

Solutions for Assignment #1 : Mathematical Statistics

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Problem 1

Let X have probability density function (p.d.f.)

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1, \\ 3/8 & 3 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function of X .

(a) Recall definition and the conditions that a cumulative distribution function F should satisfy. The definition is

$$F_X(x) = P_X(X \leq x) \text{ for all } x \quad (1)$$

and a function F is a CDF if and only if

- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$
- $F(x)$ is a nondecreasing function of x
- $F(x)$ is right-continuous; that is, for every number x_0 , $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

Thus, for $x < 1$

$$\begin{aligned} F_X(x) &= P_X(X \leq x) = \int_0^x 1/4 \, dt \\ &= x/4 \end{aligned} \quad (2)$$

and for $x \in (3, 5)$, since the density is not defined at $x = 1$ and $x = 3$, we should use the right continuity, $\lim_{x \uparrow 1} F_X(x) = \lim_{x \downarrow 3} F_X(x)$. Therefore,

$$\frac{1}{4} = \lim_{x \downarrow 3} \int_3^x 3/8 \, dt + C \quad (3)$$

which returns $C = 1/4$. Therefore,

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ x/4, & 0 < x < 1, \\ 1/4, & 1 \leq x \leq 3 \\ (3x - 7)/8, & 3 < x < 5, \\ 1, & x > 5. \end{cases} \quad (4)$$

(b) Let $Y = 1/X$. Find the probability density function $f_Y(y)$ for Y .

(b) (*Method 1*) Starting from the cumulative distribution function,

$$\begin{aligned}
 P(Y \leq y) &= P\left(\frac{1}{X} \leq y\right) \\
 &= P\left(\frac{1}{y} \leq X\right) \\
 &= \int_{x \geq 1/y} f_X(x) dx \\
 &= \begin{cases} \int_{1/y}^1 1/4 dx + \int_3^5 3/8 dx, & \text{if } 0 < 1/y < 1 \\ \int_{1/y}^5 3/8 dx, & \text{if } 3 < 1/y < 5 \end{cases} \\
 &= \begin{cases} 1/2 - 1/(4y), & \text{if } y > 1 \\ 15/8 - 3/(8y), & \text{if } 1/5 < y < 1/3 \end{cases}
 \end{aligned}$$

Therefore, the probability density function can be computed by differentiating the CDF with respect to y , i.e.,

$$f_Y(y) = \begin{cases} 1/(4y^2), & y > 1 \\ 3/(8y^2), & 1/5 < y < 1/3, \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

(*Method 2*) Using the change of variable technique given a relation $y = g(x)$,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| \quad (6)$$

immediately returns

$$f_Y(y) = \begin{cases} 1/(4y^2), & y > 1 \\ 3/(8y^2), & 1/5 < y < 1/3, \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

since $g^{-1}(y) = 1/y$ and the Jacobian becomes $1/y^2$.

Problem 2

Find $E[Y(Y-1)]$ for a geometric random variable Y by finding $\frac{d^2}{dq^2} \left(\sum_{y=1}^{\infty} q^y \right)$. Use this result to find the variance of Y . Also check if your answers are correct using the moment generating function.

Note that

$$\frac{d^2}{dq^2} \left(\sum_{y=1}^{\infty} q^y \right) = \sum_{y=1}^{\infty} \left(\frac{d^2}{dq^2} q^y \right) = \sum_{y=1}^{\infty} y(y-1)q^{y-2}$$

Therefore

$$E[Y(Y-1)] = \sum_{y=1}^{\infty} y(y-1)q^{y-1}p = pq \times \frac{d^2}{dq^2} \left(\sum_{y=1}^{\infty} q^y \right) = pq \times \frac{2}{(1-q)^3} = \frac{2q}{p^2}$$

Now that

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = E(Y(Y-1)) + E(Y) - E(Y)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}$$

Since the moment generating function of Y is

$$M_Y(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

Variance of Y can be obtained by

$$M_Y''(0) - [M_Y'(0)]^2 = \left[\frac{pe^t}{((p-1)e^t + 1)^2} - \frac{2(p-1)pe^{2t}}{((p-1)e^t + 1)^3} \right] - \left[\frac{pe^t}{(p-1)e^t + 1} \right]^2 \Bigg|_{t=0} = \frac{q}{p^2}$$

Problem 3

Let the density function of a random variable Y be given by

$$f(y) = \begin{cases} \frac{2}{\pi(1+y^2)}, & -1 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the distribution function.

(a) Using the definition

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) \\ &= \int_{-1}^y \frac{2}{\pi(1+x^2)} dx \\ &= \frac{2}{\pi} \arctan x \Big|_{-1}^y \\ &= \frac{2}{\pi} \left(\arctan y + \frac{\pi}{4} \right) \end{aligned} \tag{8}$$

(b) Find $E(Y)$.

(b) Using the definition,

$$\begin{aligned} E(Y) &= \int_{-1}^1 \frac{2y}{\pi(1+y^2)} dy \\ &= \frac{\log(y^2 + 1)}{\pi} \Big|_{-1}^1 \\ &= 0 \end{aligned} \tag{9}$$

Problem 4

Let Y denote a random variable with probability density function given by

$$f(y) = (1/2)e^{-|y|}, \quad -\infty < y < \infty.$$

Find the moment-generating function of Y and use it to find $E(Y)$.

Note that the given function is the PDF of the Double Exponential distribution or the Laplace distribution. Using the definition of the MGF,

$$\begin{aligned} E(e^{tY}) &= \int_{-\infty}^{\infty} \frac{1}{2} e^{ty - |y|} dy \\ &= \int_{-\infty}^0 \frac{1}{2} e^{(t+1)y} dy + \int_0^{\infty} \frac{1}{2} e^{(t-1)y} dy \\ &= \frac{1}{2(1-t)} + \frac{1}{2(1+t)} \\ &= \frac{1}{1-t^2}, \quad |t| < 1 \end{aligned} \tag{10}$$

Note that

$$\int_0^{\infty} \frac{1}{2} e^{(t-1)y} dy < \infty \tag{11}$$

only if $|t| < 1$. To obtain the expectation, we use the following relation:

$$E(Y) = \left. \frac{d}{dt} M_Y(t) \right|_{t=0} \tag{12}$$

which becomes

$$E(Y) = \left. \frac{2t}{(1-t^2)^2} \right|_{t=0} = 0 \tag{13}$$