

1. If G is open and P is a special polygon with $P \subset G$, prove there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

Solution: Recall the definition of a measure on an open set G :

$$\lambda(G) := \sup \{ \lambda(P) : P \subset G, P \text{ is a special polygon} \}.$$

We have two occasions:

1. $\lambda(G) < \infty$.
2. $\lambda(G) = \infty$.

Let's address the first case.

1. If $\lambda(G) < \infty$, according to the definition of *supremum*, it satisfies the following property:

$$\lambda(G) - \epsilon = \lambda(P)$$

for some $\epsilon > 0$. Then by the definition of *supremum*, there exists an element $\lambda(P') \in \{ \lambda(P) : P \subset G, P \text{ is a special polygon} \}$ such that

$$\underbrace{\lambda(G) - \epsilon < \lambda(P')}_{\lambda(P) < \lambda(P')} < \lambda(G).$$

2. If $\lambda(G) = \infty$, the set $\{ \lambda(P) : P \subset G, P \text{ is a special polygon} \}$ is not bounded above. Therefore, it immediately follows that there exists an element

$$\lambda(P') \in \{ \lambda(P) : P \subset G, P \text{ is a special polygon} \}$$

such that $\lambda(P) < \lambda(P')$.

2. (a) Prove that if G is a bounded open set, then $\lambda(G) < \infty$.

(b) In the plane \mathbb{R}^2 let

$$G = \left\{ (x, y) : 1 < x \text{ and } 0 < y < \frac{1}{x} \right\}.$$

Prove that $\lambda(G) = \infty$

(c) In the plane \mathbb{R}^2 let

$$G = \{ (x, y) : 0 < x \text{ and } 0 < y < e^{-x} \}.$$

Prove that $\lambda(G) = 1$.

(d) In the plane \mathbb{R}^2 let

$$G = \{ (x, y) : 1 < x \text{ and } 0 < y < x^{-a} \},$$

where a is a real number satisfying $a > 1$. Prove that $\lambda(G) = 1/(a-1)$.

Solution: (a) Let G' be the set of all limit points of G . Then the closure $\overline{G} = G \cup G'$ is compact because it is a closed and bounded set. Since \overline{G} is compact, there exists a finite subcover $\bigcup_{i=1}^k V_i$ of \overline{G} . If a set is closed and bounded in \mathbb{R}^n , then it can be contained in I for some n -cell I , i.e., $\overline{G} \subset I$. Because n -cell is a special rectangle, $\lambda(I)$ is well-defined to be finite. Thus,

$$\lambda(G) \leq \lambda(\overline{G}) \leq \lambda(I) < \infty.$$

Solution:

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3. Let $G_i, i \in \mathcal{I}$, be a collection of disjoint open sets in \mathbb{R}^n . Prove that only countably many of these sets are nonempty.

Solution: We can use the fact that \mathbb{Q}^n is dense in \mathbb{R}^n , which is put another way as such: every neighbourhood in \mathbb{R}^n contains a point of \mathbb{Q}^n . Therefore, if G_i is a nonempty open set, every point in G_i is an interior point with a neighbourhood with a point in \mathbb{Q}^n contained in G_i . Since G_i are disjoint, they must not share a point in \mathbb{Q}^n , which concludes that the number of nonempty open sets in the collection $\{G_i\}$ is as large as the cardinality of \mathbb{Q}^n . \mathbb{Q}^n has the same cardinality as \mathbb{N}^n thereby lending itself to the definition of *countability*.

4. *The structure of open sets in \mathbb{R} .*

Prove that every nonempty open subset of \mathbb{R} can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k (a_k, b_k),$$

where the range on k can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals.

Solution: For every point $x \in G$, we define A_x to be the largest interval contained within G . We can construct A_x by making use of supremum and infimum.

$$b_x := \sup \{b : b > x \text{ and } (x, b) \subset G\}$$

$$a_x := \inf \{a : a < x \text{ and } (a, x) \subset G\}$$

Then $A_x := (a_x, b_x)$. By construction, $G = \bigcup_{x \in G} A_x$. Now we are left with two problems: (1) A_x is uniquely defined. (2) The collection $\{A_x\}_{x \in G}$ is countable.

1. Suppose there exist two distinct points $x, x' \in G$ whose defined A_x and $A_{x'}$ overlap, i.e., $A_x \cap A_{x'} \neq \emptyset$. It follows that $A_x \cup A_{x'} \subset A_x$ since A_x is by definition the largest interval containing x within G . Likewise, $A_x \cup A_{x'} \subset A_{x'}$. Together, they suggest $A_x = A_{x'}$. This implies that if two $A_x, A_{x'}$ overlap, they do not overlap partially. They overlap entirely which makes them the same interval. Otherwise, they are disjoint, which makes each of them unique in the collection $\{A_x\}_{x \in G}$.
2. Recall that the set of rational numbers \mathbb{Q} is dense in \mathbb{R} . This states that every set in the collection $\{A_x\}_{x \in G}$ has at least one rational number. Since they are disjoint, there exists an injection $\{A_x\}_{x \in G} \mapsto \mathbb{Q}$. Therefore, the cardinality of the collection $\{A_x\}_{x \in G}$ is at its largest the same as that of \mathbb{Q} .

5. In the notation of the previous problem, prove that $\lambda(G) = \sum_k (b_k - a_k)$.

Solution:

6. Prove that the open disk $B(0, 1)$ in \mathbb{R}^2 cannot be expressed as a disjoint union of open rectangles.

Solution:

7. Prove that every nonempty open subset of \mathbb{R}^n can be expressed as a countable union of nonoverlapping special rectangles, which may be taken to be cubes:

$$G = \bigcup_{k=1}^{\infty} I_k.$$

The range on k must be infinite. Why?

(HINT: First pave \mathbb{R}^n with cubes of side 1. Select those cubes which are contained in G . Then bisect the sides of the remaining cubes to obtain cubes with side $1/2$. Select those cubes which are contained in G .)

Solution:

8. In the notation of Problem 2, calculate the measure of the totality of discarded open sets. That is, calculate $\lambda(\bigcup_{k=1}^{\infty} G_k)$.

Solution:

9. Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$. (This result will probably surprise you: Although G is open and contains every rational number, “most” of \mathbb{R} is in G^c .)

Solution:

10. Use your method of working Problem 12 to give a proof that \mathbb{R} is uncountable (cf. Section 1B).

Solution: