Assignment 2

Q1. (a) $A^{\mu} = (A^0, A^1, A^2, A^3)$

$$A_{\alpha} = g_{\mu\alpha} A^{\mu}$$

$$A_0 = g_{r0} A^r = g_{00} A^0 = A^0 \quad (\because \text{only diagonal terms are non zero.})$$

$$A_1 = g_{11} A^1 = -A^1$$

$$A_2 = g_{22} A^2 = -r^2 A^2$$

$$A_3 = g_{33} A^3 = -r^2 \sin^2\theta\, A^3$$

$$A_{\alpha} = (A^0, -A^1, -r^2 A^2, -r^2 \sin^2\theta\, A^3)$$

(b) $T^{\mu\nu} = g^{\beta\nu} g^{\alpha\mu} T_{\alpha\beta}$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{bmatrix}$$

$$\therefore T^{00} = g^{00} g^{00} T_{00} = T_{00}$$

$$T^{01} = g^{11} g^{00} T_{01} = -T_{01}$$

$$T^{02} = g^{22} g^{00} T_{02} = -\frac{1}{r^2} T_{02}$$

$$T^{03} = g^{33} g^{00} T_{03} = -\frac{1}{r^2 \sin^2\theta} T_{03}$$

$$T^{10} = g^{00} g^{11} T_{10} = -T_{10}$$

$$T^{11} = g^{11} g^{11} T_{11} = T_{11}$$

$$T^{12} = g^{22} g^{11} T_{12} = \frac{1}{r^2} T_{12}$$

$$T^{13} = g^{33} g^{11} T_{13} = \frac{1}{r^2 \sin^2\theta} T_{13}$$

$$T^{20} = g^{00} g^{22} T_{20} = \frac{-1}{r^2} T_{20}$$

$$T^{21} = g^{11} g^{22} T_{21} = \frac{1}{r^2} T_{21}$$

$$T^{22} = g^{22} g^{22} T_{22} = \frac{1}{r^4} T_{22}$$

$$T^{23} = g^{33} g^{22} T_{23} = \frac{1}{r^4 \sin^2\theta} T_{23}$$

$$T^{30} = g^{00} g^{33} T_{30} = \frac{-1}{r^2 \sin^2\theta} T_{30}$$

$$T^{31} = g^{11} g^{33} T_{31} = \frac{1}{r^2 \sin^2\theta} T_{31}$$

$$T^{32} = g^{22} g^{33} T_{32} = \frac{1}{r^4 \sin^2\theta} T_{32}$$

$$T^{33} = g^{33} g^{33} T_{33} = \frac{1}{r^4 \sin^4\theta} T_{33}$$

$$\Rightarrow T^{\mu\nu} = \begin{bmatrix} 1 & -1 & -1/r^2 & -1/r^2\sin^2\theta \\ -1 & 1 & 1/r^2 & 1/r^2\sin^2\theta \\ -1/r^2 & 1/r^2 & 1/r^4 & 1/r^4\sin^2\theta \\ -1/r^2\sin^2\theta & 1/r^2\sin^2\theta & 1/r^4\sin^2\theta & 1/r^4\sin^4\theta \end{bmatrix} T_{\alpha\beta}$$

(c) $ds^2 = a^2 d\theta^2 + \sin^2\theta \, d\phi^2$ where $a = 1$

$$\Rightarrow ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

For two sphere, $\Gamma^{\theta}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{bmatrix}$

$$\Gamma^{\phi}_{ij} = \begin{bmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{bmatrix}$$

The parallel transport eq^n is given as

$$\frac{dA^\mu}{d\lambda} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\mu}{d\lambda} A^\beta = 0$$

$A_\theta = e_\theta$ at $\phi = 0, \theta = \theta_0$

for $\mu = 1$ (i.e, $\mu = \theta$)

$$\frac{dA^\theta}{d\lambda} + \Gamma^\theta_{\phi\phi} \frac{\partial x^\phi}{d\lambda} A^\phi = 0$$

$$\therefore \frac{dA^\theta}{d\lambda} - \sin\theta_0 \cos\theta_0 \frac{d\phi}{d\lambda} A^\phi = 0 \quad \text{——— (1)}$$

for $\mu = 2$, (i.e, $\mu = \phi$)

$$\frac{dA^\phi}{d\lambda} + \Gamma^\phi_{\phi\theta} \frac{dx^\phi}{d\lambda} A^\theta + \Gamma^\phi_{\theta\phi} \frac{dx^\theta}{d\lambda} A^\phi = 0$$

