IE531 Homework 3

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Exercise 3.7

$$\mathrm{A} = \begin{matrix} r1 \\ r2 \\ \dots \end{matrix}$$

Given A has orthonormal rows: $r_i r_j = 0(i! = j), 1\ (i = j)$

Therefore, we have

$$\mathrm{A} * A^T = \begin{bmatrix} r1 \\ r2 \\ \dots \end{bmatrix} * [r1 \quad r2 \quad \dots] = \begin{bmatrix} r_1 r_1 & \cdots & r_1 r_n \\ \vdots & \ddots & \vdots \\ r_n r_1 & \cdots & r_n r_n \end{bmatrix} = I_{n*n}$$

Therefore:

$$A^T = I_{n*n} * A^{-1} = A^{-1}$$

$$A^T = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}, \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix} * [c_1 \cdots c_n] = I_{n*n}$$

Therefore $c_i c_j = 0(i! = j), 1\ (i = j)$

Columns of A are orthonormal.

Exercise 3.13

Frobenius norm for A_K

$$\left||A_K|\right|_F^2 = \sum_{i=1}^{K} \sum_{j=1}^{n} \left|a_j v_i\right|^2 = \sum_{i=1}^{K} |Av_i|^2 = \sum_{i=1}^{K} \sigma_i^2$$

2-norm for A_K

$$\left||A_K|\right|_2^2 = (\max|A_K x|)^2 (x \leq 1) = \sigma_K^2$$

Frobenius norm for $A - A_k$

$$\left||A - A_k|\right|_F^2 = \sum_{i=k+1}^{r} c_i^2 \sigma_i^2 = \sigma_{k+1}^2, Given \sum_{i=1}^{r} c_i^2 = 1 = |v|$$

$$= |top\ singular\ vector\ of\ A - A_k|$$

2 norm for $A - A_k$

$$\left||A - A_k|\right|_2^2 = \sigma_{K+1}^2$$

Exercise 3.14

$$\mathrm{A} = \mathrm{UDV^T}$$
$$\mathrm{A^T} = V\mathrm{D^T U^T} = \mathrm{VDU^T}$$

Therefore, the left singular vector of A is the right singular vector of $\mathrm{A^T}$ and vice versa.

Since A is symmetric, and thus $\mathrm{A} = \mathrm{A^T}$, $\mathrm{UDV^T} = \mathrm{VDU^T}$

According to singular value decompositions' uniqueness (due to distinct singular values), therefore, $U = V$.

Exercise 3.16

$$B = A^T A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$
$$B^6 = \begin{bmatrix} 235497536 & 334172160 \\ 334172160 & 474191936 \end{bmatrix}$$

The first singular vector is the normalized column vector of B^6

$$V_1 = \begin{bmatrix} \dfrac{235497536}{\sqrt{235497536^2 + 334172160^2}} \\ \dfrac{334172160}{\sqrt{235497536^2 + 334172160^2}} \end{bmatrix} = \begin{bmatrix} 0.5760 \\ 0.8174 \end{bmatrix}$$
$$\sigma_1 = |AV_1| = 5.4648$$
$$u_1 = \frac{AV_1}{\sigma_1} = \begin{bmatrix} 0.4046 \\ 0.9145 \end{bmatrix}$$
$$A = \sigma_1 u_1 V_1^T + \sigma_2 u_2 V_2^T$$

Therefore, we define $C = (A - \sigma_1 u_1 V_1^T)^T (A - \sigma_1 u_1 V_1^T)$

$$C = \begin{bmatrix} 0.0895 & -0.0631 \\ -0.0631 & 0.0444 \end{bmatrix}$$
$$C^6 = \begin{bmatrix} 0.3857\mathrm{e}-5 & -0.2718\mathrm{e}-5 \\ -0.2718\mathrm{e}-5 & 0.1915\mathrm{e}-5 \end{bmatrix}$$
$$V_1 = \begin{bmatrix} \dfrac{0.3857\mathrm{e}-5}{\sqrt{0.3857\mathrm{e}-5^2 + (-0.2718\mathrm{e}-5)^2}} \\ \dfrac{-0.2718\mathrm{e}-5}{\sqrt{0.3857\mathrm{e}-5^2 + (-0.2718\mathrm{e}-5)^2}} \end{bmatrix} = \begin{bmatrix} 0.8174 \\ -0.5760 \end{bmatrix}$$
$$\sigma_2 = |AV_2| = 0.3660$$
$$u_2 = \frac{AV_2}{\sigma_2} = \begin{bmatrix} -0.9143 \\ 0.4050 \end{bmatrix}$$

Q5.

a. A non-square matrix has rank $\leq$ $\min(n_1 + n_2, d_1 + d_2)$

We can disassemble the four blocks into 4 matrices with one of the four blocks in it and other elements equal to 0.

All these 4 matrices have ranks no bigger than $\min(n_i, d_j)(i, j \in (1,2))$

Therefore, the equality only holds when $\sum_{i=1}^{2}\sum_{j=1}^{2}\min\left((n_i, d_j)\right) = \min(n_1 +$

$n_2, d_1 + d_2)$, such as the case of an identity matrix.

b. Due to the best rank-k approximations of singular vectors, we have A_K, which is the projection of the rows of A onto the subspace spanned by the first k singular vectors of A.

$$||A - A_k||_F \leq ||A - B||_F$$

We can treat B_i as projection of A_i on arbitrary subspace with dimension no bigger min (n_i, d_j), with projection length of A no larger than $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ .Since for any $i \in [1,2,3,4]$, $||A_i - B_i||_F \leq \in$, we can say that $\in$ is the largest distance possible between A_i and B_i, we have the following equation holds for arbitrary A and split of A.

$$||A_i \ - A_{i,k}||_F = \sigma_{k+1}^2 \leq ||A_i - B_i||_F \leq \in$$

Therefore, the minimum value of $\in$ is $\sigma_k^2 = \left||A_{i,K}|\right|_2^2 = \left(\max\left|A_{i,K}x\right|\right)^2 (x \leq 1) \geq \left|\left||B|\right|\right|_F^2$.(B can be in subspace spanned by singular vectors number less than K, which will be the same result, will merely change $\left|\left||B|\right|\right|_F^2 = \in = \sigma_i^2 (i \leq k)$.

$$\sum_{i=1}^{4} ||A_i \ - A_{i,k}||_F \leq \sum_{i=1}^{4} ||A_i - B_i|| \leq \in$$

Therefore, we would have the following result:

$$\left\|A - \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right\| \leq 4 \in$$

And the equality holds when B_i are best rank r approximation of A_i(r is the rank of A_i).