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Stochastic Dynamic Programming

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1. Suppose that a store has two of these machines. Assume that the machines fail and are repaired independently of each other. Let Y_n be the number of machines that are working on day n. Is $\{Y_n : n \geq 1\}$ a Markov chain?

Solution: Recall X_n from the previous problem where X_n is the state of a machine that is set to 0 if it is not working and 1 if it is on day n. Since the two machines are repaired independently, we have two such $i.i.d.$ random variables $X_n^m,\ m = 1, 2$ where m indicates the machine number. The row-stochastic transition matrix of X_n^m is as follows:

$$T_X = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \tag{1}$$

Now, let's turn to Y_n which denotes the number of machines functioning on day n. The state space is obviously $S = \{0, 1, 2\}$ for we have two machines. Then, if we let $\wedge$ denote logical conjuction (`and`) and $\vee$ logical disjuction (`or`),

$$\mathrm{P}(Y_n = 2 \,|\, Y_{n-1} = 2) = \mathrm{P}((1 \to 1) \wedge (1 \to 1)) = 0.36 \tag{2}$$

$$\mathrm{P}(Y_n = 1 \,|\, Y_{n-1} = 2) = \binom{2}{1} \mathrm{P}((1 \to 1) \wedge (1 \to 0)) = 2 \times 0.6 \times 0.4 = 0.48 \tag{3}$$

$$\mathrm{P}(Y_n = 0 \,|\, Y_{n-1} = 2) = \mathrm{P}((1 \to 0) \wedge (1 \to 0)) = 0.16 \tag{4}$$

$$\mathrm{P}(Y_n = 2 \,|\, Y_{n-1} = 1) = \mathrm{P}((1 \to 1) \wedge (0 \to 1)) = 0.6 \times 0.7 = 0.42 \tag{5}$$

$$\mathrm{P}(Y_n = 1 \,|\, Y_{n-1} = 1) = \mathrm{P}((1 \to 1) \wedge (0 \to 0)) + \mathrm{P}((1 \to 0) \wedge (0 \to 1)) = 0.18 + 0.28 = 0.46 \tag{6}$$

$$\mathrm{P}(Y_n = 0 \,|\, Y_{n-1} = 1) = \mathrm{P}((1 \to 0) \wedge (0 \to 0)) = 0.4 \times 0.3 = 0.12 \tag{7}$$

$$\mathrm{P}(Y_n = 2 \,|\, Y_{n-1} = 0) = \mathrm{P}((0 \to 1) \wedge (0 \to 1)) = 0.7^2 = 0.49 \tag{8}$$

$$\mathrm{P}(Y_n = 1 \,|\, Y_{n-1} = 0) = \binom{2}{1} \mathrm{P}((0 \to 1) \wedge (0 \to 0)) = 2 \times 0.7 \times 0.3 = 0.42 \tag{9}$$

$$\mathrm{P}(Y_n = 0 \,|\, Y_{n-1} = 0) = \mathrm{P}((0 \to 0) \wedge (0 \to 0)) = 0.09 \tag{10}$$

where $i \to j$ indicates the event of an arbitrary machine state transitioning from i to j, i.e., $X_n = j \,|\, X_{n-1} = i$. Thus, the one-step transition matrix is

$$T_Y = \begin{bmatrix} 0.09 & 0.42 & 0.49 \\ 0.12 & 0.46 & 0.42 \\ 0.16 & 0.48 & 0.36 \end{bmatrix} \tag{11}$$

Every row of T_Y adds up to unity proving it a legitimate transition matrix. Thus, $\{Y_n : n \geq 1\}$ is a Markov chain.