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Real Analysis

HW2

1. Prove M6. (You must use the assumption $\lambda(A_1) < \infty$.) If the A_k's are measurable and $A_1 \supset A_2 \supset A_3 \supset \cdots$, and if $\lambda(A_1) < \infty$, then

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \lambda(A_k).$$

Solution: In order to use M5, let us define $B_k = A_1 \setminus A_k$. According to the Corollary, if $A_1, A_k \in \mathcal{L}_0$, then $A_1 \setminus A_k \in \mathcal{L}_0$ as well. Also B_k has the following property:

$$B_1 \subset B_2 \subset B_3 \subset \cdots.$$

This is because

$$\begin{aligned}
B_1 &= A_1 \setminus A_1 = \emptyset \\
B_2 &= A_1 \setminus A_2 = A_1 \cap A_2^{\mathsf{c}} \\
&\vdots \\
B_k &= A_1 \setminus A_k = A_1 \cap A_k^{\mathsf{c}}
\end{aligned}$$

and $A_1^{\mathsf{c}} \subset A_2^{\mathsf{c}} \subset \cdots$. Therefore, we can apply M5 to B_k:

$$\lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \lambda(B_k).$$

Rewriting the LHS in terms of its original representation,

$$\begin{aligned}
\bigcup_{k=1}^{\infty} B_k &= (A_1 \cap A_1^{\mathsf{c}}) \cup (A_1 \cap A_1^{\mathsf{c}}) \cup \cdots = A_1 \cap (A_1^{\mathsf{c}} \cup A_2^{\mathsf{c}} \cup \cdots) \\
&= A_1 \cap \left(\bigcup_{k=1}^{\infty} A_k^{\mathsf{c}}\right) \\
&= A_1 \cap \left(\bigcap_{k=1}^{\infty} A_k\right)^{\mathsf{c}} \qquad \text{(By DeMorgan's Law)} \\
&= A_1 \setminus \left(\bigcap_{k=1}^{\infty} A_k\right)
\end{aligned}$$

By M2, $\bigcap_{k=1}^{\infty} A_k \in \mathcal{L}$ and thus $A_1 \setminus \bigcap_{k=1}^{\infty} A_k \in \mathcal{L}$. Now we will use M11 since $\bigcap_{k=1}^{\infty} A_k \subset A_1$:

$$\lambda^*\left(\bigcap_{k=1}^{\infty} A_k\right) + \lambda_*\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) = \lambda(A_1).$$

Because they are all measurable, outer measure, inner measure, and measure are all same:

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) + \lim_{k\to\infty} \lambda(B_k) = \lambda(A_1). \tag{1}$$

Apply M11 again to $B_k = A_1 \setminus A_k$ using the fact that $A_k \subset A_1$,

$$\lambda^*(A_k) + \lambda_*(A_1 \setminus A_k) = \lambda(A_1).$$

Likewise, they are all measurable and thus inner measure, outer measure are all same. Taking the limit on both sides after rearrangement,

$$\lim_{k\to\infty} \lambda(A_k) = \lambda(A_1) - \lim_{k\to\infty} \lambda(B_k). \tag{2}$$

Plugging eqn (2) back to eqn (1), we obtain

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \lambda(A_k).$$

2. Let $A \subset \mathbb{R}^n$ be arbitrary. Prove that

$$\lambda^*(A) = \inf\left\{\sum_{k=1}^{\infty} \lambda(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k\right\}$$

where the I_k's are special rectangles.
(HINT: If $A \subset \bigcup_{k=1}^{\infty} I_k$, the $\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda(I_k)$ (why?). This will establish that $\lambda^*(A) \leq \inf\{\ \}$. To establish the reverse inequality, consider a well chosen $G \supset A$, and apply Problem 9.)

Solution: Let I_k be nonoverlapping special rectangles. Then by *3,

$$\lambda^*(A) \leq \lambda^*\left(\bigcup_{k=1}^{\infty} I_k\right).$$

Since the measure of $\bigcup_{k=1}^{\infty} I_k$ is already defined in stage 1, the outer measure has to be the measure itself:

$$\lambda^*\left(\bigcup_{k=1}^{\infty} I_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} I_k\right)$$

which is also $\sum_{k=1}^{\infty} \lambda(I_k)$. Therefore, we have obtained the first inequality

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda(I_k).$$

Next, according to Problem 9, any nonempty open set $G \subset \mathbb{R}^n$ can be expressed as

$$G = \bigcup_{k=1}^{\infty} I_k,$$

then, $A \subset G$.

