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Real Analysis

HW1

1. If G is open and P is a special polygon with $P \subset G$, prove there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

Solution: Recall the definition of a measure on an open set G:

$$\lambda(G) := \sup\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}.$$

We have two occasions:

1. $\lambda(G) < \infty$.
2. $\lambda(G) = \infty$.

Let's address the first case.

1. If $\lambda(G) < \infty$, according to the definition of *supremum*, it satisfies the following property:

$$\lambda(G) - \epsilon = \lambda(P)$$

for some $\epsilon > 0$. Then by the definition of *supremum*, there exists an element $\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ such that

$$\underbrace{\lambda(G) - \epsilon < \lambda(P')}_{\lambda(P)<\lambda(P')} < \lambda(G).$$

2. If $\lambda(G) = \infty$, the set $\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ is not bounded above. Therefore, it immediately follows that there exists an element

$$\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$$

such that $\lambda(P) < \lambda(P')$.

2. **(a)** Prove that if G is a bounded open set, then $\lambda(G) < \infty$.
(b) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < \frac{1}{x}\right\}.$$

Prove that $\lambda(G) = \infty$
(c) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 0 < x \text{ and } 0 < y < e^{-x}\right\}.$$

Prove that $\lambda(G) = 1$.
(d) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < x^{-a}\right\},$$

where a is a real number satisfying $a > 1$. Prove that $\lambda(G) = 1/(a-1)$.

Solution: **(a)** Let G' be the set of all limit points of G. Then the closure $\overline{G} = G \bigcup G'$ is compact because it is a closed and bounded set. Since $\overline{G}$ is compact, there exists a finite subcover $\bigcup_{i=1}^{k} V_i$ of $\overline{G}$. If a set is closed and bounded in $\mathbb{R}^n$, then it can be contained in I for some n-cell I, i.e., $\overline{G} \subset I$. Because n-cell is a special rectangle, $\lambda(I)$ is well-defined to be finite. Thus,

$$\lambda(G) \leq \lambda\left(\overline{G}\right) \leq \lambda(I) < \infty.$$

Solution: **(b)** Let us equidistantly split the abscissa into length $1/n$. With each piece starting from $[1, 1+1/n]$ to $[1+(k-1)/n, 1+k/n]$ for k^{th} piece, the height of the k^{th} piece should be $(1+k/n)^{-1}$ to fit in G. Then

$$\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \frac{1}{1+k/n} \cdot \frac{1}{n} = \sum_{k=1}^{\infty} \frac{1}{n+k}.$$

This sum already diverges. Therefore, the supremum is apparently ∞.

Solution: Likewise, split the abscissa equidistantly into pieces of lengths $1/n$. Then the measure of all the special rectangles are

$$\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \frac{e^{-k/n}}{n} = \frac{1}{ne^{1/n} - n}.$$

Since we need the supremum of the sum of these rectangles, we take the limit:

$$\lim_{n\to\infty} \frac{1}{ne^{1/n} - n}.$$

By power rule, we can write

$$1 \Big/ \left(\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n}\right).$$

By l'Hôpital's rule,

$$\begin{aligned} \lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} &= \lim_{n\to\infty} \frac{-e^{1/n}/n^2}{-1/n^2} \\ &= \lim_{n\to\infty} e^{1/n} \\ &= 1. \end{aligned}$$

Therefore, the supremum is 1.

Solution: **(d)** In this problem, splitting the abscissa equidistantly will make it even worse. We will cut the points where $x = r^n$ where $r > 1$. Then,

$$\begin{aligned}
\lambda\left(\bigcup_{n=1}^{\infty} I_n\right) &= r^{-a}\left(r-1\right) + \left(r^2\right)^{-a}\left(r^2 - r\right) + \cdots + \left(r^n\right)^{-a}\left(r^n - r^{n-1}\right) + \cdots \\
&= \sum_{n=1}^{\infty} r^{-na}\left(r^n - r^{n-1}\right) \\
&= \frac{r-1}{r}\sum_{n=1}^{\infty} r^{-n(a-1)} \\
&= \frac{r^{-a}\left(r-1\right)}{1-r^{-a+1}} \\
&= \frac{r-1}{r\left(r^{a-1}-1\right)}.
\end{aligned}$$

Since the supremum of the measures of these special polygons is when an arbitrary interval $r^n - r^{n-1} \to 0$, we can conclude that $r \to 1$.

$$\lim_{r\to 1}\frac{r-1}{r\left(r^{a-1}-1\right)} = \lim_{r\to 1}\frac{\cancel{r-1}}{r\cancel{(r-1)}\left(1+r+r^2+\cdots+r^{a-2}\right)}.$$

Therefore, the measure is

$$\lambda\left(G\right) = \frac{1}{1\cdot\left(\underbrace{1+1+\cdots+1}_{a-1\text{ ones}}\right)} = \frac{1}{a-1}.$$

3. Let $G_i, i \in \mathcal{I}$, be a collection of disjoint open sets in $\mathbb{R}^n$. Prove that only countably many of these sets are nonempty.

Solution: We can use the fact that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, which is put another way as such: every neighbourhood in $\mathbb{R}^n$ contains a point of $\mathbb{Q}^n$. Therefore, if G_i is a nonempty open set, every point in G_i is an interior point with a neighbourhood with a point in $\mathbb{Q}^n$ contained in G_i. Since G_i are disjoint, they must not share a point in $\mathbb{Q}^n$, which concludes that the number of nonempty open sets in the collection $\{G_i\}$ is as large as the cardinality of $\mathbb{Q}^n$. $\mathbb{Q}^n$ has the same cardinality as $\mathbb{N}^n$ thereby lending itself to the definition of *countability*.

