Cosine Basis Logistic

Daeyoung Lim[*]
Department of Statistics
Korea University

April 5, 2016

1 Logistic

1.1 Model

$$\begin{aligned}
y &\sim \mathrm{Ber}\left(\mathrm{logit}^{-1}\left(\varphi_J \theta_J\right)\right) \\
\theta_j | \tau, \psi &\sim \mathcal{N}\left(0, \tau^2 \exp\left[-j\left|\psi\right|\right]\right) \\
\tau^2 &\sim \mathrm{IG}\left(\frac{r_{0,\tau}}{2}, \frac{s_{0,\tau}}{2}\right) \\
\psi &\sim \mathrm{DE}\left(0, \omega_0\right)
\end{aligned}$$

1.2 Likelihood

$$\begin{aligned}
\ln p\left(y, \Theta\right) = & \, y^\top \varphi_J \theta_J - 1_n^\top \ln\left(1_n + \exp\left(\varphi_j \theta_J\right)\right) \\
& - \frac{1}{2}\left\{\sum_{j=1}^{J} \ln\left(2\pi\tau^2\right) - j\left|\psi\right| + \frac{\theta_j^2 e^{j|\psi|}}{\tau^2}\right\} \\
& + \frac{r_{0,\tau}}{2} \ln\left(\frac{s_{0,\tau}}{2}\right) - \ln\Gamma\left(\frac{r_{0,\tau}}{2}\right) + \left(\frac{r_{0,\tau}}{2} + 1\right) \ln\frac{1}{\tau^2} - \frac{s_{0,\tau}}{2}\frac{1}{\tau^2} \\
& - \ln\frac{\omega_0}{2} - \omega_0\left|\psi\right|
\end{aligned}$$

1.3 Getting around the intractability

$$\begin{aligned}
-\ln\left(1 + e^x\right) &= \max_{\xi \in \mathbb{R}} \left\{\lambda\left(\xi\right) x^2 - \frac{1}{2}x + \Psi\left(\xi\right)\right\}, \quad \forall x \in \mathbb{R} \\
\lambda\left(\xi\right) &= -\tanh\left(\xi/2\right)/\left(4\xi\right) \\
\Psi\left(\xi\right) &= \xi/2 - \ln\left(1 + e^{\xi}\right) + \xi\tanh\left(\xi/2\right)/4
\end{aligned}$$

then

$$\begin{aligned}
-1_n^\top \ln\left(1_n + \exp\left(\varphi_J\theta_J\right)\right) &\geq 1_n^\top\left\{\lambda\left(\xi\right) \odot \left(\varphi_J\theta_J\right)^2 - \frac{1}{2}\varphi_J\theta_J + \Psi\left(\xi\right)\right\} \\
&= \theta_J^\top \varphi_J^\top \mathrm{Dg}\left\{\lambda\left(\xi\right)\right\} \varphi_J \theta_J - \frac{1}{2} 1_n^\top \varphi_J \theta_J + 1_n^\top \Psi\left(\xi\right)
\end{aligned}$$

[*]Prof. Taeryon Choi

$$
\begin{aligned}
\ln \underline{p}\,(y,\Theta;\xi) = {} & y^\top \varphi_J\theta_J - \frac{1}{2}1_n^\top \varphi_J\theta_J + \theta_J^\top\varphi_J^\top \operatorname{Dg}\{\lambda(\xi)\}\varphi_J\theta_J + 1_n^\top \Psi(\xi) \\
& - \frac{1}{2}\left\{\sum_{j=1}^{J} \ln\left(2\pi\tau^2\right) - j\,|\psi| + \frac{\theta_j^2 e^{j|\psi|}}{\tau^2}\right\} \\
& + \frac{r_{0,\tau}}{2}\ln\left(\frac{s_{0,\tau}}{2}\right) - \ln\Gamma\left(\frac{r_{0,\tau}}{2}\right) + \left(\frac{r_{0,\tau}}{2}+1\right)\ln\frac{1}{\tau^2} - \frac{s_{0,\tau}}{2}\frac{1}{\tau^2} \\
& - \ln\frac{\omega_0}{2} - \omega_0\,|\psi|
\end{aligned}
$$

1.4 Update

1.4.1 θ_J

$$
\begin{aligned}
\Sigma^q_{\theta;\xi} &= \left(-2\varphi_J^\top \operatorname{Dg}\{\lambda(\xi)\}\varphi_J + \frac{r_{q,\tau}}{s_{q,\tau}}\operatorname{Dg}\left(Q_{1:J}\right)\right)^{-1} \\
\mu^q_{\theta;\xi} &= \Sigma^q_{\theta;\xi}\varphi_J^\top\left(y - \frac{1}{2}1_n\right)
\end{aligned}
$$

1.4.2 τ^2

$$
\begin{aligned}
r_{q,\tau} &= r_{0,\tau} + J \\
s_{q,\tau} &= s_{0,\tau} + \sum_{j=1}^{J}\left(\Sigma^q_{\theta;\xi,jj} + {\mu^q_{\theta;\xi,jj}}^2\right) Q_j\left(\mu^q_\psi, {\sigma^q_\psi}^2\right)
\end{aligned}
$$

1.4.3 ψ

$$
\begin{aligned}
{\sigma^q_\psi}^2 &= -\frac{1}{2}\left\{\frac{\partial S_1}{\partial {\sigma^q_\psi}^2} + \frac{\partial S_2}{\partial {\sigma^q_\psi}^2}\right\}^{-1} \\
\mu^q_\psi &= \mu^q_\psi + {\sigma^q_\psi}^2\left\{\frac{\partial S_1}{\partial \mu^q_\psi} + \frac{\partial S_2}{\partial \mu^q_\psi}\right\}
\end{aligned}
$$

1.4.4 ξ

$$
\xi^{\text{new}} = \sqrt{\operatorname{dg}\left\{\varphi_J\left(\Sigma^q_{\theta;\xi} + \mu^q_{\theta;\xi}{\mu^q_{\theta;\xi}}^\top\right)\varphi_J^\top\right\}}
$$

where dg results in a vector with the diagonal entries of the argument matrix. On the other hand, Dg results in a diagonal matrix with its diagonal entries being the input vector.

1.5 LB

$$\begin{aligned}\mathcal{L} &= \mu^q_{\theta;\xi}{}^\top \varphi_J^\top \left(y - \frac{1}{2} 1_n\right) + \mathrm{Tr}\left(\varphi_J^\top \mathrm{Dg}\left\{\lambda\left(\xi\right)\right\} \varphi_J \left(\Sigma^q_{\theta;\xi} + \mu^q_{\theta;\xi} \mu^q_{\theta;\xi}{}^\top\right)\right) + 1_n^\top \Psi\left(\xi\right) \\ &\quad + \frac{J}{2}\left(\mathrm{di}\left(\frac{r_{q,\tau}}{2}\right) - \ln\left(\frac{s_{q,\tau}}{2}\right) - \ln\left(2\pi\right)\right) + S_1 + S_2 \\ &\quad + \frac{r_{0,\tau}}{2} \ln\left(\frac{s_{0,\tau}}{2}\right) - \ln\Gamma\left(\frac{r_{0,\tau}}{2}\right) + \frac{r_{0,\tau}}{2}\left(\mathrm{di}\left(\frac{r_{q,\tau}}{2}\right) - \ln\left(\frac{s_{q,\tau}}{2}\right)\right) - \frac{s_{0,\tau}}{2} \frac{r_{q,\tau}}{s_{q,\tau}} - \ln \frac{\omega_0}{2} \\ &\quad + \frac{r_{q,\tau}}{2} + \ln\Gamma\left(\frac{r_{q,\tau}}{2}\right) - \frac{r_{q,\tau}}{2} \mathrm{di}\left(\frac{r_{q,\tau}}{2}\right) + \frac{J}{2}\left(1 + \ln\left(2\pi\right)\right) + \frac{1}{2} \ln\left|\Sigma^q_{\theta;\xi}\right| \\ &\quad + \frac{1}{2}\left(\ln\left(2\pi \sigma^q_\psi{}^2\right) + 1\right)\end{aligned}$$