Physical Measurements — Cheat Sheet

Based on notes by Elijan J. Mastnak for the course at FMF, University of Ljubljana

---

1. Uncertainty & Error Propagation

General Formula (First-Order)

For a function $u = f(x_1, \ldots, x_n)$, with small, positive uncertainties δx_i:

$$\delta u = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta x_i = \left(\frac{\partial f}{\partial \mathbf{x}}\right)^T \delta \mathbf{x}$$

Variance for Dependent Variables

For $u = f(x, y)$, the variance σ_u^2 is:

$$\sigma_u^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sigma_{xy}$$

where $\sigma_{xy} = \mathrm{cov}(x, y) = \langle (x - \bar{x})(y - \bar{y}) \rangle = \rho_{xy}\sigma_x\sigma_y$.

Simple Cases (Independent Variables)

- **Linear Sum:** $u = ax + by \implies \sigma_u^2 = a^2\sigma_x^2 + b^2\sigma_y^2$
- **Power Law:** $u = Ax^a y^b \implies \left(\frac{\sigma_u}{\bar{u}}\right)^2 = a^2\left(\frac{\sigma_x}{\bar{x}}\right)^2 + b^2\left(\frac{\sigma_y}{\bar{y}}\right)^2$

2. Combining Measurements

Measurements $z_a \sim N(x, \sigma_a^2)$ and $z_b \sim N(x, \sigma_b^2)$.

Optimal Combination (Two Dependent Meas.)

- **Optimal estimate** $\hat{x}$:

$$\hat{x} = z_a + K(z_b - z_a), \quad \text{where } K = \frac{\sigma_a^2 - \sigma_{ab}}{\sigma_a^2 + \sigma_b^2 - 2\sigma_{ab}}$$

- **Optimal variance** $\tilde{\sigma}^2$:

$$\tilde{\sigma}^2 = (1 - \rho_{ab}^2)\left(\frac{1}{\sigma_a^2} + \frac{1}{\sigma_b^2} - \frac{2\rho_{ab}}{\sigma_a\sigma_b}\right)^{-1} = \frac{\sigma_a^2\sigma_b^2(1 - \rho_{ab}^2)}{\sigma_a^2 + \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b}$$

- **Independent case** ($\rho_{ab} = 0, \sigma_{ab} = 0$):

$$\tilde{\sigma}^2 = \left(\frac{1}{\sigma_a^2} + \frac{1}{\sigma_b^2}\right)^{-1} = \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2 + \sigma_b^2}$$

Variance of the Mean ($\bar{z}$)

For N measurements $\{z_i\}$:

- **General (Dependent):**

$$\sigma_m^2 = \frac{1}{N^2}\left(\sum_{i=1}^{N} \sigma_i^2 + 2\sum_{i<j} \sigma_{ij}\right)$$

- **Independent:** $\sigma_m^2 = \frac{1}{N^2}\sum_{i=1}^{N} \sigma_i^2$
- **Independent, equal variance σ_z^2:** $\sigma_m^2 = \frac{\sigma_z^2}{N}$

Variance-Weighted Combination (N Indep. Meas.)

- **Optimal estimate $\bar{h}$:**

$$\bar{h} = \frac{\sum w_i h_i}{\sum w_i}, \quad \text{where weight } w_i = \frac{1}{\sigma_i^2}$$

- **Optimal variance $\tilde{\sigma}^2$:**

$$\frac{1}{\tilde{\sigma}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} = \sum_{i=1}^{N} w_i$$

3. The Kalman Filter

Constant Scalar Quantity

- **Estimate update:**

$$\hat{x}_{n+1} = \hat{x}_n + \frac{\hat{\sigma}_n^2}{\hat{\sigma}_n^2 + \sigma_{z,n+1}^2}(z_{n+1} - \hat{x}_n)$$

- **Variance update:**

$$\frac{1}{\hat{\sigma}_{n+1}^2} = \frac{1}{\hat{\sigma}_n^2} + \frac{1}{\sigma_{z,n+1}^2}$$

- **Initialization:** $\hat{x}_1 = z_1, \quad \hat{\sigma}_1^2 = \sigma_{z,1}^2$

General Scalar Dynamics

Model: $x_{n+1} = \Phi_n x_n + c_n + \Gamma_n w_n$, with process noise variance $Q = \langle w_n^2 \rangle$. Variances: $P_n = \hat{\sigma}_n^2$ (optimal), $M_{n+1} = \bar{\sigma}_{n+1}^2$ (extrapolated), $R_{n+1} = \sigma_{z,n+1}^2$ (measurement).

