Numerical Fluid Mechanics II
-Linear Stability Analysis of Plane Channel Flow-

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July 2018

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Abstract

Many problems in fluid mechanics involve some aspect of flow stability, analogous to solid mechanics. The basic question is: given a basic flow state (e.g. laminar flow through a pipe) under which conditions does the flow become unstable to certain perturbations? As a first step to determine the stability of a fluid flow problem, one often supposes that the perturbations to the basic state are of very small amplitude, which allows for a linearisation of the equations. Although this is a strong assumption, linear stability analysis has proven useful in many flow configurations.

In this assignment the stability of plane channel flow shall be analyzed. The base flow is supposed to be fully-developed, pressure-driven, laminar flow directed in the x-direction (the y and z-directions are the wall-normal and spanwise coordinates, repectively). The distance between the plates is 2h. In the following, only two-dimensional flow perturbations in the (x,y)-plane will be considered.

key words: linear stability, hydrodynamic stability, orr-sommerfeld, squire theorem, finite-difference method, spectral

Acknowledgements

I am deeply thankful to Prof. Dr. Markus Uhlmann of the Institute of Hydromechanics at the Karlsruhe Institute of Technology who was always gave me impetus although I was focused on working alone through this project. And still, if I had a question about my research project or topics not related to my exercise like turbulence in general he was always helpful and discussed about the issues. He consistently allowed this paper to be my own work, but steered me in the right direction whenever he thought I needed it.

1 History

The **"Plane Poiseuille Flow"** exercise is a stability problem which is steady and two dimensional with parallel streamlines. Flows of this type were investigated by Reynolds (1883), who observed that there are different ways in which instability could occur depending on the velocity distribution. From observations he made for different flow types he formulated two fundamental hypotheses:

First Hypothesis: The inviscid fluid may be unstable and the viscous fluid stable. The effect of viscosity is then purely stabilizing.
Second Hypothesis: The inviscid fluid maybe be stable and the viscous fluid unstable. In this case viscosity would be the cause of the instability.
Reynolds could not suggest a mechanism by which viscosity could cause instability but he refused to exclude such a possibility.

The study of inviscid flows have been initiated by Helmholtz (1868), Kelvin (1871) and Rayleigh (1880), who considered the purely inertial instability of an incompressible fluid of constant density. One of the most important theorems of that time was Rayleigh's description of inflexion points in the velocity profile.[1]

The study of viscous flows in this area was significantly influenced by Orr (1907) and Sommerfeld (1908) deriving the well known equation bearing their name. At the time the equation has been derived, existing methods for asymptotic analysis were not sufficiently well developed in order to deal with their problem. Heuristic methods of approximation have been suggested by Heisenberg (1924), Tollmien (1929, 1947) and Lin (1945, 1955).

In the following chapters, the Orr-Sommerfeld equation will be derived and together with a detailed description of the Plane Poiseuille flow case and its implementation in MATLAB as a finite-difference code the limitations, defects and physical interpretations will be examined.

2 Assignment

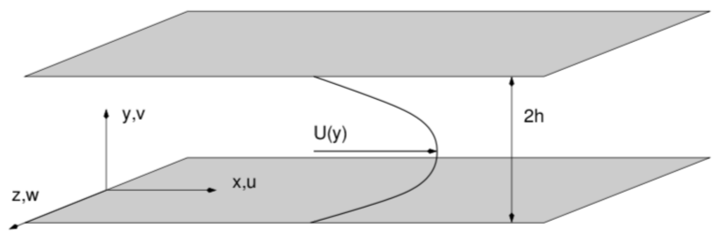

Figure 1: Schematic of plane channel flow between two parallel plates of infinite extension

As aforementioned the stability of plane channel flow shall be analyzed. For this the Navier-Stokes equations for an incompressible fluid will be considered:

$$\nabla \cdot \mathbf{u} = 0 \tag{1a}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \frac{1}{Re}\nabla^2 \mathbf{u} \tag{1b}$$

with a constant density (which has been set to unity for simplicity, and which therefore disappears from the equations), and where the Reynolds number has been defined as $Re \equiv hU_{max}/\nu$ (with being the kinematic viscosity and U_{max} the maximum velocity of the laminar velocity profile).

3 Laminar Flow Solution for U(y)

The first exercise was to determine the laminar flow solution denoted by $U(y)$ or a constant pressure gradient dp/dx.

The velocity $U(y)$ is given by:

$$U(y) = 1 - y^2 \tag{2}$$

where the maximal velocity is at $y = 0$ just for the sake of simplicity.

4 Decomposition of Quantities

Performing the decomposition of velocity and pressure into the base flow and a two-dimensional perturbation leads to the following expressions.

$$u(x, y, t) = U(y) + u'(x, y, t) \tag{3}$$

$$v(x, y, t) = v'(x, y, t) \tag{4}$$

$$p(x, y, t) = P_0 + \left(\frac{dp}{dx}\right) x + p'(x, y, t) \tag{5}$$

5 Linearization of the Navier-Stokes Equation

For exercise 3 the equations for the perturbation fields $\mathbf{u}'$ and $\mathbf{p}'$, under the hypothesis that the amplitued of the perturbations is much smaller than the one of the base flow (i.e. $|\mathbf{u}'| \ll |U|$) had to be derived. This assumption implies that terms which are quadratic in the perturbations are negligible in the equations.

The velocity and pressure in their base configuration will now be superposed with a small perturbation which yields:

$$u = u_0 + \epsilon u' \tag{6}$$

$$p = p_0 + \epsilon p' \tag{7}$$

Putting these assumptions into the Navier-Stoke Equation (1):

$$\nabla \cdot (u_0 + \epsilon u') = 0 \tag{8}$$

$$\frac{\partial (u_0 + \epsilon u')}{\partial t} + (u_0 + \epsilon u') \cdot \nabla (u_0 + \epsilon u') = -\nabla (p_0 + \epsilon p') + Re^{-1} \Delta (u_0 + \epsilon + u') \tag{9}$$

Sorting the terms for base solutions, mixed base solution and perturbed solution as well as only perturbed solutions will give us:

$$\epsilon \nabla \cdot u' = -\nabla \cdot u_0 \tag{10}$$

$$\epsilon \frac{\partial}{\partial t} u' + \epsilon u_0 \cdot \nabla u' + \epsilon u' \cdot \nabla u_0 + \epsilon^2 u' \cdot \nabla u' + \epsilon \nabla p' - \epsilon Re^{-1} \Delta u' = -\frac{\partial}{\partial t} u_0 - u_0 \cdot \nabla u_0 - \nabla p_0 + Re^{-1} \Delta u_0 \tag{11}$$

As the base solutions automatically satisfy the Navier-Stokes equations we can neglect the right hand sides of equation (10) & (11).

Dividing both equations by ϵ yields:

$$\nabla \cdot u' = 0 \tag{12}$$

$$\frac{\partial}{\partial t}u' + u_0 \cdot \nabla u' + u' \cdot \nabla u_0 + \nabla p' - Re^{-1}\Delta u' = -\epsilon u' \cdot \nabla u' \tag{13}$$

The equation or system of equations to be precise is still **non-linear** and cannot be solved by giving a general solution. But if we focus on small perturbations which means that $\epsilon \to 0$. Neglecting all terms from (12) & (13) that are associated with small perturbations ϵ as well as eliminating all the terms with higher order in u' & p' which means that the quadratic term in (13), namely $u' \cdot \nabla u'$ leading to a linear differential equation which can be used to investigate effects of small perturbations:

$$\nabla \cdot u' = 0 \tag{14}$$

$$\frac{\partial}{\partial t}u' + u_0 \cdot \nabla u' + u' \cdot \nabla u_0 = -\nabla p' + Re^{-1}\Delta u' \tag{15}$$

The method used in this case is well known as **Linearization** of equations which lead us to a general formulation of a linear differential equation for perturbations assuming we have an incompressible flow with constant density and constant substance properties. Trivially the boundary conditions for this equation have to be fulfilled in any case, namely the kinematic boundary condition and no-slip condition.
Rephrasing this equation to be easily readable one can also write:

6 Normal-Mode Ansatz & The Orr-Sommerfeld Equation

The next step is to introduce a so called normal mode ansatz (wavelike solutions):

$$u'(x, y, t) = \Re\{\hat{u}(y)exp(I\alpha(x - ct))\} \tag{16}$$

$$v'(x, y, t) = \Re\{\hat{v}(y)exp(I\alpha(x - ct))\} \tag{17}$$

$$p'(x, y, t) = \Re\{\hat{p}(y)exp(I\alpha(x - ct))\} \tag{18}$$

where I is the imaginariy unit defined as $\sqrt{-1}$, α the so called wavenumber of the perturbation (i.e. its wavelength λ given by $\lambda = 2\pi/\alpha$), and c is the phase speed (which is a complex number) where the real part of c corresponds to the propagation velocity of the perturbation and that the imaginary part of c times the wavenumber yields the growth rate of the perturbation.

As a consequence, the perturbation will increase with time whenever $\Im(c) > 0$ and it will decrease (decay) when $\Im(c) < 0$. Therefore, the limit of stability is given by the condition $\Im(c) = 0$.

In our case the spanwise wave number will be neglected thus eliminated from the equation and ω was substituted with $\omega = \alpha c$. Introducing this into the linearized equations one gets:

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} \tag{19}$$

This is the famous **Orr-Sommerfeld** equation (Orr, 1907; Sommerfeld, 1908). The frequency ω or alternatively $c = \omega/\alpha$ appears as the eigenvalue in the Orr-Sommerfeld equation - together with $\tilde{v}$ which is **complex** in general. This **homogeneous** equation can be thought of as the viscous extension of the **Rayleigh** equation which handles inviscid instabilities. [2]

Another representation of this equation can also be performed in terms of a stream function with disturbed parts. For that we first write the linearized solution of the vorticity transport equation:

$$U(y)\frac{\partial \omega_z}{\partial x} + \frac{\partial \omega_z}{\partial t} - u_y \frac{d^2 U}{dy^2} = \nu(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}) \tag{20}$$

where D^2 is defined as $\frac{\partial^2}{\partial y^2}$.

For convenience we write the disturbed flow in terms of disturbed **stream functions**:

$$u_x = \frac{\partial \psi}{\partial y} \tag{21}$$

$$u_y = -\frac{\partial \psi}{\partial x} \tag{22}$$

The disturbed vorticity is given by:

$$\omega_z = -(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}) = -\nabla^2 \psi \tag{23}$$

Putting this into (20) we get:

$$U(y)\frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \nabla^2 \psi}{\partial t} - \nu \nabla^4 \psi = \frac{\partial \psi}{\partial x}\frac{d^2 U}{dy^2} \tag{24}$$

No matter if one uses the Chebychev, Finite-Element or Finite-Difference Method one always has a **finite** possibility to calcuate disturbances because calculating infinitely many is impossible. We use sinusoidal functions of the streamwise position (in this case x) and exponential functions with time t - called **normal modes**.

