Real Analysis

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1 Lebesgue Measure

- *(Stage 3)* Let G be an open set such that $\lambda(G) = \sup\{\lambda(p) : p \subset G\}$ where p is a special polygon.
- *(State 4)* Let K be a compact set such that $\lambda(K) = \inf\{\lambda(G) : K \subset G\}$ where G is an open set. If K is a special polygon, then we have 2 definitions of $\lambda(K)$. Let $K = \bigcup_{i=1}^{n} I_i$ where I_i are non-overlapping.

$$\text{old } \lambda(K) \equiv \sum_{i=1}^{n} \lambda(I_i) = \alpha$$
$$\text{new } \lambda(K) = \beta = \inf\{\lambda(G) : K \subset G\}$$

$\alpha \le \beta \leftarrow$ If $G \supset K, \lambda(G) \ge \lambda(K) = \alpha$. To prove, $\beta \le \alpha$, we will show $\forall \epsilon > 0$, $\exists$open $G \supset K$ such that $\lambda(G) \le \alpha + \epsilon$. Choose I_k' such that $I_k \subset (I_k')^\circ$ and $\lambda(I_k') < \lambda(I_k) + \epsilon/N$, then $K = \bigcup_{k=1}^{N} I_k \subset \bigcup_{k=1}^{N} (I_k')^\circ$. Then,

$$\beta \le \overbrace{\lambda\left(\bigcup_{k=1}^{N} (I_k')^\circ\right) \le \lambda(I_k')^\circ}^{\text{O5}} = \sum_{k=1}^{N} \lambda(I_k')$$

1.1 Properties

- *(C1)* $0 \le \lambda(K) < \infty$: A measure of a compact set is by definition the infimum of the open sets that contain the compact set. Therefore, if we can define a measure that is finite with respect to one of the open sets, then the measure of the compact set also has to be finite.
- *(C2)* $K_1 \subset K_2 \implies \lambda(K_1) \le \lambda(K_2)$: $\{\lambda(G) : K_1 \subset G\} \supset \{\lambda(G) : K_2 \subset G\}$.
- *(C3)* $\lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$: enough to show $\lambda(K_1 \cup K_2) \le \lambda(G_1) + \lambda(G_2)$ where $\forall G \supset K_1$ and $\forall G \supset K_2$. <u>Proof is as follows</u>. $\lambda(K_1 \cup K_2) \le \lambda(G_1 \cup G_2)$ since the measures of K_1 and K_2 are defined to be the infimum of the measures assigned on G_1, G_2 that contain them. Further, $\lambda(G_1 \cup G_2) \le \lambda(G_1) + \lambda(G_2)$ by O5. (Q.E.D.)
- *(C4)* If K_1, K_2 are disjoint, $\lambda(K_1 \cup K_2) \ge \lambda(K_1) + \lambda(K_2)$: Recall that $\lambda(K_1 \cup K_2) = \inf\{\lambda(G) : K_1 \cup K_2 \subset G\}$. We need $\lambda(K_1) + \lambda(K_2) \le \lambda(G)$ for any open G containing $K_1 \cup K_2$. Let $K_1 \cup K_2 \subset G$. Then $\exists G_1, G_2$ disjoint open sets such that $K_1 \subset G_1 \subset G$ & $K_2 \subset G_2 \subset G$.

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1.2 Outer and Inner measures

Definition For $A \subset \mathbb{R}^n$, define

$$\lambda^*(A) = \inf\left\{\lambda(G) : A \subset G^{\text{open}}\right\} : \text{ outer measure}$$
$$\lambda_*(A) = \sup\left\{\lambda(K) : A \supset K^{\text{cpt}}\right\} : \text{ inner measure}$$

Properties

- $((*1))$ $\lambda_*(A) \leq \lambda^*(A)$ if $K^{\text{cmp}} \subset A \subset G^{\text{open}}$ because $\lambda(K) \leq \lambda(G)$.

$$\left\{\lambda(G) : A \subset G^{\text{open}}\right\} \supseteq \left\{\lambda(G) : B \subset G\right\}$$
$$\left\{\lambda(K) : K^{\text{cpt}} \subset A\right\} \subseteq \left\{\lambda(K) : K^{\text{cpt}} \subset B\right\}$$