1 Wishart Distribution

a $p \times p$ Wishart random variate has a density of the form

$$\mathcal{W}(\Omega \mid V, k) = C(k, V)|\Omega|^{(k-p-1)/2} \exp\left(-\frac{1}{2}\operatorname{Tr}\left(V^{-1}\Omega\right)\right) \tag{1}$$

$$C(k, V) = |V|^{-k/2}\left(2^{kp/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{k+1-i}{2}\right)\right)^{-1} \tag{2}$$

The log-density is

$$\log \mathcal{W}(\Omega \mid k, V) = \frac{k-p-1}{2}\log|\Omega| - \frac{1}{2}\operatorname{Tr}\left(V^{-1}\Omega\right) - \frac{k}{2}\log|V| \tag{3}$$

$$-\frac{kp}{2}\log 2 - \frac{p(p-1)}{4}\log\pi - \sum_{i=1}^{p}\log\Gamma\left(\frac{k+1-i}{2}\right) \tag{4}$$

Before jumping into the derivative of the log-density, let's do it by parts.

- $\ell = \operatorname{Tr}\left(V^{-1}\Omega\right) = \operatorname{vec}(\Omega)' \operatorname{vec}\left(V^{-1}\right)$

$$d\ell = \operatorname{vec}(\Omega)' \operatorname{vec}\left(dV^{-1}\right) \tag{5}$$

$$= \operatorname{vec}(\Omega)' \operatorname{vec}\left(-V^{-1}\, dV\, V^{-1}\right) \tag{6}$$

$$= -\operatorname{vec}(\Omega)'\left(V^{-1}\otimes V^{-1}\right) D_p\, d\operatorname{vech}(V) \tag{7}$$

$$\frac{d\operatorname{Tr}\left(V^{-1}\Omega\right)}{d\operatorname{vech}(V)'} = -D_p'\left(V^{-1}\otimes V^{-1}\right) D_p \operatorname{vech}(\Omega) \tag{8}$$

  where D_p is the unique duplication matrix such that $D_p \operatorname{vech}(A) = \operatorname{vec}(A)$.

- $\ell = \log|V|$

$$\frac{d\log|V|}{d\operatorname{vech}(V)'} = D_p \operatorname{vech}\left(V^{-1}\right) \tag{9}$$

Therefore,

$$\nabla_{\operatorname{vech}(V)} \log \mathcal{W}(\Omega \mid k, V) = \frac{1}{2}D_p'\left(V^{-1}\otimes V^{-1}\right) D_p \operatorname{vech}(\Omega) - \frac{k}{2}\operatorname{vech}\left(V^{-1}\right) \tag{10}$$

$$\nabla_k \log \mathcal{W}(\Omega \mid k, V) = \frac{1}{2}\log|\Omega| - \frac{1}{2}\log|V| - \frac{p}{2}\log 2 - \frac{1}{2}\sum_{i=1}^{p}\psi\left(\frac{k+1-i}{2}\right) \tag{11}$$

1.1 Fisher Information of Wishart

To get the Fisher information matrix, we need to go through quite a few steps. First, according to [1],

- (Var(vec(Ω)))

$$\mathrm{Var}(\mathrm{vec}(\Omega)) = k\left(\mathbf{I}_{p^2} + K_{pp}\right)(V \otimes V) \tag{12}$$

  where K_{pp} is a $p^2 \times p^2$ commutation matrix such that

$$K_{pp}\,\mathrm{vec}(C) = \mathrm{vec}(C') \tag{13}$$

  for a $p \times p$ matrix C.