An example for a normal mode with wave length **L** or

$$k = \frac{2\pi}{L} \tag{25}$$

is given by:

$$\psi(x,y,t) = \xi_r(y,t)cos(kx) + \xi_i(y,t)sin(kx) \tag{26}$$

We also introduce so called **Euler decomposition** which is defined as:

$$\psi(x,y,t) = \xi_r(y,t)cos(kx) + \xi_i(y,t)sin(kx) \tag{27}$$

We can then - with a bit of mathematical handwork - derive a function for the complex stream defined as:

$$\psi(x,y,t) = \phi(y,t)exp(ikx) \tag{28}$$

Putting this into equation (24) and keep the disturbed stream function in mind:

$$\nabla^2\psi = (-\phi k^2 + \frac{\partial^2\phi}{\partial y^2})exp(ikx) \tag{29}$$

and make sure derivations are properly carried out we get:

$$ikU(y)(-k^2\phi + \frac{\partial^2\phi}{\partial y^2}) - k^2\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial y^2\partial t} - \nu(k^4\phi - 2k^2\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^4\phi}{\partial y^2}) = ik\phi\frac{d^2U}{dy^2} \tag{30}$$

6.1 Growth Rate

In order to get the growth rate and phase velocity into our equation we have to define another function:

$$\phi(y,t) = f(y)exp(-ikct) \tag{31}$$

Where **c** is also defined as $c = \sigma/k$.We put this definition into (28) and get:

$$\psi(x,y,t) = f(y)exp[ik(x-ct)] \tag{32}$$

which can further be specificed and decomposed into real and imaginary parts by recalling the definition of the decomposition of the growth rate σ and complex phase velocity **c**.

$$\psi(x,y,t) = f(y)exp[ik(x-c_Rt)]exp(\sigma_I t) \tag{33}$$

c_R is the real phase velocity of the disturbance travelling along the x-axis. The constant σ_I is the so called **Growth Rate** of the perturbation - if this quantity is positive we say that the disturbance grows exponentially (**unstable**) in time, decays exponentially in time if it is negative (**stable**) and remains constant if $\sigma_I = 0$ (**neutrally stable**).

Putting (31) into (30) we also get our famous Orr-Sommerfeld equation:

$$k^4 f - 2k^2\frac{d^2f}{dy^2} + \frac{d^4f}{dy^4} = \frac{ik}{\nu}\left((U(y)-c)(\frac{d^2f}{dy^2} - k^2 f) - \frac{d^2U}{dy^2}f\right) \tag{34}$$

(48) and (19) can be transformed into each other which will be shown now:

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} \tag{35}$$

$$\left[(-ic\alpha + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} \tag{36}$$

$$(U - c)(D^2 - k^2)\tilde{v} - U''\tilde{v} - \frac{1}{iRe\alpha}(D^2 - k^2)^2\tilde{v} \tag{37}$$

As mentioned earlier the spanwise wave number β was neglected thus $k^2 = \alpha^2$ leading to $k = \alpha$.

$$(U - c)(D^2 - \alpha^2)\tilde{v} - U''\tilde{v} - \frac{1}{iRe\alpha}(D^2 - \alpha^2)^2\tilde{v} \tag{38}$$

$$\left[(D^2U - Uk^2 - D^2c + ck^2)\tilde{v} - \frac{1}{i\alpha Re}(D^2 - k^2)\tilde{v}\right] \tag{39}$$

$$D^2U\tilde{v} - Uk^2\tilde{v} - c\frac{d^2\tilde{v}}{dy^2} + ck^2\tilde{v} - \frac{d^2U}{dy^2}\tilde{v} - \frac{1}{i\alpha Re}\left(\frac{d^4\tilde{v}}{dy^4} - 2k^2\frac{d^2\tilde{v}}{dy^2} + \tilde{v}k^4\right) \tag{40}$$

$$U\left(\frac{d^2\tilde{v}}{dy^2}\right) - c\left(\frac{d^2\tilde{v}}{dy^2} - k^2\tilde{v}\right) - \frac{d^2U}{dy^2}\tilde{v} - \frac{1}{i\alpha Re}\left(\frac{d^4\tilde{v}}{dy^4} - 2k^2\frac{d^2\tilde{v}}{dy^2} + \tilde{v}k^4\right) \tag{41}$$

Rephrasing the Reynolds number based on the channel half-width and centre-stream velocity:

$$Re = \frac{1}{\nu} \tag{42}$$

where ν is the kinematic viscosity $[\frac{m^2}{s}]$ - one can rewrite the equation to the form we want to discretize for our finite-difference scheme which will be carried out in the next section.

6.2 Squire's theorem

We also have to explain why it is sufficient to only consider a 2D problem when someone could argue that we usually encounter 3D flows and we would have to calculate disturbances for the 3D flow instead of 2D.

The British aerospace engineer Herbert Squire derived, formulated and has proven a condition in 1933 that shows that it is sufficient to consider only two-dimensional perturbations. The first unstable waves travel

7 Finite-Difference Schemes

In this section a discretization of the equation has been performed by using a central difference formula with three grid points for the second derivative and five points for the fourth derivative which also was approximated with a central difference scheme but a fourth order formulation.

The finite-different schemes can be written as follows:

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{y_i} = \frac{f_{i-1} - 2f_i - f_{i+1}}{\Delta y^2} - \frac{\Delta y^2}{12}\left.\frac{\partial^4 f}{\partial y^4}\right|_{f_i} + ... \tag{43}$$

For the fourth derivative one can write a central finite differece scheme of second order:

$$\left.\frac{\partial^4 f}{\partial y^4}\right|_{y_i} = \frac{f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2}}{\Delta y^4} - \frac{\Delta y^2}{6}\left.\frac{\partial^4 f}{\partial y^6}\right|_{y_i} + ... \tag{44}$$

Following the exercise on the sheet one could also use a central difference scheme for the fourth derivative.

$$\phi^{iv}(y) = \frac{y_{i-2} - y_{i-1} + 2y_i - y_{i+1} + y_{i+2}}{\Delta y^2} \tag{45}$$

8 Matrix-Vector Form & Discretized OS-Equation

For the discretization I made sure that the equation was rewritten in a form to easily convert it into the form

$$\overline{\overline{A}}\overline{q} = c\overline{\overline{B}}\overline{q} \tag{46}$$

where A has a **pentadiagonal** form and B a **tridiagonal** form arising from the fact that A will be approximated with a fourth order derivative with five points and B also with a central difference scheme but with three points. The grid was chosen to be uniform at this point but no time was left to implement a non-uniform grid but the basic idea including the stretching function for the grid was given in the Bonus chapter.

As we will be using the function f the matrix-vector form can be rewritten as:

$$\overline{\overline{A}}\overline{f} = c\overline{\overline{B}}\overline{f} \tag{47}$$

As mentioned the Orr-Sommerfeld equation will be rewritten in a way that has an affinity to the matrix-vector form written above. For that let us rewrite equation (48).

$$k^4 f - 2k^2\frac{d^2 f}{dy^2} + \frac{d^4 f}{dy^4} = \frac{ik}{\nu}\left((U(y) - c)(\frac{d^2 f}{dy^2} - k^2 f) - \frac{d^2 U}{dy^2} f\right) \tag{48}$$

Multiplying the equation with **i** one gets:

$$\nu\left(k^4 f - 2k^2\frac{d^2 f}{dy^2} + \frac{d^4 f}{dy^4}\right) = ik\left(U(y)\frac{d^2 f}{dy^2} - U(y)k^2 f - c\frac{d^2 f}{dy^2} + ck^2 f - \frac{d^2 U}{dy^2 f}\right) \tag{49}$$

Expanding the equation and sorting terms:

$$i\nu k^4 f - i\nu 2k^2\frac{d^2 f}{dy^2} + i\nu\frac{d^4 f}{dy^4} = -kU(y)\frac{d^2 f}{dy^2} + U(y)k^3 f + kc\frac{d^2 f}{dy^2} - ck^3 f + \frac{d^2 U}{dy^2} fk \tag{50}$$

$$i\nu k^2\left(-2\frac{d^2 f}{dy^2} + k^2 f\right) + i\nu\frac{d^4 f}{dy^4} + kU(y)\frac{d^2 f}{dy^2} - U(y)k^3 f - \frac{d^2 U}{dy^2} fk = kc\frac{d^2 f}{dy^2} - ck^3 f \tag{51}$$

$$i\nu k^2\left(-2\frac{d^2 f}{dy^2} + k^2 f\right) + i\nu\frac{d^4 f}{dy^4} + kU(y)\left(\frac{d^2 f}{dy^2} - k^2 f\right) - \frac{d^2 U}{dy^2} fk = kc\left(\frac{d^2 f}{dy^2} - k^2 f\right) \tag{52}$$

Dividing the last expression by **k** will give us the form we need.

$$i\nu k\left(-2\frac{d^2 f}{dy^2} + k^2 f\right) + \frac{i\nu}{k}\frac{d^4 f}{dy^4} + U(y)\left(\frac{d^2 f}{dy^2} - k^2 f\right) - \frac{d^2 U}{dy^2} f = c\left(\frac{d^2 f}{dy^2} - k^2 f\right) \tag{53}$$

Inserting the finite-difference formulation and writing it in MATLAB style we get:

$$sqrt(-1)\nu k\left(-2\frac{f_{i-1}-2f_i+f_{i-1}}{\Delta y^2}+k^2 f_i\right)+\frac{sqrt(-1)\nu}{k}\underbrace{\frac{f_{i-2}-4f_{i-1}+6f_i-4f_{i+1}+f_{i+2}}{\Delta y^4}}_{PD}+ \tag{54}$$

$$U_i\left(\frac{f_{i-1}-2f_i+f_{i-1}}{\Delta y^2}-k^2 f_i\right)-U_{Diff}f_i=c\underbrace{\left(\frac{f_{i-1}-2f_i+f_{i-1}}{\Delta y^2}-k^2 f_i\right)}_{B} \tag{55}$$

where **PD** indicates the pentadiagonal matrix and B the tridiagonal matrix on the right-hand side.

