习题一

1. 位置 $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ 力 $F=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ 输出速度 $y=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}$

(a) 牛顿第二定律 $F=ma=m\ddot{x}$

$$\begin{cases}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\end{bmatrix}=\frac{1}{m}\begin{bmatrix}u_1\\u_2\end{bmatrix}\\ y=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}\end{cases}$$

定义状态变量 $x_1,\dot{x}_1,x_2,\dot{x}_2$，输入 F，输出 $y=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}$

$$\begin{bmatrix}\dot{x}_1\\\ddot{x}_1\\\dot{x}_2\\\ddot{x}_2\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\\dot{x}_1\\x_2\\\dot{x}_2\end{bmatrix}+\frac{1}{m}\begin{bmatrix}0\\u_1\\0\\u_2\end{bmatrix}$$

~~(b) 若采样周期为~~ $y=[0\ 1\ 0\ 1]\begin{bmatrix}x_1\\\dot{x}_1\\x_2\\\dot{x}_2\end{bmatrix}$

(b) 离散化表示

$$\begin{bmatrix}x_{1,t+1}\\\dot{x}_{1,t+1}\\x_{2,t+1}\\\dot{x}_{2,t+1}\end{bmatrix}-\begin{bmatrix}x_{1,t}\\\dot{x}_{1,t}\\x_{2,t}\\\dot{x}_{2,t}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_{1,t}\\\dot{x}_{1,t}\\x_{2,t}\\\dot{x}_{2,t}\end{bmatrix}\Delta t+\frac{\Delta t}{m}\begin{bmatrix}0\\u_1\\0\\u_2\end{bmatrix}$$

$$y_{t+1}-y_t=[0\ 1\ 0\ 1]\begin{bmatrix}x_1\\\dot{x}_1\\x_2\\\dot{x}_2\end{bmatrix}\Delta t$$

2. ~~E(X)=~~ 若 $\vec{x}=[x_1,\cdots,x_n]^T$ 中每个随机变量相互独立则

$E(X)=[E(X_1),\cdots,E(X_n)]^T=[\mu_1,\cdots,\mu_n]^T=m$

$$COV[X] = \begin{bmatrix} V(X_1) & COV(X_1, X_2) & \cdots & COV(X_1, X_N) \\ \vdots & & \ddots & \\ COV(X_N, X_1) & \cdots & \cdots & COV(X_N) \end{bmatrix}$$

$\forall\ i, j \in [1, N]$ $\quad Cov(X_i, X_j) = 0$

$$\therefore Cov[X] = \begin{bmatrix} V(X_1) & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & V(X_N) \end{bmatrix} = P \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix} = P$$

当随机变量不互相独立时. 存在定理1.

☆: 若存在随机向量 $\vec{x} \sim N(\vec{\mu}, \Sigma)$, 其中 $\vec{\mu} \in R^n$ 为均值向量, $\Sigma \in S_{++}^{n\times n}$ 半正定实对称矩阵为 $\vec{x}$ 的协方差矩阵, 则存在满秩矩阵 $B \in R^{n\times n}$, 使得 $\vec{z} = B^{-1}(\vec{x} - \vec{\mu})$, 而 $\vec{z} \sim N(\vec{0}, I)$

$$\therefore p(x_1, \cdots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |BB^T|^{\frac{1}{2}}} e^{-\frac{1}{2}[(\vec{x}-\vec{\mu})^T (BB^T)^{-1} (\vec{x}-\vec{\mu})]}$$

$$P = \Sigma = E[(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T] = cov(X) = BB^T$$

$$B^{-1}X - B^{-1}\vec{\mu} = Z = Y$$

$$\therefore A = B^{-1}, \quad b = -B^{-1}\vec{\mu} = -B^{-1}m$$

其中 $\quad BB^T = P$

~~$B^{-1} =$~~

3. $X \sim N(m, \sigma^2)$ 则 $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$ 则 $E[X] = m$

$$E[X^2] = Var(X) + E(X)^2 = \sigma^2 + m^2$$

$$E[X^3] = \int_{-\infty}^{+\infty} x^3 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

~~x^3 奇函数~~

~~$= \frac{1}{4}x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} = $~~

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3 \qquad \mu = m$$

$$E[(X-\mu)^3] = -\sigma^2 (x-m)^2 e^{-\frac{(x-m)^2}{2\sigma^2}} + 2\sigma^2 \int_{-\infty}^{+\infty} (x-m) e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= 0$$

3、$E[X^3]=3\mu E[X^2]-3\mu^2E[X]+\mu^3=m^3+3m\sigma^2$　其中$\mu=m$

~~$E[X^4]=m^2D[X^2]+E[X^2]^2$~~

先求四阶原点矩

$E[(X-m)^4]=\int_{-\infty}^{+\infty}t^4\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{t^2}{2\sigma^2})dt$

根据分步积分公式 $=3\sigma^2\int_{-\infty}^{+\infty}t^2\exp(-\frac{t^2}{2\sigma^2})dt$

$=3\sigma^2E[(X-m)^2]$

$=3\sigma^4$

$E[(X-m)^4]=E[X^4]-4mE[X^3]+6m^2E[X^2]-4m^3E[X]+m^4$

$E[X^4]=3\sigma^4+4m(m^3+3m\sigma^2)-6m^2(\sigma^2+m^2)+4m^4-m^4$

$=3\sigma^4+m^4+6m^2\sigma^2$

4、分别求得两个变量边缘分布的概率密度

$P_X(x)=\int_0^1 P_{X,Y}(x,y)dy=\frac{6x^2+6x+2}{7}$

$P_Y(y)=\int_0^1 P_{X,Y}(x,y)dx=\frac{6y^2+6y+2}{7}$

$E(X)=E(Y)=\frac{9}{14}$

期望为$[\frac{9}{14},\frac{9}{14}]^T$

$D(X)=D(Y)=E[X^2]-E(X)^2=\frac{199}{2940}$

$Cov(X,Y)=E(X,Y)-E(X)E(Y)$

$=\frac{3}{14}+\frac{4}{21}-\frac{9}{14}\times\frac{9}{14}$　　$E(XY)=\int_0^1\int_0^1 xy\frac{6}{7}(x+y)^2dxdy$

$=-\frac{5}{588}$

因此协方差矩阵为$\begin{bmatrix}\frac{199}{2940} & -\frac{5}{588}\\ -\frac{5}{588} & \frac{199}{2940}\end{bmatrix}$

5、引理，当相两个变量相互独立时　协方差~~矩阵~~为0 ~~为对角矩阵~~

证 $Cov(X,Y)=E[(X-E(X))(Y-E(Y))]=E[X-E(X)]\,E[Y-E(Y)]$

$=0\times0=0$

联合高斯分布

$x\sim N(0,\sigma^2)$，$y|x\sim N(x,c^2)$ $\quad f(y)=\int_x f(y|x)f(x)dx$

则 $y\sim N(0,\sigma^2+c^2)$

故v，e 分别服从期望为0的高斯分布

e的协方差矩阵 $Cov(e,e^T)$ 严格正定 故存在逆矩阵

$Cov(v-Be,e)=Cov(v,e)-B\,Cov(e,e)$

$\Rightarrow B=Cov(v,e)[Cov(e,e)^{-1}]$

唯一存在B