Real Analysis Solutions

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Chapter 10. $L^p Spaces$

Question 10-1. If $1 < p < \infty$ and $a \geq 0$, $b \geq 0$, prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1}$$

with equality if and only if $a^p = b^q$.

Proof. Let b be fixed and define a function

$$f(a) = ab - \frac{a^p}{p} \tag{2}$$

and maximize the function using elementary calculus.

$$\begin{aligned} f'(a) &= b - a^{p-1} = 0 \\ a &= b^{1/(p-1)} \end{aligned}$$

Therefore, the function is maximized when $a = b^{1/(p-1)}$. Plugging in,

$$\begin{aligned} f\left(b^{1/(p-1)}\right) &= b^{p/(p-1)} - \frac{b^{p/(1-p)}}{p} \\ &= \frac{p-1}{p} b^{\frac{p}{p-1}} \end{aligned}$$

is the maximum value. Therefore, we can conclude that

$$ab - \frac{a^p}{p} \leq \frac{b^q}{q} \tag{3}$$

by defining $q = p/(p-1)$. □

Question 10-2. Assume $1 < p_k < \infty$ for $k = 1, \ldots, N$, and $\sum_{k=1}^{N} 1/p_k = 1$. Prove that

$$\left| \int_X f_1 f_2 \cdots f_N \, d\mu \right| \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_N\|_{p_N}. \tag{4}$$

Proof. If $N = 2$, the problem is reduced to the classical Hölder's inequality. Therefore, we will assume 4 holds for some $N > 2$ and then show that it also holds for $N + 1$. First, let us define

$$q^{N+1} = \frac{p_{N+1}}{p_{N+1} - 1}$$

and for $i = 1, \ldots, N$, define

$$r_i = p_i \cdot \left(1 - \frac{1}{p_{N+1}}\right).$$

Then

$$\begin{aligned}\frac{1}{p_{N+1}} + \frac{1}{q_{N+1}} &= 1 \\ \sum_{i=1}^{N} \frac{1}{r_i} &= 1 \\ q_{N+1} \cdot r_i &= p_i.\end{aligned}$$

Applying Hölder's inequality to $f = \prod_{i=1}^{N} f_i$ and $g = f_{N+1}$, we find:

$$\int_X \left|\prod_{i=1}^{N+1} f_i\right| d\mu \leq \|f_{N+1}\|_{p_{N+1}} \cdot \left\|\prod_{i=1}^{N} f_i\right\|_{q_n} \tag{5}$$

$$= \|f_n\| \, p_n \cdots \left(\int_X \prod_{i=1}^{N} \left|f_i^{q_{N+1}}\right| d\mu\right)^{1/q_{N+1}} \tag{6}$$

$$\leq \|f_{N+1}\|_{p_{N+1}} \cdot \left(\prod_{i=1}^{N} \left\|f_i^{q_{N+1}}\right\|_{r_i}\right)^{1/q_{N+1}} \tag{7}$$

$$= \prod_{i=1}^{N+1} \|f_i\|_{p_i} \tag{8}$$

□

Question 10-3. Assume in Hölder's inequality $f \geq 0$, $g \leq 0$, and

$$\int_X fg \, d\mu = \|f\|_p \, \|g\|_q \, .$$

Prove that $f(x)^p = g(x)^q$ μ-a.e., to within a multiplicative constant.

Proof. If is true, then it also indicates

$$\int_X \bar{f}\bar{g} \, d\mu = 1$$

where $\bar{f} = \frac{f}{\|f\|_p}$ and $\bar{g} = \frac{g}{\|g\|_q}$. It also translates to

$$\int_X \bar{f}\bar{g} \, d\mu = \frac{1}{p}\int_X \bar{f}^p \, d\mu + \frac{1}{q}\int_X \bar{g}^q \, d\mu.$$

Therefore, $\bar{f}\bar{g} = \bar{f}^p/p + \bar{g}^q/q$ μ-a.e. Since this is a case of *Young's inequality*, the equality holds if and only if $\bar{f}^p = \bar{g}_q$ μ-a.e. □

Question 10-7. Suppose $1 \le p < r < q < \infty$. Prove that $L^p \cap L^q \subset L^r$.

