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Real Analysis
HW1

1. If G is open and P is a special polygon with $P \subset G$, prove there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

Solution: Recall the definition of a measure on an open set G:

$$\lambda(G) := \sup\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}.$$

We have two occasions:

1. $\lambda(G) < \infty$.
2. $\lambda(G) = \infty$.

Let's address the first case.

1. If $\lambda(G) < \infty$, according to the definition of *supremum*, it satisfies the following property:

$$\lambda(G) - \epsilon = \lambda(P)$$

for some $\epsilon > 0$. Then by the definition of *supremum*, there exists an element $\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ such that

$$\underbrace{\lambda(G) - \epsilon < \lambda(P')}_{\lambda(P)<\lambda(P')} < \lambda(G).$$

2. If $\lambda(G) = \infty$, the set $\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$ is not bounded above. Therefore, it immediately follows that there exists an element

$$\lambda(P') \in \{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$$

such that $\lambda(P) < \lambda(P')$.

2. **(a)** Prove that if G is a bounded open set, then $\lambda(G) < \infty$.
(b) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < \frac{1}{x}\right\}.$$

Prove that $\lambda(G) = \infty$
(c) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 0 < x \text{ and } 0 < y < e^{-x}\right\}.$$

Prove that $\lambda(G) = 1$.
(d) In the plane $\mathbb{R}^2$ let

$$G = \left\{(x, y) : 1 < x \text{ and } 0 < y < x^{-a}\right\},$$

where a is a real number satisfying $a > 1$. Prove that $\lambda(G) = 1/(a-1)$.

Solution: **(a)** Let G' be the set of all limit points of G. Then the closure $\overline{G} = G \bigcup G'$ is compact because it is a closed and bounded set. Since $\overline{G}$ is compact, there exists a finite subcover $\bigcup_{i=1}^{k} V_i$ of $\overline{G}$. If a set is closed and bounded in $\mathbb{R}^n$, then it can be contained in I for some n-cell I, i.e., $\overline{G} \subset I$. Because n-cell is a special rectangle, $\lambda(I)$ is well-defined to be finite. Thus,

$$\lambda(G) \leq \lambda\left(\overline{G}\right) \leq \lambda(I) < \infty.$$

Solution:

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Solution:

3. Let $G_i, i \in \mathcal{I}$, be a collection of disjoint open sets in $\mathbb{R}^n$. Prove that only countably many of these sets are nonempty.

Solution: We can use the fact that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, which is put another way as such: every neighbourhood in $\mathbb{R}^n$ contains a point of $\mathbb{Q}^n$. Therefore, if G_i is a nonempty open set, every point in G_i is an interior point with a neighbourhood with a point in $\mathbb{Q}^n$ contained in G_i. Since G_i are disjoint, they must not share a point in $\mathbb{Q}^n$, which concludes that the number of nonempty open sets in the collection $\{G_i\}$ is as large as the cardinality of $\mathbb{Q}^n$. $\mathbb{Q}^n$ has the same cardinality as $\mathbb{N}^n$ thereby lending itself to the definition of *countability*.

4. *The structure of open sets in* $\mathbb{R}$.

Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_{k} (a_k, b_k),$$

where the range on k can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals.

Solution: For every point $x \in G$, we define A_x to be the largest interval contained within G. We can construct A_x by making use of supremum and infimum.

$$b_x := \sup \{b : b > x \text{ and } (x, b) \subset G\}$$
$$a_x := \inf \{a : a < x \text{ and } (a, x) \subset G\}$$

Then $A_x := (a_x, b_x)$. By construction, $G = \bigcup_{x \in G} A_x$. Now we are left with two problems: (1) A_x is uniquely defined. (2) The collection $\{A_x\}_{x \in G}$ is countable.

