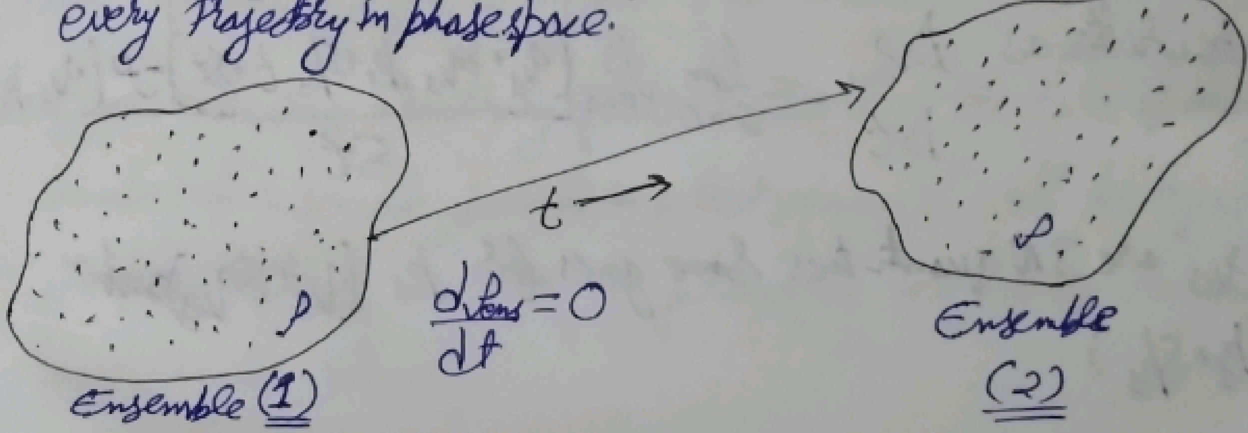
20/8/21

Assignment-1

Q1. State and prove Liouville's Theorem.

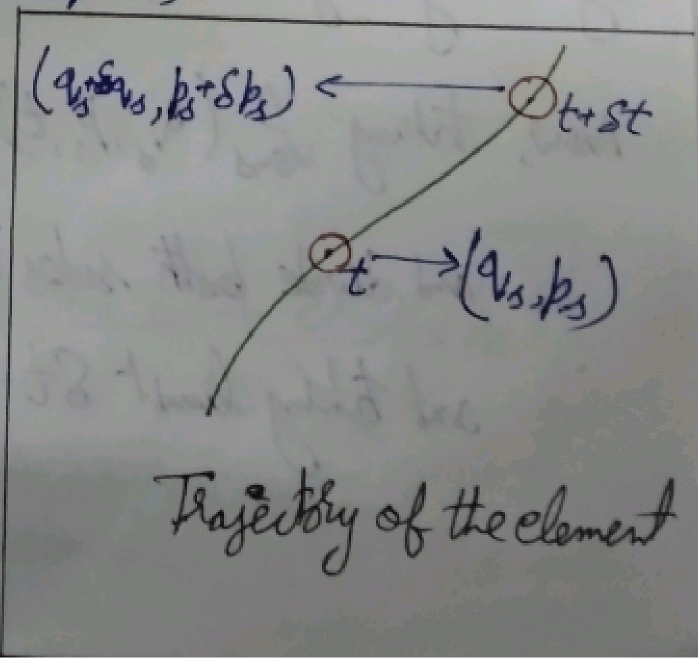
⇒ It states that, "The density of states in an ensemble of many identical states, with different initial conditions, is constant along every trajectory in phase space."



Proof – Let us say, we are in a phase space where ρ_{ens} is defined. That means, the ensemble points are very densely spaced in the phase space.

Now, Let us consider one element of the ensemble and trace its evolution with time.

If we trace its trajectory in phase space, we can calculate ρ_{ens} at every point along it.



