Assignment 2

1. The Monty Hall Problem

1. Solve the 'Monty Hall' problem given in the lectures, using Bayes' theorem.

2. The slope of the number counts of radio sources.

2. The distribution of flux densities of extragalactic radio sources are distributed as a power-law with slope $-\alpha$, say. In a non-evolving Euclidean universe $\alpha = 3/2$ (can you prove this?) and departure of α from the value 3/2 is evidence for cosmological evolution of radio sources. This was the most telling argument against the steady-state cosmology in the early 1960s (even though they got the value of α wrong by quite a long way).

Given error-free observations of radio sources with flux densities $S_i; i = 1, \ldots, n$ above a known, fixed measurement limit S_0, what is the posterior for α? What is the MAP (maximum a posteriori) value of α?

If a single source is observed with flux $S_1 = 2S_0$, what is the most probable value of α?

Can you deduce the uncertainty on α?

Hints:
1. You will use the distribution as a probability distribution function (pdf), $p(S)$ and will have to normalise it.
2. A pdf is not a probability, but $p(S)\,\Delta S$ is, for a (small) interval ΔS. You will need to introduce an arbitrary (but small) interval around each source.

3. You are the barrister

3. A known petty thief was lodging in a house when a theft of a piece of cheese takes place. The defence lawyer has argued that only 1/2500 of known thieves (T) who are in lodgings steal cheese from their hosts, *so the information that he is a known thief is irrelevant and must be ignored.* $p(S|T) = 0.0004$. You are the prosecution barrister, and you are fairly sure that the thief has stolen the cheese (this possible event we call S). How do you counter this argument? You are in possession of the knowledge that the probability of a theft of cheese (C) from lodgings is 1/20000.

What is the real probability, on the basis only of the information supplied here, that the lodger thief was the culprit? It is not 0.0004... Hint: what piece of information has the defence lawyer ignored (probably deliberately)?

4. The Lighthouse Problem

4. This problem was set by Steve Gull to first year Cambridge students many years ago. It contrasts the Bayesian approach with an estimator-based approach.

A lighthouse is situated at unknown coordinates (x_0, y_0) with respect to a straight coastline $y = 0$. It sends a series of N flashes in random directions, and these are recorded on the coastline at positions x_i; $i = 1 \ldots N$. Only the positions of the arrivals of the flashes, not the directions, nor the intensities, are recorded. Using a Bayesian approach, find the posterior distribution of x_0, y_0.

Now focus only on the unknown x_0. Define a suitable estimator, $\hat{x}$, for x_0 from the observed x_i. Work out the probability distribution for $\hat{x}$. You may need to refer to a proof of the Central Limit Theorem, for the pdf of repeated trials of the same experiment. You may also find this useful:

$$\int_{-\infty}^{\infty} e^{ikx} \frac{1}{\left[1 + \frac{(x-x_0)^2}{y_0^2}\right]} \, dx = e^{ikx_0 - |k| y_0}$$

Comment.

You can get some extra points if you simulate this process, compute the posterior distribution, and also show the estimator.