VB Negative Binomial Regression with Cosine Basis Expansion

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1. Derivation

- Likelihood:

$$f\left(y_i \,|\, \mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\theta}_J, \psi\right) = \frac{\Gamma\left(y_i + \psi\right)}{y_i! \Gamma\left(\psi\right)} \left[\frac{\lambda_i}{\lambda_i + \psi}\right]^{y_i} \left[\frac{\psi}{\lambda_i + \psi}\right]^{\psi}, \qquad \lambda_i = \exp\left(\mathbf{w}_i' \boldsymbol{\beta} + \boldsymbol{\varphi}_i' \boldsymbol{\theta}_J\right) \tag{1}$$

$$\ell\left(\boldsymbol{\beta}, \boldsymbol{\theta}_J, \psi \,|\, \mathbf{y}, \mathbf{X}\right) = n \log \Gamma(\psi) + n\psi \log \psi \tag{2}$$

$$+ \sum_{i=1}^{n} \log \Gamma\left(y_i + \psi\right) - \log y_i! - \left(y_i + \psi\right) \log\left(\psi + \exp\left(\mathbf{w}_i' \boldsymbol{\beta} + \boldsymbol{\varphi}_i' \boldsymbol{\theta}_J\right)\right) \tag{3}$$

$$+ \mathbf{y}' \mathbf{W} \boldsymbol{\beta} + \mathbf{y}' \boldsymbol{\varphi}_J \boldsymbol{\theta}_J \tag{4}$$

- $f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \theta_\ell \varphi_\ell\left(x_i\right)$
- Priors
  - $\boldsymbol{\beta} \,|\, \sigma^2 \sim \mathcal{N}\left(\boldsymbol{\mu}_\beta^0, \sigma^2 \boldsymbol{\Sigma}_\beta^0\right)$
  - $\theta_j \,|\, \sigma, \tau, \gamma \sim \mathcal{N}\left(0, \sigma^2 \tau^2 e^{-j\gamma}\right)$
  - $\sigma^2 \sim \text{InvGam}\left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right)$
  - $\tau^2 \sim \text{InvGam}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right)$
  - $\psi \sim \text{InvGam}(a, b)$
  - $\gamma \sim \text{Exp}\left(w_0\right)$
- Transformations:
  - $\zeta = \log(\exp(\psi) - 1)$
  - $\alpha = \log(\exp(\sigma^2) - 1)$
  - $\eta = \log(\exp(\tau^2) - 1)$
  - $\xi = \log(\exp(\gamma) - 1)$
- Parameters: $\Theta = \left(\boldsymbol{\beta}', \boldsymbol{\theta}_J', \zeta, \alpha, \eta, \xi\right)$
- Variational distribution: $q(\Theta) = \mathcal{N}(\boldsymbol{\mu}, LL')$

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- Derivative of the transformation (Jacobian):

$$\frac{d}{dx} \log(\exp(x) + 1) = \frac{e^x}{1 + e^x} \tag{5}$$

- Derivative of the log-Jacobian:

$$\frac{d}{dx} \left(\log \frac{e^x}{1 + e^x} \right) = \frac{1}{1 + e^x} \tag{6}$$