Recalling the matrix-vector formulation:

$$\overline{\overline{A}}\overline{q}=c\overline{\overline{B}}\overline{q} \tag{56}$$

Having that in mind we can define **A** as:

$$A=\frac{i\nu}{k}\cdot PD+T1-2ik\nu TD \tag{57}$$

and in discretized form:

$$A(i,i)=A(i,i)-U(i+1)k^2-U_{Diff}(i+1)+sqrt(-1)\nu k^3 \tag{58}$$

And **B** is just the tridiagonal matrix **TD** we already defined - writing this matrix in discretized form we get:

$$B(i,i)=B(i,i)-k^2 \tag{59}$$

Once these matrices are defined we can use the blackbox from MATLAB to calculate our eigenvalues, namely:

$$egvl=eig(A,B)$$

8.1 Boundary Conditions

In general the boundary conditions are such that $f(-1)=f(+1)=0$ are $f'(-1)=f'(+1)=0$. The latter boundary condition is a bit harder to implement - we rewrite the derivative boundary condition as follows:

$$f'_0=\frac{f_1-f_{-1}}{2y} \tag{60}$$

which can be rewritten into the following form:

$$f_{-1}=f_1-2yf'_0 \tag{61}$$

As we know that $f'(-1)=f'(+1)=0$, we can say that the last term of (61) vanishes and we can write:

$$f_{-1}=f_1 \tag{62}$$

For the second order central difference scheme the no-slip condition holds:

$$f_1 = f_{N+1} = 0 \tag{63}$$

where i = 2,...,N and is exactly the same definition as derived in equation (62) but rewritten in a form that suits our problem. One can immediately see that putting i = 2 into the FDM formulation for the second derivative will cause the first column to vanish resulting in a general system of (N-1) x (N-1).

For the fourth order central difference scheme with 5 points it is necessary that the no-penetration condition as well as the no-slip condition are fulfilled. That means:

$$f_1 = f_{N+1} = 0 \tag{64}$$

and for the no slip we force:

$$f_0 = f_2 \tag{65}$$

and

$$f_{N+2} = f_N \tag{66}$$

With that the matrices have the following form:

$$PD = \frac{1}{\Delta y^4} \begin{bmatrix} 7.0 & -4.0 & 1.0 & & & & & & & \\ -4.0 & 6.0 & -4.0 & 1.0 & & & & \mathbf{0} & & \\ 1.0 & -4.0 & 6.0 & -4.0 & 1.0 & & & & & \\ & 1.0 & -4.0 & 6.0 & -4.0 & 1.0 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & 1.0 & -4.0 & 6.0 & -4.0 \\ & \mathbf{0} & & & & & & 1.0 & -4.0 & 7.0 \end{bmatrix}$$

$$TD = B = \frac{1}{\Delta y^2} \begin{bmatrix} -2.0 & 1.0 & & & & & & & \\ 1.0 & -2.0 & 1.0 & & & & \mathbf{0} & & \\ & 1.0 & -2.0 & 1.0 & & & & & \\ & & 1.0 & -2.0 & 1.0 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & 1.0 & -2.0 & 1.0 \\ & \mathbf{0} & & & & & & 1.0 & -2.0 \end{bmatrix}$$

For the second tridiagonal matrix we simply have to multiply the **TD** matrix with the discretized velocity U(i+1) because U(1) is zero.

$$TD1 = \frac{1}{\Delta y^2} \begin{bmatrix} -2.0*U(2) & 1.0*U(2) & & & & 0 & \\ 1.0*U(3) & -2.0*U(3) & 1.0*U(3) & & & & \\ & 1.0*U(4) & -2.0*U(4) & 1.0*U(4) & & & \\ & & 1.0*U(5) & -2.0*U(5) & 1.0*U(5) & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & 1.0*U(N) & -2.0*U(N) & \end{bmatrix}$$

9 Parameter Study - Variation of Reynolds Number and Alpha

In this section a parameter space is going to be investigated with the following parameter for the Reynolds number Re and the wave number α:

$$Re = 1000, 3000, 6000 \tag{67}$$

$$\alpha = 0.1, 0.5, 1, 2, 5 \tag{68}$$

The results for each configuration for a discretization of $N = 50$ is shown below.

