

Cosine basis

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1 Normal

1.1 Model specifications

$$\begin{aligned}y_i &= w_i^\top \beta + f(x_i) + \epsilon_i, & \epsilon_i &\sim \mathcal{N}(0, \sigma^2) \\ \theta_j | \sigma, \tau, \gamma &\sim \mathcal{N}(0, \sigma^2 \tau^2 \exp[-j\gamma]) \\ \tau^2 &\sim \text{IG}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right) \\ \sigma^2 &\sim \text{IG}\left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right) \\ \beta &\sim \mathcal{N}(\mu_\beta^0, \Sigma_\beta^0) \\ \gamma &\sim \text{Exp}(\omega_0) \\ |\psi| &= \gamma, \quad \psi \sim \text{DE}(0, \omega_0) \\ \varphi_j(x) &= \sqrt{2} \cos(\pi j x)\end{aligned}$$

Joint density:

$$\begin{aligned}p(y, \Theta) &= \mathcal{N}(y | W\beta + f_J, \sigma^2 I_n) \left\{ \prod_{j=1}^J \mathcal{N}(\theta_j | 0, \sigma^2 \tau^2 \exp[-j|\psi|]) \right\} \text{IG}\left(\tau^2 \middle| \frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right) \text{IG}\left(\sigma^2 \middle| \frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2}\right) \mathcal{N}(\beta | \mu_\beta^0, \Sigma_\beta^0) \\ &\quad \text{DE}(\psi | 0, \omega_0)\end{aligned}$$

We will use the joint density to derive the LB and updating algorithm. The variational distributions are

$$\begin{aligned}q_1(\beta) &= \mathcal{N}(\mu_\beta^q, \Sigma_\beta^q) \\ q_2(\theta_J) &= \mathcal{N}(\mu_\theta^q, \Sigma_\theta^q) \\ q_3(\sigma^2) &= \text{IG}\left(\frac{r_{q,\sigma}}{2}, \frac{s_{q,\sigma}}{2}\right) \\ q_4(\tau^2) &= \text{IG}\left(\frac{r_{q,\tau}}{2}, \frac{s_{q,\tau}}{2}\right) \\ q_5(\psi) &= \mathcal{N}(\mu_\psi^q, \sigma_\psi^{2q}) \quad (\text{NCVMP}).\end{aligned}$$

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1.2 Lower bound

1.2.1 LB: $\mathbb{E} [\ln p(y|\Theta)]$

$$\begin{aligned}\mathbb{E} [\ln p(y|\Theta)] &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2} \mathbb{E} \left[(y - W\beta - \varphi_J \theta)^\top (y - W\beta - \varphi_J \theta) \right] \\ &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2} \left(y - W\mu_\beta^q - \varphi_J \mu_\theta^q \right)^\top \left(y - W\mu_\beta^q - \varphi_J \mu_\theta^q \right) - \frac{1}{2} \left(\text{Tr} \left(W^\top W \Sigma_\beta^q \right) + \text{Tr} \left(\varphi_J^\top \varphi_J \Sigma_\theta^q \right) \right)\end{aligned}$$

1.3 LB: $\mathbb{E} [\ln p(\theta_j|\sigma, \tau, \psi)]$

$$\sum_{j=1}^J \mathbb{E} [\ln p(\theta_j|\sigma, \tau, \psi)] = \sum_{j=1}^J \mathbb{E} \left[-\frac{1}{2} \ln(2\pi) + \ln \frac{1}{\sigma} + \ln \frac{1}{\tau} + \frac{j}{2} |\psi| - \frac{\theta_j^2 e^{j|\psi|}}{2\sigma^2 \tau^2} \right]$$