Proof. Since $\frac{1}{q} < \frac{1}{r} < \frac{1}{p}$, there exists a unique θ such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

The number θ satisfies $0 < \theta < 1$ and equals

$$\theta = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}, \qquad 1 - \theta = \frac{\frac{1}{r} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}}.$$

It follows that $1 = \frac{r\theta}{p} + \frac{r(1-\theta)}{q}$ which indicates that $p/(r\theta)$ and $q/(r(1-\theta))$ are conjugate exponents. Thus by Hölder's inequality,

$$\begin{aligned}
\|f\|_r &= \left\| f^\theta f^{1-\theta} \right\|_r && (9)\\
&= \left\| f^{r\theta} f^{r(1-\theta)} \right\|_1^{1/r} && (10)\\
&\le \left(\left\| f^{r\theta} \right\|_{\frac{p}{r\theta}} \left\| f^{r(1-\theta)} \right\|_{\frac{q}{r(1-\theta)}} \right)^{1/r} && (11)\\
&= \left(\|f\|_p^{r\theta} \|f\|_q^{r(1-\theta)} \right)^{1/r} && (12)\\
&= \|f\|_p^\theta \|f\|_q^{1-\theta} < \infty. && (13)
\end{aligned}$$

Therefore, L^r norm is also finite for all $f \in L^p \cap L^q$. $\square$

Question 10-13. Suppose $1 \le p < r < \infty$. Prove that $L^p \cap L^\infty \subset L^r$. Moreover, show that if $f \in L^p \cap L^\infty$, then

$$\|f\|_r \le \|f\|_p^{p/r} \|f\|_\infty^{1-p/r}.$$

Proof. By the definition of $\|f\|_\infty$, $|f(x)| \le \|f\|_\infty$ μ-a.e.. Therefore,

$$\begin{aligned}
|f(x)|^{r-p} &\le \|f\|_\infty^{r-p}\\
|f|^{r-p} |f|^p &\le \|f\|_\infty^{r-p} |f|^p\\
\int |f|^{r-p} |f|^p \, d\mu &\le \|f\|_\infty^{r-p} \int |f|^p \, d\mu\\
\left(\int |f|^r \, d\mu \right)^{1/r} &\le \|f\|_\infty^{1-p/r} \left\{ \left(\int |f|^p \, d\mu \right)^{1/p} \right\}^{p/r}\\
\|f\|_r &\le \|f\|_\infty^{1-p/r} \|f\|_p^{p/r}.
\end{aligned}$$

$\square$

For the journal

We have similar environments for Theorems, Exercises, and Challenge Problems.

Theorem 1.32. You should restate the theorem here.

Proof. Insert your proof here. □

Exercise 1.22.

Proof. □

Challenge Problem 1.14.

Proof. □

Writing in math mode

When you want to write something in "math mode", you should enclose it in dollar signs: $x + y = a^2 + b^2$. If it's an important equation and you want to set it off, you can do it like this:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If you have a chain of deductions, you can line them up like this. (The ampersand & tells it where to line up—usually you will want it right before the equals sign. You should have an ampersand in each line. The double-backslash \\ tells it to move to the next line.)

$$\begin{aligned} 2x &= 2y - 10 \\ x &= y - 5 \\ z + x &= y - 5 + z \end{aligned}$$

If you want to include justifications, one way is like this:

$$\begin{aligned} 2x &= 2y - 10 && (A5) \\ x &= y - 5 && (A4) \\ z + x &= y - 5 + z && (W) \end{aligned}$$

You can also include words:

$$\begin{aligned} 2x &= 2y - 10 && \text{by commutativity} \\ x &= y - 5 && \text{since } y \text{ is the GCD} \\ z + x &= y - 5 + z && \text{because 2 is prime} \end{aligned}$$

Here are some symbols that might come up in the homeworks and journals (but remember you can view the source of the scripts and homeworks on the course webpage).

Let $x \in \mathbb{Z}[i]$ and $y \in \mathbb{Q}[i]$. Then $x^2 + y^2 \in \mathbb{Q}[i]$. If you have two numbers $n_1 \in \mathbb{Z}$ and $n_2 \in \mathbb{Z}$, then you should be satisfied. Many students don't have it so good. You can ask whether $n_1 < n_2$, or even if $n_1 \leq n_2$. I wonder if $a|b$? Greek letters: α, β, γ, π, Γ, Δ.

Maybe you want to talk about the set $S = \{x \in \mathbb{Z}|x^2 = 2\}$, or maybe the set $T = \{x \in \mathbb{Z}|x^2 \text{ is even}\}$. Did you notice that $S \subseteq T$? It's also true that $S \neq T$.

You might need to write fractions like $\frac{p}{q}$, or even $\frac{a}{a^2+b^2}$. If this fractions are too small, you can display them like this (see how I made those big parentheses?):

$$\left(\frac{p}{q}\right) \cdot \left(\frac{t}{u}\right) = \frac{a}{a^2 + b^2}$$

If you can't figure out a symbol, you should google "Detexify". All you have to do is draw the symbol using your mouse, and it tells you the latex command.