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**Real Analysis**
**HW1**

1. If $G$ is open and $P$ is a special polygon with $P \subset G$, prove there exists a special polygon $P'$ such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

**Solution:** Recall the definition of a measure on an open set $G$:

$$\lambda(G) := \sup\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}.$$

We have two occasions:

1. $\lambda(G) < \infty$.
2. $\lambda(G) = \infty$.

Let's address the first case.

1. If $\lambda(G) < \infty$, according to the definition of *supremum*, it satisfies the following property:

$$\lambda(G) - \epsilon = \lambda(P)$$

for some $\epsilon > 0$. Then by the definition of *supremum*, there exists an element $\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ such that

$$\underbrace{\lambda(G) - \epsilon < \lambda(P')}_{\lambda(P)<\lambda(P')} < \lambda(G).$$

2. If $\lambda(G) = \infty$, the set $\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ is not bounded above. Therefore, it immediately follows that there exists an element

$$\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$$

such that $\lambda(P) < \lambda(P')$.

2. **(a)** Prove that if $G$ is a bounded open set, then $\lambda(G) < \infty$.
**(b)** In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < \frac{1}{x}\right\}.$$

Prove that $\lambda(G) = \infty$
**(c)** In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 0 < x \text{ and } 0 < y < e^{-x}\right\}.$$

Prove that $\lambda(G) = 1$.
**(d)** In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < x^{-a}\right\},$$

where $a$ is a real number satisfying $a > 1$. Prove that $\lambda(G) = 1/(a-1)$.

**Solution:** **(a)** Let $G'$ be the set of all limit points of $G$. Then the closure $\overline{G} = G \bigcup G'$ is compact because it is a closed and bounded set. Since $\overline{G}$ is compact, there exists a finite subcover $\bigcup_{i=1}^{k} V_i$ of $\overline{G}$. If a set is closed and bounded in $\mathbb{R}^n$, then it can be contained in $I$ for some $n$-cell $I$, i.e., $\overline{G} \subset I$. Because $n$-cell is a special rectangle, $\lambda(I)$ is well-defined to be finite. Thus,

$$\lambda(G) \le \lambda\left(\overline{G}\right) \le \lambda(I) < \infty.$$

**Solution:** **(b)** Let us equidistantly split the abscissa into length $1/n$. With each piece starting from $[1, 1+1/n]$ to $[1+(k-1)/n, 1+k/n]$ for $k^{\text{th}}$ piece, the height of the $k^{\text{th}}$ piece should be $(1+k/n)^{-1}$ to fit in $G$. Then

$$\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \frac{1}{1+k/n} \cdot \frac{1}{n} = \sum_{k=1}^{\infty} \frac{1}{n+k}.$$

This sum already diverges. Therefore, the supremum is apparently $\infty$.

**Solution:** Likewise, split the abscissa equidistantly into pieces of lengths $1/n$. Then the measure of all the special rectangles are

$$\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \frac{e^{-k/n}}{n} = \frac{1}{ne^{1/n} - n}.$$

Since we need the supremum of the sum of these rectangles, we take the limit:

$$\lim_{n\to\infty} \frac{1}{ne^{1/n} - n}.$$

By power rule, we can write

$$1 \Big/ \left(\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n}\right).$$

By l'Hôpital's rule,

$$\begin{aligned} \lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} &= \lim_{n\to\infty} \frac{-e^{1/n}/n^2}{-1/n^2} \\ &= \lim_{n\to\infty} e^{1/n} \\ &= 1. \end{aligned}$$

Therefore, the supremum is 1.

**Solution:** **(d)** In this problem, splitting the abscissa equidistantly will make it even worse. We will cut the points where $x = r^k$. Then,

$$\begin{aligned}\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) &= \sum_{k=1}^{\infty}\left(r^k\right)^{-a}\left(r^k - r^{k-1}\right)\\ &= \sum_{k=1}^{\infty} r^{k(-a+1)} - r^{k(-a+1)-1}\end{aligned}$$

3. Let $G_i, i \in \mathcal{I}$, be a collection of disjoint open sets in $\mathbb{R}^n$. Prove that only countably many of these sets are nonempty.

**Solution:** We can use the fact that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, which is put another way as such: every neighbourhood in $\mathbb{R}^n$ contains a point of $\mathbb{Q}^n$. Therefore, if $G_i$ is a nonempty open set, every point in $G_i$ is an interior point with a neighbourhood with a point in $\mathbb{Q}^n$ contained in $G_i$. Since $G_i$ are disjoint, they must not share a point in $\mathbb{Q}^n$, which concludes that the number of nonempty open sets in the collection $\{G_i\}$ is as large as the cardinality of $\mathbb{Q}^n$. $\mathbb{Q}^n$ has the same cardinality as $\mathbb{N}^n$ thereby lending itself to the definition of *countability*.

4. *The structure of open sets in* $\mathbb{R}$.
Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k (a_k, b_k),$$

where the range on $k$ can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals.

