Kishlay Singh

30/01/21 Mid Semester Examination

Kishlay Singh
2003121005

Ques 1 →

(a) Probability is the branch of Mathematics concerning numerical descriptions of how likely an event is to occur, or how likely it is that a proposition is true. The analysis of events governed by probability is called Statistics

~~Bayesian~~ Bayesian Probability →

Probability is a 'degree of belief' in a proposition; allocated by an observer given the available information ~~or~~ (data); Uncertainity arising from incomplete data or noise. This is radically different to the frequentist approach, but allows us to deal with situations where no ~~an~~ ensemble can be even be imagined: eg. 'what is the probability that there is life on Mars?'.

Frequentist Probability →

Probabilities are measurable frequencies, assigned to objects or events. The relative frequency (probability) of an event arises from the number of times this event would occur relative to an 'infinite ensemble' of 'identical' experiments. This is intutively linked to games of chance but breaks down in some obvious situations eg., for single events, or in situations where we cannot in practice measure the frequency, we ~~have~~ have to invent a hypothetical ensemble of events. Mathematically it requires notions of infinity and randomness which are not well defined in general.

Kshitiz Singh

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(b). Baye's theorem is stated mathematically as the following equation.

$$P(A|B) = \frac{P(B|A)\ P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$

$P(A|B)$ is called the posterior probability which represents the state of knowledge about the system in light of data.

$P(A)$ is called the prior as it contains all the information we know about the probability.

$P(B)$ is called the evidence

$P(B|A)$ is called the ~~likely~~ likelihood.

Que 3 → P(positive result, given patient is allergic)
= P(positive / allergy) = 0.8

P(positive / no allergy) = 0.1

Normalizing :-

P(allergy / positive) + P(no allergy / negative) = 1

Kishlay Singh
2003121005

~~Then~~

$\cancel{P(\text{allergy} | \text{positive}) = \frac{P(\text{positive} | \text{allergy})}{}}$

$$\cancel{P(\text{allergy} | +ve) = \frac{P(+ve | \text{allergy})}{P(+ve | \text{allergy}) + P(+ve | \text{no allergy}) \times \left[\frac{P(\text{no allergy})}{P(\text{allergy})}\right]}}$$

~~Then~~

~~P~~

Then

$$P(\text{allergy} | +ve) = \frac{P(+ve | \text{allergy})}{P(+ve | \text{allergy}) + P(+ve | \text{no allergy}) \left[\frac{P(\text{no allergy})}{P(\text{allergy})}\right]}$$

Given $P(\text{allergy}) = 0.01$, $P(\text{no allergy}) = 1 - 0.01$
$= 0.99$

$$P(\text{allergy} | +ve) = \frac{0.8}{0.8 + 0.1 \times \frac{0.99}{0.01}} = \frac{8}{8 + 99} = 0.0747$$

$$P(\text{allergy} | +ve) = \cancel{0.747}\ \underline{0.0747} \sim 0.075$$

Rishabh Singh
2003121005

Ques 2 →

Our prior information is that the coin has a 99% chance of being fair (single sided), ~~indicating~~ indicating a 1% chance of being unfair (double-headed). These are the two models and their priors P(B) are $P(M_{OK}) = 0.99$; $P(M_{dh}) = 0.01$.

Supposing we get n heads in a row, i.e. O_{nh} for the first n tosses. Now:-

$$p(A) = P(O_{nh}) = P(O_{nh}/M_{OK}) \times P(M_{OK}) + P(O_{nh}/M_{dh}) \times P(M_{dh})$$

with $P(O_{nh}/M_{OK}) = \left(\frac{1}{2}\right)^n$

and $P(O_{nh}/M_{dh}) = 1.00$

Plugging into Bayes' Theorem, remembering that A refers to the data and B to the model:-

$$P(B/A) = \frac{P(A/B)\,P(B)}{P(A)} = \frac{P(O_{nh}/M_{OK}) \times P(M_{OK})}{P(O_{nh})}$$

We see for n=2, model is correct, 99% chance the coin is good, is $P(B/A) = 0.961$.

But by n=7, we find 0.436 which is less than a 50% chance that the model is acceptable.
The inference is influenced by the given prior, more coin tosses might give a better judgement of the fairness of the coin.

Kishlay Singh

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Que 4 →

We want to know $P(x_0, y_0 \mid \{x_i\})$

Using Baye's Theorem, we write this as:-

$$P(x_0, y_0 \mid \{x_i\}) \propto P(\{x_i\} \mid x_0, y_0) P(x_0, y_0) \propto \prod_i P(x_i \mid x_0, y_0)$$

if we assume a uniform prior for x_0, y_0.

Let the angle of the direction of the flash to the normal to the coast line be φ. Then by trigonometry, the position that the flash arrives at is given by

$$\frac{x_i - x_0}{y_0} = \tan \varphi_i$$

So,

$$P(x_i \mid x_0, y_0) = P(\varphi_i \mid x_0, y_0) \left| \frac{d\varphi_i}{dx_i} \right|$$

and for signals that are recieved on the shore, φ is uniformly distributed in $-\pi/2 < \varphi < \pi/2$, so $P(\varphi_i) = \frac{1}{\pi}$ in this range, independent of x_0, y_0. Also,

$$\sec^2 \varphi_i \frac{d\varphi_i}{dx_i} = \frac{1}{y_0} \Rightarrow \left[1 + \frac{(x_i - x_0)^2}{y_0^2} \right] \frac{d\varphi_i}{dx_i} = \frac{1}{y_0}$$

and the likelihood of x_i is a ~~tact~~ Cauchy distribution.

$$P(x_i \mid x_0, y_0) = \frac{1}{\pi y_0 \left[1 + \frac{(x_i - x_0)^2}{y_0^2} \right]}$$

Kishlay Singh
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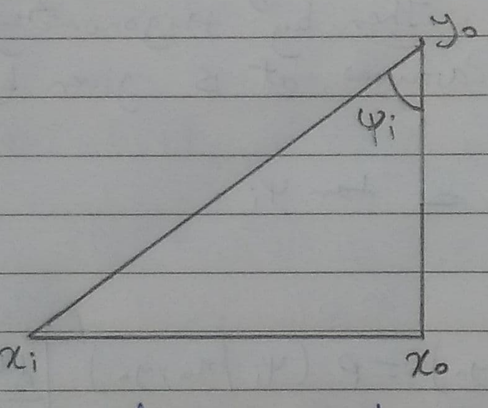
Here the (unnormalised) posterior for x_0, y_0 is

$$P(x_0, y_0 \mid \{x_i\}) \propto \prod_{i=1}^{N} \frac{1}{\pi y_0 \left[1 + \frac{(x_i - x_0)^2}{y_0^2}\right]}$$

which is our desired outcome.



Reference diagram for the given question.

Que 5 → Probability of murder is $P(M)$
Probability the person involved is $P(V)$
Probability of the person murdering is $P(S)$.

Given ⟹ $P(S/V) = \frac{1}{2500} = 0.0004$

and $P(M \mid S, V) = 1$

Using Baye's theorem

$$P(S \mid M, V) = \frac{P(M \mid S, V)\, P(S \mid V)}{P(M \mid S, V)\, P(S \mid V) + P(M \mid S^*, V)\, P(S^* \mid V)}$$

Kishlay Singh
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Now,

$$P(M \mid S, V) = 1,$$

$$P(S \mid V) = 4 \times 10^{-4}$$

$$P(S^* \mid V) = 0.9996 \quad \text{as} \quad P(S \mid V) + P(S^* \mid V) = 1$$

$P(M \mid S^*, V)$ is the probability that there is a muder given that the criminal is not a violent murderer.

For this, we take the probability of a murder in the general population, i.e. $P(M \mid S^*, V) = 0.00005$.

So, we have

$$P(S \mid M, V) = \frac{1 \times 0.0004}{1 \times 0.0004 + 5 \times 10^{-5} \times 0.9996} = 0.8889$$

$$P(S \mid M, V) = 0.8889 \approx 0.89$$

Hence $P(S \mid M, V) = 0.89$

Hence according to the given data and by Bayesian Statistics, there is an 89% chance that O.J. Simpson murdered his wife.