Rosen's paper review

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1 Model Specifications

$$\boldsymbol{y}_i\left(t_{ij}\right)=\boldsymbol{X}_{ij}\boldsymbol{\mu}\left(t_{ij}\right)+\boldsymbol{Z}_{ij}\boldsymbol{g}_i\left(t_{ij}\right)+\boldsymbol{\delta}_i\left(t_{ij}\right) \tag{1}$$

where

$$\begin{aligned}
\boldsymbol{y}_i\left(t_{ij}\right) &: \ p\times 1 \qquad i=1,\ldots,n,\ j=1,\ldots,m_i && (2)\\
\boldsymbol{\mu}\left(t\right) &= \left(\boldsymbol{\mu}_1'\left(t\right),\ldots,\boldsymbol{\mu}_p'\left(t\right)\right)' && (3)\\
\boldsymbol{\mu}_k'\left(t\right) &= \left(\mu_{k1}\left(t\right),\ldots,\mu_{kr}\left(t\right)\right)' : \qquad r\times 1 && (4)\\
\boldsymbol{g}_i\left(t\right) &= \left(\boldsymbol{g}_{i1}'\left(t\right),\ldots,\boldsymbol{g}_{ip}'\left(t\right)\right)' && (5)\\
\boldsymbol{g}_{ik}\left(t\right) &= \left(g_{ij1}\left(t\right),\ldots,g_{iks}\left(t\right)\right)' : \qquad s\times 1 && (6)\\
\boldsymbol{x}_{ij} &: \ r\times 1 && (7)\\
\boldsymbol{z}_{ij} &: \ s\times 1 && (8)\\
\boldsymbol{X}_{ij} &= \boldsymbol{I}_p\otimes\boldsymbol{x}_{ij}' && (9)\\
\boldsymbol{Z}_{ij} &= \boldsymbol{I}_p\otimes\boldsymbol{z}_{ij}' && (10)\\
\boldsymbol{\delta}_i\left(t_{ij}\right) &\sim \text{Ornstein-Uhlenbeck process} && (11)
\end{aligned}$$

2 Ornstein-Uhlenbeck process

An Ornstein-Uhlenbeck process is the second-order stationary process $\{X_t\}$, that satisfies the following differential equation:

$$dX\left(t\right)=-aX\left(t\right)\,dt+\sigma\,dB\left(t\right),\quad t\geq 0 \tag{12}$$

where $\{B\left(t\right)\}$ is standard Brownian motion, and a and $\sigma>0$ are parameters and X_0 is a random variable that is independent of $\{B\left(t\right)\}$.

2.1 Multivariate Ornstien-Uhlenbeck process

The univariate version of the OU process naturally evolves to a multivariate form which reads as follows:

$$d\boldsymbol{X}\left(t\right)=-\boldsymbol{A}\boldsymbol{X}\left(t\right)\,dt+\boldsymbol{B}\,d\boldsymbol{W}\left(t\right), \tag{13}$$

where $\boldsymbol{A},\boldsymbol{B}$ are constant matrices. The solution for this SDE is

$$\boldsymbol{X}\left(t\right)=\exp\left(-\boldsymbol{A}t\right)\boldsymbol{X}_0+\int_0^t\exp\left\{-\boldsymbol{A}\left(t-t'\right)\right\}\boldsymbol{B}\,d\boldsymbol{W}\left(t\right). \tag{14}$$

The properties of such an OU process are as folows:

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- If $\boldsymbol{A}$ has only eigenvalues with positive real part, a stationary solution exists of the form

$$\boldsymbol{X}_s(t) = \int_{-\infty}^{t} \exp\left\{-\boldsymbol{A}\left(t - t'\right)\right\} \boldsymbol{B}\, d\boldsymbol{W}(t). \tag{15}$$

  The expected value $\mathrm{E}\left[\boldsymbol{X}_s(t)\right] = 0$ and the covariance matrix

$$\boldsymbol{\Sigma} = \mathrm{Cov}\left(\boldsymbol{X}_s(t), \boldsymbol{X}_s'(s)\right) = \int_{-\infty}^{\min(t,s)} \exp\left\{-\boldsymbol{A}\left(t - t'\right)\right\} \boldsymbol{B}\boldsymbol{B}' \exp\left\{-\boldsymbol{A}'\left(s - t'\right)\right\}\, dt'. \tag{16}$$

- The stationary covariance matrix satisfies the following equation:

$$\boldsymbol{A}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{A}' = \boldsymbol{B}\boldsymbol{B}'. \tag{17}$$

  The solution to this equation is given by

$$\boldsymbol{\Sigma} = \frac{|\boldsymbol{A}|\,\boldsymbol{B}\boldsymbol{B}' + \left[\boldsymbol{A} - \mathrm{Tr}(\boldsymbol{A})\,\boldsymbol{I}\right]\boldsymbol{B}\boldsymbol{B}'\left[\boldsymbol{A} - \mathrm{Tr}(\boldsymbol{A})\,\boldsymbol{I}\right]}{2\left(\mathrm{Tr}(\boldsymbol{A})\right)|\boldsymbol{A}|} \tag{18}$$

- *(Time Correlation Matrix in the Stationary State)* The following relations hold.

$$\begin{aligned}
\mathrm{Cov}\left(\boldsymbol{X}_s(t), \boldsymbol{X}_s'(s)\right) &= \exp\left\{-\boldsymbol{A}(t-s)\right\}\boldsymbol{\Sigma}, && t > s && (19)\\
&= \boldsymbol{\Sigma}\exp\left\{-\boldsymbol{A}'(s-t)\right\}, && t < s && (20)
\end{aligned}$$

- If all its transition densities depend only on the time differences, then the Markov process is *homogeneous*. OU process is homogeneous. Therefore, the transition probabiliy of an OU process is given by

$$\mathrm{P}\left(\boldsymbol{X}_s(t) \,|\, \boldsymbol{X}_s(s), t-s\right) = |2\pi\boldsymbol{\Omega}|^{-1/2} \exp\left\{-\frac{1}{2}\gamma'\boldsymbol{\Omega}^{-1}\gamma\right\} \tag{21}$$

  where

$$\begin{aligned}
\gamma &= \boldsymbol{X}_s(t) - \exp\left(-\boldsymbol{A}(t-s)\right)\boldsymbol{X}_s(s) && (22)\\
\boldsymbol{\Omega} &= \boldsymbol{\Sigma} - \exp\left\{-\boldsymbol{A}(t-s)\right\}\boldsymbol{\Sigma}\exp\left\{-\boldsymbol{A}'(t-s)\right\}. && (23)
\end{aligned}$$

3 Cubic Spline