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Statistics for Economics

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1. The sample space Ω is $\Omega = [0, 1]$ and the probability can be expressed as follows.

$$p(x) = 2xI\left\{0 \le x \le 1\right\} \tag{1}$$

Let random variables X and Y be defined as below.

$$X(\omega) = \begin{cases} 1, & \text{if } 0 \le \omega < 1/4 \\ 0, & \text{if } 1/4 \le \omega < 1/2 \\ -1, & \text{if } 1/2 \le \omega < 3/4 \\ 0, & \text{if } 3/4 \le \omega \le 1 \end{cases}, \qquad Y(\omega) = \begin{cases} 1, & \text{if } 0 \le \omega < 1/2 \\ 0, & \text{if } 1/2 \le \omega \le 1 \end{cases} \tag{2}$$

(1) Compute the joint density of X and Y.

(2) Compute the marginal densities of X and Y respectively.

(3) Compute the expectations of X and Y respectively.

(4) Compute the conditional expectation $\mathbf{E}(X \mid Y)$.

Solution:

(1) If we get every combination of possible values of X and Y, then we can say that we have obtained the joint probability mass function of X and Y.

- $\Pr(X = 0, Y = 0)$

$$\begin{aligned}
&= \Pr\left[\left(\left\{\omega \mid \frac{1}{4} \le \omega < \frac{1}{2}\right\} \cup \left\{\omega \mid \frac{3}{4} \le \omega \le 1\right\}\right) \cap \left\{\omega \mid \frac{1}{2} \le \omega \le 1\right\}\right] && (3) \\
&= \Pr\left[\left\{\omega \mid \frac{3}{4} \le \omega \le 1\right\}\right] && (4) \\
&= \int_{3/4}^{1} 2x\,dx = 1 - \frac{9}{16} && (5) \\
&= \frac{7}{16} && (6)
\end{aligned}$$

- $\Pr(X=0, Y=1)$

$$
\begin{aligned}
&= \Pr\left[\left(\left\{\omega \mid \frac{1}{4} \le \omega < \frac{1}{2}\right\} \cup \left\{\omega \mid \frac{3}{4} \le \omega \le 1\right\}\right) \cap \left\{\omega \mid 0 \le \omega < \frac{1}{2}\right\}\right] && (7)\\
&= \Pr\left[\left\{\omega \mid \frac{1}{4} \le \omega < \frac{1}{2}\right\}\right] && (8)\\
&= \int_{1/4}^{1/2} 2x\,dx = \frac{1}{4} - \frac{1}{16} && (9)\\
&= \frac{3}{16} && (10)
\end{aligned}
$$

- $\Pr(X=1, Y=0)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid 0 \le \omega < \frac{1}{4}\right\} \cup \left\{\omega \mid \frac{1}{2} \le \omega \le 1\right\}\right] && (11)\\
&= \Pr\left[\emptyset\right] && (12)\\
&= 0 && (13)
\end{aligned}
$$

- $\Pr(X=1, Y=1)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid 0 \le \omega < \frac{1}{4}\right\} \cup \left\{0 \le \omega < \frac{1}{2}\right\}\right] && (14)\\
&= \Pr\left[\left\{\omega \mid 0 \le \omega < \frac{1}{2}\right\}\right] && (15)\\
&= \int_{0}^{1/4} 2x\,dx = \frac{1}{16} && (16)
\end{aligned}
$$

- $\Pr(X=-1, Y=0)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid \frac{1}{2} \le \omega < \frac{3}{4}\right\} \cup \left\{\omega \mid \frac{1}{2} \le \omega \le 1\right\}\right] && (17)\\
&= \Pr\left[\left\{\omega \mid \frac{1}{2} \le \omega < \frac{3}{4}\right\}\right] && (18)\\
&= \int_{1/2}^{3/4} 2x\,dx = \frac{9}{16} - \frac{1}{4} && (19)\\
&= \frac{5}{16} && (20)
\end{aligned}
$$

- $\Pr(X = -1, Y = 1)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid \frac{1}{2} \leq \omega < \frac{3}{4}\right\} \cup \left\{\omega \mid 0 \leq \omega \leq \frac{1}{2}\right\}\right] && (21)\\
&= \Pr\left(\left\{\frac{1}{2}\right\}\right) && (22)\\
&= 0 && (23)
\end{aligned}
$$

because the probability for a singleton with respect to a continuous random variable is always zero.

