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1. (5 points) Consider the regression model with usual assumptions of the errors

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

   (a) Suppose that the observation y_i falls directly on the fitted regression line (i.e., $y_i = \hat{y}_i$). If this case is deleted, would the least squares estimators with the remaining $n-1$ cases be changed?

   (b) Suppose that we have fit the straight-line regression model $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$, but the true regression function is

$$\mathbb{E}\left[y\right] = \beta_0 + \beta_1 x_1 + \beta_2 x_2.$$

   Is the estimator $\hat{\beta}_1$ biased?

   (c) Show that $\hat{\boldsymbol{\beta}}$ and $\boldsymbol{e}$ are statistically independent, where $\hat{\boldsymbol{\beta}}$ is the least squares estimator of $\boldsymbol{\beta}$ and $\boldsymbol{e}$ is the corresponding residuals.

Solution:

Solution:

Solution: The least squares estimator of $\boldsymbol{\beta}$ can be written as follows:

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{y}.$$

And for notational conveniece, let $\boldsymbol{H} = \boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'$, called the *hat matrix* in linear regression terminology, or *projection matrix* in linear algebra literature. Then

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{y} \\
&= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\left(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}\right) \\
&= \boldsymbol{\beta} + \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{\epsilon} \\
\boldsymbol{e} &= \boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{y} \\
&= \left(\boldsymbol{I} - \boldsymbol{H}\right)\left(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}\right) \\
&= \left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{X}\boldsymbol{\beta} - \left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{\epsilon}.
\end{aligned}$$

Since both $\hat{\boldsymbol{\beta}}$ and $\boldsymbol{e}$ are linear combinations of normal variates, they also follow normal distributions. For normal distributions, having no correlation is the same as being independent.

$$\begin{aligned}
\mathrm{Cov}\left[\hat{\boldsymbol{\beta}}, \boldsymbol{e}\right] &= \mathrm{Cov}\left[\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{\epsilon}, \left(\boldsymbol{H} - \boldsymbol{I}\right)\boldsymbol{\epsilon}\right] \\
&= \sigma^2 \cancel{\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}} \cancel{\boldsymbol{X}'\boldsymbol{X}} \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}' - \sigma^2\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'
\end{aligned}$$

2. (5 points) Consider the regression model with k regressor variables, $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \boldsymbol{V}\right),\ \boldsymbol{V} \neq \boldsymbol{I}$.

   (a) Obtain the generalized least squares estimator of $\boldsymbol{\beta}$.

   (b) Find the expected value and variance of $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ in the part (a).

Solution:

Solution:

3. (5 points) Consider the model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where $\boldsymbol{\epsilon}$ are i.i.d. normally distributed with mean $\mathbf{0}$ and variance $\sigma^2 \boldsymbol{I}$. Assume that we have a multicollinearity problem.

   (a) Define the ridge estimator of $\boldsymbol{\beta}$.

   (b) Show that the ridge estimator is the solution to the problem

$$\text{Minimize } \left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)' \boldsymbol{X}'\boldsymbol{X} \left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right) \text{ subject to } \boldsymbol{\beta}'\boldsymbol{\beta} \leq c$$

Solution:

Solution:

4. (4 points) Let S denote the sample space. Suppose that $C \subset S$ and $\mathbb{P}(C) > 0$ and define $Q(A) = \mathbb{P}(A|C)$. Show that Q satisfies the axiom of probability.

Solution:

5. (3 points) Construct an example where $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ but $X_n + Y_n$ does not converge in distribution to $X + Y$.

Solution:

6. (6 points) Suppose that we observe $X_1, \dots, X_n$ from $\mathcal{N}\left(\mu, \sigma^2\right)$. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\eta = \mu/\sigma^2$.

Solution:

7. (4 points) Let $X \sim \mathrm{Exp}\,(1)$, and define Y to be the integer part of $X+1$, that is

$$Y = i+1 \text{ if and only if } i \leq X < i+1, \quad i = 0, 1, 2, \dots$$

Find the conditional distirbution of $X - 4$ given $Y \leq 5$.

Solution:

8. (6 points) Let $Y \sim \mathrm{Poisson}\,(\lambda)$ and $X|Y = y \sim \mathrm{Bin}\,(y, p)$.

   (a) Determine the distribution of X.

   (b) Find the conditional pdf of Y given $X = x$.

Solution:

Solution:

9. (6 points) Suppose that we observe X having a distribution $P \in \{P_0, P_1\}$, and that we wish to test the hypothesis $H_0 : P = P_0$ versus $H_1 : P = P_1$, based on an observation of X. Let P_0 and P_1 be given by

x	0	1	2	3
P_0	0.03	0.05	0.12	0.80
P_1	0.15	0.10	0.30	0.45

   (a) Find non-randomized most powerful test(s) at level $\alpha = 0.05$.

   (b) Find randomized most powerful test(s) at level $\alpha = 0.05$.

Solution:

Solution:

10. (6 points) Let X_n be a random variable with $\mathbb{P}(X_n = i/n) = 1/n, 1 \leq i \leq n$. Find the limit distribution of X_n as $n \to \infty$.

Solution: