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Assignment - 1

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Q1. Given: the state $|\alpha\rangle$ has a mean momentum operator $\langle P\rangle$

To prove: $e^{\frac{ip_0\hat{x}}{\hbar}}|\alpha\rangle$ has the mean momentum $\langle P\rangle + p_0$

Ans The mean momentum operator $\langle P\rangle$ can be defined as

$$\langle\alpha|\hat{P}|\alpha\rangle = \langle P\rangle \text{ ——— (eq}^n\text{)}$$

Now, for the given state, $e^{\frac{ip_0\hat{x}}{\hbar}}|\alpha\rangle$, we can define the momentum operator as

$$\langle\alpha|e^{\frac{-ip_0\hat{x}}{\hbar}}\ \hat{P}\ e^{\frac{ip_0\hat{x}}{\hbar}}|\alpha\rangle$$

This can be written as,

$$\langle\alpha|e^{\frac{-ip_0\hat{x}}{\hbar}}\left\{e^{\frac{ip_0\hat{x}}{\hbar}}\hat{P} + p_0\cdot e^{\frac{ip_0\hat{x}}{\hbar}}\right\}|\alpha\rangle$$

$$= \langle\alpha|e^{\frac{-ip_0\hat{x}}{\hbar}}\cdot e^{\frac{ip_0\hat{x}}{\hbar}}\cdot\hat{P}|\alpha\rangle + \langle\alpha|e^{\frac{-ip_0\hat{x}}{\hbar}}\cdot p_0\cdot e^{\frac{ip_0\hat{x}}{\hbar}}|\alpha\rangle$$

Now, as $e^{\frac{-ip_0\hat{x}}{\hbar}}\cdot e^{\frac{ip_0\hat{x}}{\hbar}} = e^0$, The above equation reduces to

$$= \langle\alpha|\hat{P}|\alpha\rangle + p_0\langle\alpha|\alpha\rangle$$

$$= \langle P\rangle + p_0 \quad (\because \langle\alpha|\alpha\rangle = 1)$$

Hence proved.

Q2. $L_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$L_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$

$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) If we take $\lambda_1, \lambda_2, \lambda_3$ to be the eigenvalues,

then $\text{Trace}(L_z) = \lambda_1 + \lambda_2 + \lambda_3 = 0$ (Trace of matrix)

$\text{Det}(L_z) = \lambda_1 \lambda_2 \lambda_3 = 0$ (Determinant)

The possible values, we may obtain for measuring L_z are -1, 0 and 1

[As the other values like $(-2,0,2)$, $(-3,0,3)$ all can be written as $a(-1,0,1)$ as well. Thus, this is the most basic.]

b) Let us take the eigenkets of L_z for the different values of λ -

for $\lambda = 1$, let the eigenket be $|1\rangle$

$\lambda = 0$, " " " " $|0\rangle$

$\lambda = -1$, " " " " $|-\rangle$

Now, for $\lambda = 1$, $|1\rangle$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -z \end{bmatrix} = 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, for $|0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and $|-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now, we may represent the three observables in terms of these eigenkets.

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\Big[|0\rangle\langle 1| + |-1\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle -1|\Big]$$

$$L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{i}{\sqrt{2}}\Big[-1\,|0\rangle\langle 1| - 1\,|-1\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle -1|\Big]$$

$$L_z = |1\rangle\langle 1| - |-1\rangle\langle -1|$$

(c) For the value of $\lambda = -1$,

the eigenket is $|-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1. \end{bmatrix}$

Now, we have to calculate $\langle L_x \rangle$ and $\langle L_x^2 \rangle$.

$\langle L_x \rangle$ can be written as $\langle -1|L_x|-1\rangle$

$$\langle -1|L_x|-1\rangle = \frac{1}{\sqrt{2}}\langle -1|\Big[|-1\rangle\langle 0| + |0\rangle\langle -1| + |0\rangle\langle 1| + |1\rangle\langle 0|\Big]|-1\rangle$$

$$= \frac{1}{\sqrt{2}}\Big[\langle -1|-1\rangle\langle 0| + \langle -1|0\rangle\langle -1| + \langle -1|0\rangle\langle 1| + \langle -1|1\rangle\langle 0|\Big]|-1\rangle$$

$$= \frac{1}{\sqrt{2}}\Big[\langle 0|-1\rangle + \langle 0|-1\rangle\Big] = 0 = \langle L_x \rangle$$

Now,

L_x^2 can be written as $L_x \cdot L_x$.

Thus, The expectation value of L_x^2, $\langle L_x^2 \rangle$ can be expressed as $\langle L_x^2 \rangle = \langle -1|L_x^2|-1\rangle$

$$\langle L_x^2\rangle = \frac{1}{2}\langle -1|\left\{L_x \cdot L_x\right\}|-1\rangle$$

$$= \frac{1}{2}\langle -1|\left\{\begin{matrix}(|-1\rangle\langle 0| + |0\rangle\langle -1| + |0\rangle\langle 1| + |1\rangle\langle 0|)\times \\ (|-1\rangle\langle 0| + |0\rangle\langle -1| + |0\rangle\langle 1| + |1\rangle\langle 0|)\end{matrix}\right\}|-1\rangle$$

Solving, we have

$$= \frac{1}{2}\left\{(\langle 0| + 0 + 0 + 0)(|0\rangle + 0 + 0 + 0)\right\}$$

$$= \frac{1}{2}(\langle 0|0\rangle) = \frac{1}{2}$$

$$\Rightarrow \boxed{\begin{matrix}\langle L_x\rangle = 0 \\ \langle L_x^2\rangle = 1/2\end{matrix}}$$

(d) For the eigen values of L_y, the eigenkets can be written as (3)

$$|-\rangle = \begin{bmatrix} 1 \\ i\sqrt{2} \\ -1 \end{bmatrix} \; ; \; |0_y\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \; ; \; |+\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{bmatrix}$$

Normalising these eigenkets we have

$$|-\rangle = \begin{bmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{bmatrix} \; ; \; |0_y\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \text{ and } |+\rangle = \begin{bmatrix} 1/2 \\ -i/\sqrt{2} \\ -1/2. \end{bmatrix}$$

In the L_z basis, these become

$$|-\rangle = \frac{-1}{2}|-1\rangle - \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{2}|+1\rangle$$

($|-1\rangle$; $|0\rangle$; $|+1\rangle$ are eigenkets of L_z)

$$|0_y\rangle = \frac{1}{\sqrt{2}}|-1\rangle + \frac{1}{\sqrt{2}}|+1\rangle$$

$$|+1\rangle = \frac{-1}{2}|-1\rangle + \frac{i}{\sqrt{2}}|0_z\rangle + \frac{1}{2}|+1\rangle$$

Q3. $\langle\alpha|S_x|\alpha\rangle = \left(\langle+|a^* + \langle-|b^*\right)\left[\frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\right]\left(a|+\rangle + b|-\rangle\right)$

$$= \left(\langle+|a^* + \langle-|b^*\right)\left[\frac{\hbar}{2}\left(a|-\rangle + b|+\rangle\right)\right]$$

$$\langle S_x\rangle = \frac{\hbar}{2}\left(a^*b + b^*a\right)$$

$$\langle S_y\rangle = \langle\alpha|S_y|\alpha\rangle = \langle\alpha|\frac{\hbar}{2}\left(i|-\rangle\langle+| - i|+\rangle\langle-|\right)|\alpha\rangle$$

$$= \frac{\hbar}{2}\left(-i\,ba^* + i\,ab^*\right) = \langle S_y\rangle$$

$$\langle S_z\rangle = \langle\alpha|\frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)|\alpha\rangle$$

$$= \langle\alpha|\frac{\hbar}{2}\left[a|+\rangle - b|-\rangle\right]$$

$$= \frac{\hbar}{2}\left[aa^* - bb^*\right] = \langle S_z\rangle$$

$\Rightarrow$ Clearly $\langle S_x\rangle \neq \langle S_y\rangle \neq \langle S_z\rangle$

Hence proved.

