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Dynamic Nelson Siegel Model

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1 Model

- Initial model

$$y_t(\tau) = L_t + \frac{1 - e^{-\tau\lambda}}{\tau\lambda} S_t + \left(\frac{1 - e^{-\tau\lambda}}{\tau\lambda} - e^{-\tau\lambda} \right) C_t + \eta_t(\tau) \tag{1.1}$$

- State space model

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & (1 - e^{-\tau_1\lambda})(\lambda\tau_1)^{-1} & (1 - e^{-\tau_1\lambda})(\lambda\tau_1)^{-1} - e^{\lambda\tau_1} \\ 1 & (1 - e^{-\tau_2\lambda})(\lambda\tau_2)^{-1} & (1 - e^{-\tau_2\lambda})(\lambda\tau_2)^{-1} - e^{\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & (1 - e^{-\tau_N\lambda})(\lambda\tau_N)^{-1} & (1 - e^{-\tau_N\lambda})(\lambda\tau_N)^{-1} - e^{\lambda\tau_N} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \eta_t(\tau_1) \\ \eta_t(\tau_2) \\ \vdots \\ \eta_t(\tau_N) \end{pmatrix} \tag{1.2}$$

The evolution of the factors (state equation):

$$\begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} = \begin{pmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \mu_{3,t} \end{pmatrix} + \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} \\ S_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix} \tag{1.3}$$

- Compact notation (Dynamic linear model)

$$
\begin{aligned}
y_t &= \Gamma \boldsymbol{\beta}_t + \eta_t, \quad \eta_t \sim N(\mathbf{0}, \Sigma) \\
\boldsymbol{\beta}_t &= \mu + \mathbf{G}\boldsymbol{\beta}_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(\mathbf{0}, \Omega)
\end{aligned} \tag{1.4}
$$

2 Forward Filtering Backward Sampling (FFBS) algorithm

2.1 Kalman Filter

Algorithm 1 Kalman Filter

procedure KALMANFILTER

$f_{0|0} \leftarrow (I_3 - \mathbf{G})^{-1}\,\mu \quad (\because \mathrm{E}(\boldsymbol{\beta}) = \mu + \mathbf{G}\,\mathrm{E}(\boldsymbol{\beta}))$

$\mathrm{vec}\left(P_{0|0}\right) \leftarrow (I_9 - \mathbf{G} \otimes \mathbf{G})^{-1}\,\mathrm{vec}(\Omega) \quad \left(\because \mathrm{Var}(\boldsymbol{\beta}) = \mathbf{G}\,\mathrm{Var}(\boldsymbol{\beta})\mathbf{G}^T + \Omega\right)$

for $t = 1 : T$ **do**

$f_{t|t-1} \leftarrow \mu + \mathbf{G} f_{t-1|t-1}$

$P_{t|t-1} \leftarrow \mathbf{G} P_{t-1|t-1}\mathbf{G}^T + \Omega$

$K_t \leftarrow P_{t|t-1}\Gamma^T \left(\Gamma P_{t|t-1}\Gamma^T + \Sigma\right)^{-1}$

$\boldsymbol{\beta}_{t|t} \leftarrow f_{t|t-1} + K_t \left(y_t - \Gamma f_{t|t-1}\right)$

$P_{t|t} \leftarrow (I_3 - K_t\Gamma)\, P_{t|t-1}$

Note: $\mathrm{vec}(ABC) = \left(C^T \otimes A\right)\mathrm{vec}(B)$

2.2 Backward Sampling

Algorithm 2 Backward Sampling

procedure BackwardSampling

$\boldsymbol{\beta}_T \sim N\left(f_{T|T}, P_{T|T}\right)$

for $t = (T-1) : 1$ **do**

$f_{t+1|t} \leftarrow \mu + \mathbf{G} f_{t|t}$

$P_{t+1|t} = \mathbf{G} P_{t|t} \mathbf{G}^T + \Omega$

$f_{t|t,\beta_{t+1}} \leftarrow f_{t|t} + P_{t|t} \mathbf{G}^T P_{t+1|t}^{-1} \left(\boldsymbol{\beta}_{t+1} - f_{t+1|t}\right)$

$P_{t|t,\beta_{t+1}} \leftarrow P_{t|t} - P_{t|t} \mathbf{G}^T P_{t+1|t}^{-1} \mathbf{G} P_{t|t}$

$\boldsymbol{\beta}_t \sim N\left(f_{t|t,\beta_{t+1}}, P_{t|t,\beta_{t+1}}\right)$

$$\begin{pmatrix} \boldsymbol{\beta}_t \\ \boldsymbol{\beta}_{t+1} \end{pmatrix} \mid Y^t \sim N\left(\begin{pmatrix} f_{t|t} \\ f_{t+1|t} \end{pmatrix}, \begin{pmatrix} P_{t|t} & P_{t|t}\mathbf{G}^T \\ \mathbf{G} P_{t|t} & P_{t+1|t} \end{pmatrix}\right) \tag{2.1}$$