

Solutions for Assignment #1 : Topics in Mathematical Statistics

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Problem 1

Suppose that k events B_1, \dots, B_k form a partition of the sample space Ω . For $i = 1, \dots, k$, let $P(B_i)$ denote the prior probability of B_i . Also, for each event A such that $P(A) > 0$, let $P(B_i|A)$ denote the posterior probability of B_i given that the event A has occurred. Prove that if $P(B_1|A) < P(B_1)$, then $P(B_i|A) > P(B_i)$ for at least one value of i ($i = 2, \dots, k$).

Recall that

$$\sum_{i=1}^k P(B_i) = 1 \quad \text{and} \quad \sum_{i=1}^k P(B_i | A) = 1 \quad (1)$$

(Proof by contradiction) Suppose that $P(B_i | A) < P(B_i), \forall i = 1, \dots, k$. Then

$$\sum_{i=1}^k P(B_i | A) < \sum_{i=1}^k P(B_i) \quad (= 1 < 1) \quad (2)$$

Thus, contradiction.

Problem 2

Suppose that a box contains five coins, and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the i th coin is tossed ($i = 1, \dots, 5$), and suppose that $p_1 = 0, p_2 = 1/4, p_3 = 1/2, p_4 = 3/4$, and $p_5 = 1$.

(a) Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the i th coin was selected ($i = 1, \dots, 5$)?

(b) If the same coin were tossed again, what would be the probability of obtaining another head?

(c) If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

Problem 3

The *skewness* of a random variable X can be defined as $\gamma_1 = \mu_3/(\mu_2)^{3/2}$ where

$$\mu_n = E(X - E(X))^n$$

Find the skewness of a random variable X with a binomial distribution $B(n, \pi)$ of index n and parameter π .

(Method 1: Direct calculation)

Expanding,

$$\begin{aligned} E\{(X - E(X))^3\} &= E(X^3 - 3n\pi X^2 + 3(n\pi)^2 X - (n\pi)^3) \\ &= E(X^3) - 3n\pi E(X^2) + 3n^2\pi^2 E(X) - n^3\pi^3 \end{aligned} \quad (3)$$

Since $E(X) = n\pi$ and $E(X^2) = n\pi(1 - \pi + n\pi)$, $E(X^3)$ should be computed through MGF (or PGF).

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^n e^{tx} \binom{n}{x} \pi^x (1 - \pi)^{n-x} = (1 - \pi + e^t \pi)^n \quad (\because \text{Binominal theorem}) \quad (4)$$

Then

$$M_X^{(3)}(t) = \pi n e^t (\pi(e^t - 1) + 1)^{n-3} (\pi^2(n^2 e^{2t} + (1 - 3n)e^t + 1) + \pi((3n - 1)e^t - 2) + 1) \quad (5)$$

and

$$E(X^3) = M_X^{(3)}(0) = \pi n (\pi^2(n - 2)(n - 1) + 3\pi(n - 1) + 1) \quad (6)$$

which yields

$$\mu_3 = n\pi(1 - \pi)(1 - 2\pi) \quad (7)$$

Therefore,

$$\gamma_1 = \frac{1 - 2\pi}{\sqrt{n\pi(1 - \pi)}} \quad (8)$$

(Method 2: Change of variable)

Consider $X \stackrel{D}{=} \sum_{i=1}^n Y_i$ where $Y_i \sim \text{Bernoulli}(\pi)$. Thus

$$X - n\pi \stackrel{D}{=} \sum_{i=1}^n Y_i - n\pi = \sum_{i=1}^n (Y_i - \pi) \quad (9)$$

If we again let $Z_i = Y_i - \pi$,

$$X - n\pi \stackrel{D}{=} \sum_{i=1}^n Z_i, \quad \text{where } Z_i = \begin{cases} 1 - \pi, & \text{with } \pi \\ -\pi, & \text{with } 1 - \pi \end{cases} \quad (10)$$

Thus,

$$\begin{aligned} (X - n\pi)^3 &= \left(\sum_{i=1}^n Z_i \right)^3 = \sum_{i=1}^n Z_i^3 + \sum_{i \neq j} Z_i^2 Z_j + \sum_{i \neq j \neq k} Z_i Z_j Z_k \\ E\{(X - n\pi)^3\} &= \sum_{i=1}^n E(Z_i^3) = n E(Z_1^3) \quad (\because Z_i \perp\!\!\!\perp Z_j \ \forall i \neq j \text{ and } E(Z_i) = 0) \\ &= n\pi(1 - \pi)(1 - 2\pi) \implies \gamma_1 = \frac{1 - 2\pi}{\sqrt{n\pi(1 - \pi)}} \end{aligned} \quad (11)$$

Problem 4

Define

$$I = \int_0^\infty \exp\left(-\frac{1}{2}z^2\right) dz$$

and show that

$$I = \int_0^\infty \exp\left(-\frac{1}{2}(xy)^2\right) y \, dx = \int_0^\infty \exp\left(-\frac{1}{2}(zx)^2\right) z \, dx.$$

Deduce that

$$I^2 = \int_0^\infty \int_0^\infty \exp\left\{-\frac{1}{2}(x^2 + 1)z^2\right\} z \, dz \, dx.$$

and show that $I = \sqrt{\pi/2}$. (This method is due to Laplace, 1812, Section 24.)

Using the transformation $z = yx$,

$$dz = y \, dx \implies \int_0^\infty \exp\left(-\frac{1}{2}z^2\right) dz = \int_0^\infty \exp\left(-\frac{1}{2}(xy)^2\right) y \, dx \quad (12)$$

Since y is just a dummy variable, we can replace it with z :

$$I = \int_0^\infty \exp\left(-\frac{1}{2}(zx)^2\right) z \, dx \quad (13)$$

Then

$$\begin{aligned} I^2 &= \left(\int_0^\infty \exp\left(-\frac{1}{2}z^2\right) dz\right) \left(\int_0^\infty \exp\left(-\frac{1}{2}(zx)^2\right) z \, dx\right) \\ &= \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}(x^2 + 1)z^2\right) z \, dz \, dx \end{aligned} \quad (14)$$

To evaluate the double integral, let

$$\begin{aligned} u &= -\frac{1}{2}(x^2 + 1)z^2 \implies du = -(x^2 + 1)z \, dz \\ \int_0^\infty \exp\left(-\frac{1}{2}(x^2 + 1)z^2\right) z \, dz &= -\frac{1}{x^2 + 1} \int_0^{-\infty} e^u \, du = \frac{1}{x^2 + 1} \end{aligned} \quad (15)$$

Thus,

$$I^2 = \int_0^\infty \frac{1}{x^2 + 1} \, dx = \arctan x \Big|_0^\infty = \frac{\pi}{2} \implies I = \sqrt{\frac{\pi}{2}} \quad (16)$$

Problem 5

Let X_1, X_2 be two independent random variables each with p.d.f. $f_1(x) = e^{-x}$ for $x > 0$ and $f_1(x) = 0$ for $x \leq 0$. Let $Z = X_1 - X_2$ and $W = X_1/X_2$.

(Notice that $\{Z = 0\} = \{W = 1\}$, but the conditional distribution of X_1 given $Z = 0$ is not the same as the conditional distribution of given $W = 1$. This discrepancy is known as the *Borel paradox*: The conditional p.d.f.'s are not like conditioning on events of probability 0. We can see that “ Z is very close to 0” is not the same as “ W is very close to 1”.)

(a) Prove that the conditional p.d.f of X_1 given $Z = 0$ is

$$g_1(x_1|0) = \begin{cases} 2e^{-2x_1} & x_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the change of variable,

$$\begin{aligned} \begin{cases} X_1 = X_1 \\ X_2 = X_1 - Z \end{cases} &\implies |J| = \left| \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right| = 1 \\ f_{X_1, Z}(x_1, z) &= e^{-2x_1 + z} \end{aligned} \quad (17)$$

For Z ,

$$\begin{aligned} P(Z \leq z) &= P(X_1 - X_2 \leq z) = P(X_2 \geq X_1 - z) \\ &= \begin{cases} \int_0^\infty \int_{x_1 - z}^\infty e^{-x_1 - x_2} dx_2 dx_1, & \text{if } z \leq 0 \\ 1 - \int_z^\infty \int_0^{x_1 - z} e^{-x_1 - x_2} dx_2 dx_1, & \text{if } z > 0 \end{cases} \\ &= \begin{cases} \frac{e^z}{2}, & \text{if } z \leq 0 \\ 1 - \frac{e^{-z}}{2}, & \text{if } z > 0 \end{cases} \\ f_Z(z) &= \frac{1}{2} e^{-|z|}, \quad -\infty < z < \infty \end{aligned} \quad (18)$$

Thus,

$$g_1(x_1 | z) = 2e^{-2x_1 + z + |z|} \implies g_1(x_1 | 0) = 2e^{-2x_1}, \quad x_1 > 0 \quad (19)$$

