# Solutions for Assignment #3 : Mathematical Statistics

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## Problem 1

Let $X_r$ have a $t$-distribution with $r(\geq 3)$ degrees of freedom : $X_r \sim t(r)$. Compute $E(X_r^2)$.

Note that

$$X_r \equiv \frac{Z}{\sqrt{W/r}}, \quad Z \sim \mathrm{N}(0,1) \perp\!\!\!\perp W \sim \chi^2(r)$$

Then,

$$\begin{aligned} E(X_r^2) &= E\left(\frac{Z^2}{W/r}\right) = r \times E\left(Z^2\right) E\left(\frac{1}{W}\right) && \left(\because Z^2 \perp\!\!\!\perp 1/W\right) \\ &= r \times E\left(\frac{1}{W}\right) = \frac{r}{r-2} && \left(\because Z^2 \sim \chi^2(1)\right) \end{aligned}$$

From the fact that

$$E\left(\frac{1}{W}\right) = \int_0^\infty \frac{1}{w}\frac{1}{2^{r/2}\Gamma(r/2)} w^{r/2-1}e^{-w/2}dw = \frac{1}{2\cdot(r/2-1)}\underbrace{\int_0^\infty \frac{1}{2^{(r-2)/2}\Gamma((r-2)/2)} w^{(r-2)/2-1}e^{-w/2}dw}_{=1}$$

$$= \frac{1}{r-2}$$

## Problem 2

Suppose that $W_1$ and $W_2$ are independent $\chi^2$ distributed random variables with $\nu_1$ and $\nu_2$ degrees of freedom, respectively. According to Definition 7.3,

$$F = \frac{W_1/\nu_1}{W_2/\nu_2}$$

has an $F$ distribution with $\nu_1$ and $\nu_2$ numerator and denominator degrees of freedom, respectively. Use the preceding structure of $F$, the independence of $W_1$ and $W_2$, and the result summarized in Exercise 7.30(b) to show
(a) $E(F) = \nu_2/(\nu_2 - 2)$, if $\nu_2 > 2$.

(b) $V(F) = [2\nu_2^2(\nu_1 + \nu_2 - 2)]/[\nu_1(\nu_2 - 2)^2(\nu_2 - 4)]$, if $\nu_2 > 4$.

# Problem 3

Suppose that $Z$ has a standard normal distribution, $Z \sim N(0,1)$, $Y$ has an exponential distribution with mean 2, $Y \sim \text{Exp}(2)$, and that $Z$ and $Y$ are independent. Let $T = Z/\sqrt{Y/2}$. Find the p.d.f (probability density function) of $T$.

# Problem 4

Let $X_1, \dots, X_n$ be a random sample from a $N(\mu_1, \sigma^2)$ and $Y_1, \dots, Y_m$ be a random sample from a $N(\mu_2, \sigma^2)$, independent with $X_1, \dots, X_n$.
(a) Define

$$W = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{\sum_{i=1}^{m}(Y_i - \overline{Y}_m)^2}$$

Compute the expected value of $W$, $\text{E}(W)$.

(b) Let $\overline{X}_{n-1} = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} X_i$, $\overline{Y}_m = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} Y_i$ and $T_m = \left[\dfrac{1}{n}\displaystyle\sum_{i=1}^{m}(Y_i - \overline{Y}_m)^2\right]^{1/2}$. Determine the value of a constant $k$ such that the random variable $k(X_n - \overline{X}_{n-1})/T_m$ will have a $t$ distribution with an appropriate degree of freedom.

# Problem 5

Let $X_1, \dots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ with $-\infty < \mu < \infty$ and $\sigma^2 > 0$ unknown. Define $Y = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$. Compute the expected value of $Y^2$, $\text{E}(Y^2)$.