Solutions for Assignment #1 : Mathematical Statistics

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Problem 1

Let X have probability density function (p.d.f.)

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1, \\ 3/8 & 3 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function of X.

(a) Recall definition and the conditions that a cumulative distribution function F should satisfy. The definition is

$$F_X(x) = P_X(X \le x) \text{ for all } x \tag{1}$$

and a function F is a CDF if and only if

- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$
- $F(x)$ is a nondecreasing function of x
- $F(x)$ is right-continuous; that is, for every number x_0, $\lim_{x\downarrow x_0} F(x) = F(x_0)$.

Thus, for $x < 1$

$$\begin{aligned} F_X(x) = P_X(X \le x) &= \int_0^x 1/4 \,\mathrm{d}t \\ &= x/4 \end{aligned} \tag{2}$$

and for $x \in (3, 5)$, since the density is not defined at $x = 1$ and $x = 3$, we should use the right continuity, $\lim_{x\uparrow 1} F_X(x) = \lim_{x\downarrow 3} F_X(x)$. Therefore,

$$\frac{1}{4} = \lim_{x\downarrow 3} \int_3^x 3/8 \,\mathrm{d}t + C \tag{3}$$

which returns $C = 1/4$. Therefore,

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ x/4, & 0 < x < 1, \\ 1/4, & 1 \le x \le 3 \\ (3x-7)/8, & 3 < x < 5, \\ 1, & x > 5. \end{cases} \tag{4}$$

Problem 1 continued on next page. . .

(b) Let $Y = 1/X$. Find the probability density function $f_Y(y)$ for Y.

(b) *(Method 1)* Starting from the cumulative distribution function,

$$\begin{aligned}
P(Y \le y) &= P\left(\frac{1}{X} \le y\right) \\
&= P\left(\frac{1}{y} \le X\right) \\
&= \int_{x \ge 1/y} f_X(x)\,\mathrm{d}x \\
&= \begin{cases} \int_{1/y}^{1} 1/4\,\mathrm{d}x + \int_3^5 3/8\,\mathrm{d}x, & \text{if } 0 < 1/y < 1 \\ \int_{1/y}^{5} 3/8\,\mathrm{d}x, & \text{if } 3 < 1/y < 5 \end{cases} \\
&= \begin{cases} 1/2 - 1/(4y), & \text{if } y > 1 \\ 15/8 - 3/(8y), & \text{if } 1/5 < y < 1/3 \end{cases}
\end{aligned}$$

Therefore, the probability density function can be computed by differentiating the CDF with respect to y, i.e.,

$$f_Y(y) = \begin{cases} 1/(4y^2), & y > 1 \\ 3/(8y^2), & 1/5 < y < 1/3, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

(Method 2) Using the change of variable technique given a relation $y = g(x)$,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| \tag{6}$$

immediately returns

$$f_Y(y) = \begin{cases} 1/(4y^2), & y > 1 \\ 3/(8y^2), & 1/5 < y < 1/3, \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

since $g^{-1}(y) = 1/y$ and the Jacobian becomes $1/y^2$.

Problem 2

Find $E[Y(Y-1)]$ for a geometric random variable Y by finding $\frac{d^2}{dq^2}\left(\sum_{y=1}^{\infty} q^y\right)$. Use this result to find the variance of Y. Also check if your answers are correct using the moment generating function.

Note that

$$\frac{d^2}{dq^2}\left(\sum_{y=1}^{\infty} q^y\right) = \sum_{y=1}^{\infty}\left(\frac{d^2}{dq^2}q^y\right) = \sum_{y=1}^{\infty} y(y-1)q^{y-2}$$

Therefore

$$\mathrm{E}[Y(Y-1)] = \sum_{y=1}^{\infty} y(y-1)q^{y-1}p = pq \times \frac{d^2}{dq^2}\left(\sum_{y=1}^{\infty} q^y\right) = pq \times \frac{2}{(1-q)^3} = \frac{2q}{p^2}$$

Now that

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 = \mathrm{E}(Y(Y-1)) + \mathrm{E}(Y) - \mathrm{E}(Y)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}$$

Since the moment generating function of Y is

$$M_Y(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

Variance of Y can be obtained by

$$M_Y''(0) - [M_Y'(0)]^2 = \left[\frac{pe^t}{((p-1)e^t+1)^2)} - \frac{2(p-1)pe^{2t}}{((p-1)e^t+1)^3}\right] - \left[\frac{pe^t}{(p-1)e^t+1)^2}\right]^2\Bigg|_{t=0} = \frac{q}{p^2}$$

Problem 3

Let the density function of a random variable Y be given by

$$f(y) = \begin{cases} \frac{2}{\pi(1+y^2)}, & -1 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the distribution function.

