STAT 412 Fall 2017 Assignment One

Due no later than 5:00 p.m. on Thursday, October 12th, at the beginning of the class.

1. Suppose that k events $B_1, \dots, B_k$ form a partition of the sample space Ω. For $i = 1, \dots, k$, let $P(B_i)$ denote the prior probability of B_i. Also, for each event A such that $P(A) > 0$, let $P(B_i|A)$ denote the posterior probability of B_i given that the event A has occurred. Prove that if $P(B_1|A) < P(B_1)$, then $P(B_i|A) > P(B_i)$ for at least one value of i $(i = 2, \dots, k)$.

2. Suppose that a box contains five coins, and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed $(i = 1, \dots, 5)$, and suppose that $p_1 = 0, p_2 = 1/4, p_3 = 1/2, p_4 = 3/4$, and $p_5 = 1$.

   (a) Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected $(i = 1, \dots, 5)$?

   (b) If the same coin were tossed again, what would be the probability of obtaining another head?

   (c) If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

3. The *skewness* of a random variable X can be defined as $\gamma_1 = \mu_3/(\mu_2)^{\frac{3}{2}}$ where

$$\mu_n = \mathrm{E}(X - \mathrm{E}(X))^n$$

   Find the skewness of a random variable X with a binomial distribution $B(n, \pi)$ of index n and parameter π.

4. Define

$$I = \int_0^\infty \exp(-\tfrac{1}{2}z^2)\,dz$$

   and show that

$$I = \int_0^\infty \exp(-\tfrac{1}{2}(xy)^2)\,y\,dx = \int_0^\infty \exp(-\tfrac{1}{2}(zx)^2)\,z\,dx.$$

   Deduce that

$$I^2 = \int_0^\infty \int_0^\infty \exp\{-\tfrac{1}{2}(x^2+1)z^2\}\,z\,dz\,dx.$$

   and show that $I = \sqrt{\pi/2}$. (This method is due to Laplace, 1812, Section 24.)

5. Let X_1, X_2 be two independent random variables each with p.d.f. $f_1(x) = e^{-x}$ for $x > 0$ and $f_1(x) = 0$ for $x \leq 0$. Let $Z = X_1 - X_2$ and $W = X_1/X_2$.

   (a) Prove that the conditional p.d.f of X_1 given $Z = 0$ is

$$g_1(x_1|0) = \begin{cases} 2e^{-2x_1} & x_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Prove that the conditional p.d.f. of X_1 given $W = 1$ is

$$h_1(x_1|1) = \begin{cases} 4x_1e^{-2x_1} & x_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Notice that $\{Z = 0\} = \{W = 1\}$, but the conditional distribution of X_1 given $Z = 0$ is not the same as the conditional distribution of given $W = 1$. This discrepancy is known as the *Borel paradox* : The conditional p.d.f.'s are not like conditioning on events of probability 0. We can see that "Z is very close to 0" is not the same as "W is very close to 1".

6. Suppose that the random variable K has a logarithmic series distribution with parameter θ (where $0 < \theta < 1$), so that

$$P(K = k) = \frac{\alpha\theta^k}{k} \qquad (k = 1, 2, \ldots)$$

   where $\alpha = -[\log(1 - \theta)]^{-1}$. Find the mean and variance of K.

7. A random variable X is sub-Gaussian if there is some $c > 0$ such that

$$\mathrm{E}\left(e^{tX}\right) \leq e^{c^2t^2/2}$$

   for all t.

   (a) Suppose that X is sub-Gaussian. Show that $\mathrm{E}(X) = 0$ and $\mathrm{Var}(X) \leq c^2$.

   (b) Let $X \sim \mathrm{Unif}(0, 1)$. Show that $X - 1/2$ is sub-Gaussian and check if (a) holds under $X - 1/2$.

8. Let X denote a random variable with a standard Laplace distribution with p.d.f

$$f(x) = \frac{1}{2}\exp\{-|x|\}, \quad -\infty < x < \infty.$$

   Find the first, second and the third cumulants, i.e. κ_1, κ_2 and κ_3.