9.1 Reynolds Number = 1,000

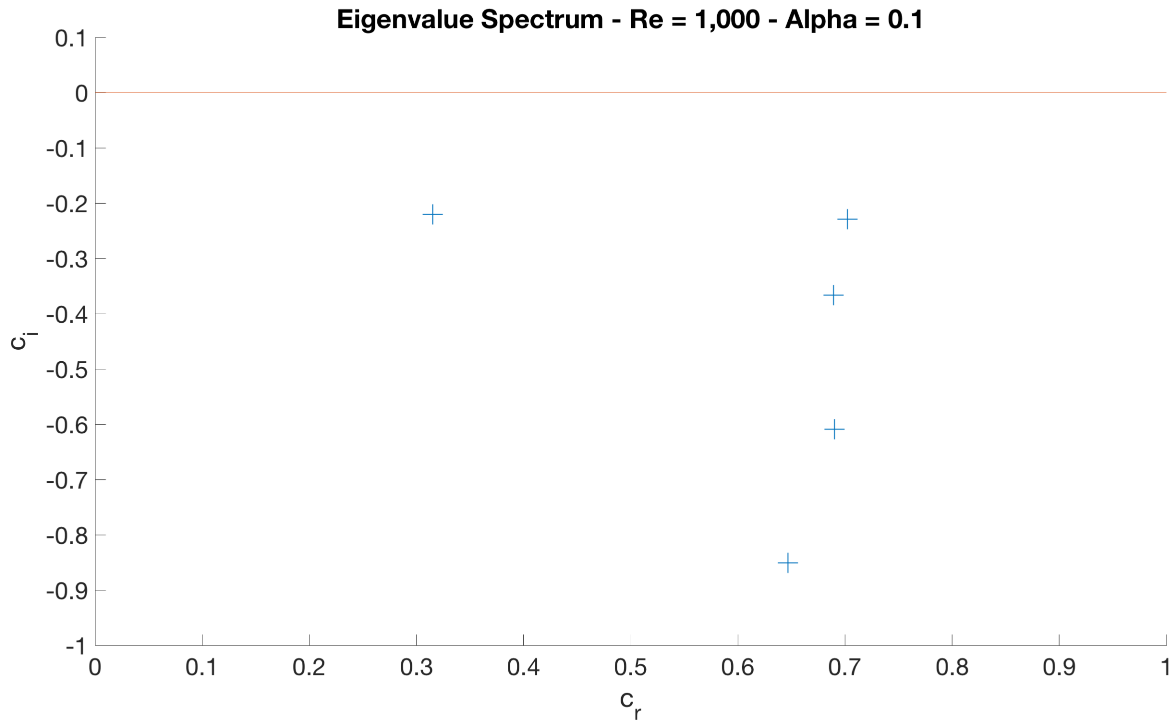

Figure 2: Re = 1,000 - Alpha = 0.1

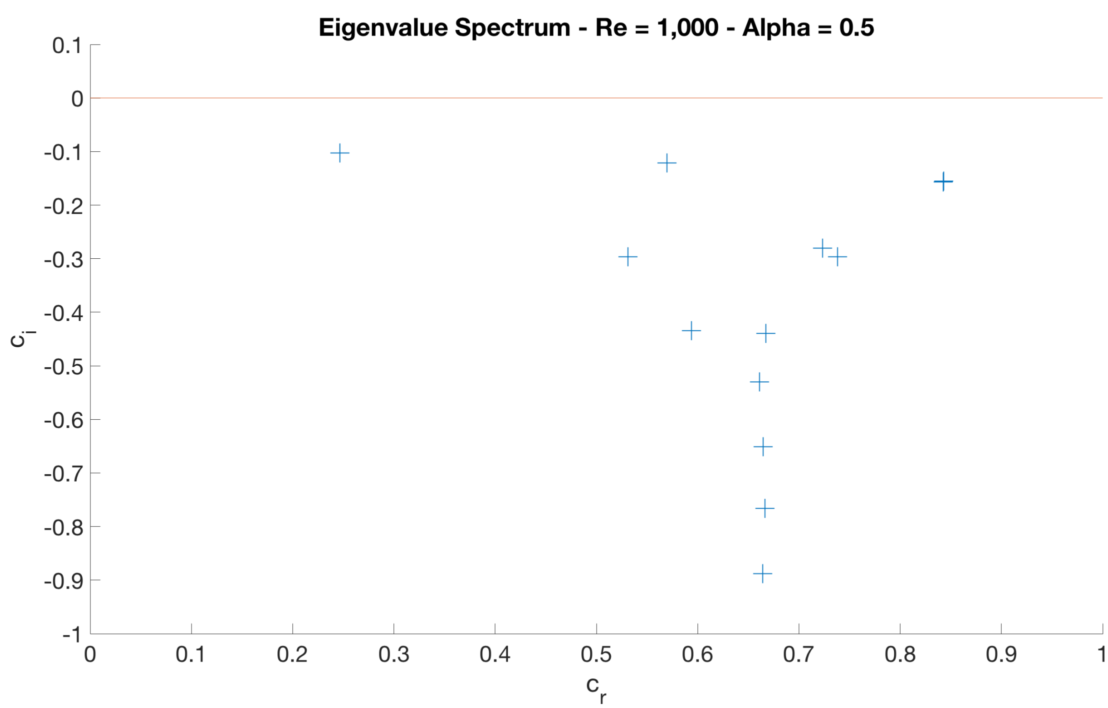

Figure 3: Re = 1,000 - Alpha = 0.5

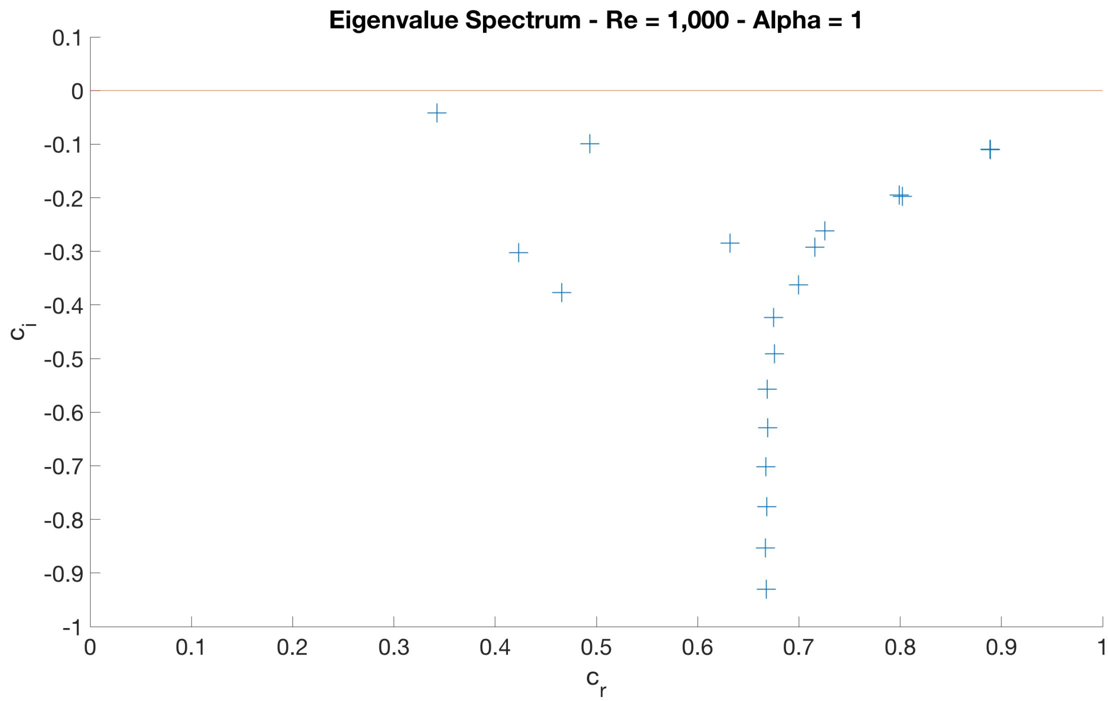

Figure 4: Re = 1,000 - Alpha = 1

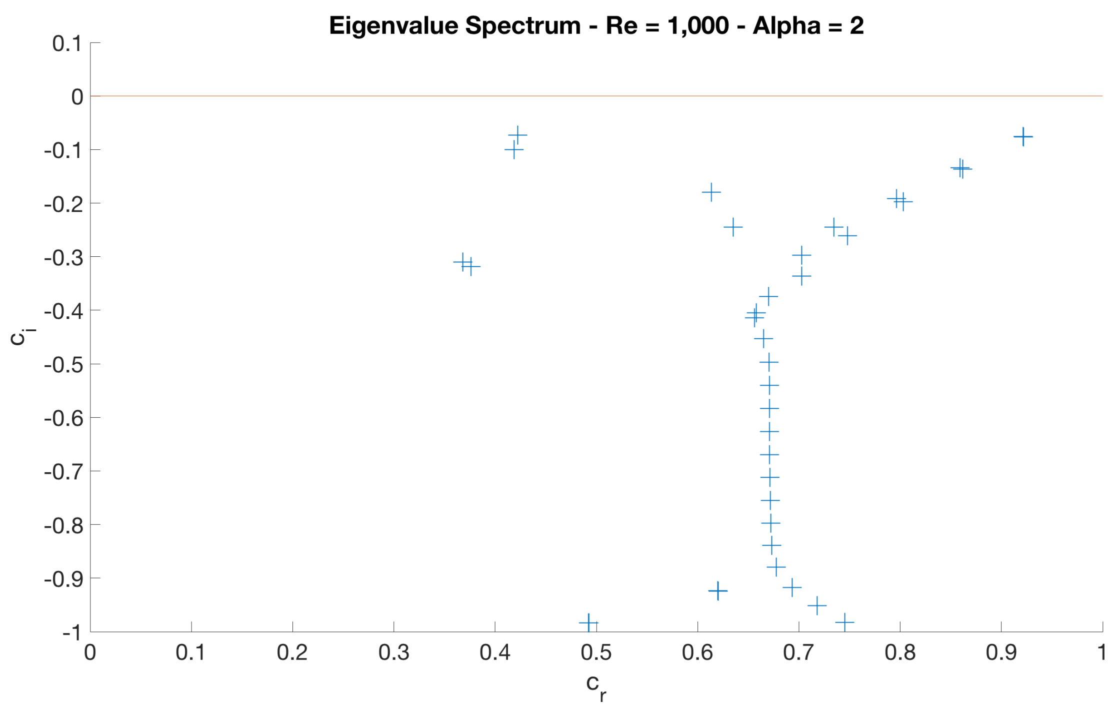

Figure 5: Re = 1,000 - Alpha = 2

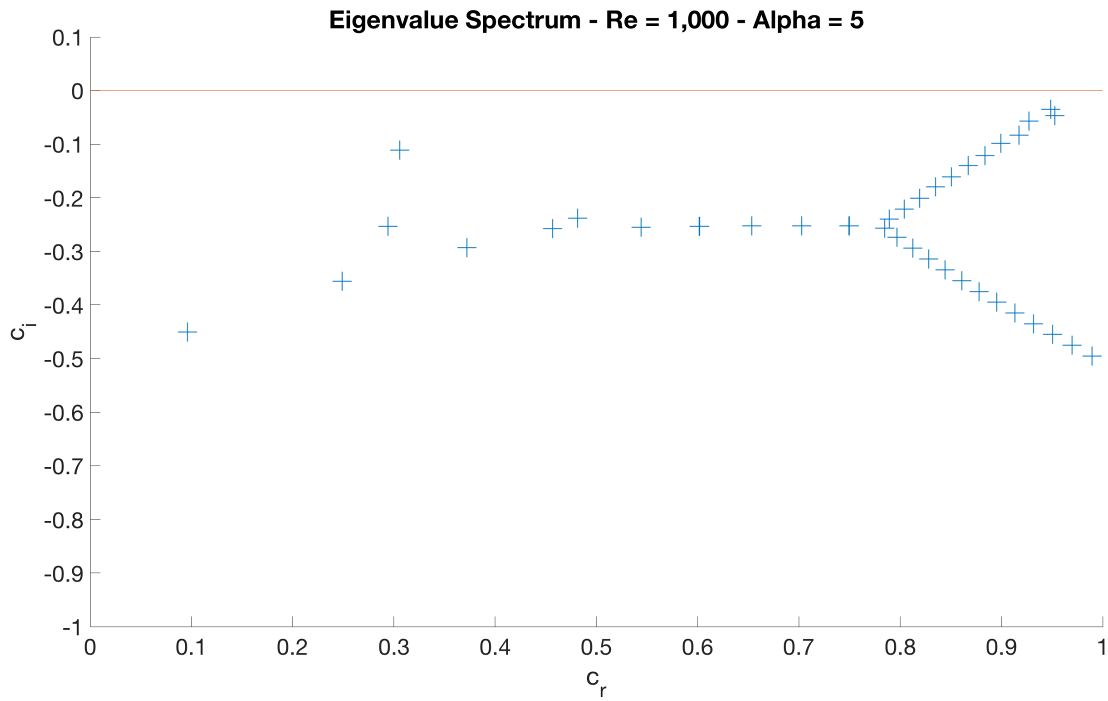

Figure 6: Re = 1,000 - Alpha = 5

9.2 Reynolds Number = 3,000

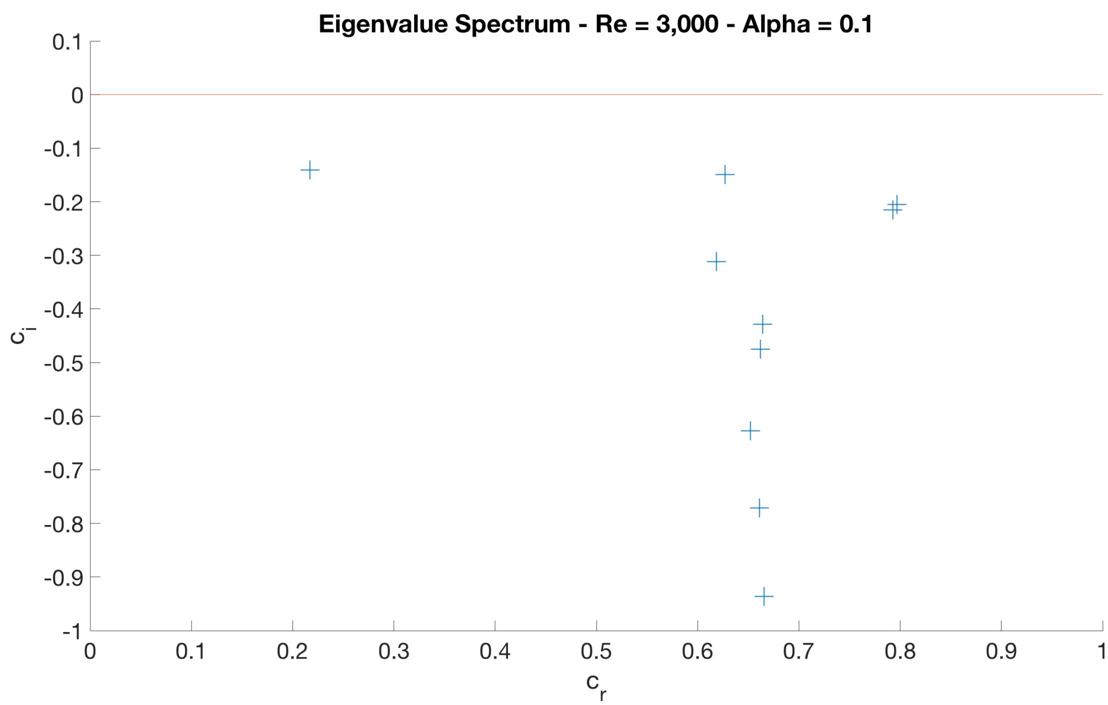

Figure 7: Re = 3,000 - Alpha = 0.1

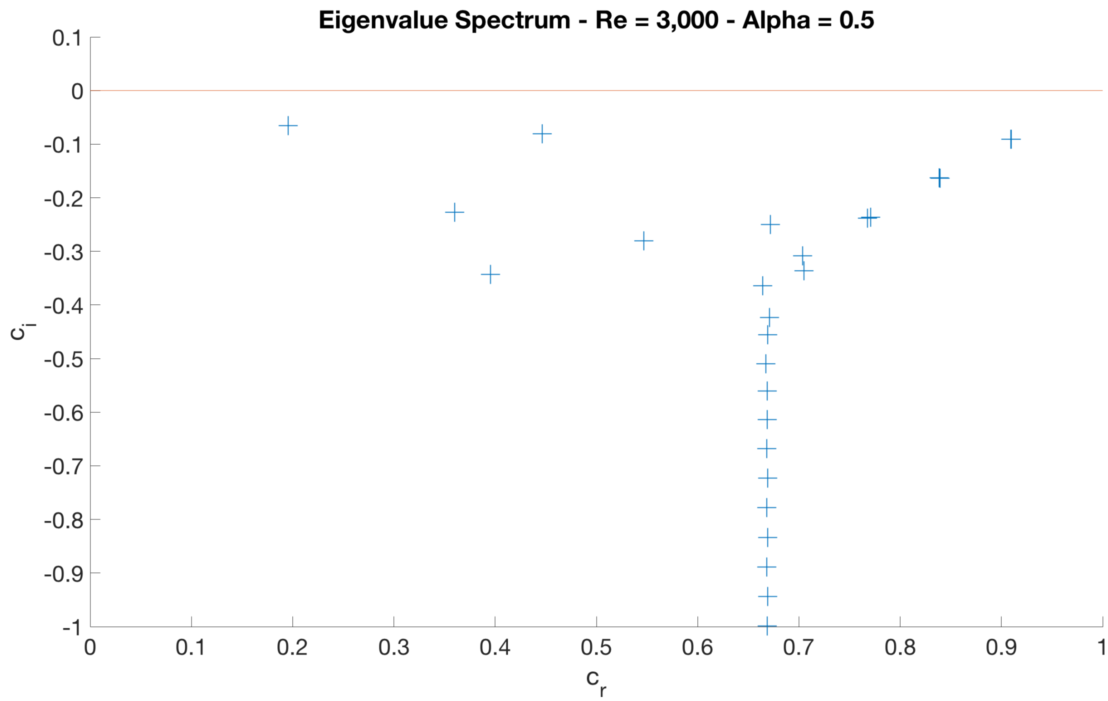

Figure 8: Re = 3,000 - Alpha = 0.5

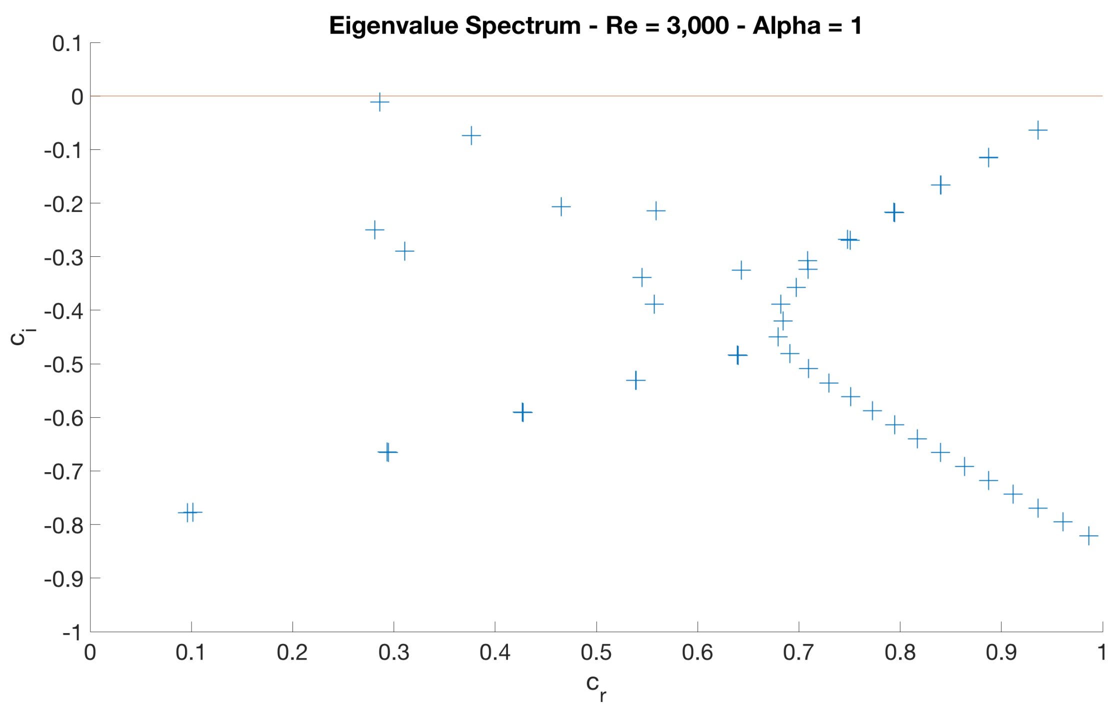