Suppose, we take two infinitesimally close points t and $t+\delta t$

their states are given by (q_s, p_s) at t and

$(q_s+\delta q_s, p_s+\delta p_s)$ at $(t+\delta t)$

Now, we want to calculate $\frac{D\rho_{ens}}{Dt}$.

This can be written as $$\frac{D\rho_{ens}}{Dt} = \lim_{\delta\to 0} \frac{\rho_{ens}\left[q_s+\delta q_s, p_s+\delta p_s, t+\delta t\right] - \rho_{ens}(q_s, p_s, t)}{\delta t}$$

where, q_s' are $3N$ quantities, Same goes for p_s $(q_s+\delta q_s)$ and $(p_s+\delta p_s)$

$$\rho_{ens}(q_s+\delta q_s, p_s+\delta p_s, t+\delta t) = \rho_{ens}(q_s, p_s, t) + \sum_{s=1}^{3N}\left[\frac{\partial \rho_{ens}}{\partial q_s}\delta q_s + \frac{\partial \rho_{ens}}{\partial p_s}\delta p_s\right] + \frac{\partial \rho_{ens}}{\partial t}\partial t$$

by Using Taylor's expansion.

Now, taking $\rho_{ens}(q_s, p_s, t)$ to the left hand side,

and divide both sides by δt

and taking limit $\delta t \to 0$

$$\frac{D\rho_{ens}}{Dt} = \frac{\partial \rho_{ens}}{\partial t} + \sum_s \left[\frac{\partial \rho_{ens}}{\partial q_s}\dot{q}_s + \frac{\partial \rho_{ens}}{\partial p_s}\dot{p}_s\right]$$

where, $\dot{q}_s = \frac{Dq_s}{Dt}$ and $\dot{p}_s = \frac{Dp_s}{Dt}$

Now, we have to prove that this equals to zero.

If we take any orbitrary fluid element, then the rate of change of number of ensemble points on a test volume V, will be equal to the flux rate of the particles through the boundary surfaces enclosing the volume.

$$\frac{d}{dt}\int \rho_{ens}\, d\tau = -\oint \rho_{ens}\, \mathbf{u}\cdot d\mathbf{S} = -\int_\tau \nabla\cdot(\rho_{ens}\mathbf{u})\, d\tau$$

⇒ If the change of mass is negative, it should come out of the surface and vice versa.

as The surface integral, using Gauss' divergence theorem, is

Differentiating this wrt time according to Leibniz rule, we have

$$\frac{\partial \rho_{ens}}{\partial t} + \nabla\cdot(\rho_{ens}\mathbf{u}) = 0$$

The divergence term can be written as –

$$\nabla\cdot(\rho_{ens}\mathbf{u}) = \sum_s \left[\frac{\partial(\rho_{ens}\dot{q}_s)}{\partial q_s} + \frac{\partial(\rho_{ens}\dot{p}_s)}{\partial p_s}\right]$$

where $\mathbf{u}$ is the phase space velocity for 6N dimensions with point position coordinates (q_s, p_s)

Expanding the divergence, we have,

$$\frac{\partial \rho_{ens}}{\partial t} + \sum_s \left[\frac{\partial \rho_{ens}}{\partial q_s}\dot{q}_s + \frac{\partial \rho_{ens}}{\partial p_s}\dot{p}_s\right] + \sum_s \rho_{ens}\left[\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s}\right] = 0$$

This can be now rewritten as

$$\frac{D\rho_{ens}}{Dt} + \sum_s \rho_{ens}\left[\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s}\right] = 0$$

For every position and for every momentum, a Hamiltonian of a conservative system can be defined.

$$\frac{\partial H}{\partial p_s} = \dot{q}_s \quad ; \quad \frac{\partial H}{\partial q_s} = -\dot{p}_s$$

$$\frac{\partial \dot{q}_s}{\partial q_s} = \frac{\partial^2 H}{\partial p_s \partial q_s} \qquad \frac{\partial \dot{p}_s}{\partial p_s} = \frac{-\partial^2 H}{\partial p_s \partial q_s}$$

$$\Rightarrow \quad \frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s} = 0$$

$$\Rightarrow \quad \boxed{\frac{D\rho_{ens}}{Dt} = 0}$$

Hence proved.

