VB nonparametric Regression with Cosine Basis Expansion

Daeyoung Lim[1]

1. Derivation

- Likelihood:

$$\ell\left(\boldsymbol{\beta},\boldsymbol{\theta}\mid\mathbf{y}\right)=-\frac{n}{2}\log\left(2\pi\right)-\frac{1}{2\sigma^2}\left(\mathbf{y}-\boldsymbol{\varphi}\boldsymbol{\theta}\right)'\left(\mathbf{y}-\boldsymbol{\varphi}\boldsymbol{\theta}\right) \tag{1}$$

- $f(\mathbf{x})=\sum_{\ell=0}^{\infty}\theta_\ell\varphi_\ell\left(x_i\right)$
- Priors
  - $\theta_j\mid\sigma,\tau,\gamma\sim\mathcal{N}\left(0,\sigma^2\tau^2e^{-j\gamma}\right)$
  - $\sigma^2\sim\text{InvGam}\left(\frac{r_{0,\sigma}}{2},\frac{s_{0,\sigma}}{2}\right)$
  - $\tau^2\sim\text{InvGam}\left(\frac{r_{0,\tau}}{2},\frac{s_{0,\tau}}{2}\right)$
  - $\psi\sim\text{InvGam}(a,b)$
  - $\gamma\sim\text{Exp}\left(w_0\right)$
- Transformations:
  - $\zeta=\log(\exp(\psi)-1)$
  - $\alpha=\log(\exp(\sigma^2)-1)$
  - $\eta=\log(\exp(\tau^2)-1)$
  - $\xi=\log(\exp(\gamma)-1)$
- Parameters: $\Theta=\left(\boldsymbol{\beta}',\boldsymbol{\theta}'_J,\zeta,\alpha,\eta,\xi\right)$
- Variational distribution: $q(\Theta)=\mathcal{N}(\boldsymbol{\mu},LL')$
- Derivative of the transformation (Jacobian):

$$\frac{d}{dx}\log(\exp(x)+1)=\frac{e^x}{1+e^x} \tag{2}$$

- Derivative of the log-Jacobian:

$$\frac{d}{dx}\left(\log\frac{e^x}{1+e^x}\right)=\frac{1}{1+e^x} \tag{3}$$

[1]Department of Statistics, Korea University

- Generating synthetic data

$$y_i = \begin{cases} 1, & \text{if } \mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta} + \epsilon_i > 0 \text{ where } \epsilon \sim \text{Logistic}(0,1) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

2. Monotone Shape Restriction (VBM)

Basic regression model:

$$y_i = f(x_i) + \epsilon_i \tag{5}$$

We use the derivative representation for the monotone function written in terms of integrals:

$$f(x) = \delta \left[\int_0^x Z^2(s)\, ds - \int_0^1 \int_0^x Z^2(s)\, ds\, dx \right] \tag{6}$$

$$\delta = \begin{cases} 1, & \text{for non-decreasing function} \\ -1, & \text{for non-increasing function} \end{cases} \tag{7}$$

Using the spectral representatino of $Z(x)$,

$$f(x) = \delta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \theta_j \theta_k \varphi_{j,k}^a(x) \tag{8}$$

$$\varphi_{j,k}^a(x) = \int_0^x \varphi_j(s)\overline{\varphi}_k(s)\, ds - \int_0^1 \int_0^s \varphi_j(t)\overline{\varphi}_k(t)\, dt\, ds \quad \text{for } j,k \geq 0 \tag{9}$$

(Omit the cosine basis functions)

$$y_i = \delta \boldsymbol{\theta}' \boldsymbol{\varphi}(x_i) \boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{10}$$

- Priors
  - $\theta_0 \mid \sigma \sim \mathcal{N}\left(0, \sigma\sigma_0^2\right)$ where σ_0^2 is known
  - $\theta_j \mid \sigma, \tau, \gamma \sim \mathcal{N}\left(0, \sigma\tau^2 e^{-j\gamma}\right)$ for $j \geq 1$
  - $\sigma^2 \sim \text{InvGam}\left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right)$
  - $\tau^2 \sim \text{InvGam}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right)$
  - $\psi \sim \text{InvGam}(a, b)$
  - $\gamma \sim \text{Exp}(w_0)$