Figure 9: Re = 3,000 - Alpha = 1

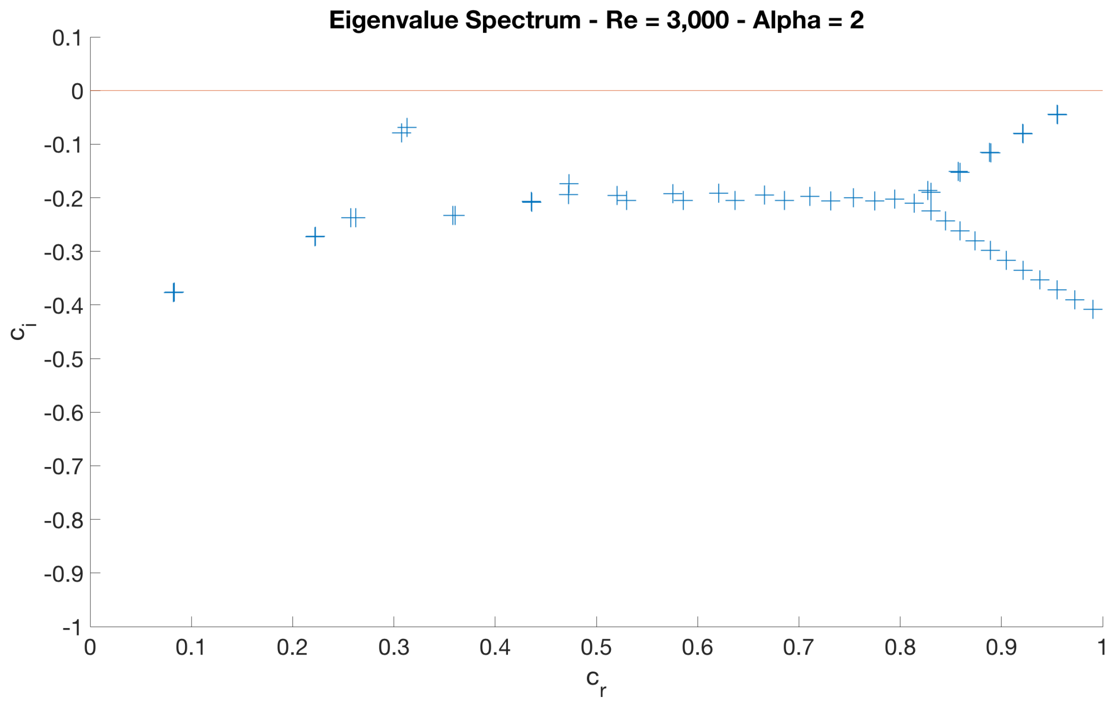

Figure 10: Re = 3,000 - Alpha = 2

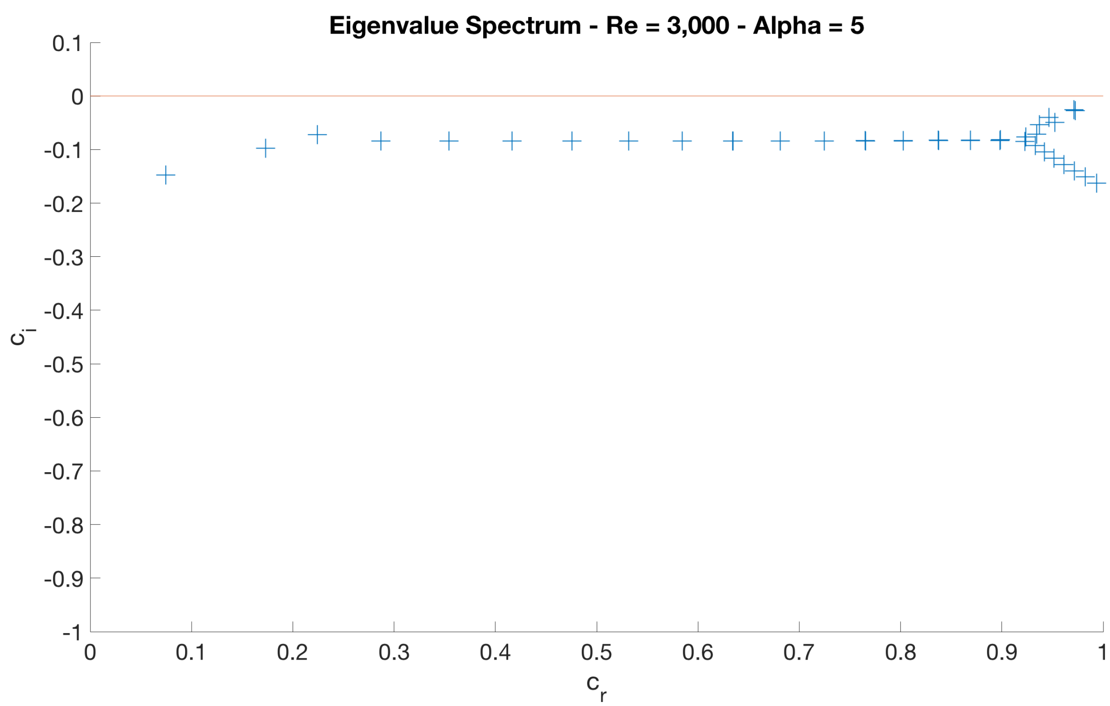

Figure 11: Re = 3,000 - Alpha = 5

9.3 Reynolds Number = 6,000

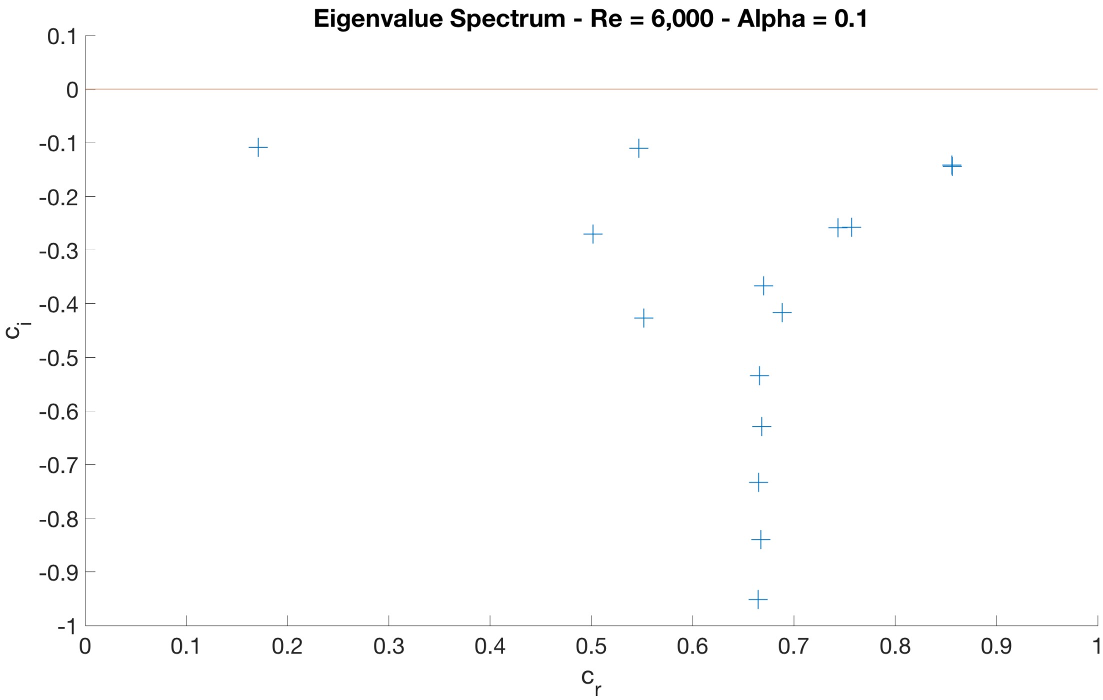

Figure 12: Re = 6,000 - Alpha = 0.1

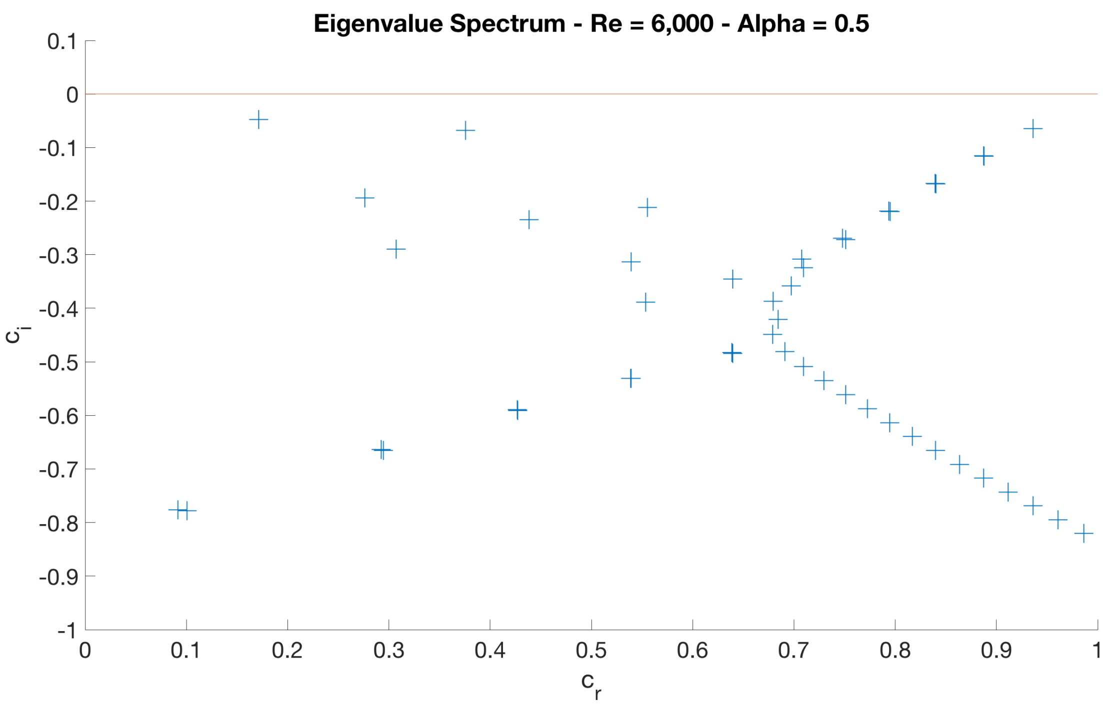

Figure 13: Re = 6,000 - Alpha = 0.5

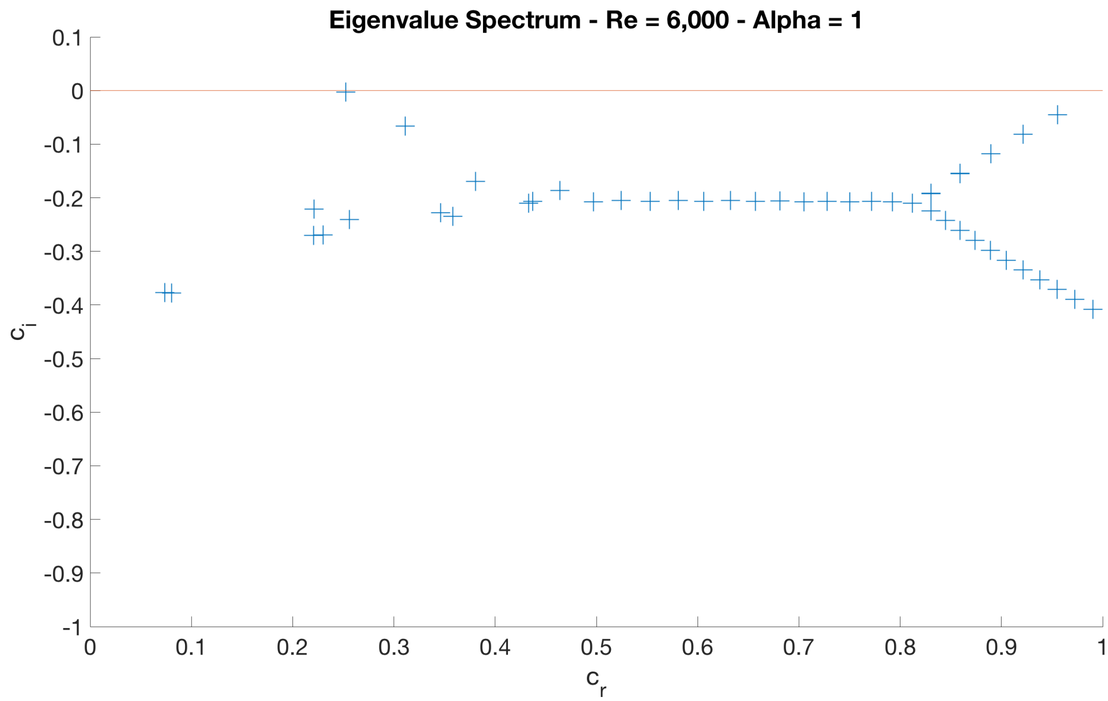

Figure 14: Re = 6,000 - Alpha = 1

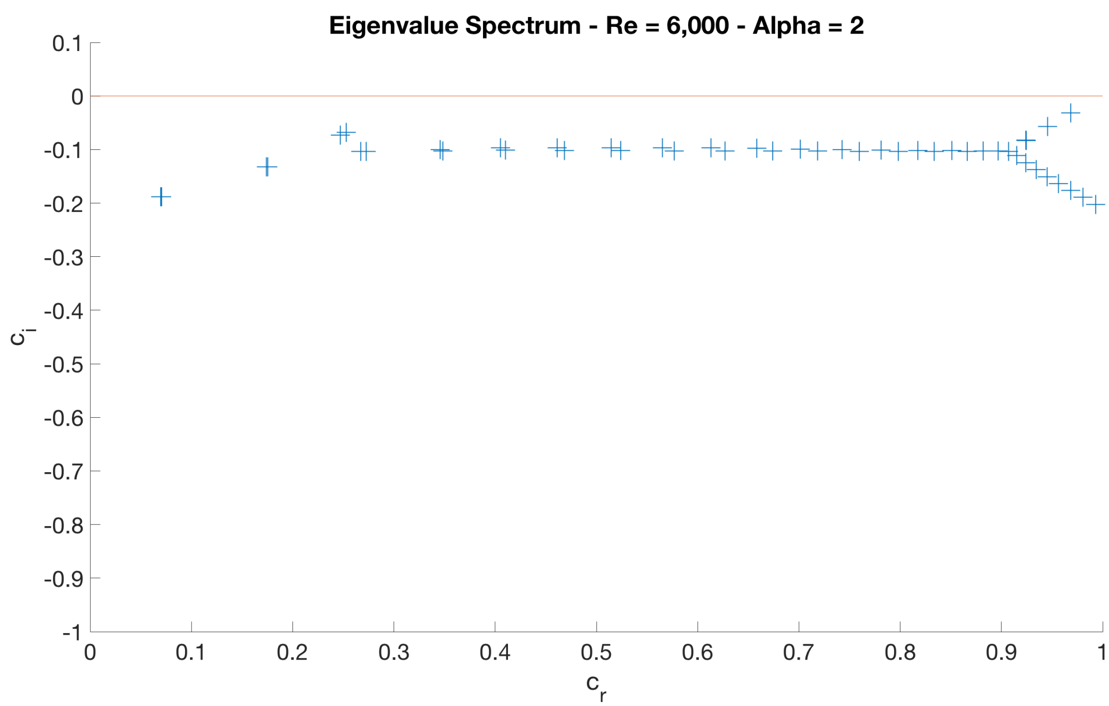

Figure 15: Re = 6,000 - Alpha = 2

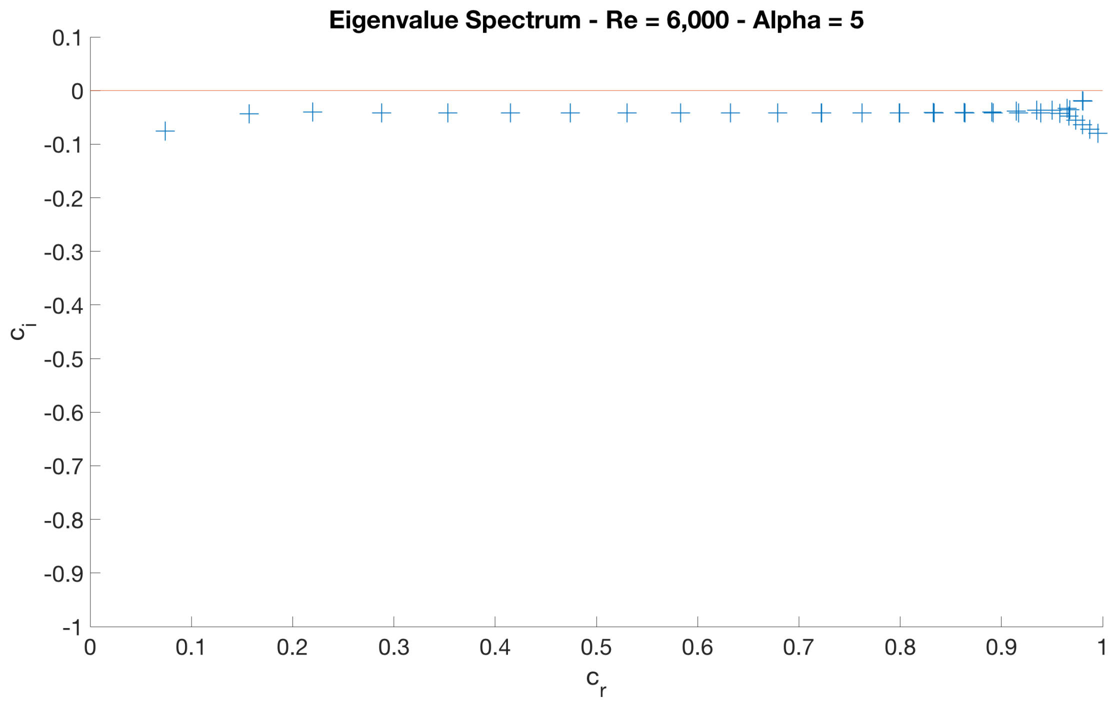

Figure 16: Re = 6,000 - Alpha = 5

As can be seen every configuration investigated has an imaginary part that is smaller than zero thus implying that the base flow is stable for those parameter combinations. However I was quite sceptical and assumed that the small resolution ”exterminated” the critical imaginary parts. This will be discussed below!

It is important to note the different branches for instance in figure 8. We can see the vertical branch of eigenvalues, known as the **continuous spectrum** of the Orr-Sommerfeld equation, and the two branches coming off the vertical branch at an angle. These branches are known as the **discrete spectrum** of the Orr-Sommerfeld equation.

