VB Logistic Regression with Cosine Basis Expansion

Daeyoung Lim[1]

1. Derivation

- Likelihood:

$$\prod_{i=1}^{n} \Pr\left(Y_i = y_i \mid \boldsymbol{\beta}, \boldsymbol{\theta}\right) = \prod_{i=1}^{n} \left[\frac{\exp\left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)}{1 + \exp\left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)}\right]^{y_i} \left[\frac{1}{1 + \exp\left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)}\right]^{1-y_i} \tag{1}$$

$$= \prod_{i=1}^{n} \left[\frac{\exp\left\{y_i \left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)\right\}}{1 + \exp\left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)}\right] \tag{2}$$

$$\ell\left(\boldsymbol{\beta}, \boldsymbol{\theta} \mid \mathbf{y}\right) = \sum_{i=1}^{n} y_i \left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right) - \log\left(1 + \exp\left(\mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta}\right)\right) \tag{3}$$

- $f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \theta_\ell \varphi_\ell\left(x_i\right)$
- Priors
  - $\boldsymbol{\beta} \mid \sigma^2 \sim \mathcal{N}\left(\boldsymbol{\mu}_\beta^0, \sigma^2 \boldsymbol{\Sigma}_\beta^0\right)$
  - $\theta_j \mid \sigma, \tau, \gamma \sim \mathcal{N}\left(0, \sigma^2\tau^2 e^{-j\gamma}\right)$
  - $\sigma^2 \sim \text{InvGam}\left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right)$
  - $\tau^2 \sim \text{InvGam}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right)$
  - $\psi \sim \text{InvGam}(a, b)$
  - $\gamma \sim \text{Exp}\left(w_0\right)$
- Transformations:
  - $\zeta = \log(\exp(\psi) - 1)$
  - $\alpha = \log(\exp(\sigma^2) - 1)$
  - $\eta = \log(\exp(\tau^2) - 1)$
  - $\xi = \log(\exp(\gamma) - 1)$
- Parameters: $\Theta = \left(\boldsymbol{\beta}', \boldsymbol{\theta}_J', \zeta, \alpha, \eta, \xi\right)$
- Variational distribution: $q(\Theta) = \mathcal{N}(\boldsymbol{\mu}, LL')$
- Derivative of the transformation (Jacobian):

$$\frac{d}{dx}\log(\exp(x) + 1) = \frac{e^x}{1 + e^x} \tag{4}$$

[1]Department of Statistics, Korea University

- Derivative of the log-Jacobian:

$$\frac{d}{dx}\left(\log\frac{e^x}{1+e^x}\right) = \frac{1}{1+e^x} \tag{5}$$

- Generating synthetic data

$$y_i = \begin{cases} 1, & \text{if } \mathbf{w}_i'\boldsymbol{\beta} + \boldsymbol{\varphi}_i'\boldsymbol{\theta} + \epsilon_i > 0 \text{ where } \epsilon \sim \text{Logistic}(0, 1) \\ 0, & \text{otherwise} \end{cases} \tag{6}$$