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Economic Time Series

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1. Consider the following MA(2) process:

$$x_t = (1 + 2.4L + 0.8L^2)e_t \tag{1}$$

where $e_t \sim \text{WN}(0, 1)$.

(a) Is this process stationary? Prove it using autocovariance function.

(b) Is it invertible? If so, prove. Otherwise, find the invertible representation for the process. Calculate the autocovariance of the invertible representation and verify that they are the same as those obtained in (a).

Solution:

(a) Since e_t is a white noise, its mean is zero and its autocovariance function is

$$R_e(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0, & \text{otherwise} \end{cases} = \mathbb{I}_{\{0\}}(h) \tag{2}$$

Moreover, if we define $\psi(L) = 1 + 2.4L + 0.8L^2$, it follows that

$$\sum_{j=-\infty}^{\infty} \left|\psi_j^2\right| < \infty. \tag{3}$$

By proposition 1 in p.50 in the lecture note, x_t is also stationary whose covariance function is

$$\begin{aligned} R_x(h) &= \sum_{j,k=-\infty}^{\infty} \psi_j \psi_k R_e(h - j + k) &(4)\\ &= \sum_{j,k=1}^{2} \psi_j \psi_k \mathbb{I}_{\{0\}}(h - j + k) &(5)\\ &= \begin{cases} 1 + \psi_1^2 + \psi_2^2, & \text{if } h = 0 \\ \psi_1^2 + \psi_1\psi_2, & \text{if } |h| = 1 \\ \psi_2, & \text{if } |h| = 2 \\ 0, & \text{if } |h| > 2 \end{cases} &(6) \end{aligned}$$

(b) By theorem 2 in p.52 of the lecture note, $A(L) = 1$ and $\Psi(L) = 1 + \psi_1 L + \psi_2 L^2$ have no common zeros since

$$\Psi(z) = 1 + 2.4z + 0.8z^2 = 0 \tag{7}$$

to which the solutions are $z = -2.5$ and $z = -0.5$. Therefore, the process is non-invertible. To get the invertible representation, we should reciprocate the problematic root, which yields

$$\tilde{x}_t = \left(1 - \tilde{\psi}_1 L - \tilde{\psi}_2 L^2\right) \tilde{e}_t, \quad \tilde{e}_t \sim \text{WN}\left(0, \sigma^2\right) \tag{8}$$

where $\tilde{\psi}_1 = -4.5$, $\tilde{\psi}_2 = -5$, and $\sigma^2 = 0.16$. This comes from the factorized equation

$$(1 + 2z)\left(1 + \frac{z}{0.4}\right) \tag{9}$$

so as to push the roots outside of the unit circle. For any general MA(2) model, the autocovariance function becomes as follows,

$$R_{\tilde{x}}(h) = \begin{cases} \sigma^2\left(1 + \tilde{\psi}_1^2 + \tilde{\psi}_2^2\right) = 7.4, & \text{if } h = 0 \\ -\sigma^2 \tilde{\psi}_1\left(1 - \tilde{\psi}_2\right) = 4.32, & \text{if } |h| = 1 \\ -\sigma^2 \tilde{\psi}_2 = 0.8, & \text{if } |h| = 2 \\ 0, & \text{if } |h| > 2 \end{cases} \tag{10}$$

Thus, $\tilde{x}_t = \left(1 + 4.5L + 5L^2\right) \tilde{e}_t$ becomes the invertible counterpart.