As it turns out, flows which are bounded (Poiseuille flow) have an infinite many number of eigenvalues in the discrete spectrum. However it turns out that flows on unbounded domains have only finitely many eigenvalues in the discrete spectrum! [3]

To reinforce my argument that the discretization is a necessary and important factor to determine slight instabilites - imaginary values bigger than zero - I showed that for a Reynolds number of 6000 and an α of 1. The cases that will be compared are with discretizations of $N = 50$ and $N = 200$.

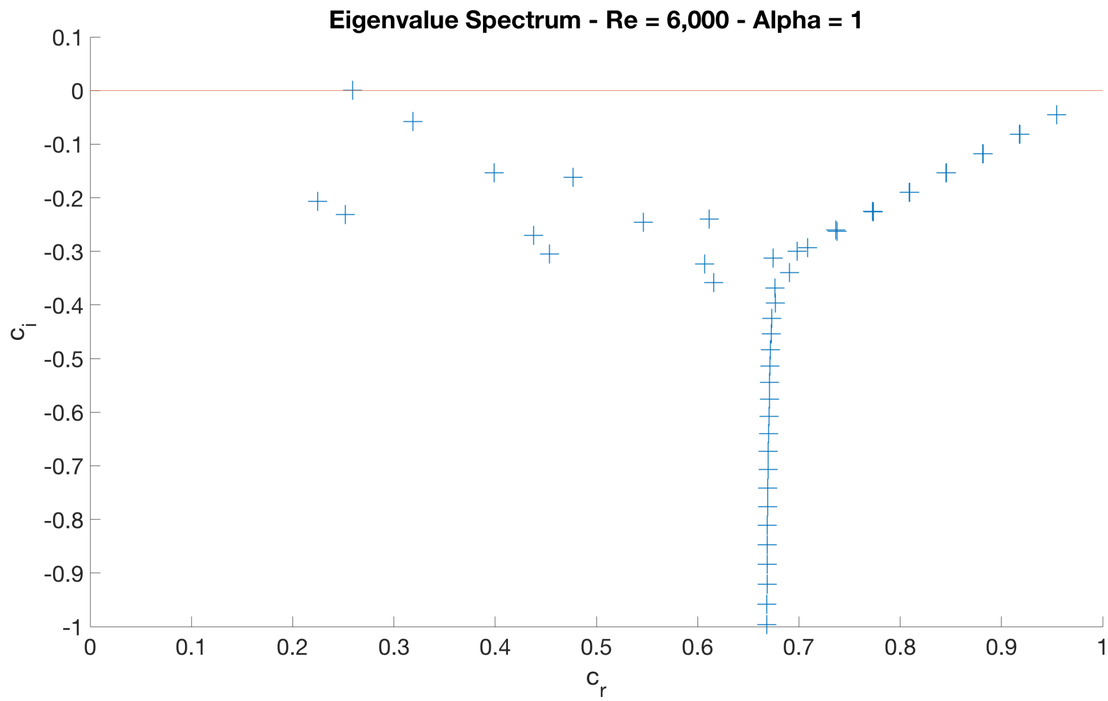

Figure 17: Re = 6,000 - Alpha = 1 - N = 200

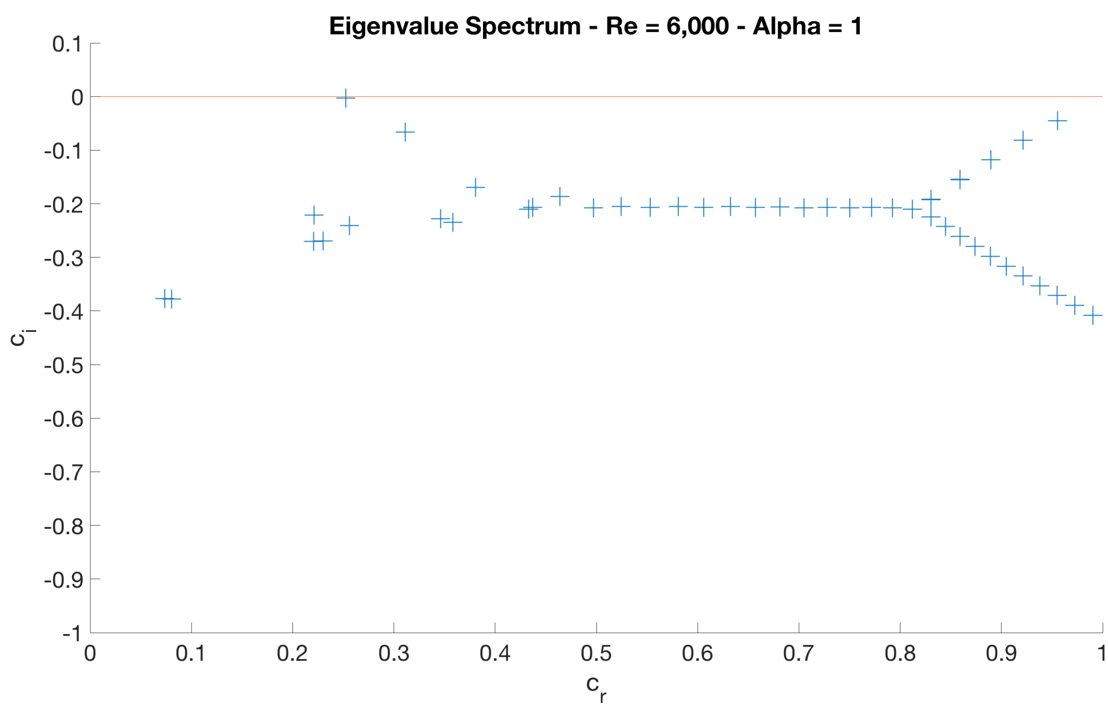

Figure 18: Re = 6,000 - Alpha = 1 - N = 50

When investigating the extremum of the eigenvalue spectrum inside of MATLAB one can see that for a discretization of $N = 200$ the base flow is indeed unstable. Not only do both plots differ in that way but one can also observe some kind of "metamorphosis" of the branches that look completely different, especially the continuous spectrum.

10 Results and Discussion

Hint: All the calculations done in MATLAB have been carried out by the following machine configuration:

- Macbook Pro 2016
- Processor: 2,9 GHz Intel Core i5
- Memory: 8 GB 2133 MHz LPDDR3
- Graphic Card: Intel Iris Graphics 550 1536 MB

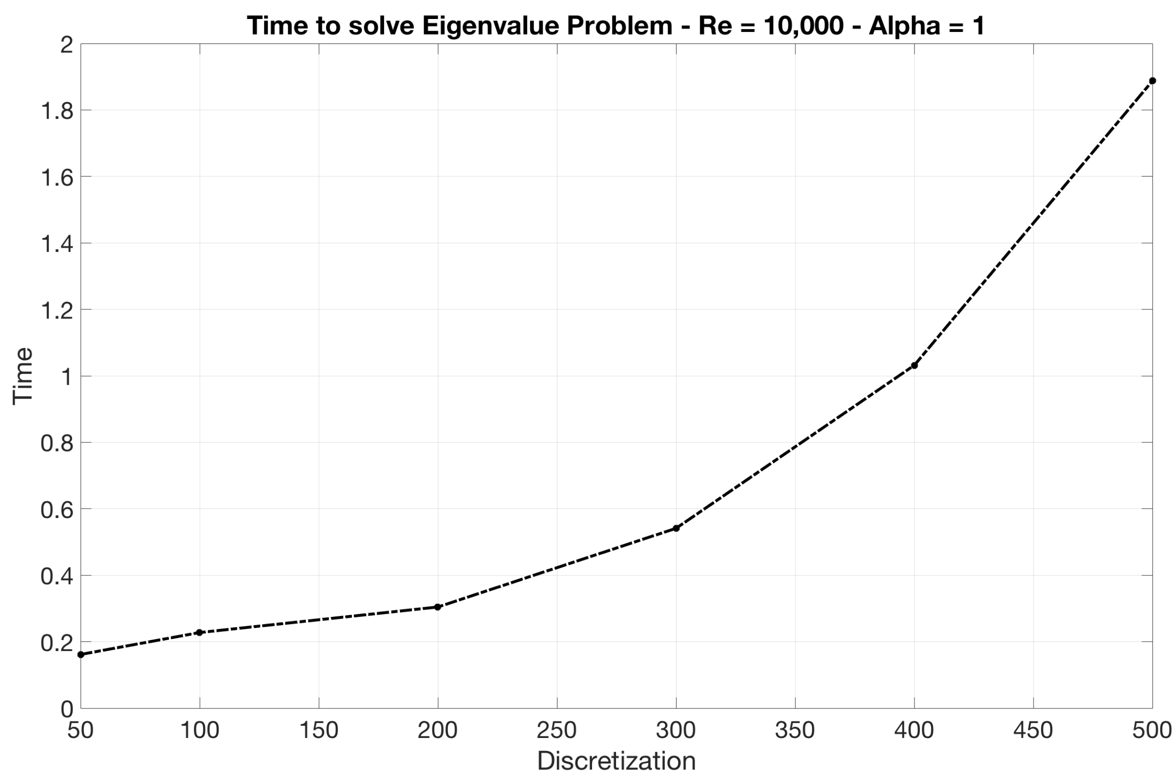

Figure 19: Time to solve Eigenvalue problem

The plot above shows the speed of the finite difference method outlined. The method is solving for the most unstable eigenvalue of plane Poiseuille flow at a Reynolds number 10,000, and $\alpha = 1$. On the x-axis is the discretization, while the y-axis is the time to solve for the solution which has been measured with the **tic-toc** function inside MATLAB.