(a) Using the definition

$$\begin{aligned} F_Y(y) &= P_Y(Y \le y) \\ &= \int_{-1}^{y} \frac{2}{\pi(1+x^2)}\,\mathrm{d}x \\ &= \frac{2}{\pi}\arctan x\Big|_{-1}^{y} \\ &= \frac{2}{\pi}\left(\arctan y + \frac{\pi}{4}\right) \end{aligned} \tag{8}$$

(b) Find $E(Y)$.

(b) Using the definition,

$$\begin{aligned} \mathrm{E}(Y) &= \int_{-1}^{1} \frac{2y}{\pi(1+y^2)}\,\mathrm{d}y \\ &= \frac{\log(y^2+1)}{\pi}\Big|_{-1}^{1} \\ &= 0 \end{aligned} \tag{9}$$

Problem 4

Let Y denote a random variable with probability density function given by

$$f(y) = (1/2)e^{-|y|}, \quad -\infty < y < \infty.$$

Find the moment-generating function of Y and use it to find $E(Y)$.

Note that the given function is the PDF of the Double Exponential distribution or the Laplace distribution. Using the definition of the MGF,

$$\begin{aligned}
\mathrm{E}(e^{tY}) &= \int_{-\infty}^{\infty} \frac{1}{2} e^{ty-|y|}\,\mathrm{d}y \\
&= \int_{-\infty}^{0} \frac{1}{2} e^{(t+1)y}\,\mathrm{d}y + \int_{0}^{\infty} \frac{1}{2} e^{(t-1)y}\,\mathrm{d}y \\
&= \frac{1}{2(1-t)} + \frac{1}{2(1+t)} \\
&= \frac{1}{1-t^2}, \quad |t| < 1
\end{aligned} \tag{10}$$

Note that

$$\int_{0}^{\infty} \frac{1}{2} e^{(t-1)y}\,\mathrm{d}y < \infty \tag{11}$$

only if $|t| < 1$. To obtain the expectation, we use the following relation:

$$\mathrm{E}(Y) = \frac{d}{dt} M_Y(t)\bigg|_{t=0} \tag{12}$$

which becomes

$$\mathrm{E}(Y) = \frac{2t}{(1-t^2)^2}\bigg|_{t=0} = 0 \tag{13}$$

Problem 5

Let (Y_1, Y_2) denote the coordinates of a point chosen at random inside a unit circle whose center is at the origin. That is, Y_1 and Y_2 have a joint density function given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{\pi}, & y_1^2 + y_2^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(Y_1 \le Y_2)$.

Problem 5 continued on next page...

The problem is equivalent to calculating $P(Y_1 - Y_2 \leq 0)$ and since the integration region is a circle on the xy-plane, it is more comfortable to use the polar coordinate. The region satisfying

$$y_1 \leq y_2 \quad \text{and} \quad y_1^2 + y_2^2 \leq 1 \tag{14}$$

is the circle below the line $y_2 = y_1$. Thus,

$$\begin{aligned} P(Y_1 \leq Y_2) &= \int_{-3\pi/4}^{\pi/4} \int_0^1 \frac{r}{\pi} \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_{-3\pi/4}^{\pi/4} \frac{1}{2\pi} \, \mathrm{d}\theta \\ &= \frac{1}{2} \end{aligned} \tag{15}$$

Problem 6

Let (X, Y) have a uniform distribution over the unit square, i.e. the joint p.d.f. of (X, Y) is given by

$$f(x, y) = \begin{cases} 1 & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

Find the moment generating function $Z = -\log(X) - \log(Y)$.

Note that the given function is the joint PDF of two independent standard uniform random variables, that is, $X, Y \overset{\text{iid}}{\sim} \text{Unif}(0, 1)$. Thus, we can use the relations

- $-\log U \sim \text{Exp}(1)$ where $U \sim \text{Unif}(0, 1)$.

  Proof. Let $g(u) = -\log u$. Then, $g^{-1}(x) = e^{-x}$. By the variable transformation,

  $$f_X(x) = f_U(g^{-1}(x)) \left| \frac{dg^{-1}(x)}{dx} \right| = e^{-x} \tag{17}$$

  which is the density of the exponential distribution with mean 1. $\square$

- If $E_1, E_2 \overset{\text{iid}}{\sim} \text{Exp}(1)$, $E_1 + E_2 \sim \text{Gamma}(2, 1)$.

Thus, the MGF of $Z \sim \text{Gamma}(2, 1)$ becomes

$$\begin{aligned} \mathrm{E}(e^{tZ}) &= \int_0^\infty z e^{(t-1)z} \, \mathrm{d}z \\ &= \frac{1}{(t-1)^2}, \quad t < 1 \end{aligned} \tag{18}$$