1. **Extrapolation (Prediction):**

$$\bar{x}_{n+1} = \Phi_n \hat{x}_n + c_n$$
$$M_{n+1} = \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

2. **Optimization (Update):**

$$K_{n+1} = \frac{M_{n+1}}{M_{n+1} + R_{n+1}} \quad \text{(Kalman Gain)}$$
$$\hat{x}_{n+1} = \bar{x}_{n+1} + K_{n+1}(z_{n+1} - \bar{x}_{n+1})$$
$$P_{n+1} = (1 - K_{n+1})M_{n+1}$$

Vector Kalman Filter

Model: $\mathbf{x}_{n+1} = \mathbf{\Phi}_n \mathbf{x}_n + \mathbf{c}_n + \mathbf{\Gamma}_n \mathbf{w}_n$. Measurement model $\mathbf{z}_n = \mathbf{H}_n \mathbf{x}_n + \mathbf{v}_n$. Covariance matrices: $\mathbf{P}$ (state), $\mathbf{Q}$ (process noise), $\mathbf{R}$ (measurement noise).

1. **Extrapolation (Prediction):**

$$\bar{\mathbf{x}}_{n+1} = \mathbf{\Phi}_n \hat{\mathbf{x}}_n + \mathbf{c}_n$$
$$\mathbf{M}_{n+1} = \mathbf{\Phi}_n \mathbf{P}_n \mathbf{\Phi}_n^T + \mathbf{\Gamma}_n \mathbf{Q}_n \mathbf{\Gamma}_n^T$$

2. **Optimization (Update):**

$$\mathbf{K}_{n+1} = \mathbf{M}_{n+1}\mathbf{H}_{n+1}^T(\mathbf{H}_{n+1}\mathbf{M}_{n+1}\mathbf{H}_{n+1}^T + \mathbf{R}_{n+1})^{-1}$$
$$\hat{\mathbf{x}}_{n+1} = \bar{\mathbf{x}}_{n+1} + \mathbf{K}_{n+1}(\mathbf{z}_{n+1} - \mathbf{H}_{n+1}\bar{\mathbf{x}}_{n+1})$$
$$\mathbf{P}_{n+1} = (\mathbf{I} - \mathbf{K}_{n+1}\mathbf{H}_{n+1})\mathbf{M}_{n+1}$$

Continuous-time dynamics: $\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{\Gamma}\mathbf{Q}\mathbf{\Gamma}^T - \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P}$.

4. Distributions & Statistics

Estimators (from sample $\{z_i\}_{i=1}^{N}$)

- **Unbiased Mean:** $\bar{z} = \frac{1}{N}\sum z_i$
- **Unbiased Variance:** $s^2 = \frac{1}{N-1}\sum(z_i - \bar{z})^2$

Key Distributions

- **Normal (Gaussian) $N(\mu, \sigma^2)$:** Standardized variable $u = (z-\mu)/\sigma \sim N(0,1)$.

$$P(\mu - n\sigma < z < \mu + n\sigma) = 2\Phi(n) - 1$$

  where $\Phi(n)$ is the standard normal CDF.
- **Chi-Square $\chi^2(\nu)$:** Tests variance.

$$\chi^2 = \frac{(N-1)s^2}{\sigma_0^2} \sim \chi^2(N-1)$$

  Used for confidence interval on σ^2: $\sigma^2_\pm = (N-1)s^2/\chi^2_\mp$.
- **Student's t-distribution $t(\nu)$:** Tests mean when σ is unknown.

$$T = \frac{\bar{z} - \mu_0}{s/\sqrt{N}} \sim t(N-1)$$

  Used for confidence interval on μ: $\mu_\pm = \bar{z} \mp T_c \frac{s}{\sqrt{N}}$.

Hypothesis Testing

1. State null hypothesis H_0 (e.g., $\mu = \mu_0, \sigma^2 = \sigma_0^2$).
2. Choose significance level α.
3. Compute test statistic (T_0, χ_0^2) from data under H_0.
4. Find critical value(s) (T_c, χ_c^2) from tables for α and ν.
5. If $|T_0| > T_c$ (or χ_0^2 outside interval), reject H_0.

Pearson χ^2 Goodness-of-Fit Test

Tests if sample follows a proposed distribution.

- Divide data into B bins. N_k = observed count in bin k.
- p_k = theoretical probability for bin k. Np_k = expected count.
- Test statistic: $\chi_P^2 = \sum_{k=1}^{B} \frac{(N_k - Np_k)^2}{Np_k} \sim \chi^2(B-1)$.
- Requirement: $Np_k \geq 5$ for all bins.

