Daeyoung Lim

Real Analysis

HW2

1. Prove M6. (You must use the assumption $\lambda(A_1) < \infty$.) If the A_k's are measurable and $A_1 \supset A_2 \supset A_3 \supset \cdots$, and if $\lambda(A_1) < \infty$, then

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \lambda(A_k).$$

Solution: In order to use M5, let us define $B_k = A_1 \setminus A_k$. According to the Corollary, if $A_1, A_k \in \mathcal{L}_0$, then $A_1 \setminus A_k \in \mathcal{L}_0$ as well. Also B_k has the following property:

$$B_1 \subset B_2 \subset B_3 \subset \cdots.$$

This is because

$$\begin{aligned}
B_1 &= A_1 \setminus A_1 = \emptyset \\
B_2 &= A_1 \setminus A_2 = A_1 \cap A_2^{\mathsf{c}} \\
&\vdots \\
B_k &= A_1 \setminus A_k = A_1 \cap A_k^{\mathsf{c}}
\end{aligned}$$

and $A_1^{\mathsf{c}} \subset A_2^{\mathsf{c}} \subset \cdots$. Therefore, we can apply M5 to B_k:

$$\lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \lambda(B_k).$$

Rewriting the LHS in terms of its original representation,

$$\begin{aligned}
\bigcup_{k=1}^{\infty} B_k = (A_1 \cap A_1^{\mathsf{c}}) \cup (A_1 \cap A_1^{\mathsf{c}}) \cup \cdots &= A_1 \cap (A_1^{\mathsf{c}} \cup A_2^{\mathsf{c}} \cup \cdots) \\
&= A_1 \cap \left(\bigcup_{k=1}^{\infty} A_k^{\mathsf{c}}\right) \\
&= A_1 \cap \left(\bigcap_{k=1}^{\infty} A_k\right)^{\mathsf{c}} \qquad \text{(By DeMorgan's Law)} \\
&= A_1 \setminus \left(\bigcap_{k=1}^{\infty} A_k\right)
\end{aligned}$$

By M2, $\bigcap_{k=1}^{\infty} A_k \in \mathcal{L}$ and thus $A_1 \setminus \bigcap_{k=1}^{\infty} A_k \in \mathcal{L}$. Now we will use M11 since $\bigcap_{k=1}^{\infty} A_k \subset A_1$:

$$\lambda^*\left(\bigcap_{k=1}^{\infty} A_k\right) + \lambda_*\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) = \lambda(A_1).$$

Because they are all measurable, outer measure, inner measure, and measure are all same:

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) + \lim_{k\to\infty} \lambda(B_k) = \lambda(A_1). \tag{1}$$

Apply M11 again to $B_k = A_1 \setminus A_k$ using the fact that $A_k \subset A_1$,

$$\lambda^* (A_k) + \lambda_* (A_1 \setminus A_k) = \lambda (A_1) .$$

Likewise, they are all measurable and thus inner measure, outer measure are all same. Taking the limit on both sides after rearrangement,

$$\lim_{k\to\infty} \lambda (A_k) = \lambda (A_1) - \lim_{k\to\infty} \lambda (B_k) . \tag{2}$$

Plugging eqn (2) back to eqn (1), we obtain

$$\lambda \left(\bigcap_{k=1}^{\infty} A_k \right) = \lim_{k\to\infty} \lambda (A_k) .$$

2. Let $A \subset \mathbb{R}^n$ be arbitrary. Prove that

$$\lambda^* (A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda (I_k) : A \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

where the I_k's are special rectangles.
(HINT: If $A \subset \bigcup_{k=1}^{\infty} I_k$, the $\lambda^* (A) \le \sum_{k=1}^{\infty} \lambda (I_k)$ (why?). This will establish that $\lambda^* (A) \le \inf \{ \}$. To establish the reverse inequality, consider a well chosen $G \supset A$, and apply Problem 9.)

Solution:

3. Suppose that $A \cup B$ is measurable and that

$$\lambda (A \cup B) = \lambda^* (A) + \lambda^* (B) < \infty.$$

Prove that A and B are measurable.

Solution:

4. Prove that if A and B are measurable , then

$$\lambda (A) + \lambda (B) = \lambda (A \cup B) + \lambda (A \cap B) .$$

Solution:

5. Prove that in general

$$\lambda^*(A) + \lambda^*(B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B)$$

and

$$\lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) + \lambda_*(A \cap B).$$

Solution:

6. Prove that if A is countable, then $\lambda(A) = 0$. (In particular, $\lambda(\mathbb{Q}) = 0$.)

Solution: Let $A \subset \mathbb{R}^n$. Since a countable set can be indexed as a sequence, let $\{x_k\}_{k=1}^{\infty}$ be the indexed sequence of the set A. Then for every x_k, there exists a neighbourhood $B\left(x_k, 2^{-k}\epsilon\right)$ such that $B\left(x_k, 2^{-k}\epsilon\right) \subset I_k$ where

$$I_k = \left[x_{k1} - 2^{-k}\epsilon, x_{k1} + 2^{-k}\epsilon\right] \times \left[x_{k2} - 2^{-k}\epsilon, x_{k2} + 2^{-k}\epsilon\right] \times \cdots \times \left[x_{kn} - 2^{-k}\epsilon, x_{kn} + 2^{-k}\epsilon\right].$$

It holds that $A \subset \bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)$ and $\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right) \subset \bigcup_{k=1}^{\infty} I_k$. Therefore,

$$\lambda(A) \leq \lambda\left(\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} \lambda(I_k)$$

where

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda(I_k) &= 2^n\epsilon \sum_{k=1}^{\infty} 2^{-nk} \\
&= 2^n\epsilon \frac{2^{-n}}{1 - 2^{-n}} \\
&= \epsilon \frac{2^n}{2^n - 1} < (2\epsilon)^n.
\end{aligned}$$

Since ϵ is arbitrary,

$$\lambda(A) \leq \lambda\left(\bigcup_{k=1}^{\infty} B\left(x_k, 2^{-k}\epsilon\right)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} \lambda(I_k) \leq (2\epsilon)^n \leq 0.$$

Therefore, $\lambda(A) = 0$.

7. Let $A \in \mathbb{R}$ be fixed. Prove that

$$\lambda\left(\{a\} \times \mathbb{R}^{n-1}\right) = 0.$$

Solution:

8. Prove that if $E \subset \mathbb{R}^n$ and $\lambda^*(E) < \infty$, then a measurable set A exists such that

$$E \subset A \text{ and } \lambda^*(E) = \lambda(A).$$

(The set A is called a *measurable hull* of E.)
(HINT: Choose open sets G_k such that $\lambda(G_k) < \lambda^*(E) + k^{-1}$ and $E \subset G_k$. Then let $A = \bigcap_{k=1}^{\infty} G_k$.)

Solution:

9. Let $\lambda^*(E) < \infty$, $E \subset A \in \mathcal{L}$. Prove that A is a measurable hull of $E \iff \lambda_*(A \setminus E) = 0$.

Solution:

10. Prove that if $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} \lambda^*(E_k).$$

(HINT: Let A_k be a measurable hull of E_k, and define $B_k = \bigcap_{j=k}^{\infty} A_j$. Then B_k is also a measurable hull of E_k, and furthermore $B_1 \subset B_2 \subset B_3 \subset \cdots$. Apply Property M5.)

Solution:

11. Suppose $A_1, A_2, A_3, \ldots$ are measurable sets and

$$\sum_{k=1}^{\infty} \lambda(A_k) < \infty.$$

Prove that $\lambda(\limsup A_k) = 0$.

Solution: