Cosine basis

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1 Model specifications

$$
\begin{aligned}
y_i &= w_i^\top \beta + f\left(x_i\right) + \epsilon_i, \qquad \epsilon_i \sim \mathcal{N}\left(0, \sigma^2\right) \\
\theta_j | \sigma, \tau, \gamma &\sim \mathcal{N}\left(0, \sigma^2 \tau^2 \exp\left[-j\gamma\right]\right) \\
\tau^2 &\sim \mathrm{IG}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right) \\
\sigma^2 &\sim \mathrm{IG}\left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right) \\
\beta &\sim \mathcal{N}\left(\mu_\beta^0, \Sigma_\beta^0\right) \\
\gamma &\sim \mathrm{Exp}\left(\omega_0\right) \\
|\psi| &= \gamma, \quad \psi \sim \mathrm{DE}\left(0, \omega_0\right) \\
\varphi_j\left(x\right) &= \sqrt{2}\cos\left(\pi j x\right)
\end{aligned}
$$

Joint density:

$$
\begin{aligned}
p\left(y, \Theta\right) = {} & \mathcal{N}\left(y \middle| W\beta + f_J, \sigma^2 I_n\right) \left\{\prod_{j=1}^{J} \mathcal{N}\left(\theta_j \middle| 0, \sigma^2\tau^2 \exp\left[-j\left|\psi\right|\right]\right)\right\} \mathrm{IG}\left(\tau^2 \middle| \frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right) \mathrm{IG}\left(\sigma^2 \middle| \frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right) \mathcal{N}\left(\beta \middle| \mu_\beta^0, \Sigma_\beta^0\right) \\
& \mathrm{DE}\left(\psi | 0, \omega_0\right)
\end{aligned}
$$

We will use the joint density to derive the LB and updating algorithm. The variational distributions are

$$
\begin{aligned}
q_1\left(\beta\right) &= \mathcal{N}\left(\mu_\beta^q, \Sigma_\beta^q\right) \\
q_2\left(\theta_J\right) &= \mathcal{N}\left(\mu_\theta^q, \Sigma_\theta^q\right) \\
q_3\left(\sigma^2\right) &= \mathrm{IG}\left(\frac{r_{q,\sigma}}{2}, \frac{s_{q,\sigma}}{2}\right) \\
q_4\left(\tau^2\right) &= \mathrm{IG}\left(\frac{r_{q,\tau}}{2}, \frac{s_{q,\tau}}{2}\right) \\
q_5\left(\psi\right) &= \mathcal{N}\left(\mu_\psi^q, {\sigma_\psi^2}^q\right) \quad \text{(NCVMP)}.
\end{aligned}
$$

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2 Lower bound

2.1 LB: $\mathsf{E}\left[\ln p\left(y|\Theta\right)\right]$

$$\begin{aligned}
\mathsf{E}\left[\ln p\left(y|\Theta\right)\right] &= -\frac{n}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2}\mathsf{E}\left[\left(y - W\beta - \varphi_J^\top\theta\right)^\top\left(y - W\beta - \varphi_J^\top\theta\right)\right] \\
&= -\frac{n}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2}\left(y - W\mu_\beta^q - \varphi_J^\top\mu_\theta^q\right)^\top\left(y - W\mu_\beta^q - \varphi_J^\top\mu_\theta^q\right) - \frac{1}{2}\left(\mathrm{Tr}\left(W^\top W\Sigma_\beta^q\right) + \mathrm{Tr}\left(\varphi_J\varphi_J^\top\Sigma_\theta^q\right)\right)
\end{aligned}$$

2.2 LB: $\mathsf{E}\left[\ln p\left(\theta_j|\sigma,\tau,\psi\right)\right]$

$$\sum_{j=1}^J \mathsf{E}\left[\ln p\left(\theta_j|\sigma,\tau,\psi\right)\right] = \sum_{j=1}^J \mathsf{E}\left[-\frac{1}{2}\ln\left(2\pi\right) + \ln\frac{1}{\sigma} + \ln\frac{1}{\tau} + \frac{j}{2}\left|\psi\right| - \frac{\theta_j^2 e^{j|\psi|}}{2\sigma^2\tau^2}\right]$$

Let's note the following fact: if $X \sim \mathcal{N}\left(\mu,\sigma^2\right)$, then $|X| \sim$ folded-Normal $\left(\mu,\sigma^2\right)$. Then,

$$\begin{aligned}
\mathsf{E}\left|X\right| &= \sigma\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} + \mu\left(1 - 2\Phi\left(\frac{-\mu}{\sigma}\right)\right) \\
&= \sigma\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} - \mu\,\mathrm{erf}\left(\frac{-\mu}{\sqrt{2\sigma^2}}\right) \\
\mathsf{E}e^{t|X|} &= \exp\left\{\frac{\sigma^2t^2}{2} + \mu t\right\}\left[1 - \Phi\left(-\frac{\mu}{\sigma} - \sigma t\right)\right] + \exp\left\{\frac{\sigma^2t^2}{2} - \mu t\right\}\left[1 - \Phi\left(\frac{\mu}{\sigma} - \sigma t\right)\right].
\end{aligned}$$