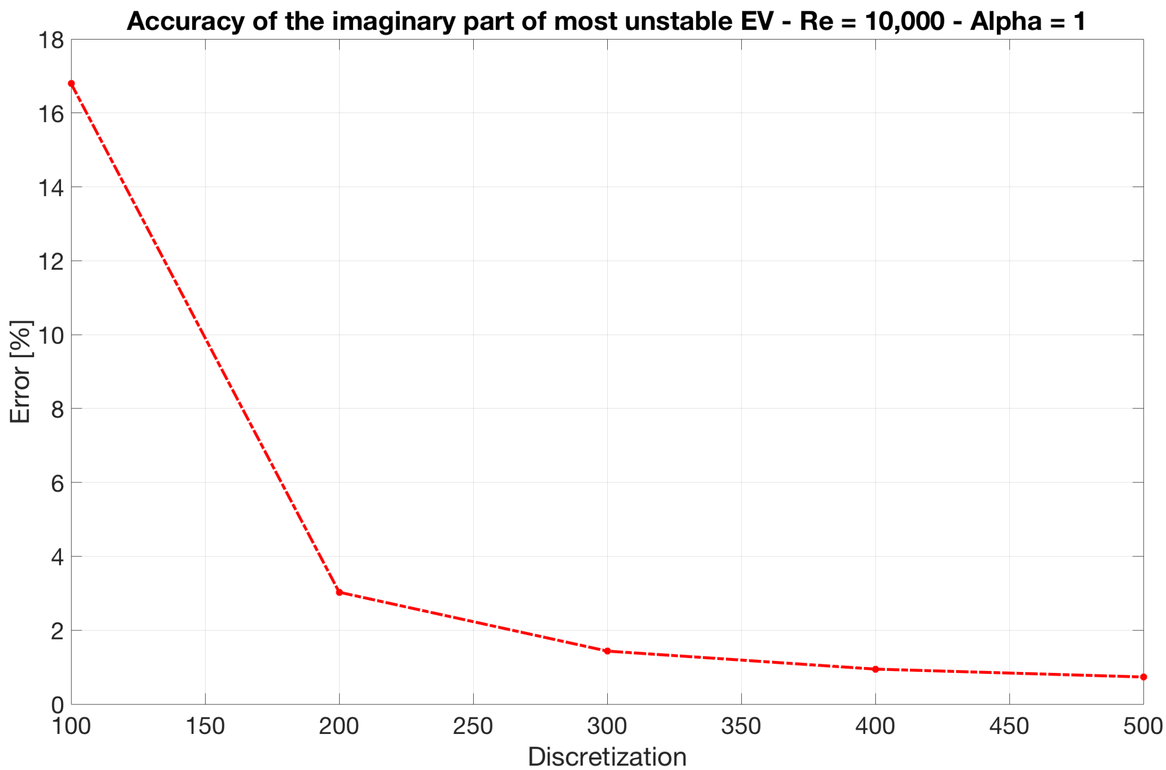

Figure 20: Accuracy of the imaginary part

The method is solving for the most unstable eigenvalue of plane Poiseuille flow at a Reynolds number 10,000, and $\alpha = 1$. On the x-axis is the grid size, while the y-axis is the percent error from the correct solution given in [4]. Here we see that at 500 mesh points, the maximum error obtained in the eigenvalue was "only" 0.7%.

I am sure that an increase of the mesh points as well as introducing a non-uniform grid with an optimal parameter for the point distribution will bring down the error even more and this can indeed be done in further investigations to see what impact this error has because it is know that machine precision creates an error as well as the cut-off error of the method itself also implies an error that is not insignificant!

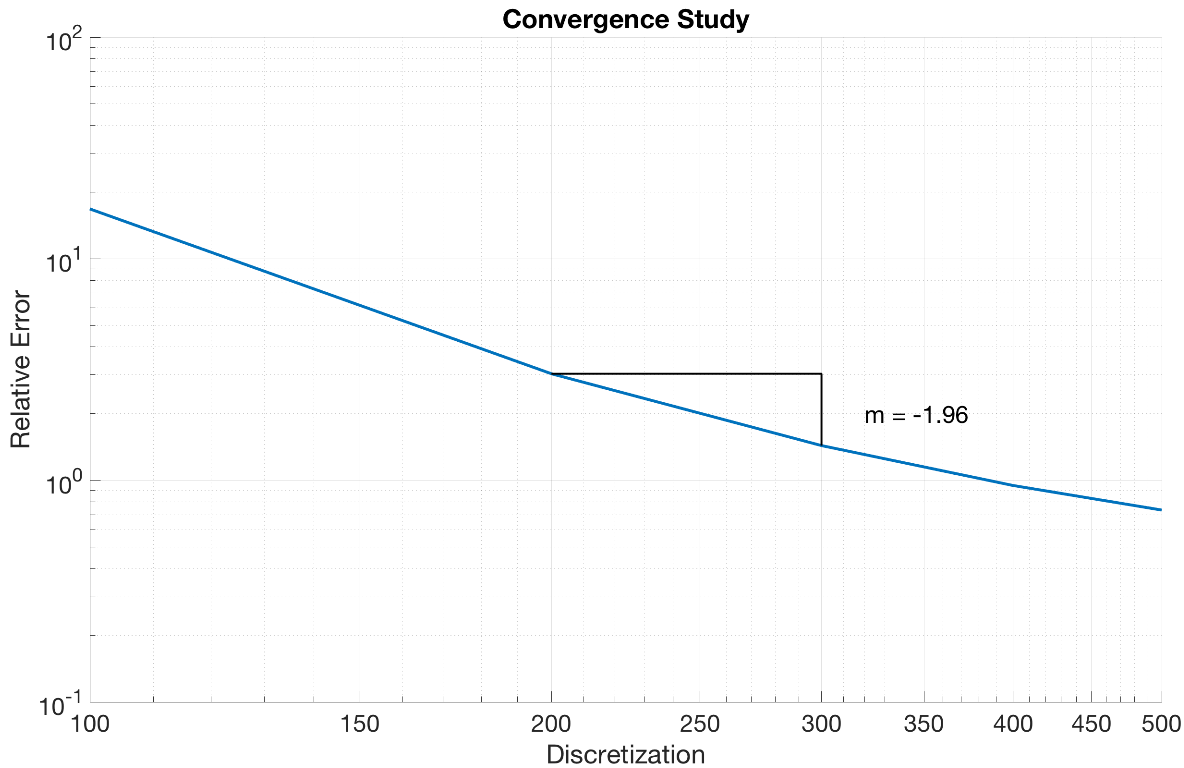

Figure 21: Convergence Study

21 shows the convergence study performed with a grid discretization up to 500 points. One can show by hand ((43) & (44)) that the error should decrease with Δy^2.

11 Stability

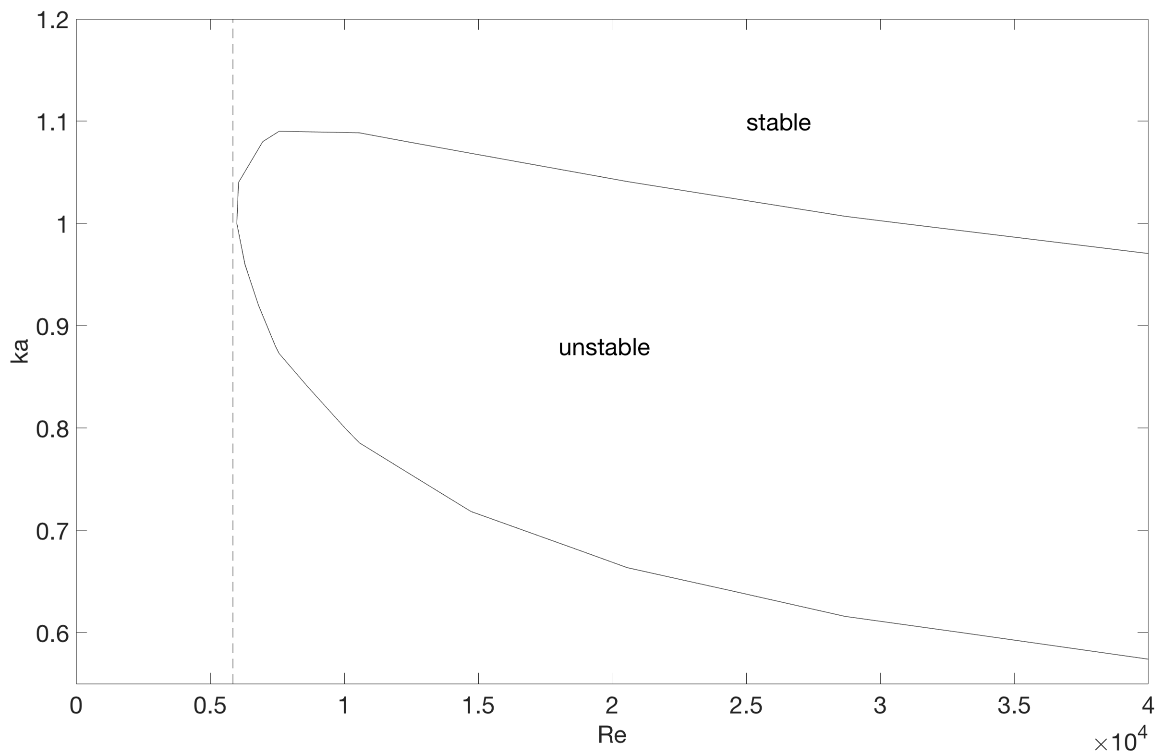

Figure 22: Contour Plot of Stability

In the stability plot for a discretization of $N = 200$ one can see the regions of stability and instability for which parameter combinations one gets stable and unstable configurations. α in this case was 1 and that is one of the reasons the critical Reynolds number is at approximately $Re_c = 5850$ and not $Re_c = 5722.22$ as stated in Orszag's paper which assumed an α of 1.02. Additionally discretization errors also play a significant role here but no further studies have been performed also due to the long processing time of the plot.

12 Bonus Exercises

12.1 Bonus 1

The first bonus exercise was to use a non-uniform grid formulation. However at the end of the project I had the ideas of implementing the non-uniform grid but was not able to finish the task in time.
However S.Orszag gave an idea in [4] how a non-uniform grid implementation or at least stretching function for the grid could look like. He cited the work of Gary & Helgason (1970) who were also using a finite-difference scheme with various orders of accuracy. The stretching function was:

$$z = ye^{y^2-1} \tag{69}$$

for $0 \leq z \leq 1$ where only symmetric modes were taken into account.

Asai & Nakasuji have also found out that there is an optimal stretching factor for a given converstion of coordinates that yield the most efficient scheme of centered differencing. [5]

12.2 Bonus 2

The second bonus task was to implement a higher order finite-difference method with 5 grid points on a fourth order derivative. This was done directly at the beginning without using second order formulation for the fourth derivative as it is present in the Orr-Sommerfeld equation.
Problems and accuracy improvements with this approach have been discussed above.

13 Outlook

Next steps for this project would be to extend the functionalities for spanwise wave numbers β that are not zero for further investigations and how that affects the instabilities inside the flow. On top of that instabilities travelling through the domain with an angle of inclanation γ can also be investigated to see the effect. Coupling of Spatio-temporal investgations can also be carried out.

Simulations to show the perfect initial conditions for the plane poiseuille flow to be stable as long as possible would be another idea that can be implemented and added to the code.

The code, the report as well as my PowerPoint presentation will be available at GitHub from Jousef Murad.

References

[1] P.G. Drazin and W.H. Reid. "Hydrodynamic Stability". In: *Cambridge U. Press* (1981).

[2] Peter J. Schmid and Dan S. Henningson. "Stability and Transition in Shear Flows". In: *Applied Mathematical Sciences 142* (2005).

[3] M. Miklavcic. "Eigenvalues of the orr-sommerfeld equation in an unbounded domain". In: *Archive for rational mechanics and analysis, 83(3):221228* (1983).

[4] S.A. Orszag. "Accurate solution of the Orr-Sommerfeld equation". In: *J. Fluid Mech., 50:689703* (1971).

[5] Tomio Asai and Isao Nakasuji. *A NOTE ON APPLICATION OF FINITE-DIFFERENCE METHOD TO STABILITY ANALYSIS OF BOUNDARY LAYER FLOWS.*