3. Suppose that $A \cup B$ is measurable and that

$$\lambda(A \cup B) = \lambda^*(A) + \lambda^*(B) < \infty.$$

Prove that A and B are measurable.

Solution: By M11,

$$\begin{aligned}\lambda^*(B) + \lambda_*((A \cup B) \setminus B) &= \lambda(A \cup B) \\ &= \lambda^*(A) + \lambda^*(B).\end{aligned}$$

Note that $(A \cup B) \setminus B = A \setminus B$ using elementary set operations:

$$\begin{aligned}(A \cup B) \setminus B &= (A \cup B) \cap B^{\mathsf{c}} \\ &= (A \cap B^{\mathsf{c}}) \cup (B \cap B^{\mathsf{c}}) \\ &= (A \cap B^{\mathsf{c}}) \\ &= A \setminus B.\end{aligned}$$

Therefore, $\lambda_*(A \setminus B) = \lambda^*(A)$ which by definition is

$$\sup\{\lambda(K) : K \subset A \setminus B,\ K \text{ compact}\} = \inf\{\lambda(G) : A \subset G,\ G \text{ open}\}.$$

For every compact set $K \subset A \setminus B$, it is obvious that $K \subset A$. Therefore, $K \subset A \subset G$, which lends itself directly to the theorem on approximation. Therefore, A is measurable. Now using M11 once more as

$$\begin{aligned}\lambda^*(A) + \lambda_*((A \cup B) \setminus A) &= \lambda(A \cup B) \\ &= \lambda^*(A) + \lambda^*(B),\end{aligned}$$

we can arrive at the same conclusion for B.

4. Prove that if A and B are measurable , then

$$\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B).$$

Solution: By M3, $A \setminus B, B \setminus A \in \mathcal{L}$ and by corollary, $A \cap B \in \mathcal{L}$. According to the theorem of countable additivity,

$$\lambda(A) = \lambda(A \setminus B) + \lambda(A \cap B)$$
$$\lambda(B) = \lambda(B \setminus A) + \lambda(A \cap B)$$

holds for disjoint sets $A \setminus B, A \cap B$ and $B \setminus A, A \cap B$. Therefore,

$$\lambda(A) + \lambda(B) = \lambda(A \setminus B) + \lambda(B \setminus A) + 2\lambda(A \cap B).$$

Because $A \setminus B, A \cap B, B \setminus A$ are all disjoint, the following holds:

$$\lambda((A \setminus B) \cup (B \setminus A) \cup (A \cap B)) = \lambda(A \setminus B) + \lambda(B \setminus A) + \lambda(A \cap B).$$

Note that $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$. Therefore,

$$\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B).$$

5. Prove that in general

$$\lambda^*(A) + \lambda^*(B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$$

and

$$\lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) + \lambda_*(A \cap B).$$

Solution: Recall the definition of an outer measure:

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G,\ G \text{ open}\}.$$

Then since $A \subset G$ and G is open, *5 states that G is measurable. Therefore, we can take 2 open sets G_1, G_2 for which

$$\begin{aligned} A &\subset G_1 \\ B &\subset G_2. \end{aligned}$$

Then, it is obvious that $A \cup B \subset G_1 \cup G_2$ and $A \cap B \subset G_1 \cap G_2$. Now we will use the result from the previous problem for measurable sets that

$$\lambda(G_1) + \lambda(G_2) = \lambda(G_1 \cup G_2) + \lambda(G_1 \cap G_2).$$

6. Prove that if A is countable, then $\lambda(A) = 0$. (In particular, $\lambda(\mathbb{Q}) = 0$.)

Solution: Let $A \subset \mathbb{R}^n$. Since a countable set can be indexed as a sequence, let $\{x_k\}_{k=1}^{\infty}$ be the indexed sequence of the set A. Then for every x_k, there exists a neighbourhood $B\left(x_k, 2^{-k}\epsilon\right)$ such that $B\left(x_k, 2^{-k}\epsilon\right) \subset I_k$ where

$$I_k = \left[x_{k1} - 2^{-k}\epsilon, x_{k1} + 2^{-k}\epsilon\right] \times \left[x_{k2} - 2^{-k}\epsilon, x_{k2} + 2^{-k}\epsilon\right] \times \cdots \times \left[x_{kn} - 2^{-k}\epsilon, x_{kn} + 2^{-k}\epsilon\right].$$

It holds that $A \subset \bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)$ and $\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right) \subset \bigcup_{k=1}^{\infty} I_k$. Therefore,

$$\lambda(A) \leq \lambda\left(\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} \lambda(I_k)$$

where

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda(I_k) &= 2^n \epsilon \sum_{k=1}^{\infty} 2^{-nk} \\ &= 2^n \epsilon \frac{2^{-n}}{1-2^{-n}} \\ &= \epsilon \frac{2^n}{2^n - 1} < (2\epsilon)^n .\end{aligned}$$

Since ϵ is arbitrary,

$$\lambda(A) \leq \lambda\left(\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} \lambda(I_k) \leq (2\epsilon)^n \leq 0.$$

Therefore, $\lambda(A) = 0$.

