inflation is thus:

$$\sum_{i=0}^{\infty} \beta^i \widehat{\pi}_{H,t+i} = \sum_{i=0}^{\infty} \beta^i L^{-i} \widehat{\pi}_{H,t} \quad = \quad \frac{1}{1-\beta\lambda_1}\widehat{\pi}_{H,t} + \frac{\sigma^{-1}\kappa}{\beta(\lambda_2-\rho_M)}\frac{1}{1-\beta\lambda_1\rho_M}\varepsilon_t^M$$

Integrating this result in Equation (107), the equilibrium is defined by:

$$\begin{aligned}
& \frac{1}{1-\beta\lambda_1}\widehat{\pi}_{H,t} + \frac{\sigma^{-1}\kappa}{\beta(\lambda_2-\rho_M)}\frac{1}{1-\beta\lambda_1\rho_M}\varepsilon_t^M \\
= \quad & \frac{1}{1-\beta\alpha_\pi}\left[\widehat{b}_{t-1}^r - (1-\beta)\sum_{i=0}^{\infty}\beta^i \widehat{d}_{t+i}^r + \beta\sum_{i=0}^{\infty}\beta^i \varepsilon_{t+i}^M\right] \\
= \quad & \frac{1}{1-\beta\alpha_\pi}\left[\widehat{b}_{t-1}^r - \frac{1-\beta}{1-\beta L^{-1}}\widehat{d}_t^r + \frac{\beta}{1-\beta\rho_M}\varepsilon_t^M\right] \\
\widehat{\pi}_{H,t} \quad = \quad & \frac{1-\beta\lambda_1}{1-\beta\alpha_\pi}\left[\widehat{b}_{t-1}^r - \frac{1-\beta}{1-\beta L^{-1}}\widehat{d}_t^r\right] \\
& +(1-\beta\lambda_1)\left(\frac{\beta}{(1-\beta\alpha_\pi)(1-\beta\rho_M)} - \frac{\sigma^{-1}\kappa}{\beta(1-\beta\lambda_1\rho_M)(\lambda_2-\rho_M)}\right)\varepsilon_t^M
\end{aligned}$$

If the fiscal policy is active, as for instance,$\gamma = 0$, then $\widehat{d}_t^r = \varepsilon_t^F$ and thus:

$$\begin{aligned}
\widehat{\pi}_{H,t} \quad = \quad & \frac{1-\beta\lambda_1}{1-\beta\alpha_\pi}\left[\widehat{b}_{t-1}^r - \frac{1-\beta}{1-\beta\rho_F}\varepsilon_t^F\right] \\
& +(1-\beta\lambda_1)\left(\frac{\beta}{(1-\beta\alpha_\pi)(1-\beta\rho_M)} - \frac{\sigma^{-1}\kappa}{\beta(1-\beta\lambda_1\rho_M)(\lambda_2-\rho_M)}\right)\varepsilon_t^M
\end{aligned}$$

We deduce the equilibrium path of the output gap using the Phillips curve:

$$\widehat{y}_t \quad = \quad \frac{1}{\kappa}(1-\beta L^{-1})\widehat{\pi}_{H,t},$$

where $t_t = t_t^* = 0$ leading to $\widehat{s}_t = 0$, $\forall t$, for simplicity. The dynamics of Foreign aggregates y_t^* and $\widehat{\pi}_{F,t}^*$ are deduced from Equations (6) and $\widehat{\pi}_{F,t}^* \equiv \widehat{\pi}_{H,t}$, given that $t_t = t_t^* = 0$ implying $\widehat{s}_t = 0$, $\forall t$. Therefore, γ^* must be larger than $\beta^{-1} - 1$ (fiscal brake) in order to ensure the stability of the Foreign real debt, given the stationarity of the others aggregates $\{\widehat{y}, \widehat{y}_t^*, \widehat{\pi}_{H,t}, \widehat{\pi}_{F,t}^*, \widehat{b}_t^r\}$.

Now, assuming that the ZLB binds at the initial period. The debt dynamic is then:

$$
\begin{aligned}
\widehat{b}^r_{t-1} &= \beta \widetilde{b}^r_t + \widetilde{\pi}_{H,t} + (1-\beta)\widehat{d}^r_t \\
\widehat{b}^r_{t-1} &= \beta \left[\begin{array}{l} \beta p \widetilde{b}^r_{t+1} + \beta(1-p)\widehat{b}^r_{t+1} + p\widetilde{\pi}_{H,t+1} + (1-\beta\alpha_\pi)(1-p)\widehat{\pi}_{H,t+1} \\ +(1-\beta)p\widetilde{d}^r_{t+1} + (1-\beta)(1-p)\widehat{d}^r_{t+1} - (1-p)\beta\varepsilon^M_{t+1} \end{array} \right] \\
&\quad + \widetilde{\pi}_{H,t} + (1-\beta)\widehat{d}^r_t \\
&= \beta^2 p \widetilde{b}^r_{t+1} + \beta^2(1-p)\widehat{b}^r_{t+1} + \sum_{i=0}^{1}(\beta p)^i \widetilde{\pi}_{H,t+i} + (1-\beta)\sum_{i=0}^{1}(1-p)^i \widehat{d}^r_{t+i} \\
&\quad +(1-\beta)p\widetilde{d}^r_{t+1} + (1-p)\left[(1-\beta\alpha_\pi)\widehat{\pi}_{H,t+1} - \beta\varepsilon^M_{t+1}\right] \\
&= \beta^2 p \left[\begin{array}{l} \beta p \widetilde{b}^r_{t+2} + \beta(1-p)\widehat{b}^r_{t+2} + p\widetilde{\pi}_{H,t+2} + (1-\beta\alpha_\pi)(1-p)\widehat{\pi}_{H,t+2} \\ +(1-\beta)p\widetilde{d}^r_{t+2} + (1-\beta)(1-p)\widehat{d}^r_{t+2} - (1-p)\beta\varepsilon^M_{t+2} \end{array} \right] \\
&\quad +\beta^2(1-p)\left[\beta\widehat{b}^r_{t+2} + (1-\beta\alpha_\pi)\widehat{\pi}_{H,t+2} + (1-\beta)\widehat{d}^r_{t+2} - \beta\varepsilon^M_{t+2}\right] \\
&\quad + \sum_{i=0}^{1}(\beta p)^i \widetilde{\pi}_{H,t+i} + (1-\beta)\sum_{i=0}^{1}(1-p)^i \widehat{d}^r_{t+i} \\
&\quad +(1-\beta)p\widetilde{d}^r_{t+1} + (1-p)\left[(1-\beta\alpha_\pi)\widehat{\pi}_{H,t+1} - \beta\varepsilon^M_{t+1}\right] \\
&= \sum_{i=0}^{\infty}(\beta p)^i \widetilde{\pi}_{H,t+i} + (1-\beta)\beta p \sum_{i=0}^{\infty}(\beta p)^i \widetilde{d}_{t+1+i} + (1-\beta)\widehat{d}_{t+i} \\
&\quad +(1-p)\beta^2 \sum_{i=0}^{\infty}(\beta p)^i \widehat{b}^r_{t+1+i}
\end{aligned}
$$

This gives the value of expected inflation:

$$
\sum_{i=0}^{\infty}(\beta p)^i \widetilde{\pi}_{H,t+i} = \widehat{b}^r_{t-1} - \frac{(1-\beta)\beta p}{1-\beta p L^{-1}}\widetilde{d}^r_{t+1} + (1-\beta)\widehat{d}^r_t - \frac{(1-p)\beta^2}{1-\beta p L^{-1}}\widehat{b}^r_{t+1}
$$

After the shock, the dynamics of inflation must also consider the stochastic end of the ZLB:

$$
\begin{aligned}
&\text{No ZLB:} && \widehat{\pi}_{H,t+2} - \frac{1+\beta+\sigma^{-1}\kappa}{\beta}\widehat{\pi}_{H,t+1} + \frac{1+\alpha_\pi\sigma^{-1}\kappa}{\beta}\widehat{\pi}_{H,t} = -\frac{\sigma^{-1}\kappa}{\beta}\varepsilon^M_t \\
&\text{with ZLB:} && p^2\widetilde{\pi}_{H,t+2} + (p(1-p)+(1-p)^2)\widehat{\pi}_{H,t+2} \\
& && -\frac{1+\beta+\sigma^{-1}\kappa}{\beta}p\widetilde{\pi}_{H,t+1} - \frac{1+\beta+\sigma^{-1}\kappa}{\beta}(1-p)\widehat{\pi}_{H,t+1} + \frac{1}{\beta}\widetilde{\pi}_{H,t} = -\frac{\sigma^{-1}\kappa}{\beta}\varepsilon^M_t \\
& &\Leftrightarrow\ & \widetilde{\pi}_{H,t+2} - \frac{1+\beta+\sigma^{-1}\kappa}{\beta p}\widetilde{\pi}_{H,t+1} + \frac{1}{\beta p^2}\widetilde{\pi}_{H,t} \\
& && = \underbrace{\frac{1-p}{p^2}\left[\frac{1+\alpha_\pi\sigma^{-1}\kappa}{\beta}\widehat{\pi}_{H,t} + \frac{\sigma^{-1}\kappa}{\beta}\varepsilon^M_t\right]}_{x_t}
\end{aligned}
$$

The two roots of the second-order differential equation for inflation are $\lambda_i = \frac{1}{2}\frac{1}{p\beta}(a \pm \sqrt{a^2 - 4\beta})$ with $a = 1 + \beta + \sigma^{-1}$, with $|\lambda_1| < 1 < |\lambda_2|$, for $p \in]0.5; 1[$. Using $(L^{-1} - \lambda_1)(L^{-1} - \lambda_2)\widetilde{\pi}_t = x_t$, we deduce the solution of the inflation dynamic when the ZLB binds:

$$\widetilde{\pi}_{H,t+1} = \lambda_1 \widetilde{\pi}_{H,t} - \lambda_1 \beta p \sum_{i=0}^{\infty} \lambda_2^{-i} x_{t+i}$$

The expected value of inflation is then:

$$\sum_{i=0}^{\infty} (\beta p)^i \widetilde{\pi}_{H,t+i} = \frac{1}{1 - \beta p \lambda_1} \widetilde{\pi}_{H,t} - \frac{\lambda_1 \beta p}{1 - \beta p \lambda_1 L^{-1}} x_t$$

After equalizing the two expressions of the expected inflation, we obtain:

$$\frac{1}{1 - \beta p \lambda_1} \widetilde{\pi}_{H,t} - \frac{\lambda_1 \beta p}{1 - \beta p \lambda_1 L^{-1}} x_t = \widehat{b}^r_{t-1} - \frac{(1-\beta)\beta p}{1 - \beta p L^{-1}} \widetilde{d}^r_{t+1} + (1-\beta)\widehat{d}^r_t - \frac{(1-p)\beta^2}{1 - \beta p L^{-1}} \widehat{b}^r_{t+1}$$

When the fiscal policy is active, for instance, $\gamma = 0$, the equilibrium inflation is:

$$\widetilde{\pi}_{H,t} = (1 - \beta p \lambda_1) \left[\widehat{b}^r_{t-1} - \frac{(1-\beta)(2\beta p \rho - 1)}{1 - \beta p \rho} \varepsilon^F_t - \frac{(1-p)\beta^2}{1 - \beta p L^{-1}} \widehat{b}^r_{t+1} + \frac{\lambda_1 \beta p}{1 - \beta p \lambda_1 L^{-1}} x_t \right],$$

where the sign of the impact of a redistributive shock ε^F_t depends on the inequality $\beta p \rho \lesseqgtr 1/2$. As for the previous case, the equilibrium path of the output gap is deduced from the Phillips curve:

$$\widehat{y}_t = \frac{1}{\kappa}(1 - \beta L^{-1}) \widehat{\pi}_{H,t},$$

where $t_t = t^*_t = 0$ leading to $\widehat{s}_t = 0$, $\forall t$, for simplicity. The dynamics of Foreign aggregates y^*_t and $\widehat{\pi}^*_{F,t}$ are deduced from Equations (6) and $\widehat{\pi}^*_{F,t} \equiv \widehat{\pi}_{H,t}$, given that $t_t = t^*_t = 0$ implying $\widehat{s}_t = 0$, $\forall t$.