1 Wishart Distribution

a $p \times p$ Wishart random variate has a density of the form

$$\mathcal{W}(\Omega \mid V, k) = C(k, V)|\Omega|^{(k-p-1)/2} \exp\left(-\frac{1}{2}\operatorname{Tr}\left(V^{-1}\Omega\right)\right) \tag{1}$$

$$C(k, V) = |V|^{-k/2}\left(2^{kp/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{k+1-i}{2}\right)\right)^{-1} \tag{2}$$

The log-density is

$$\begin{aligned}\log \mathcal{W}(\Omega \mid k, V) = {} & \frac{k-p-1}{2}\log|\Omega| - \frac{1}{2}\operatorname{Tr}\left(V^{-1}\Omega\right) - \frac{k}{2}\log|V| && (3)\\ & -\frac{kp}{2}\log 2 - \frac{p(p-1)}{4}\log\pi - \sum_{i=1}^{p}\log\Gamma\left(\frac{k+1-i}{2}\right) && (4)\end{aligned}$$

Before jumping into the derivative of the log-density, let's do it by parts.

- $\ell = \operatorname{Tr}\left(V^{-1}\Omega\right) = \operatorname{vec}(\Omega)'\operatorname{vec}\left(V^{-1}\right)$

$$\begin{aligned} d\ell &= \operatorname{vec}(\Omega)'\operatorname{vec}\left(dV^{-1}\right) && (5)\\ &= \operatorname{vec}(\Omega)'\operatorname{vec}\left(-V^{-1}\,dV\,V^{-1}\right) && (6)\\ &= -\operatorname{vec}(\Omega)'\left(V^{-1}\otimes V^{-1}\right)D_p\,d\operatorname{vech}(V) && (7)\\ \frac{d\operatorname{Tr}\left(V^{-1}\Omega\right)}{d\operatorname{vech}(V)'} &= -D_p'\left(V^{-1}\otimes V^{-1}\right)D_p\operatorname{vech}(\Omega) && (8)\end{aligned}$$

  where D_p is the unique duplication matrix such that $D_p\operatorname{vech}(A) = \operatorname{vec}(A)$.

- $\ell = \log|V|$

$$\frac{d\log|V|}{d\operatorname{vech}(V)'} = D_p\operatorname{vech}\left(V^{-1}\right) \tag{9}$$

Therefore,

$$\begin{aligned}\nabla_{\operatorname{vech}(V)}\log\mathcal{W}(\Omega \mid k, V) &= \frac{1}{2}D_p'\left(V^{-1}\otimes V^{-1}\right)D_p\operatorname{vech}(\Omega) - \frac{k}{2}D_p\operatorname{vech}\left(V^{-1}\right) && (10)\\ \nabla_k\log\mathcal{W}(\Omega \mid k, V) &= \frac{1}{2}\log|\Omega| - \frac{1}{2}\log|V| - \frac{p}{2}\log 2 - \frac{1}{2}\sum_{i=1}^{p}\psi\left(\frac{k+1-i}{2}\right) && (11)\end{aligned}$$