As the motion is along $\theta = \theta_0$ (constant), we have

$$\frac{dx^\theta}{d\lambda} = 0$$

$$\frac{dA^\phi}{d\lambda} + \cot\theta_0 \frac{d\phi}{d\lambda} A^\theta = 0$$

$$\frac{dA^\phi}{d\phi} \cdot \frac{d\phi}{d\lambda} + \cot\theta_0 \frac{d\phi}{d\lambda} A^\theta = 0$$

$$\frac{dA^\phi}{d\phi} + \cot\theta_0 A^\theta = 0 \quad \text{——— (2)}$$

(1) can be written as

$$\frac{dA^\theta}{d\phi} - \sin\theta_0 \cos\theta_0 A^\phi = 0 \quad \text{——— (3)}$$

Decoupling (2) and (3) by differentiation, we have

from 2 -
$$\frac{d^2A^\phi}{d\phi^2} - \sin\theta_0\cos\theta_0\left(-\cot\theta_0 A^\phi\right) = 0$$

$$\frac{d^2A^\phi}{d\phi^2} + \cos^2\theta_0 A^\phi = 0 \quad \text{——— (4)}$$

from 3 -
$$\frac{d^2A^\theta}{d\phi^2} + \cot\theta_0\left(\sin\theta_0\cos\theta_0 A^\theta\right) = 0$$

$$\frac{d^2A^\theta}{d\phi^2} + \cos^2\theta_0 A^\theta = 0 \quad \text{——— (5)}$$

The general solution for (4) and (5), are

$$A^\phi = c_1\cos(\phi\cos\theta_0) + c_2\sin(\phi\cos\theta_0) \quad \text{——— (6)}$$

$$A^\theta = c_3\cos(\phi\cos\theta_0) + c_4\sin(\phi\cos\theta_0) \quad \text{——— (7)}$$

Now, we know that, $A_\theta = c_\theta$ at $\phi = 0$

i.e $A^\theta = g^{\theta\theta}A_\theta$, $A^\phi = g^{\theta\phi}A_\theta$

We have, $g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{bmatrix}$$

$\Rightarrow$ $A^\theta = A_\theta = c_\theta$, $A^\phi = 0$

Using this in (6) and (7), we have,

$c_1 = 0, c_3 = 1$

Also $\left.\frac{dA^\theta}{d\phi}\right|_{\phi=0} - \sin\theta_0 \cos\theta_0 A^\phi = 0$

$\Rightarrow \left.\frac{dA^\theta}{d\phi}\right|_{\phi=0} = 0$

and $\left.\frac{dA^\phi}{d\phi}\right|_{\phi=0} = -\cot\theta_0$

Using (6), $\frac{dA^\phi}{d\phi} = \left[-c_1 \sin(\phi\cos\theta_0) + c_2\cos(\phi\cos\theta_0)\right]\cos\theta_0$

At $\phi = 0$, $c_4 = 0$

$A^\theta = \cos(\phi\cos\theta_0)$

$A^\phi = \frac{-\sin(\phi\cos\theta_0)}{\sin\theta_0}$

After parallel transport, $\phi = 2\Omega$

$A = (A^\theta, A^\phi) = \left(\cos(2\Omega\cos\theta_0), \frac{-\sin(\phi\cos\theta_0)}{\sin\theta_0}\right)$

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$$\sqrt{A\cdot A} = \sqrt{g_{\mu\nu} A^{\mu} A^{\nu}}$$

$$= \sqrt{g_{\theta\theta} A^{\theta} A^{\theta} + g_{\phi\phi} A^{\phi} A^{\phi}}$$

$$= \sqrt{\cos^2(2\Omega\cos\theta_0) + \frac{\sin^2\theta \sin^2(2\Omega\cos\theta_0)}{\sin^2\theta_0}}$$

$$= \sqrt{\cos^2(2\Omega\cos\theta_0) + \sin^2(2\Omega\cos\theta_0)}$$

$$= \underline{\underline{1}}$$ Answer

Q4. (a) $\gamma_{SR} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

For general relativity, velocity can be reduced to the gravitational potential.

$$\gamma_{GR} = \frac{1}{\sqrt{1-\frac{2\phi}{c^2}}} = \frac{1}{\sqrt{1+\frac{2GM}{c^2 r}}}$$

(b) $T^2 = \frac{4\pi^2}{GM} r^3$

$$\Rightarrow r^3 = \frac{T^2 GM}{4\pi^2} = \frac{(12\times60\times60)(6.67\times10^{-11}\,N\,m^2/kg^2)(5.98\times10^{24}\,kg)}{4\pi^2}$$

$$r^3 = 1.88\times10^{22}\,m^3 \Rightarrow \boxed{r = 2.659\times10^{7}\,m}$$

$$V = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.67\times10^{-11})(5.98\times10^{24})}{2.59\times10^{7}}}$$

$$= \sqrt{15.4\times10^{6}}$$

$$= \boxed{3.92\times10^{3}\ m/s}$$

$r = 2.59\times10^{7}\ m$

$V = 3.92\times10^{3}\ m/s.$

Also, $x = r - R_e = (2.59\times10^{7} - 6.38\times10^{6})\ m$

($\because R_e = 6.38\times10^{6}\ m$ earth radius)