Q2. Explain what is meant by Eulerian and Lagrangian derivatives. Show that the Lagrangian derivative of a fluid quantity $G(x, t)$ can be expressed as a sum of the Eulerian rate of change and the convective rate of change.
Also write the expressions for Lagrangian derivative in cylindrical and spherical coordinate systems.

⇒ **Eulerian Approach** - It can be defined as fluid flow as seen by an observer in the lab frame. It involves observing a spatial point or region. The properties like density, velocity are expressed as field functions of time and space.
Eulerian derivative denoted by $\partial/\partial t$ implies differentiation wrt time at a fixed point.

Lagrangian approach - When we focus on a single particle of a fluid and track its movement. It can also be defined as fluid flow as seen by an observer sitting on a fluid parcel. The properties of the particle are a function of time.

One can think of moving with a fluid element with fluid velocity v and time differentiating some quantity associated with this moving fluid element. This is called Lagrangian derivative and is denoted by d/dt

If x and $x+v\delta t$ are the positions of a fluid element at times t and $t+\delta t$, then the Lagrangian time derivative of some quantity $Q(x,t)$ can be defined as

$$\frac{dQ}{dt} = \lim_{\delta t \to 0} \frac{Q(\mathbf{x}+\mathbf{v}\delta t, t+\delta t) - Q(x,t)}{\delta t} \quad \text{—} (1)$$

Using Taylor's expansion and keeping the first order terms, we have

$$\phi(\mathbf{x}+\mathbf{v}\delta t, t+\delta t) = \phi(\mathbf{x},t) + \delta t \frac{\partial \phi}{\partial t} + \delta t\, \mathbf{v}\cdot\nabla\phi \quad \text{—} (2)$$

From (1) and (2), we have

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi$$

For a fluid quantity $G(x,t)$, the equation can be written as

$$\underbrace{\frac{dG}{dt}}_{A} = \underbrace{\frac{\partial G}{\partial t}}_{B} + \underbrace{\mathbf{v}\cdot\nabla G}_{C}$$

where

A = Lagrangian derivative, the total rate of change in quantity G felt by Lagrangian observer.

B = Eulerian derivative

C = Convective change term or Convective rate of change.

Fluid is incompressible if its density remains constant or nearly constant as a function of time and distance.

Flow: It is incompressible if the total derivative of ρ with time is constant.

The fundamental relationship b/w Lagrangian and Eulerian time derivatives can also be written as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} \equiv \frac{\partial}{\partial t} + (u\cdot\nabla).$$

or in its cartesian components as

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and $$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In the cylindrical coordinates, (r, θ, z) with velocities (u_r, u_θ, u_z)

the Lagrangian derivative becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

In the spherical coordinates, (r, θ, ϕ) with velocities (u_r, u_θ, u_ϕ)

the Lagrangian derivative can be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + u_\theta\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Q3. Find the equations of the pathlines for a fluid flow with velocity field (23)
$V = ay\hat{i} + bt\hat{j}$ where a and b are +ve constants.
Sketch the pathlines for the fluid particles which pass through the points
$(X, 0)$ at time $t = 0$. for $X = -1, 0, 1, 2, 3$.

$\Rightarrow$ The vector differential equation for the pathline of a particle is

$$\frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\Rightarrow \frac{dx}{dt} = ay. \qquad \frac{dy}{dt} = bt$$

Integrating $\frac{dy}{dt}$, we have $y = \frac{1}{2}bt^2 + C_1$

Using this value, we have $\frac{dx}{dt} = \frac{abt^2}{2} + aC_1$

Integrating $\Rightarrow$ $x = \frac{abt^3}{6} + aC_1 t + C_2$.

At $t = 0$, $x = X$ and $y = 0$

$\Rightarrow$ we have $x = \frac{1}{6}abt^3 + X$; $y = \frac{1}{2}bt^2$

Eliminating t from these, we have

$$(x - X)^2 = \frac{2a^2}{9b}y^3$$

or, $y^3 = \frac{9b}{2a^2}(x - X)^2$

for $x=-1$ $\quad y^3 = \frac{9b}{2a^2}(x+1)^2$

for $x=0$ $\quad y^3 = \frac{9b}{2a^2}(x)^2$

for $x=1$ $\quad y^3 = \frac{9b}{2a^2}(x-1)^2$

for $x=2$ $\quad y^3 = \frac{9b}{2a^2}(x-2)^2$

for $x=3$ $\quad y^3 = \frac{9b}{2a^2}(x-3)^2$

Figures attached-

① Figure 1 considers $b/2a^2$ to be equal to 1. and plots the pathlines

② Figure 2 considers $\frac{b}{a^2} = c$, another constant.

③ Figure 3 considers $\frac{b}{a^2}$ as it is and varies both of them.

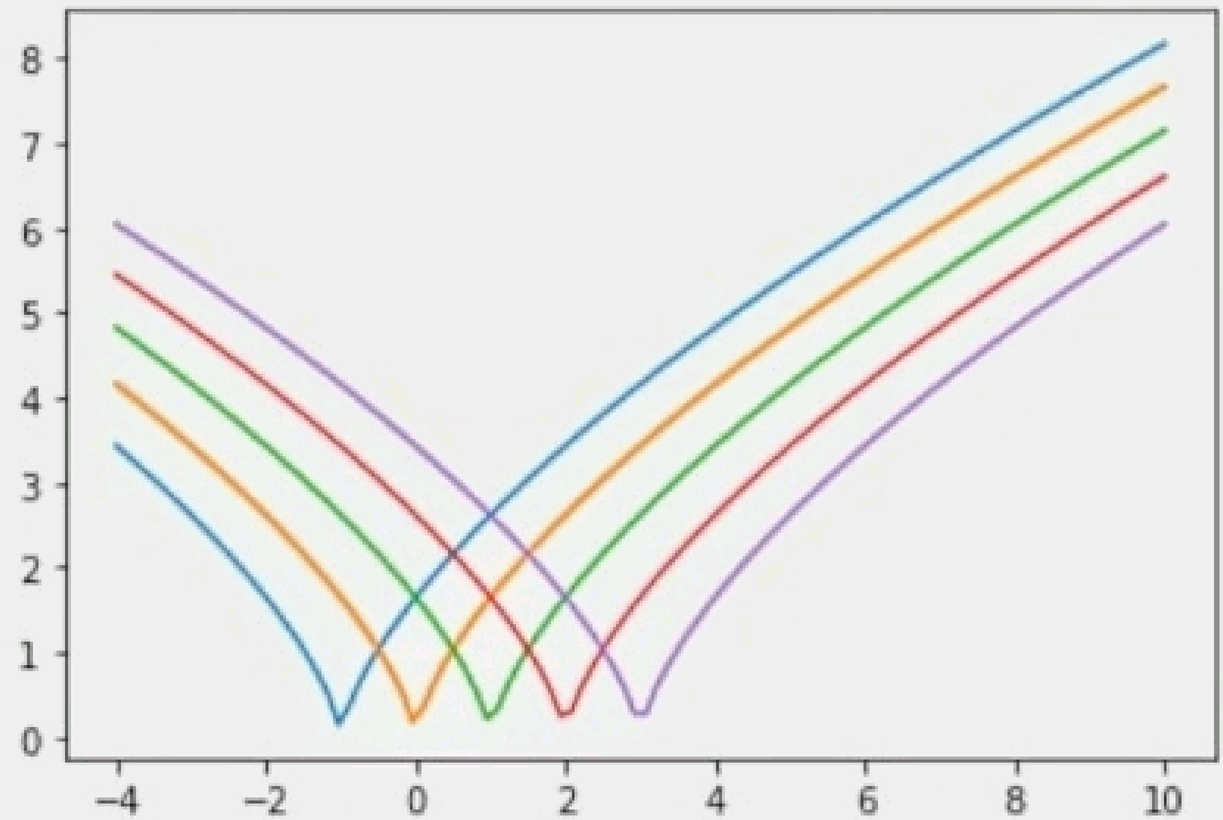
Figure 1

Taking b/((a^2)) as equal to 1

X takes given values of -1,0,1,2,3

[32]
```python
import numpy as np
import matplotlib.pyplot as plt
x = np. linspace(-4,10,100)
for X in range(-1,4,1):
    y = ((9*(x-X)**2)/2)**(1/3)
    plt.plot(x,y)

plt.show()
```



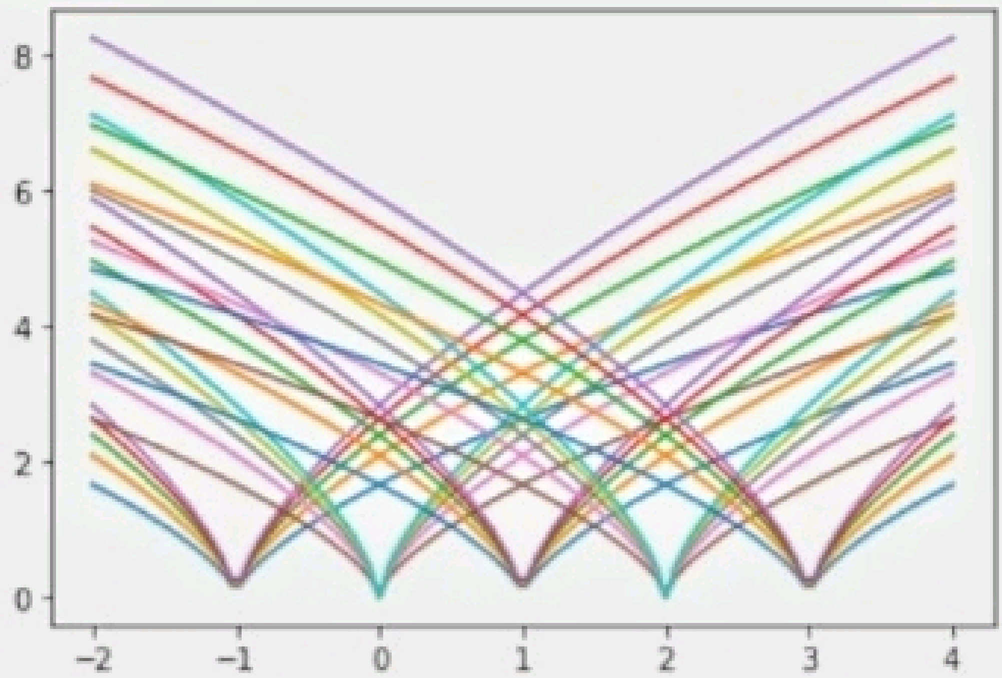
Taking b/((a^2)) as equal to a new constant c and varying it from 1 to 5

X takes given values of -1,0,1,2,3

Figure 2

```
import numpy as np
import matplotlib.pyplot as plt
x = np. linspace(-2,4,100)
for X in range(-1,4,1):
    for c in range(1,6,1):
        y = ((9*c*(x-X)**2)/2)**(1/3)
        plt.plot(x,y)

plt.show()
```



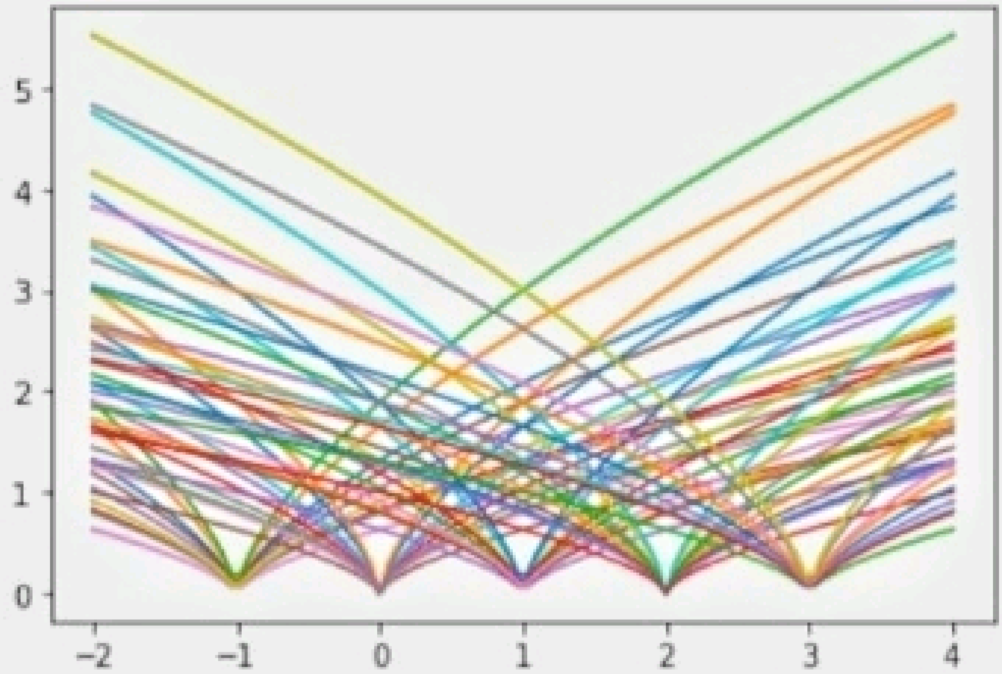
Taking b/((a^2)) as it is and varying both a and b from 1 to 3

X takes given values of -1,0,1,2,3

Figure 3

```
[34] import numpy as np
     import matplotlib.pyplot as plt
     x = np. linspace(-2,4,100)
     for X in range(-1,4,1):
         for a in range(1,4,1):
             for b in range(1,4,1):
                 y = ((9*(b/(2*(a**2)))*(x-X)**2)/2)**(1/3)
                 plt.plot(x,y)

     plt.show()
```



Q4. A velocity field in a plane flow is given by $V = 2yt\hat{i} + x\hat{j}$. Find the equation of the streamline passing through (4,2) at $t=4$.

$\Rightarrow$ For the given $V = 2yt\hat{i} + x\hat{j}$

$v_x = 2yt$
$v_y = x$

(Unsteady flow as time dependent)

$$\Rightarrow \frac{dy}{dx} = \frac{v_y}{v_x}$$

$$\frac{dy}{dx} = \frac{x}{2yt}$$

$$\Rightarrow 2yt\,dy = x\,dx.$$

Integrating both sides, we have

$$2t\frac{y^2}{2} = \frac{x^2}{2} + C$$

or $ty^2 = \frac{x^2}{2} + C$

at $t=4$, $(x,y) = (4,2)$. Using these conditions, we have

$$\underline{C = 8}$$

$\Rightarrow$ The eq^n of the streamline becomes $y^2 = \dfrac{x^2 + 16}{2t}$

at time $t = 4$.

Q5. Discuss the validity of fluid approximation. Based on these criteria, show whether the fluid approximation is valid in

(a) Core of a sun like star.

(b) Solar Corona

(c) Earth's magnetosphere

(d) ISM Molecular clouds.

for typical physical parameters of such systems.
Assume $\sigma = 10^{-4}(T/K)^{-2}\,\text{cm}^2$ and mean free path, $\lambda = 1/n\sigma$

$\Rightarrow$ Fluid approximation involves considering elements of fluid of sizes which are:

→ Small enough to keep the numbers large.

→ Large enough to contain sufficient no of particles and avoid discreteness noise

→ and large enough so that the constituent particles know about local conditions through collisions.

This approach is only valid where the mean flight time $\langle T \rangle$ of microscopic particles is not comparable to the characteristic timescale. As/If they become comparable, this approach does not remain valid.

Alternatively, in cases when the mean free path is very very small as compared to the characteristic length scale of the system, this approach is valid. ie, $L_{scale} \gg \lambda = \frac{1}{n\sigma}$ should be true.

(a) <u>Core of a sun like star-</u>

$$n = \frac{10^{2}}{10^{-24}} = 10^{26}\ cm^{-3}$$

$$\sigma = 10^{-4}(10^{7})^{-2}\ cm^{2} = 10^{-18}\ cm^{2}$$

$$\Rightarrow \lambda = \frac{1}{n\sigma} = \frac{1}{10^{-18}\times 10^{26}} = 10^{-8}\ cm.$$

$$L_{scale} = 0.05\, R_{sun} = 0.05 \times 6.96 \times 10^{10}\ cm = 0.348 \times 10^{10}\ cm.$$

as $L_{scale} \ggg \lambda$

$\Rightarrow$ Fluid approximation is valid.

(b) <u>Solar Corona-</u>

$$n = \frac{10^{-15}}{10^{-24}} = 10^{9}\ cm^{-3}.$$

$$\sigma = 10^{-4}(10^{6})^{-2}\ cm^{2} = 10^{-16}\ cm^{2}$$

$$\Rightarrow \lambda = \frac{1}{n\sigma} = \frac{1}{(10^{9})(10^{-16})} = 10^{7}\ cm = 10^{5}\ m.$$

$$L_{scale} = 10^{7}\ m.$$

as $L_{scale} \gg \lambda$

$\Rightarrow$ Fluid approximation is valid.

(c) ISM molecular clouds – $n = 10^{3}\,cm^{-3}$

~~Earth's magnetosphere.~~ $\sigma = 10^{-4}(10)^{-2} = 10^{-6}\,cm^{2}$

$\Rightarrow \lambda = \frac{1}{n\sigma} = \frac{1}{10^{3} \times 10^{-6}} = 10^{3}\,cm$

$L_{scale} = 2.4 \times 10^{20}\,cm$

as $L_{scale} \gg \lambda$

$\Rightarrow$ Fluid approximation is valid.

(d) ISM ionised medium – $n = 10^{-3}\,cm^{-3}$

$\sigma = 10^{-4}(10^{6})^{-2} = 10^{-16}\,cm^{2}$

$\Rightarrow \lambda = \frac{1}{n\sigma} = \frac{1}{10^{-3} \times 10^{-16}} = 10^{19}\,cm \approx 3pc$

$L_{scale} = 1000 - 3000\,pc$

as $L_{scale} \gg \lambda$

$\Rightarrow$ Fluid approximation is valid.

26. Find the equations of streamlines and pathlines for a fluid flow with velocity field $V = \left(\frac{m}{r}\right)\hat{r}$ where m is a positive constant and $r \neq 0$. Plot the streamlines and the pathlines and discuss the conditions under which the two will be identical. Also, find the pathline of the particle (fluid parcel) which passes through the point $(r=1, \theta=\pi/4)$ at $t=0$. Does the particle speed up or slow down with time?

⇒ Let us start with pathlines,

The polar form of the diff. eq^n for the pathlines is

$$\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = u = \left(\frac{m}{r}\right)\hat{r}$$

$$\Rightarrow \frac{dr}{dt} = \frac{m}{r}, \quad r\frac{d\theta}{dt} = 0 \qquad (r \neq 0).$$

$$\Rightarrow r\frac{dr}{dt} = m \quad \text{and} \quad \frac{d\theta}{dt} = 0$$

Integrating, we have

$$r^2 = 2mt + C_1 \quad \text{and} \quad \theta = C_2$$

Pathline of the particle passing through $(r=1, \theta=\pi/4)$ at $t=0$

$$\Rightarrow C_1 = 1 \quad \text{and} \quad C_2 = \pi/4$$

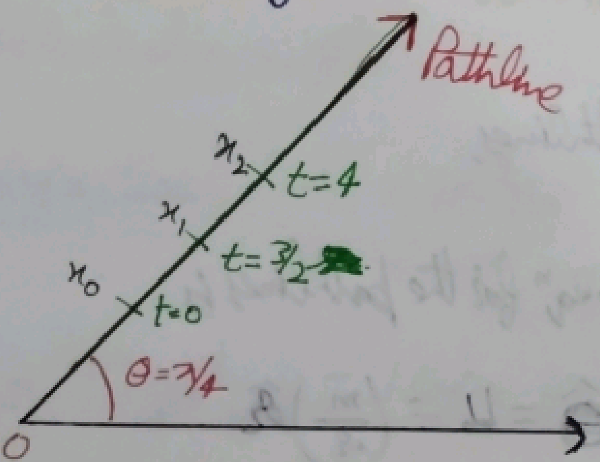
$$\Rightarrow r^2 = 2mt + 1$$

As θ is constant, we don't need to eliminate t.

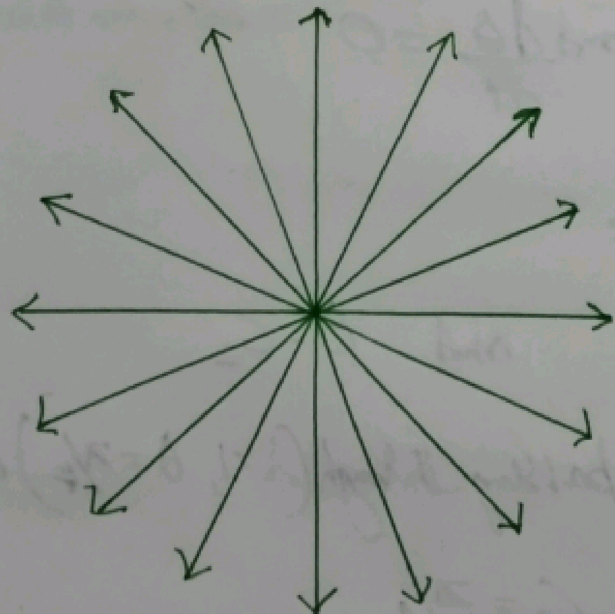
The direction of flow along the pathline is outwards, at a constant angle $\pi/4$.

As time goes by, the particle will slow down.

The pathline followed by the particle is



This equation can be further used as general equation of pathline of the fluid if we vary θ. In this case, the flow pathline for the fluid can be represented as.



Now, for streamlines, we have.

$$r\frac{d\theta}{dr} = \frac{u_\theta}{u_r} = \frac{0}{m/r} = 0.$$

$$\Rightarrow d\theta = 0$$

$$\Rightarrow \theta = \text{constant}.$$

⇒ The streamlines are rays from the origin and are identical to the above figure of the pathlines for the fluid.

The plot for streamlines/pathlines for the particle

(I tried to plot the line but couldn't so I just added this as it does show the points)

[36]

```
import numpy as np
import matplotlib.pyplot as plt

plt.axes(projection='polar')



rads = np.arange(0, (1/2)* np.pi,1)
r = np.sqrt(2*m*t+1)

for rad in rads:

  for t in range(0,10,100):
    for m in range(1,100,100):
      rads = np.arange(r, (1/2)* np.pi,1)
      plt.polar(rad, r)

# display the polar plot
plt.show()
```