**Solution:** For every point $x \in G$, we define $A_x$ to be the largest interval contained within $G$. We can construct $A_x$ by making use of supremum and infimum.

$$\begin{aligned} b_x &:= \sup\{b : b > x \text{ and } (x,b) \subset G\}\\ a_x &:= \inf\{a : a < x \text{ and } (a,x) \subset G\}\end{aligned}$$

Then $A_x := (a_x, b_x)$. By construction, $G = \bigcup_{x\in G} A_x$. Now we are left with two problems: (1) $A_x$ is uniquely defined. (2) The collection $\{A_x\}_{x\in G}$ is countable.

1. Suppose there exist two distinct points $x, x' \in G$ whose defined $A_x$ and $A_{x'}$ overlap, i.e., $A_x \bigcap A_{x'} \neq \emptyset$. It follows that $A_x \bigcup A_{x'} \subset A_x$ since $A_x$ is by definition the largest interval containing $x$ within $G$. Likewise, $A_x \bigcup A_{x'} \subset A_{x'}$. Together, they suggest $A_x = A_{x'}$. This implies that if two $A_x, A_{x'}$ overlap, they do not overlap partially. They overlap

entirely which makes them the same interval. Otherwise, they are disjoint, which makes each of them unique in the collection $\{A_x\}_{x\in G}$.

2. Recall that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. This states that every set in the collection $\{A_x\}_{x\in G}$ has at least one rational number. Since they are disjoint, there exists an injection $\{A_x\}_{x\in G} \mapsto \mathbb{Q}$. Therefore, the cardinality of the collection $\{A_x\}_{x\in G}$ is at its largest the same as that of $\mathbb{Q}$.

5. In the notation of the previous problem, prove that $\lambda(G) = \sum_k (b_k - a_k)$.

**Solution:** By property O6, for disjoint sets $A_x$,

$$\lambda(G) = \lambda\left(\bigcup_{x\in G} A_x\right) = \sum_{x\in G} \lambda(A_x).$$

Since $\lambda(A_x)$ is defined to be $b_x - a_x$, it follows that

$$\lambda(G) = \sum_x (b_x - a_x).$$

6. Prove that the open disk $B(0,1)$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles.

**Solution:** Recall the fact that a unit ball is a *connected* set, i.e., it is not a union of two nonempty separated sets. However, if we suppose nonempty disjoint open rectangles $\{G_k\}_{k=1}^{\infty}$ can be coupled to create the unit ball $B(0,1)$, then it violates the connectedness of a unit ball. Let $A_1 = G_1$ and $A_2 = \bigcup_{i=2}^{\infty} G_i$. Then $A_1 \bigcap A_2 = \emptyset$ but $A_1 \bigcup A_2 = B(0,1)$, which is a contradiction.

7. Prove that every nonempty open subset of $\mathbb{R}^n$ can be expressed as a countable union of nonoverlapping special rectangles, which may be taken to be cubes:

$$G = \bigcup_{k=1}^{\infty} I_k.$$

The range on $k$ must be infinite. Why?
(HINT: First pave $\mathbb{R}^n$ with cubes of side 1. Select those cubes which are contained in $G$. Then bisect the sides of the remaining cubes to obtain cubes with side $1/2$. Select those cubes which are contained in $G$.)

**Solution:** As the hint states, first imagine a lattice with grids of side 1 — grids are special rectangles —. Pick all the rectangles that are contained in $G$. Of the remaining grids, halve the length of each rectangle. Again, pick the ones that are contained in $G$. Because of the property proven in Problem 3($P \subset P' \subset G$), we can keep doing this while making all the special rectangles non-overlapping.

Now since $G$ is an open set, each point in $G$ has a neighbourhood (open ball) contained in $G$. As we proceed the process constructing a fine lattice, there must exist a rectangle that contains the point while being contained in the neighbourhood.

8. Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$. (This result will probably surprise you: Although $G$ is open and contains every rational number, "most" of $\mathbb{R}$ is in $G^{\mathsf{c}}$.)

**Solution:** From the fact that $\mathbb{Q}$ is countable, we can construct a sequence $\{x_n\}, n \in \mathbb{Z}^+$ for which a bijection $f$ exists with $\mathbb{Q}$. Therefore, if we take sequence to be equivalent to the set of rational numbers, take a neighbourhood around each element in $\{x_n\}$ and denote it by $B(x_n, \epsilon/3^n)$ for some $\epsilon > 0$. In $\mathbb{R}$, a neighbourhood is an open interval. Once we set $G$ to be the union of these neighbourhoods, it is then followed by

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right).$$

An infinite union of open sets is also an open set. Therefore, $G$ is also open. Since

$$\lambda(G) = \lambda\left(\bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right)\right) \le \sum_{n=1}^{\infty} \lambda\left(B\left(x_n, \frac{\epsilon}{3^n}\right)\right) = \sum_{n=1}^{\infty} \frac{\epsilon}{3^n},$$

$\lambda(G) \le \epsilon/2$.

9. Use your method of working Problem 12 to give a proof that $\mathbb{R}$ is uncountable (cf. Section 1B).

**Solution:** Assume $\mathbb{R}$ is countable. Then using the method in the previous problem, we can construct a sequence with a bijection with $\mathbb{Z}^+$. Equally, there exists an open cover of countably many open intervals for $\mathbb{R}$. From this, there exists $\epsilon > 0$ which satisfies the following relation:

$$\lambda(\mathbb{R}) < \epsilon.$$

This is a violation of the property O3, stating $\lambda(\mathbb{R}) = \infty$.