(b) Prove that the conditional p.d.f. of X_1 given $W = 1$ is

$$h_1(x_1 | 1) = \begin{cases} 4x_1 e^{-2x_1} & x_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the change of variable,

$$\begin{aligned} \begin{cases} X_1 = X_1 \\ X_2 = \frac{X_1}{W} \end{cases} &\implies |J| = \left| \det \begin{pmatrix} 1 & 0 \\ 1/w & -x_1/w^2 \end{pmatrix} \right| = \frac{x_1}{w^2} \\ f_{X_1, W}(x_1, w) &= e^{-(1+1/w)x_1} \frac{x_1}{w^2} \end{aligned} \quad (20)$$

and

$$P\left(\frac{X_1}{X_2} \leq w\right) = P\left(X_2 \geq \frac{X_1}{w}\right) = \int_0^\infty \int_{x_1/w}^\infty e^{-x_1 - x_2} dx_2 dx_1 = \frac{w}{w+1} \implies f_W(w) = \frac{1}{(1+w)^2} \quad (21)$$

Thus,

$$h_1(x_1 | w) = \left(\frac{w+1}{w}\right)^2 x_1 e^{-(1+1/w)x_1} \implies h_1(x_1 | 1) = 4x_1 e^{-2x_1}, \quad x_1 > 0 \quad (22)$$

Problem 6

Suppose that the random variable K has a logarithmic series distribution with parameter θ (where $0 < \theta < 1$), so that

$$P(K = k) = \frac{\alpha \theta^k}{k} \quad (k = 1, 2, \dots)$$

where $\alpha = -[\log(1 - \theta)]^{-1}$. Find the mean and variance of K .

(Method 1: Direct calculation)

$$\begin{aligned}
 E(K) &= \alpha \sum_{k=1}^{\infty} \theta^k = \frac{\alpha\theta}{1-\theta} \\
 E(K^2) &= \alpha \sum_{k=1}^{\infty} k \theta^k = \alpha\theta \sum_{k=1}^{\infty} k \theta^{k-1} = \alpha\theta \sum_{k=1}^{\infty} \frac{d}{d\theta} \theta^k = \alpha\theta \cdot \frac{d}{d\theta} \frac{\theta}{1-\theta} = \frac{\alpha\theta}{(1-\theta)^2} \\
 \text{Var}(K) &= \frac{\alpha\theta(1-\alpha\theta)}{(1-\theta)^2}
 \end{aligned} \tag{23}$$

(Method 2: Probability generating function)

Recall that

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \tag{24}$$

It follows that

$$\begin{aligned}
 E(s^K) &= G_K(s) = \alpha \sum_{k=1}^{\infty} \frac{(s\theta)^k}{k} = -\alpha \log(1-s\theta) \\
 E(K) &= G'_K(1) = -\frac{\alpha\theta}{s\theta-1} \Big|_{s=1} = \frac{\alpha\theta}{1-\theta} \\
 \text{Var}(K) &= G''_K(1) + G'_K(1) - \{G'_K(1)\}^2 = \frac{\alpha\theta^2}{(1-s\theta)^2} \Big|_{s=1} + \frac{\alpha\theta}{1-\theta} - \left(\frac{\alpha\theta}{1-\theta}\right)^2 \\
 &= \frac{\alpha\theta(1-\alpha\theta)}{(1-\theta)^2} \\
 (\because G_K^{(m)}(1) &= E\{K(K-1)\cdots(K-m+1)\} \quad (\text{"mth factorial moment of K"})
 \end{aligned} \tag{25}$$

(Method 3: Moment generating function)

$$\begin{aligned}
 E(e^{tK}) &= M_K(t) = \alpha \sum_{k=1}^{\infty} \frac{(e^t\theta)^k}{k} = -\alpha \log(1-\theta e^t) \\
 E(K) &= M'_K(0) = -\frac{\alpha\theta e^t}{\theta e^t-1} \Big|_{t=0} = \frac{\alpha\theta}{1-\theta} \\
 E(K^2) &= M''_K(0) = \frac{\alpha\theta e^t}{(1-\theta e^t)^2} \Big|_{t=0} = \frac{\alpha\theta}{(1-\theta)^2} \\
 \text{Var}(K) &= \frac{\alpha\theta(1-\alpha\theta)}{(1-\theta)^2}
 \end{aligned} \tag{26}$$

(Method 4: Cumulant generating function)

$$\begin{aligned}
 K(t) &= \log M_K(t) = \log(-\alpha \log(1-\theta e^t)) \\
 E(K) &= K'(0) = \frac{\theta e^t}{(\theta e^t-1) \log(1-\theta e^t)} \Big|_{t=0} = \frac{-\theta}{(1-\theta) \log(1-\theta)} = \frac{\alpha\theta}{1-\theta} \quad \left(\because \alpha = -\frac{1}{\log(1-\theta)}\right) \\
 \text{Var}(K) &= K''(0) = \frac{\theta e^t(\theta e^t + \log(1-\theta e^t))}{(\theta e^t-1)^2 \{\log(1-\theta e^t)\}^2} \Big|_{t=0} = -\frac{\theta(\theta + \log(1-\theta))}{(\theta-1)^2 \{\log(1-\theta)\}^2} \\
 &= -\frac{\alpha^2\theta(\theta-1/\alpha)}{(1-\theta)^2} = \frac{\alpha\theta(1-\alpha\theta)}{(1-\theta)^2}
 \end{aligned} \tag{27}$$

Problem 7

A random variable X is sub-Gaussian if there is some $c > 0$ such that

$$\mathbb{E}(e^{tX}) \leq e^{c^2 t^2 / 2}$$

for all t .

(a) Suppose that X is sub-Gaussian. Show that $\mathbb{E}(X) = 0$ and $\text{Var}(X) \leq c^2$.

The inequality translates to

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}(X^k) \leq \sum_{k=0}^{\infty} \frac{c^{2k} t^{2k}}{2^k k!} \quad (28)$$

Observe that up to the quadratic terms,

$$t \mathbb{E}(X) + \frac{t^2}{2} \mathbb{E}(X^2) + o(t^2) \leq \frac{c^2 t^2}{2} + o(t^2) \quad \text{as } t \rightarrow 0 \quad (29)$$

Note that $f(t) \in o(t^2) \iff f(t)/t^2 \rightarrow 0$ as $t \rightarrow 0$. Thus, for $t > 0$, dividing by t ,

$$\lim_{t \downarrow 0} \left(\mathbb{E}(X) + \frac{t}{2} \mathbb{E}(X^2) + \frac{o(t^2)}{t} \right) \leq \lim_{t \downarrow 0} \left(\frac{c^2 t}{2} + \frac{o(t^2)}{t} \right) \implies \mathbb{E}(X) \leq 0 \quad (30)$$

Again for $t < 0$, dividing by t ,

$$\lim_{t \uparrow 0} \left(\mathbb{E}(X) + \frac{t}{2} \mathbb{E}(X^2) + \frac{o(t^2)}{t} \right) \geq \lim_{t \uparrow 0} \left(\frac{c^2 t}{2} + \frac{o(t^2)}{t} \right) \implies \mathbb{E}(X) \geq 0 \quad (31)$$

Thus, $\mathbb{E}(X) = 0$.

Now dividing both sides by t^2 ,

$$\lim_{t \rightarrow 0} \frac{\mathbb{E}(X^2)}{2} \leq \lim_{t \rightarrow 0} \left(\frac{c^2}{2} + \frac{o(t^2)}{t^2} \right) \implies \text{Var}(X) \leq c^2 \quad (32)$$

(b) Let $X \sim \text{Unif}(0, 1)$. Show that $X - 1/2$ is sub-Gaussian and check if (a) holds under $X - 1/2$.

$$\mathbb{E}(e^{t(X-1/2)}) = \int_0^1 e^{tx-t/2} dx = \frac{e^{t/2} - e^{-t/2}}{t} \quad (33)$$

Observe that

$$\begin{aligned} e^{t/2} &= \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} \\ e^{-t/2} &= \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^k}{k!} \\ e^{t/2} - e^{-t/2} &= 2 \sum_{k=0}^{\infty} \frac{(t/2)^{2k+1}}{(2k+1)!} \\ \mathbb{E}(e^{t(X-1/2)}) &= \sum_{k=0}^{\infty} \frac{(t/2)^{2k}}{(2k+1)!} \leq \sum_{k=0}^{\infty} \frac{(t/2)^{2k}}{k! 2^k} = \sum_{k=0}^{\infty} \frac{(t^2/8)^k}{k!} = e^{t^2/8} \quad (\because (2k+1)! \geq k! 2^k) \end{aligned} \quad (34)$$

Thus, $X - 1/2$ is sub-Gaussian and $c = 1/2$.

Problem 8

Let X denote a random variable with a standard Laplace distribution with p.d.f

$$f(x) = \frac{1}{2} \exp\{-|x|\}, \quad -\infty < x < \infty.$$

Find the first, second and the third cumulants, i.e. κ_1 , κ_2 and κ_3 .

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