5. Sensors & Filters (Laplace)

Laplace Transform: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$. **Derivative Property:** $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$. **Transfer Function:** $H(s) = \frac{X(s)}{Z(s)}$, where $Z(s)$ is input, $X(s)$ is output.

First-Order Sensor

- **DE:** $z(t) = \tau\dot{x}(t) + x(t)$
- **Transfer Function:** $H(s) = \frac{1}{1+\tau s}$

Second-Order Sensor

- **DE:** $\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \omega_0^2 z$
- **Transfer Function:** $H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
- **Damping:** ζ is damping coefficient, ω_0 resonant frequency.
- **Optimal Damping:** $\zeta = 1/\sqrt{2}$.

Response to Periodic Signals

For a sinusoidal input $z(t) = z_0 e^{i\omega t}$, the steady-state output is:

$$x(t) = H(i\omega)z(t) = |H(i\omega)|e^{i\delta}z_0 e^{i\omega t}$$

- **Gain (Amplification):** $A = |H(i\omega)|$
- **Phase Shift δ:** $\tan\delta = \frac{\mathrm{Im}\{H(i\omega)\}}{\mathrm{Re}\{H(i\omega)\}}$

Bode Plots

Graphical plot of filter response.

- **Y-axis (Gain):** $20\log_{10}|H(i\omega)|$ [dB]
- **X-axis (Frequency):** $\log_{10}(\omega\tau)$ or $\log_{10}(\omega)$

6. Common Filter Circuits

Passive RC/CR Filters

- **RC Low-Pass:** Resistor in series, Capacitor to ground.

$$H(s) = \frac{1}{1+sRC}$$

- **CR High-Pass:** Capacitor in series, Resistor to ground.

$$H(s) = \frac{sRC}{1+sRC}$$

Ideal Op-Amp Rules

1. No current flows into input terminals: $I_+ = I_- = 0$.
2. Input terminals are at the same voltage: $V_+ = V_-$.

Op-Amp Circuits (Gain $G = V_{out}/V_{in}$)

- **Inverting Amplifier:** $G = -\frac{R_f}{R_{in}}$
- **Non-Inverting Amplifier:** $G = 1 + \frac{R_f}{R_{in}}$
- **Voltage Follower (Buffer):** $G = 1$
- **Ideal Differentiator (periodic):** $H(i\omega) = -i\omega R_f C_{in}$
- **Ideal Integrator (periodic):** $H(i\omega) = \frac{1}{-i\omega R_{in} C_f}$

Band-Pass & Band-Stop Filters

- **Quality Factor Q:** $Q = \frac{\omega_0}{\Delta\omega_{-3\text{dB}}}$. High Q = narrow band.
- **Band-Pass Transfer Function:**

$$H(s) = \frac{s\omega_c}{s^2 + s\omega_c + \omega_0^2}, \quad \omega_c = \frac{1}{RC}, \omega_0^2 = \frac{1}{LC}$$

- **Band-Stop Transfer Function:**

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + s\omega_c + \omega_0^2}, \quad \omega_c = \frac{R}{L}, \omega_0^2 = \frac{1}{LC}$$

7. Linear Least Squares (LLS)

Model: $\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{r}$, where $\mathbf{z}$ are N measurements, $\mathbf{x}$ are M model parameters.

- **Goal:** Minimize $\chi^2 = (\mathbf{z} - \mathbf{H}\mathbf{x})^T \mathbf{R}^{-1} (\mathbf{z} - \mathbf{H}\mathbf{x})$.
- **Noise Covariance:** $\mathbf{R} = \langle \mathbf{r}\mathbf{r}^T \rangle$. For independent noise with different variances, $\mathbf{R}$ is a diagonal matrix of σ_i^2. If all variances are equal, $\mathbf{R} = \sigma^2 \mathbf{I}$.
- **Optimal Parameters $\hat{\mathbf{x}}$:**

$$\hat{\mathbf{x}} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}^{-1} \mathbf{z}$$

- **Parameter Covariance Matrix P:**

$$\mathbf{P} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}$$

- **Minimum χ^2 value:** distributed as $\chi^2(N - M)$.

LLS for $z_i = x_0 t_i + x_1$ (equal variance)

- **Structure Matrix H**:

$$\mathbf{H} = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_N & 1 \end{pmatrix}$$

- **Slope $\hat{x}_0$:**

$$\hat{x}_0 = \frac{\overline{tz} - \bar{t}\bar{z}}{\overline{t^2} - (\bar{t})^2}$$

- **Intercept $\hat{x}_1$:**

$$\hat{x}_1 = \bar{z} - \hat{x}_0 \bar{t}$$