Let's note the following fact: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $|X| \sim \text{folded-Normal}(\mu, \sigma^2)$. Then,

$$\begin{aligned}\mathbb{E} |X| &= \sigma \sqrt{\frac{2}{\pi}} \exp \left\{ -\frac{\mu^2}{2\sigma^2} \right\} + \mu \left(1 - 2\Phi \left(\frac{-\mu}{\sigma} \right) \right) \\ &= \sigma \sqrt{\frac{2}{\pi}} \exp \left\{ -\frac{\mu^2}{2\sigma^2} \right\} - \mu \text{erf} \left(\frac{-\mu}{\sqrt{2}\sigma} \right) \\ \mathbb{E} e^{t|X|} &= \exp \left\{ \frac{\sigma^2 t^2}{2} + \mu t \right\} \left[1 - \Phi \left(-\frac{\mu}{\sigma} - \sigma t \right) \right] + \exp \left\{ \frac{\sigma^2 t^2}{2} - \mu t \right\} \left[1 - \Phi \left(\frac{\mu}{\sigma} - \sigma t \right) \right].\end{aligned}$$

2 Probit

2.1 Model specifications

$$\begin{aligned}\Pr(y_i = 1|f, \beta) &= \Phi(w_i^\top \beta + f(x_i)) \\ y_i^* &= w_i^\top \beta + f(x_i) + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, 1) \\ y_i &= \begin{cases} 1, & \text{if } y_i^* \geq 0 \\ 0, & \text{if } y_i^* < 0 \end{cases} \\ \theta_j|\sigma, \tau, \gamma &\sim \mathcal{N}(0, \sigma^2 \tau^2 \exp[-j\gamma]) \\ \tau^2 &\sim \text{IG} \left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2} \right) \\ \sigma^2 &\sim \text{IG} \left(\frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2} \right) \\ \beta &\sim \mathcal{N}(\mu_\beta^0, \sigma^2 \Sigma_\beta^0) \quad (p \times 1) \\ \gamma &\sim \text{Exp}(\omega_0) \\ |\psi| &= \gamma, \quad \psi \sim \text{DE}(0, \omega_0) \\ \varphi_j(x) &= \sqrt{2} \cos(\pi j x)\end{aligned}$$

Joint density:

$$\begin{aligned}p(y, y^*, \Theta) &= C \left\{ \prod_{j=1}^J \mathcal{N}(\theta_j | 0, \sigma^2 \tau^2 \exp[-j|\psi|]) \right\} \text{IG} \left(\tau^2 \middle| \frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2} \right) \text{IG} \left(\sigma^2 \middle| \frac{r_{0,\sigma}}{2}, \frac{s_{0,\sigma}}{2} \right) \mathcal{N}(\beta | \mu_\beta^0, \sigma^2 \Sigma_\beta^0) \\ &\quad \text{DE}(\psi | 0, \omega_0) \left\{ \prod_{i=1}^n (1[y_i^* \geq 0] 1[y_i = 1] + 1[y_i^* < 0] 1[y_i = 0]) \phi(y_i^* - w_i^\top \beta - \varphi_i^\top \theta_J) \right\}\end{aligned}$$

where C is the normalizing constant. The variational distributions are

$$\begin{aligned}
q_1(\beta) &= \mathcal{N}(\mu_\beta^q, \Sigma_\beta^q) \\
q_2(\theta_J) &= \mathcal{N}(\mu_\theta^q, \Sigma_\theta^q) \\
q_3(\sigma^2) &= \text{IG}\left(\frac{r_{q,\sigma}}{2}, \frac{s_{q,\sigma}}{2}\right) \\
q_4(\tau^2) &= \text{IG}\left(\frac{r_{q,\tau}}{2}, \frac{s_{q,\tau}}{2}\right) \\
q_5(\psi) &= \mathcal{N}(\mu_\psi^q, \sigma_\psi^{2q}) \quad (\text{NCVMP}) \\
q_6(y^*) &= \mathcal{TN}(\mu_{y^*}^q, I_n, 0)
\end{aligned}$$

2.2 Lower bound

2.2.1 LB: $\mathbb{E}[\ln p(y^*|\text{rest})] + \mathbb{H}[y^*]$

$$\begin{aligned}
\mathbb{E}[\ln p(y^*|\text{rest})] + \mathbb{H}[y^*] &= \sum_{i=1}^n \mathbb{E} \left[\ln \phi(y_i^* - w_i^\top \beta - \varphi_i^\top \theta_J) - \ln \phi(y_i^* - w_i^\top \mu_\beta^q - \varphi_i^\top \mu_\theta^q) \right] \\
&\quad + \sum_{i=1}^n \ln \left(\left\{ \Phi(w_i^\top \mu_\beta^q + \varphi_i^\top \mu_\theta^q) \right\}^{y_i} \left\{ 1 - \Phi(w_i^\top \mu_\beta^q + \varphi_i^\top \mu_\theta^q) \right\}^{1-y_i} \right) \\
&= -\frac{1}{2} \left(\text{Tr}(W^\top W \Sigma_\beta^q) + \text{Tr}(\varphi_J^\top \varphi_J \Sigma_\theta^q) \right) \\
&\quad + \sum_{i=1}^n \ln \left(\left\{ \Phi(w_i^\top \mu_\beta^q + \varphi_i^\top \mu_\theta^q) \right\}^{y_i} \left\{ 1 - \Phi(w_i^\top \mu_\beta^q + \varphi_i^\top \mu_\theta^q) \right\}^{1-y_i} \right)
\end{aligned}$$

2.2.2 LB: $\mathbb{E}[\ln p(\theta_j|\sigma, \tau, \psi)]$

Let's note the following fact: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $|X| \sim \text{folded-Normal}(\mu, \sigma^2)$. Then,

$$\begin{aligned}
\mathbb{E}|X| &= \sigma \sqrt{\frac{2}{\pi}} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} + \mu \left(1 - 2\Phi\left(\frac{-\mu}{\sigma}\right)\right) \\
&= \sigma \sqrt{\frac{2}{\pi}} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} - \mu \text{erf}\left(\frac{-\mu}{\sqrt{2}\sigma}\right) \\
\mathbb{E}e^{t|X|} &= \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\} \left[1 - \Phi\left(-\frac{\mu}{\sigma} - \sigma t\right)\right] + \exp\left\{\frac{\sigma^2 t^2}{2} - \mu t\right\} \left[1 - \Phi\left(\frac{\mu}{\sigma} - \sigma t\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^J \mathbb{E}[\ln p(\theta_j|\sigma, \tau, \psi)] + \mathbb{H}[\theta_J] &= \sum_{j=1}^J \mathbb{E} \left[-\frac{1}{2} \ln(2\pi) + \frac{1}{2} \ln \frac{1}{\sigma^2} + \frac{1}{2} \ln \frac{1}{\tau^2} + \frac{j}{2} |\psi| - \frac{\theta_j^2 e^{j|\psi|}}{2\sigma^2 \tau^2} \right] + \mathbb{H}[\theta_J] \\
&= -\frac{J}{2} \left\{ \ln(2\pi) - \left(\text{di}\left(\frac{r_{q,\sigma}}{2}\right) - \ln\left(\frac{s_{q,\sigma}}{2}\right) \right) - \left(\text{di}\left(\frac{r_{q,\tau}}{2}\right) - \ln\left(\frac{s_{q,\tau}}{2}\right) \right) \right\} \\
&\quad + \frac{J(J+1)}{4} \left\{ \sigma_\psi^q \sqrt{\frac{2}{\pi}} \exp\left(-\frac{\mu_\psi^{q2}}{2\sigma_\psi^{q2}}\right) + \mu_\psi^q \left(1 - 2\Phi\left(\frac{-\mu_\psi^q}{\sigma_\psi^q}\right)\right) \right\} \\
&\quad - \frac{1}{2} \frac{r_{q,\sigma}}{s_{q,\sigma}} \frac{r_{q,\tau}}{s_{q,\tau}} \left(\text{Tr}(\Sigma_\theta^q) + \mu_\theta^{q\top} \mu_\theta^q \right) \sum_{j=1}^J Q_j(\mu_\psi^q, \sigma_\psi^{q2}) + \frac{J}{2} (1 + \ln(2\pi)) + \frac{1}{2} \ln |\Sigma_\theta^q|
\end{aligned}$$

where

$$\begin{aligned} Q_j \left(\mu_\psi^q, \sigma_\psi^{q^2} \right) &= \mathbb{E} e^{j|\psi|} \\ &= \exp \left\{ \frac{\sigma_\psi^{q^2} j^2}{2} + \mu_\psi^q j \right\} \left[1 - \Phi \left(-\frac{\mu_\psi^q}{\sigma_\psi^q} - \sigma_\psi^q j \right) \right] + \exp \left\{ \frac{\sigma_\psi^{q^2} j^2}{2} - \mu_\psi^q j \right\} \left[1 - \Phi \left(\frac{\mu_\psi^q}{\sigma_\psi^q} - \sigma_\psi^q j \right) \right]. \end{aligned}$$

2.2.3 LB: $\mathbb{E} [\ln p(\tau^2)] + \mathbb{H}[\tau^2]$

$$\begin{aligned} \mathbb{E} [\ln p(\tau^2)] + \mathbb{H}[\tau^2] &= \frac{r_{0,\tau}}{2} \ln \frac{s_{0,\tau}}{2} - \ln \Gamma \left(\frac{r_{0,\tau}}{2} \right) + \left(\frac{r_{0,\tau}}{2} + 1 \right) \left\{ \text{di} \left(\frac{r_{q,\tau}}{2} \right) - \ln \left(\frac{s_{q,\tau}}{2} \right) \right\} - \frac{s_{0,\tau}}{2} \frac{r_{q,\tau}}{s_{q,\tau}} \\ &\quad + \frac{r_{q,\tau}}{2} + \ln \frac{s_{q,\tau}}{2} + \ln \Gamma \left(\frac{r_{q,\tau}}{2} \right) - \left(1 + \frac{r_{q,\tau}}{2} \right) \text{di} \left(\frac{r_{q,\tau}}{2} \right) \end{aligned}$$

2.2.4 LB: $\mathbb{E} [\ln p(\sigma^2)] + \mathbb{H}[\sigma^2]$

$$\begin{aligned} \mathbb{E} [\ln p(\sigma^2)] + \mathbb{H}[\sigma^2] &= \frac{r_{0,\sigma}}{2} \ln \frac{s_{0,\sigma}}{2} - \ln \Gamma \left(\frac{r_{0,\sigma}}{2} \right) + \left(\frac{r_{0,\sigma}}{2} + 1 \right) \left\{ \text{di} \left(\frac{r_{q,\sigma}}{2} \right) - \ln \left(\frac{s_{q,\sigma}}{2} \right) \right\} - \frac{s_{0,\sigma}}{2} \frac{r_{q,\sigma}}{s_{q,\sigma}} \\ &\quad + \frac{r_{q,\sigma}}{2} + \ln \frac{s_{q,\sigma}}{2} + \ln \Gamma \left(\frac{r_{q,\sigma}}{2} \right) - \left(1 + \frac{r_{q,\sigma}}{2} \right) \text{di} \left(\frac{r_{q,\sigma}}{2} \right) \end{aligned}$$

2.2.5 LB: $\mathbb{E} [\ln p(\beta)] + \mathbb{H}[\beta]$

$$\begin{aligned} \mathbb{E} [\ln p(\beta)] + \mathbb{H}[\beta] &= \frac{p+1}{2} + \frac{1}{2} \left(\text{di} \left(\frac{r_{q,\sigma}}{2} \right) - \ln \left(\frac{s_{q,\sigma}}{2} \right) \right) + \frac{1}{2} \ln \left| \Sigma_\beta^{0-1} \Sigma_\beta^q \right| \\ &\quad - \frac{1}{2} \frac{r_{q,\sigma}}{s_{q,\sigma}} \left\{ \text{Tr} \left(\Sigma_\beta^{0-1} \Sigma_\beta^q \right) + \left(\mu_\beta^q - \mu_\beta^0 \right)^\top \Sigma_\beta^{0-1} \left(\mu_\beta^q - \mu_\beta^0 \right) \right\} \end{aligned}$$

2.2.6 LB: $\mathbb{E} [\ln p(\psi)] + \mathbb{H}[\psi]$

$$\begin{aligned} \mathbb{E} [\ln p(\psi)] + \mathbb{H}[\psi] &= \ln \frac{\omega_0}{2} - \omega_0 \left\{ \sigma_\psi^q \sqrt{\frac{2}{\pi}} \exp \left(-\frac{\mu_\psi^{q^2}}{2\sigma_\psi^{q^2}} \right) + \mu_\psi^q \left(1 - 2\Phi \left(-\frac{\mu_\psi^q}{\sigma_\psi^q} \right) \right) \right\} \\ &\quad + \frac{1}{2} \ln \left(2\pi \sigma_\psi^{q^2} \right) - \frac{1}{2} \end{aligned}$$

2.3 Update

2.3.1 θ_j

$$\begin{aligned} \Sigma_\theta^q &\leftarrow \left(\varphi_J^\top \varphi_J + \frac{r_{q,\sigma}}{s_{q,\sigma}} \frac{r_{q,\tau}}{s_{q,\tau}} \text{Dg}(Q_{1:J}) \right)^{-1} \\ \mu_\theta^q &\leftarrow \Sigma_\theta^q \varphi_J^\top \left(\mu_{y^*}^q - W \mu_\beta^q \right) \end{aligned}$$

2.3.2 τ^2

$$\begin{aligned} r_{q,\tau} &\leftarrow r_{0,\tau} + J \\ s_{q,\tau} &\leftarrow s_{0,\tau} + \frac{r_{q,\sigma}}{s_{q,\sigma}} \sum_{j=1}^J \left(\Sigma_{\theta,jj}^q + \mu_{\theta,j}^{q^2} \right) Q_j \end{aligned}$$

2.3.3 σ^2

$$r_{q,\sigma} \leftarrow r_{0,\sigma} + J + p + 1$$

$$s_{q,\sigma} \leftarrow s_{0,\sigma} + \frac{r_{q,\tau}}{s_{q,\tau}} \sum_{j=1}^J \left(\Sigma_{\theta,jj}^q + \mu_{\theta,j}^{q^2} \right) Q_j + \text{Tr} \left(\Sigma_{\beta}^{0^{-1}} \Sigma_{\beta}^q \right) + \left(\mu_{\beta}^q - \mu_{\beta}^0 \right)^{\top} \Sigma_{\beta}^{0^{-1}} \left(\mu_{\beta}^q - \mu_{\beta}^0 \right)$$

2.3.4 β

$$\begin{aligned} \Sigma_{\beta}^q &\leftarrow \frac{s_{q,\sigma}}{r_{q,\sigma}} \left(W^{\top} W + \Sigma_{\beta}^{0^{-1}} \right)^{-1} \\ \mu_{\beta}^q &\leftarrow \frac{r_{q,\sigma}}{s_{q,\sigma}} \Sigma_{\beta}^q \left(\Sigma_{\beta}^{0^{-1}} \mu_{\beta}^0 + W^{\top} \left(\mu_{y^*}^q - \varphi_J \mu_{\theta}^q \right) \right) \end{aligned}$$

2.3.5 ψ

$$\begin{aligned} \frac{\partial S_1}{\partial \mu_{\psi}^q} &= -\omega_0 \left\{ -\frac{1}{\sigma_{\psi}^q} \sqrt{\frac{2}{\pi}} \exp \left(-\frac{\mu_{\psi}^{q^2}}{2\sigma_{\psi}^{q^2}} \right) + 1 - 2\Phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} \right) + 2\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} \phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} \right) \right\} \\ \frac{\partial S_1}{\partial \sigma_{\psi}^{q^2}} &= -\omega_0 \left\{ \left(\frac{1}{\sqrt{2\pi}\sigma_{\psi}^q} + \frac{\mu_{\psi}^{q^2}}{\sqrt{\pi}(\sigma_{\psi}^{q^2})^{3/2}} \right) \exp \left(-\frac{\mu_{\psi}^{q^2}}{2\sigma_{\psi}^{q^2}} \right) - \phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} \right) \frac{\mu_{\psi}^{q^2}}{(\sigma_{\psi}^{q^2})^{3/2}} \right\} \\ \frac{\partial Q_j}{\partial \mu_{\psi}^q} &= j \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} + \mu_{\psi}^q j \right) \left\{ 1 - \Phi \left(\frac{-\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \right\} + \frac{1}{\sigma_{\psi}^q} \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} + \mu_{\psi}^q j \right) \phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \\ &\quad - j \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} - \mu_{\psi}^q j \right) \left\{ 1 - \Phi \left(\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \right\} - \frac{1}{\sigma_{\psi}^q} \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} - \mu_{\psi}^q j \right) \phi \left(\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \\ \frac{\partial Q_j}{\partial \sigma_{\psi}^{q^2}} &= j \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} + \mu_{\psi}^q j \right) \left\{ 1 - \Phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \right\} + \left(\frac{1}{2\sigma_{\psi}^q} j - \frac{\mu_{\psi}^q}{2(\sigma_{\psi}^{q^2})^{3/2}} \right) \phi \left(-\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} + \mu_{\psi}^q j \right) \\ &\quad + j \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} - \mu_{\psi}^q j \right) \left\{ 1 - \Phi \left(\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \right\} + \left(\frac{1}{2\sigma_{\psi}^q} j + \frac{\mu_{\psi}^q}{2(\sigma_{\psi}^{q^2})^{3/2}} \right) \phi \left(\frac{\mu_{\psi}^q}{\sigma_{\psi}^q} - \sigma_{\psi}^q j \right) \exp \left(\frac{\sigma_{\psi}^{q^2} j^2}{2} - \mu_{\psi}^q j \right) \\ \frac{\partial S_2}{\partial \mu_{\psi}^q} &= -\frac{1}{2} \frac{r_{q,\sigma}}{s_{q,\sigma}} \frac{r_{q,\tau}}{s_{q,\tau}} \left(\text{Tr}(\Sigma_{\theta}^q) + \mu_{\theta}^{q\top} \mu_{\theta}^q \right) \sum_{j=1}^J \frac{\partial Q_j}{\partial \mu_{\psi}^q} - \frac{J(J+1)}{4\omega_0} \frac{\partial S_1}{\partial \mu_{\psi}^q} \\ \frac{\partial S_2}{\partial \sigma_{\psi}^{q^2}} &= -\frac{1}{2} \frac{r_{q,\sigma}}{s_{q,\sigma}} \frac{r_{q,\tau}}{s_{q,\tau}} \left(\text{Tr}(\Sigma_{\theta}^q) + \mu_{\theta}^{q\top} \mu_{\theta}^q \right) \sum_{j=1}^J \frac{\partial Q_j}{\partial \sigma_{\psi}^{q^2}} - \frac{J(J+1)}{4\omega_0} \frac{\partial S_1}{\partial \sigma_{\psi}^{q^2}} \\ \sigma_{\psi}^{q^2} &\leftarrow -\frac{1}{2} \left\{ \frac{\partial S_1}{\partial \sigma_{\psi}^{q^2}} + \frac{\partial S_2}{\partial \sigma_{\psi}^{q^2}} \right\}^{-1} \\ \mu_{\psi}^q &\leftarrow \mu_{\psi}^q + \sigma_{\psi}^{q^2} \left\{ \frac{\partial S_1}{\partial \mu_{\psi}^q} + \frac{\partial S_2}{\partial \mu_{\psi}^q} \right\} \end{aligned}$$