- (Var($\log|\Omega|$)) To get the variance of the log-determinant, we will rely on the following relation.

$$\frac{|\Omega|}{|V|} = \chi_k^2 \chi_{k-1}^2 \cdots \chi_{k-p+1}^2 \tag{14}$$

  where every chi-squared random variables are independent of each other. We need to do variable transformation to get the density of log chi-squared random variate. If we say $X \sim \log \chi_\nu^2$, the density is

$$p(x) = \left(2^{\nu/2}\Gamma(\nu/2)\right)^{-1} \exp\left(\frac{1}{2}\nu x - \frac{1}{2}\exp(x)\right), \qquad -\infty < x < \infty \tag{15}$$

  Then, performing the integration, we obtain the following central moments

$$\mathbf{E}(X) = \log 2 + \psi(\nu/2) \tag{16}$$

$$\mathrm{Var}(X) = \psi_1(\nu/2) \tag{17}$$

  where ψ_1 is the tri-gamma function. Thus, since $\log|\Omega| - \log|V| = \sum_{i=1}^{p} \log \chi_{k-i+1}^2$,

$$\mathrm{Var}(\log|\Omega|) = \sum_{i=1}^{p} \psi_1\left(\frac{k-i+1}{2}\right) \tag{18}$$

- The block-diagonal matrices of the Fisher information have been obtained in the above items. However, it is quite difficult to compute the following off-diagonal covariance term:

$$\mathrm{Cov}(\mathrm{vec}(\Omega), \log|\Omega|) \tag{19}$$

2 SUR

Seemingly Unrelated Regression model is constructed as follows.

$$\mathbf{y}_t = X_i'\beta + \mathbf{e}_t, \quad \mathbf{e}_t \sim \mathcal{N}\left(\mathbf{0}, \Omega^{-1}\right) \tag{20}$$

- $\beta \sim \mathcal{N}\left(\mu_\beta^0, \Sigma_\beta^0\right), \quad (m \times 1)$
- $\Omega \sim \mathcal{W}(k, V), \quad (p \times p)$

The varational posteriors are

- $q(\beta) = \mathcal{N}\left(\mu_\beta^q, \Sigma_\beta^q\right)$
- $q(\Omega) = \mathcal{W}\left(k_q, V_q\right)$

Therefore,

$$\begin{aligned}
\log h(\theta) = &\frac{T}{2}\log\det\Omega - \frac{1}{2}\sum_{t=1}^{T}\left(\mathbf{y}_t - X_t'\beta\right)'\Omega\left(\mathbf{y}_t - X_t'\beta\right) - \frac{Tp}{2}\log(2\pi) && (21)\\
&-\frac{1}{2}\log\det\Sigma_\beta^0 - \frac{1}{2}\left(\beta - \mu_\beta^0\right)'\Sigma_\beta^{0\,-1}\left(\beta - \mu_\beta^0\right) && (22)\\
&+\frac{k-p-1}{2}\log\det\Omega - \frac{1}{2}\operatorname{Tr}\left(V^{-1}\Omega\right) - \frac{k}{2}\log\det V - \frac{kp}{2}\log 2 && (23)\\
&-\sum_{i=1}^{p}\log\Gamma\left(\frac{k+1-i}{2}\right) && (24)\\
\log q_\lambda(\theta) = &-\frac{1}{2}\log\det\Sigma_\beta^q - \frac{1}{2}\left(\beta - \mu_\beta^q\right)'\Sigma_\beta^{q\,-1}\left(\beta - \mu_\beta^q\right) && (25)\\
&+\frac{k_q-p-1}{2}\log\det\Omega - \frac{1}{2}\operatorname{Tr}\left(V_q^{-1}\Omega\right) - \frac{k_q}{2}\log\det V_q - \frac{k_q p}{2}\log 2 && (26)\\
&-\sum_{i=1}^{p}\log\Gamma\left(\frac{k_q+1-i}{2}\right) && (27)
\end{aligned}$$

where $\theta = (\beta, \Omega)$ and $\lambda = \left(\mu_\beta^q, \Sigma_\beta^q, k_q, V_q\right)$.

References

[1] Muirhead, R. J. *Aspects of multivariate statistical theory*. Wiley series in probability and mathematical statistics. Probability and mathematical statistics. John Wiley & Sons, New York, 1982.