It is possible for some (a,b) for which at least two expectation values vanish.

If a and b are chosen such that $\underline{\underline{a=b}}$ and $\underline{\underline{a^*=b^*}}$

In this case,

$$\langle S_x\rangle = \frac{\hbar}{2}(aa^* + aa^*) = \hbar(aa^*)$$

$$\langle S_y\rangle = \frac{\hbar}{2}(-iaa^* + i\,aa^*) = \underline{\underline{0}}.$$

$$\langle S_z\rangle = \frac{\hbar}{2}(aa^* - aa^*) = \underline{\underline{0}}.$$

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Q4.

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$S_x + S_y = \begin{pmatrix} 0 & 1+i \\ -i+1 & 0 \end{pmatrix}\frac{\hbar}{2}$$

Eigenvalues of this matrix λ_1, λ_2 (say)

trace - $\lambda_1 + \lambda_2 = 0$

determinant - $\Rightarrow \lambda_1 \lambda_2 = 0 - (1+i)(1-i) = -(1-i^2) = -2$

Possible eigenvalues : $\sqrt{2}, -\sqrt{2}$

The corresponding eigenkets can be found out as

$$\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} (1+i)y \\ (1-i)x \end{pmatrix} = \begin{pmatrix} \sqrt{2}x \\ \sqrt{2}y \end{pmatrix}$$

If $y = 1$; $x = \frac{1+i}{\sqrt{2}}$

one eigenket can be $\begin{pmatrix} \frac{(1+i)}{\sqrt{2}} \\ 1 \end{pmatrix}$

The normalised eigenket is $\begin{pmatrix} \frac{1+i}{2} \\ 1/\sqrt{2} \end{pmatrix}$ and it corresponds to $\sqrt{2}$.

Similarly for $-\sqrt{2}$, $(1+i)y = -\sqrt{2}x$

$(1-i)x = -\sqrt{2}y$

If $y = 1$; $x = -\frac{1+i}{\sqrt{2}}$

hence the normalised eigenket corresponding to $-\sqrt{2}$ is $\begin{pmatrix} -\frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

They also satisfy the orthogonality condition,

$$\begin{pmatrix} -\frac{1-i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1+i}{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{-1-i^2}{4} + \frac{1}{2} = \frac{-1}{2} + \frac{1}{2} = 0.$$

So, the eigenvalues for $S_x + S_y$ are $\frac{\hbar}{\sqrt{2}}$ and $\frac{-\hbar}{\sqrt{2}}$

and the corresponding eigenvectors are

$$\begin{pmatrix} \frac{1+i}{2} \\ 1/\sqrt{2} \end{pmatrix} \text{ and } \begin{pmatrix} -1+i/2 \\ 1/\sqrt{2} \end{pmatrix}$$

It is given that on measurement of $S_x + S_y$ operator, the system was found in the state $\begin{pmatrix} \frac{1+i}{2} \\ 1/\sqrt{2} \end{pmatrix}$

Eigenvectors of S_z are: for $\hbar/2 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $-\frac{\hbar}{2} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Expressing in three eigenvectors

$$\begin{pmatrix} \frac{1+i}{2} \\ 1/\sqrt{2} \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow A = \frac{1+i}{2} ; B = \frac{1}{\sqrt{2}}.$$

Probability that measurement of S_z yields a spin value $+\hbar/2$ is given by $|A|^2 = \frac{1^2+1^2}{4} = \frac{2}{4} = \frac{1}{2}$

$\frac{1}{2}$ is as expected since on measurement of S_z the system forgets about being in a state of

$|S_x + S_y ; +\rangle$, and so, $|S_z +\rangle$ and $|S_z -\rangle$ are equally probable.