(2) For X,

- $\Pr(X = 1)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid 0 \leq \omega < \frac{1}{4}\right\}\right] && (24)\\
&= \int_{0}^{1/4} 2x\, dx = \frac{1}{16} && (25)
\end{aligned}
$$

- $\Pr(X = 0)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid \frac{1}{4} \leq \omega < \frac{1}{2}\right\} \cup \left\{\omega \mid \frac{3}{4} \leq \omega \leq 1\right\}\right] && (26)\\
&= \int_{1/4}^{1/4} 2x\, dx + \int_{3/4}^{1} 2x\, dx = \left(\frac{1}{4} - \frac{1}{16}\right) + \left(1 - \frac{9}{16}\right) && (27)\\
&= \frac{5}{8} && (28)
\end{aligned}
$$

- $\Pr(X = -1)$

$$
\begin{aligned}
&= \Pr\left[\left\{\omega \mid \frac{1}{2} \leq \omega < \frac{3}{4}\right\}\right] && (29)\\
&= \int_{1/2}^{3/4} 2x\, dx = \frac{9}{16} - \frac{1}{4} && (30)\\
&= \frac{5}{16} && (31)
\end{aligned}
$$

For Y,

- $\Pr(Y = 0)$

$$= \Pr\left[\left\{\omega \mid \frac{1}{2} \leq \omega \leq 1\right\}\right] \tag{32}$$

$$= \int_{1/2}^{1} 2x \, dx = 1 - \frac{1}{4} \tag{33}$$

$$= \frac{3}{4} \tag{34}$$

- $\Pr(Y = 1)$

$$= \Pr\left[\left\{\omega \mid 0 \leq \omega < \frac{1}{2}\right\}\right] \tag{35}$$

$$= \int_{0}^{1/2} 2x \, dx \tag{36}$$

$$= \frac{1}{4} \tag{37}$$

(3) Since we know the marginal distributions of X and Y, we in full position to compute the expectations.

- For $\mathbf{E}(X)$,

$$\mathbf{E}(X) = 1 \times \frac{1}{16} + 0 \times \frac{5}{8} - 1 \times \frac{5}{16} \tag{38}$$

$$= -\frac{1}{4} \tag{39}$$

- For $\mathbf{E}(Y)$,

$$\mathbf{E}(Y) = 0 \times \frac{3}{4} + 1 \times \frac{1}{4} \tag{40}$$

$$= \frac{1}{4} \tag{41}$$

(4) By definition of the conditional expectation,

$$\mathbf{E}(X \mid Y) = \sum_{x \in \mathcal{X}} x \Pr(X = x \mid Y = y) \tag{42}$$

which means we have 2 different cases of $Y = y$ to be conditioned on.

- $\mathbf{E}(X \mid Y = 1)$

$$
\begin{aligned}
&= \sum_{x \in \mathcal{X}} x \Pr(X = x \mid Y = 1) && (43) \\
&= \sum_{x \in \mathcal{X}} x \frac{\Pr(X = x, Y = 1)}{\Pr(Y = 1)} && (44) \\
&= 1 \times \frac{\Pr(X = 1, Y = 1)}{\Pr(Y = 1)} + 0 \times \frac{\Pr(X = 0, Y = 1)}{\Pr(Y = 1)} - 1 \times \frac{\Pr(X = -1, Y = 1)}{\Pr(Y = 1)} && (45)
\end{aligned}
$$

- $\mathbf{E}(X \mid Y = 0)$

$$
\begin{aligned}
&= \sum_{x \in \mathcal{X}} x \Pr(X = x \mid Y = 0) && (46) \\
&= \sum_{x \in \mathcal{X}} x \frac{\Pr(X = x, Y = 0)}{\Pr(Y = 0)} && (47) \\
&= 1 \times \frac{\Pr(X = 1, Y = 0)}{\Pr(Y = 0)} + 0 \times \frac{\Pr(X = 0, Y = 0)}{\Pr(Y = 0)} - 1 \times \frac{\Pr(X = -1, Y = 0)}{\Pr(Y = 0)} && (48)
\end{aligned}
$$