7. Let $a \in \mathbb{R}$ be fixed. Prove that

$$\lambda\left(\{a\} \times \mathbb{R}^{n-1}\right) = 0.$$

Solution: We will use problem 9 and problem 27 to solve this. According problem 9, $\mathbb{R}^n$ can be covered with special rectangles having a side length of 1, putting each on the lattice of integer length cubes. Since the set of integers is countable, we can recast it into a sequence $\{x_s\}_{s=1}^{\infty}$.

8. Prove that if $E \subset \mathbb{R}^n$ and $\lambda^*(E) < \infty$, then a measurable set A exists such that

$$E \subset A \text{ and } \lambda^*(E) = \lambda(A).$$

(The set A is called a *measurable hull* of E.)
(HINT: Choose open sets G_k such that $\lambda(G_k) < \lambda^*(E) + k^{-1}$ and $E \subset G_k$. Then let $A = \bigcap_{k=1}^{\infty} G_k$.)

Solution: Using the hint, since G_k are open sets, we can use property *5 and say

$$\lambda(G_k) = \lambda^*(G_k) = \lambda_*(G_k).$$

Because we chose G_k such that $\lambda(G_K) < \lambda^*(E) + k^{-1}$, it is obvious that $\lambda(G) < \infty$. By definition, $G_k \in \mathcal{L}_0$ and thus $G_k \in \mathcal{L}$. Now we rely on the fact that $\mathcal{L}$ is a σ-algebra. Then,

$$A = \bigcap_{k=1}^{\infty} G_k \in \mathcal{L}.$$

Since $E \subset G_k$ for all k, it is obvious that $E \subset \bigcap_{k=1}^{\infty} G_k$. Therefore,

$$\lambda\left(\bigcap_{k=1}^{\infty} G_k\right) \geq \lambda^*(E).$$

For the other direction, we should note that $\lambda\left(\bigcap_{k=1}^{\infty} G_k\right) \leq \lambda(G_k)$ since $\bigcap_{k=1}^{\infty} G_k \subset G_k$ and *2. We have chosen G_k such that $\lambda(G_k) < \lambda^*(E) + k^{-1}$, the following holds:

$$\lambda\left(\bigcap_{k=1}^{\infty} G_k\right) \leq \lambda(G_k) < \lambda^*(E) + k^{-1}.$$

Since $k^{-1} \to 0$ as $k \to \infty$, we can conclude that

$$\lambda\left(\bigcap_{k=1}^{\infty} G_k\right) \leq \lambda^*(E).$$

Combining both inequalities,

$$\lambda(A) = \lambda^*(E).$$

9. Let $\lambda^*(E) < \infty$, $E \subset A \in \mathcal{L}$. Prove that A is a measurable hull of $E \iff \lambda_*(A \setminus E) = 0$.

Solution: $(\Rightarrow)$

If A is a measurable hull of E, by M11 and the result of the previous problem,

$$\lambda^*(E) + \lambda_*(A \setminus E) = \lambda(A)$$
$$\lambda(A) = \lambda^*(E).$$

Combining two equations, we obtain that

$$\lambda_*(A \setminus E) = 0.$$

$(\Leftarrow)$

If $\lambda_*(A \setminus E) = 0$, by M11

$$\lambda^*(E) = \lambda(A).$$

By the result from the previous problem, we call this set A, the measurable hull of E.

10. Prove that if $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \lambda^*\left(E_k\right).$$

(HINT: Let A_k be a measurable hull of E_k, and define $B_k = \bigcap_{j=k}^{\infty} A_j$. Then B_k is also a measurable hull of E_k, and furthermore $B_1 \subset B_2 \subset B_3 \subset \cdots$. Apply Property M5.)

Solution:

11. Suppose $A_1, A_2, A_3, \ldots$ are measurable sets and

$$\sum_{k=1}^{\infty} \lambda\left(A_k\right) < \infty.$$

Prove that $\lambda\left(\limsup A_k\right) = 0$.

Solution: