Solutions for Assignment #2 : Mathematical Statistics

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Problem 1

Let (Y_1, Y_2) denote the coordinates of a point chosen at random inside a unit circle whose center is at the origin. That is, Y_1 and Y_2 have a joint density function given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{\pi}, & y_1^2 + y_2^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(Y_1 \le Y_2)$.

(Method 1) Directly dealing with the circle on the Cartesian plane, we should divide the integration region.

$$\begin{aligned} P(Y_1 \le Y_2) &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{-\sqrt{1-x^2}}^{x} \frac{1}{\pi} \,\mathrm{d}y \,\mathrm{d}x + \int_{\sqrt{2}/2}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \,\mathrm{d}y \,\mathrm{d}x \\ &= \frac{1}{2\pi} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2\pi} = \frac{1}{2} \end{aligned} \tag{1}$$

(Method 2)
The problem is equivalent to calculating $P(Y_1 - Y_2 \le 0)$ and since the integration region is a circle on the xy-plane, it is more comfortable to use the polar coordinate where

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{2}$$

The Jacobian term for the transformation becomes

$$|J| = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = r \tag{3}$$

The region satisfying

$$y_1 \le y_2 \quad \text{and} \quad y_1^2 + y_2^2 \le 1 \tag{4}$$

is the circle below the line $y_2 = y_1$. Thus, the integration region for r is obvious, $0 \le r \le 1$, and the integration region for θ varies depending on where you start drawing the circle. Assume $\theta \in (-\pi, \pi)$. Then

$$\begin{aligned} P(Y_1 \le Y_2) &= \int_{-3\pi/4}^{\pi/4} \int_0^1 \frac{r}{\pi} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \int_{-3\pi/4}^{\pi/4} \frac{1}{2\pi} \,\mathrm{d}\theta \\ &= \frac{1}{2} \end{aligned} \tag{5}$$

(Method 3)
Intuitively, the answer is trivially $1/2$ since the probability density is uniform on the circle and the integration region is exactly half the circle.

Problem 2

Let (X, Y) have a uniform distribution over the unit square, i.e. the joint p.d.f. of (X, Y) is given by

$$f(x, y) = \begin{cases} 1 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Find the moment generating function $Z = -\log(X) - \log(Y)$.

(Method 1)

$$\begin{aligned} \mathrm{E}(e^{tZ}) &= \mathrm{E}(e^{t(-\log X - \log Y)}) = \mathrm{E}\left[(XY)^{-t}\right] \\ &= \int_0^1 \int_0^1 (xy)^{-t}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y^{-t} \left(\frac{x^{1-t}}{1-t}\Big|_0^1 \right) \mathrm{d}y \\ &= \int_0^1 \frac{y^{-t}}{1-t}\,\mathrm{d}y = \frac{y^{1-t}}{(1-t)^2}\Big|_0^1 = \frac{1}{(t-1)^2} \quad t < 1 \end{aligned} \tag{7}$$

(Method 2)

Note that the given function is the joint PDF of two independent standard uniform random variables, that is, $X, Y \overset{\text{iid}}{\sim} \text{Unif}(0, 1)$. Thus, we can use the relations

- $-\log U \sim \text{Exp}(1)$ where $U \sim \text{Unif}(0, 1)$.

  Proof. Let $g(u) = -\log u$. Then, $g^{-1}(x) = e^{-x}$. By the variable transformation,

  $$f_X(x) = f_U(g^{-1}(x)) \left| \frac{dg^{-1}(x)}{dx} \right| = e^{-x} \tag{8}$$

  which is the density of the exponential distribution with mean 1. □

- If $E_1, E_2 \overset{\text{iid}}{\sim} \text{Exp}(1)$, $E_1 + E_2 \sim \text{Gamma}(2, 1)$.

  Proof. Using the convolution, set $Z = E_1 + E_2$,

  $$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{E_1}(z - e_2) f_{E_2}(e_2)\,\mathrm{d}e_2 \\ &= \int_0^z e^{-(z-e_2)} e^{-e_2}\,\mathrm{d}e_2, \qquad (\because z - e_2 > 0) \\ &= \int_0^z e^{-z}\,\mathrm{d}e_2 \\ &= z e^{-z} \end{aligned} \tag{9}$$

  □

Thus, the MGF of $Z \sim \text{Gamma}(2, 1)$ becomes

$$\begin{aligned} \mathrm{E}(e^{tZ}) &= \int_0^{\infty} z e^{(t-1)z}\,\mathrm{d}z \\ &= \frac{1}{(t-1)^2}, \quad t < 1 \end{aligned} \tag{10}$$

Problem 3

Let Y have a distribution function given by

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0 \end{cases}$$

Find a transformation $G(U)$ such that, if U has a uniform distribution on the interval $(0, 1)$, $G(U)$ has the same distribution as Y.

(Method 1)
Let $X = G(U)$. Then

$$P(X \leq x) = P(G(U) \leq x) = P(U \leq G^{-1}(x)) = G^{-1}(x) \quad (\because F_U(u) = u) \tag{11}$$

Thus, $F = G^{-1} \implies G = F^{-1}$.

$$F^{-1}(u) = \sqrt{-\log(1-u)} \tag{12}$$

(Method 2)
This is called the *probability integral transform* which states that we can sample a random variable by plugging a standard uniform random variable U into the *quantile function* or the *inverse CDF* of the desired distribution. That is, $F^{-1}(U) \sim F$.

Proof. Let $U \sim \text{Unif}(0, 1)$ and F be the distribution function that we want to sample from. If we define $Y = F^{-1}(U)$,

$$\begin{aligned} P(Y \leq y) &= P(F^{-1}(U) \leq y) \\ &= P(U \leq F(y)) \quad \text{(F is bijective)} \\ &= F(y) \end{aligned} \tag{13}$$

Thus, the CDF of the newly defined variable $F^{-1}(U)$ recovers the CDF of interest. □

Then, we now know that $G = F^{-1}$ where

$$F^{-1}(u) = \sqrt{-\log(1-u)} \tag{14}$$

Problem 4

Let $Y_1, Y_2, \ldots, Y_n$ be independent Poisson random variables with means $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Find the
(a) probability function of $\sum_{i=1}^n Y_i$.

Note that the moment generating function (MGF) of Y_i is

$$M_{Y_i}(t) = \exp\left(\lambda_i(\exp(t) - 1)\right)$$

Then,

$$\begin{aligned} M_{\sum_{i=1}^n Y_i}(t) &= E\left(\exp\left(t\sum_{i=1}^n Y_i\right)\right) = E\left(\prod_{i=1}^n \exp(tY_i)\right) = \prod_{i=1}^n E\left(\exp(tY_i)\right) \qquad (\because\ \forall Y_i : \text{ind}) \\ &= \prod_{i=1}^n M_{Y_i}(t) = \exp\left(\sum_{i=1}^n \lambda_i(\exp(t)-1)\right) \end{aligned}$$

By the uniqueness theorem of MGF,

$$\sum_{i=1}^n Y_i \sim \text{Poi}\left(\sum_{i=1}^n \lambda_i\right)$$

(b) conditional probability function of Y_1, given that $\sum_{i=1}^n Y_i = m$.

Proposition. *If* $X_i \overset{ind}{\sim} \text{Poi}(\lambda_i), (i = 1, 2)$*, then given* $X_1 + X_2 = m$*,* $X_1 \mid (X_1 + X_2 = m) \sim \text{Binom}\left(m, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

Proof. Note that $X_1 + X_2 \sim \text{Poi}(\lambda_1 + \lambda_2)$ by (a).

$$\begin{aligned} P(X_1 = s | X_1 + X_2 = m) &= \frac{P\left((X_1 = s) \cap (X_1 + X_2 = m)\right)}{P(X_1 + X_2 = m)} = \frac{P\left((X_1 = s) \cap (X_2 = m - s)\right)}{P(X_1 + X_2 = m)} \\ &= \frac{P\left(X_1 = s\right) P\left(X_2 = m - s\right)}{P(X_1 + X_2 = m)} \qquad (\because X_1 \perp\!\!\!\perp X_2) \\ &= \frac{\lambda_1^s e^{-\lambda_1}}{s!} \frac{\lambda_2^{(m-s)} e^{-\lambda_2}}{(m-s)!} \frac{m!}{(\lambda_1+\lambda_2)^m e^{-(\lambda_1+\lambda_2)}} \\ &= \frac{m!}{s!(m-s)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^s \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-s}, \quad (s = 0, \cdots, m) \end{aligned}$$

□

Take $X_1 = Y_1$, and $X_2 = \sum_{i=2}^n Y_i \sim \text{Poi}\left(\sum_{i=2}^n \lambda_i\right)$ by (a). Then $X_1 \mid (X_1 + X_2 = m) \sim \text{Binom}\left(m, \frac{\lambda_1}{\sum_{i=1}^n \lambda_i}\right)$.

(c) conditional probability function of $Y_1 + Y_2$ given that $\sum_{i=1}^n Y_i = m$.

Take $X_1 = Y_1 + Y_2 \sim \text{Poi}(\lambda_1 + \lambda_2)$ by (a), and $X_2 = \sum_{i=3}^n Y_i \sim \text{Poi}\left(\sum_{i=3}^n \lambda_i\right)$ by (a). From (b),

$$Y_1 + Y_2 \mid \sum_{i=1}^n Y_i = m \sim \text{Binom}\left(m, \frac{\lambda_1+\lambda_2}{\sum_{i=1}^n \lambda_i}\right)$$

Problem 5

Suppose that $W = Y_1 + Y_2$ where Y_1 and Y_2 are independent. If W has a χ^2 distribution with ν degrees of freedom and Y_1 has a χ^2 distribution with $\nu_1 < \nu$ degrees of freedom, show that Y_2 has a χ^2 distribution with $\nu - \nu_1$ degrees of freedom.

Since we do not yet know whether $W \perp\!\!\!\perp Y_1$, it is impossible to use the convolution. Therefore, we should use the MGF instead.

$$\begin{aligned} \mathrm{E}\left(e^{tW}\right) &= \mathrm{E}\left(e^{t(Y_1+Y_2)}\right) \\ &= \mathrm{E}\left(e^{tY_1}\right)\mathrm{E}\left(e^{tY_2}\right) \end{aligned} \tag{15}$$

If we explicitly compute the MGF of $X \sim \chi^2$-distribution with r degrees of freedom,

$$\begin{aligned} \mathrm{E}(e^{tX}) &= \int_0^\infty e^{tx}\frac{(1/2)^{r/2}}{\Gamma(r/2)}x^{r/2-1}\exp\left(-\frac{1}{2}x\right)\mathrm{d}x \\ &= \int_0^\infty \frac{(1/2)^{r/2}}{\Gamma(r/2)}x^{r/2-1}\exp\left\{-\left(\frac{1}{2}-t\right)x\right\}\mathrm{d}x \\ &= \frac{(1/2)^{r/2}}{(1/2-t)^{r/2}}\underbrace{\int_0^\infty \frac{(1/2-t)^{r/2}}{\Gamma(r/2)}x^{r/2-1}\exp\left\{-\left(\frac{1}{2}-t\right)x\right\}\mathrm{d}x}_{=1} \\ &= \left(\frac{1}{1-2t}\right)^{r/2}, \quad t < \frac{1}{2} \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left(e^{tY_2}\right) &= \mathrm{E}\left(e^{tW}\right)/\,\mathrm{E}\left(e^{tY_1}\right) \\ &= (1-2t)^{-\nu/2}/(1-2t)^{-\nu_1/2} \\ &= (1-2t)^{-(\nu-\nu_1)/2}, \quad t < \frac{1}{2} \end{aligned} \tag{17}$$

Since the MGF completely determines the distribution if it exists, the MGF of Y_2 is of a χ^2-distribution with $\nu - \nu_1$ degrees of freedom.

Problem 7

Suppose that Y_1 and Y_2 are independent exponentially distributed random variables, both with mean β, and define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1/Y_2$.

a. Show that the joint density of (U_1, U_2) is

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} \frac{1}{\beta^2} u_1 e^{-u_1/\beta} \frac{1}{(1+u_2)^2}, & 0 < u_1,\ 0 < u_2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that the joint probability distribution of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\beta^2} \exp\left(-\frac{y_1 + y_2}{\beta}\right) \tag{18}$$

The inverse relations are

$$\begin{aligned} y_1 &= \frac{u_1 u_2}{1 + u_2} \\ y_2 &= \frac{u_1}{1 + u_2} \end{aligned} \tag{19}$$

and the Jacobian is

$$|J| = \left| \det \begin{pmatrix} u_2/(1+u_2) & 1/(1+u_2) \\ u_1/(1+u_2)^2 & -u_1/(1+u_2)^2 \end{pmatrix} \right| = \frac{u_1}{(1+u_2)^2} \tag{20}$$

Using the change of variable,

$$\begin{aligned} f_{U_1,U_2}(u_1,u_2) &= f_{Y_1,Y_2}(y_1,y_2)|J| \\ &= \frac{1}{\beta^2} \exp\left(-\frac{u_1}{\beta}\right) \frac{u_1}{(1+u_2)^2} \\ &= \left(\frac{1}{\beta^2} u_1 \exp\left(-\frac{u_1}{\beta}\right)\right) \times \left(\frac{1}{(1+u_2)^2}\right), \quad u_1 > 0,\ u_2 > 0 \end{aligned} \tag{21}$$

b. Are U_1 and U_2 are independent? Why?

The joint probability distribution of (U_1, U_2) is separable, that is,

$$f_{U_1,U_2}(u_1,u_2) = \text{Gamma}\left(U_1 \mid 2, \frac{1}{\beta}\right) \times \left(\frac{1}{(1+u_2)^2}\right), \quad u_1 > 0, u_2 > 0 \tag{22}$$

It is clear that the gamma distribution component is a valid density. We should check if $1/(1+x)^2$ is a valid probability density function as well.

$$\int_0^\infty \frac{1}{(1+x)^2}\,\mathrm{d}x = -\frac{1}{1+x}\bigg|_0^\infty = 1 \tag{23}$$

It satisfies the conditions for a probability densify function. Thus, they are independent.