4. *The structure of open sets in* $\mathbb{R}$.

Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k \left(a_k, b_k\right),$$

where the range on k can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals.

Solution: For every point $x \in G$, we define A_x to be the largest interval contained within G. We can construct A_x by making use of supremum and infimum.

$$b_x := \sup \{b : b > x \text{ and } (x, b) \subset G\}$$
$$a_x := \inf \{a : a < x \text{ and } (a, x) \subset G\}$$

Then $A_x := (a_x, b_x)$. By construction, $G = \bigcup_{x \in G} A_x$. Now we are left with two problems: (1) A_x is uniquely defined. (2) The collection $\{A_x\}_{x \in G}$ is countable.

1. Suppose there exist two distinct points $x, x' \in G$ whose defined A_x and $A_{x'}$ overlap, i.e., $A_x \bigcap A_{x'} \neq \emptyset$. It follows that $A_x \bigcup A_{x'} \subset A_x$ since A_x is by definition the largest interval containing x within G. Likewise, $A_x \bigcup A_{x'} \subset A_{x'}$. Together, they suggest $A_x = A_{x'}$. This implies that if two $A_x, A_{x'}$ overlap, they do not overlap partially. They overlap entirely which makes them the same interval. Otherwise, they are disjoint, which makes each of them unique in the collection $\{A_x\}_{x \in G}$.
2. Recall that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. This states that every set in the collection $\{A_x\}_{x \in G}$ has at least one rational number. Since they are disjoint, there exists an injection $\{A_x\}_{x \in G} \mapsto \mathbb{Q}$. Therefore, the cardinality of the collection $\{A_x\}_{x \in G}$ is at its largest the same as that of $\mathbb{Q}$.

5. In the notation of the previous problem, prove that $\lambda(G) = \sum_k (b_k - a_k)$.

Solution: By property O6, for disjoint sets A_x,

$$\lambda(G) = \lambda\left(\bigcup_{x \in G} A_x\right) = \sum_{x \in G} \lambda(A_x).$$

Since $\lambda(A_x)$ is defined to be $b_x - a_x$, it follows that

$$\lambda(G) = \sum_x (b_x - a_x).$$

6. Prove that the open disk $B(0,1)$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles.

Solution: Recall the fact that a unit ball is a *connected* set, i.e., it is not a union of two nonempty separated sets. However, if we suppose nonempty disjoint open rectangles $\{G_k\}_{k=1}^{\infty}$ can be coupled to create the unit ball $B(0,1)$, then it violates the connectedness of a unit ball. Let $A_1 = G_1$ and $A_2 = \bigcup_{i=2}^{\infty} G_i$. Then $A_1 \bigcap A_2 = \emptyset$ but $A_1 \bigcup A_2 = B(0,1)$, which is a contradiction.

7. Prove that every nonempty open subset of $\mathbb{R}^n$ can be expressed as a countable union of nonoverlapping special rectangles, which may be taken to be cubes:

$$G = \bigcup_{k=1}^{\infty} I_k.$$

The range on k must be infinite. Why?
(HINT: First pave $\mathbb{R}^n$ with cubes of side 1. Select those cubes which are contained in G. Then bisect the sides of the remaining cubes to obtain cubes with side 1/2. Select those cubes which are contained in G.)

Solution: As the hint states, first imagine a lattice with grids of side 1 — grids are special rectangles —. Pick all the rectangles that are contained in G. Of the remaining grids, halve the length of each rectangle. Again, pick the ones that are contained in G. Because of the property proven in Problem 3($P \subset P' \subset G$), we can keep doing this while making all the special rectangles non-overlapping.

Now since G is an open set, each point in G has a neighbourhood (open ball) contained in G. As we proceed the process constructing a fine lattice, there must exist a rectangle that contains the point while being contained in the neighbourhood.

8. Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$. (This result will probably surprise you: Although G is open and contains every rational number, "most" of $\mathbb{R}$ is in G^{c}.)

Solution: From the fact that $\mathbb{Q}$ is countable, we can construct a sequence $\{x_n\}, n \in \mathbb{Z}^+$ for which a bijection f exists with $\mathbb{Q}$. Therefore, if we take sequence to be equivalent to the set of rational numbers, take a neighbourhood around each element in $\{x_n\}$ and denote it by $B(x_n, \epsilon/3^n)$ for some $\epsilon > 0$. In $\mathbb{R}$, a neighbourhood is an open interval. Once we set G to be the union of these neighbourhoods, it is then followed by

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right).$$

An infinite union of open sets is also an open set. Therefore, G is also open. Since

$$\lambda(G) = \lambda\left(\bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right)\right) \le \sum_{n=1}^{\infty} \lambda\left(B\left(x_n, \frac{\epsilon}{3^n}\right)\right) = \sum_{n=1}^{\infty} \frac{\epsilon}{3^n},$$

$\lambda(G) \le \epsilon/2$.

9. Use your method of working Problem 12 to give a proof that $\mathbb{R}$ is uncountable (cf. Section 1B).

Solution: Assume $\mathbb{R}$ is countable. Then using the method in the previous problem, we can construct a sequence with a bijection with $\mathbb{Z}^{+}$. Equally, there exists an open cover of countably many open intervals for $\mathbb{R}$. From this, there exists $\epsilon > 0$ which satisfies the following relation:

$$\lambda\left(\mathbb{R}\right) < \epsilon.$$

This is a violation of the property O3, stating $\lambda\left(\mathbb{R}\right) = \infty$.