1. Suppose there exist two distinct points $x, x' \in G$ whose defined A_x and $A_{x'}$ overlap, i.e., $A_x \bigcap A_{x'} \neq \emptyset$. It follows that $A_x \bigcup A_{x'} \subset A_x$ since A_x is by definition the largest interval containing x within G. Likewise, $A_x \bigcup A_{x'} \subset A_{x'}$. Together, they suggest $A_x = A_{x'}$. This implies that if two $A_x, A_{x'}$ overlap, they do not overlap partially. They overlap entirely which makes them the same interval. Otherwise, they are disjoint, which makes each of them unique in the collection $\{A_x\}_{x \in G}$.
2. Recall that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$. This states that every set in the collection $\{A_x\}_{x \in G}$ has at least one rational number. Since they are disjoint, there exists an injection $\{A_x\}_{x \in G} \mapsto \mathbb{Q}$. Therefore, the cardinality of the collection $\{A_x\}_{x \in G}$ is at its largest the same as that of $\mathbb{Q}$.

5. In the notation of the previous problem, prove that $\lambda (G) = \sum_k (b_k - a_k)$.

Solution: By property O6, for disjoint sets A_x,

$$\lambda (G) = \lambda \left(\bigcup_{x \in G} A_x \right) = \sum_{x \in G} \lambda (A_x) .$$

Since $\lambda (A_x)$ is defined to be $b_x - a_x$, it follows that

$$\lambda (G) = \sum_x (b_x - a_x) .$$

6. Prove that the open disk $B (0, 1)$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles.

Solution: Recall the fact that a unit ball is a *connected* set, i.e., it is not a union of two nonempty separated sets. However, if we suppose nonempty disjoint open rectangles $\{G_k\}_{k=1}^{\infty}$ can be coupled to create the unit ball $B (0, 1)$, then it violates the connectedness of a unit ball. Let $A_1 = G_1$ and $A_2 = \bigcup_{i=2}^{\infty} G_i$. Then $A_1 \bigcap A_2 = \emptyset$ but $A_1 \bigcup A_2 = B (0, 1)$, which is a contradiction.

7. Prove that every nonempty open subset of $\mathbb{R}^n$ can be expressed as a countable union of nonoverlapping special rectangles, which may be taken to be cubes:

$$G = \bigcup_{k=1}^{\infty} I_k.$$

The range on k must be infinite. Why?
(HINT: First pave $\mathbb{R}^n$ with cubes of side 1. Select those cubes which are contained in G. Then bisect the sides of the remaining cubes to obtain cubes with side 1/2. Select those cubes which are contained in G.)

Solution:

8. Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$. (This result will probably surprise you: Although G is open and contains every rational number, "most" of $\mathbb{R}$ is in G^{c}.)

Solution: From the fact that $\mathbb{Q}$ is countable, we can construct a sequence $\{x_n\}, n \in \mathbb{Z}^+$ for which a bijection f exists with $\mathbb{Q}$. Therefore, if we take sequence to be equivalent to the set of rational numbers, take a neighbourhood around each element in $\{x_n\}$ and denote it by $B(x_n, \epsilon/3^n)$ for some $\epsilon > 0$. In $\mathbb{R}$, a neighbourhood is an open interval. Once we set G to be the union of these neighbourhoods, it is then followed by

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right).$$

An infinite union of open sets is also an open set. Therefore, G is also open. Since

$$\lambda(G) = \lambda\left(\bigcup_{n=1}^{\infty} B\left(x_n, \frac{\epsilon}{3^n}\right)\right) \le \sum_{n=1}^{\infty} \lambda\left(B\left(x_n, \frac{\epsilon}{3^n}\right)\right) = \sum_{n=1}^{\infty} \frac{\epsilon}{3^n},$$

$\lambda(G) \le \epsilon/2$.

9. Use your method of working Problem 12 to give a proof that $\mathbb{R}$ is uncountable (cf. Section 1B).

Solution: Assume $\mathbb{R}$ is countable. Then using the method in the previous problem, we can construct a sequence with a bijection with $\mathbb{Z}^+$. Equally, there exists an open cover of countably many open intervals for $\mathbb{R}$. From this, there exists $\epsilon > 0$ which satisfies the following relation:

$$\lambda(\mathbb{R}) < \epsilon.$$

This is a violation of the property O3, stating $\lambda(\mathbb{R}) = \infty$.