**Daeyoung Lim**
**Real Analysis**
**HW1**

1. If $G$ is open and $P$ is a special polygon with $P \subset G$, prove there exists a special polygon $P'$ such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

> **Solution:** Recall the definition of a measure on an open set $G$:
>
> $$\lambda(G) := \sup \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}.$$
>
> We have two occasions:
>
> 1. $\lambda(G) < \infty$.
> 2. $\lambda(G) = \infty$.
>
> Let's address the first case.
>
> 1. If $\lambda(G) < \infty$, according to the definition of *supremum*, it satisfies the following property:
>
> $$\lambda(G) - \epsilon = \lambda(P)$$
>
> for some $\epsilon > 0$. Then by the definition of *supremum*, there exists an element $\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ such that
>
> $$\underbrace{\lambda(G) - \epsilon < \lambda(P')}_{\lambda(P) < \lambda(P')} < \lambda(G).$$
>
> 2. If $\lambda(G) = \infty$, the set $\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ is not bounded above. Therefore, it immediately follows that there exists an element
>
> $$\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$$
>
> such that $\lambda(P) < \lambda(P')$.

2. **(a)** Prove that if $G$ is a bounded open set, then $\lambda(G) < \infty$.
**(b)** In the plane $\mathbb{R}^2$ let
$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < \frac{1}{x}\right\}.$$
Prove that $\lambda(G) = \infty$
**(c)** In the plane $\mathbb{R}^2$ let
$$G = \left\{(x, y) : 0 < x \text{ and } 0 < y < e^{-x}\right\}.$$
Prove that $\lambda(G) = 1$.
**(d)** In the plane $\mathbb{R}^2$ let
$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < x^{-a}\right\},$$
where $a$ is a real number satisfying $a > 1$. Prove that $\lambda(G) = 1/(a-1)$.

**Solution:**

**Solution:**

**Solution:**

**Solution:**

3. Let $G_i, i \in \mathcal{I}$, be a collection of disjoint open sets in $\mathbb{R}^n$. Prove that only countably many of these sets are nonempty.

**Solution:** We can use the fact that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, which is put another way as such: every neighbourhood in $\mathbb{R}^n$ contains a point of $\mathbb{Q}^n$. Therefore, if $G_i$ is a nonempty open set, every point in $G_i$ is an interior point with a neighbourhood with a point in $\mathbb{Q}^n$ contained in $G_i$. Since $G_i$ are disjoint, they must not share a point in $\mathbb{Q}^n$, which concludes that the number of nonempty open sets in the collection $\{G_i\}$ is as large as the cardinality of $\mathbb{Q}^n$. $\mathbb{Q}^n$ has the same cardinality as $\mathbb{N}^n$ thereby lending itself to the definition of *countability*.

4. *The structure of open sets in* $\mathbb{R}$.

Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k \left(a_k, b_k\right),$$

where the range on $k$ can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals.

**Solution:**

5. In the notation of the previous problem, prove that $\lambda\left(G\right) = \sum_k \left(b_k - a_k\right)$.

**Solution:**

6. Prove that the open disk $B\left(0,1\right)$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles.

**Solution:**

7. Prove that every nonempty open subset of $\mathbb{R}^n$ can be expressed as a countable union of nonoverlapping special rectangles, which may be taken to be cubes:

$$G = \bigcup_{k=1}^{\infty} I_k.$$

The range on $k$ must be infinite. Why?
(HINT: First pave $\mathbb{R}^n$ with cubes of side 1. Select those cubes which are contained in $G$. Then bisect the sides of the remaining cubes to obtain cubes with side $1/2$. Select those cubes which are contained in $G$.)

**Solution:**

8. In the notation of Problem 2, calculate the measure of the totality of discarded open sets. That is, calculate $\lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$.

**Solution:**

9. let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$. (This result will probably surprise you: Although $G$ is open and contains every rational number, "most" of $\mathbb{R}$ is in $G^c$.)

**Solution:**

10. Use your method of working Problem 12 to give a proof that $\mathbb{R}$ is uncountable (cf. Section 1B).

**Solution:**