Math For Social Sciences

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1 Functions

1.1 Definition

Given two sets A, B, a rule that maps $\forall x \in A$ to $y \in B$ is called a *function*. The standard notation is

$$y = f(x), f : x \mapsto y \text{ or } f : A \mapsto B.$$

1.2 Inverse function

For an inverse function to be defined, the original function should be one-to-one by which switching the domain and range still satisfies the definition of a function.

$$\begin{aligned} f(x) &= \underbrace{\frac{f(x)+f(-x)}{2}}_{f_1(x)} + \underbrace{\frac{f(x)-f(-x)}{2}}_{f_2(x)} \\ &= f_1(x) + f_2(x) \\ f_1(-x) &= \frac{f(-x)+f(x)}{2} = f_1(x) \\ f_2(-x) &= \frac{f(-x)-f(x)}{2} = -f_2(x) \end{aligned}$$

1.3 Polynomials

$$P(x) = a_n x^n + a_{n-1} a^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad n \in \mathbb{Z}_+$$

There are several properties of polynomials.

1. $P(x) = x^n$ is an even function if $x = 2k$ and an odd function if $n = 2k+1$.
2. $P(a) = 0 \implies (x-a) \,|\, P(x)$.
3. *(Weierstrauss approximation theorem)* If a function $f(x)$ is continuous on a closed interval $[a, b]$, then there exists an approximation for $f(x)$ on the same closed interval $[a, b]$.

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Suppose there are two circles with radii r_1, r_2 respectively such that $r_1 > r_2$. If the same length Δ was added to the circumferences of the two circles, how larger have the radii of both circles become?

$$\begin{aligned}
\frac{2\pi r_1 + \Delta}{2\pi} &= r_1' \\
\frac{2\pi r_2 + \Delta}{2\pi} &= r_2' \\
r_1' - r_1 &= \frac{\Delta}{2\pi} \\
r_2' - r_2 &= \frac{\Delta}{2\pi}
\end{aligned}$$

Thus the lengths that are added to both radii are equal.

- The formula for finding the roots of a polynomial exists up to the 4^{th} order. Starting from 5^{th} order, Galoi and Abel proved that there does not exist a fixed formula that expresses the roots with respect to the coefficients of a polynomial.

- Suppose there are $(n+1)$ data points. How can we find a n^{th} order polynomial that passes through all the data points? Formally in mathematical notation,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is the set of all data points. Let's think of the following:

$$\begin{aligned}
L_{n,0}(x) &= \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} \\
L_{n,0}(x_0) &= 1 \\
L_{n,0}(x_1) &= 0 \\
L_{n,0}(x_2) &= 0 \\
\therefore L_{n,0}(x_i) &= \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases} \\
L_{n,1}(x_i) &= \begin{cases} 1, & \text{if } i = 1 \\ 0, & \text{if } i \neq i \end{cases} \\
L_{n,j}(x_i) &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
\end{aligned}$$

Now let's define the following:

$$\begin{aligned}
L_n(x) &= y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x) \\
&= \sum_{i=0}^{n} L_{n,i}(x)\, y_i
\end{aligned}$$

This polynomial has been named after *Lagrange*.

$$P(x)\,|_{x=a} = 0$$