$x = 19.52\times10^{6}\ m$

$$\boxed{x = 1.952\times10^{7}\ m}$$

Here x is the distance from the surface of the earth

(c) $\gamma_{SR} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{15.4\times10^{6}}{9\times10^{16}}}} = 0.9999999992$ — ①

(d) $\gamma_{GR} = \frac{1}{\sqrt{1+\frac{2GM}{c^2 r}}} = \frac{1}{\sqrt{1+\frac{2\times6.67\times10^{-11}\times5.98\times10^{24}}{9\times10^{16}\times2.59\times10^{7}}}} = 1.0000000017$ — ②

(e) For SR, $\frac{\Delta t'}{\Delta t} = \sqrt{1-\frac{v^2}{c^2}} = 1 - \frac{v^2}{2c^2}$ (for $v \ll c$)

Thus for 1 sec of proper time, moving clock loses $\frac{v^2}{2c^2}$.

$$\frac{\Delta t'}{\tau} = \frac{v^2}{2c^2}$$

For GR, the velocity is replaced by gravitational potential, which is negative.

$$\frac{\Delta t}{\tau} = -\frac{GM}{c^2 r}$$

The -ive sign indicates that the time is gained by the clock at high altitudes. The gain of time for 1 second on earth, the clock gains $\frac{GM}{c^2 r}$.

(f) From ①, $\gamma_{SR} = \underline{0.99999999992}$

From ②, $\gamma_{GR} = \underline{1.00000000017}$

We can clearly see that $\gamma_{SR} < \gamma_{GR}$.

$\Rightarrow$ GR effect is larger than SR effect.

(g) GR effect in time precision -

For 1 sec, GR effect will be $= \frac{GM}{c^2 r} = 1.66 \times 10^{-10}$ sec. $= \underline{t_1}$

1 min, $= t_1 \times 60 = 1.00002 \times 10^{-8}$ sec $= \underline{t_2}$

1 hour, $= t_2 \times 60 = 6.00012 \times 10^{-7}$ sec $= \underline{t_3}$

1 day, $= t_3 \times 24 = 1.4400288 \times 10^{-5}$ sec $= \underline{t_4}$

1 week, $= t_4 \times 7 = 1.00802 \times 10^{-4}$ sec $= \underline{t_5}$

1 year, $= t_4 \times 365 = 5.256 \times 10^{-3}$ sec $= \underline{t_6}$

(h) GR effect in position accuracy -

For 1 sec $= c \times t_1 = 4.98 \times 10^{-2}\ \text{m}$

1 min $= c \times t_2 = 3.00006 \times 10^{0}\ \text{m}$

1 hour $= c \times t_3 = 180.036 \times 10^{0}\ \text{m} = 1.8036 \times 10^{2}\ \text{m}$

1 day $= c \times t_4 = 4320.0864 \times 10^{0}\ \text{m} = 4.32 \times 10^{3}\ \text{m}$

1 week $= c \times t_5 = 30240 \times 10^{0}\ \text{m} = 3.0240 \times 10^{4}\ \text{m}$

1 year $= c \times t_6 = 1576800 \times 10^{0}\ \text{m} = 1.5768 \times 10^{6}\ \text{m}$

Duration	GR effect in Time precision	GR effect in Position accuracy
1 sec	1.66×10^{-10} sec	4.98×10^{-2} m
1 min	1.00002×10^{-8} sec	3.00006 m
1 hour	6.00012×10^{-7} sec	1.8036×10^{2} m
1 day	1.4400288×10^{-5} sec	4.32×10^{3} m
1 week	1.00802×10^{-4} sec	3.0240×10^{4} m
1 year	5.256×10^{-3} sec	1.5768×10^{6} m

Q2 (b) The Schwarzchild metric is given by

$$ds^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 - \left(1+\frac{2GM}{c^2r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

The metric is independent of t and ϕ $(\alpha = 0, 3)$

$$g_{\alpha\nu}\frac{dx^\nu}{d\tau} = g_{00}\frac{dx^0}{d\tau} \qquad \left[g_{00}\frac{dx^0}{d\tau} = \text{constant}\right]$$

$$\therefore K_1 = c^2\left(1-\frac{2GM}{c^2r}\right)\dot{t}$$

For $\alpha = 3$, $g_{3\nu}\frac{dx^0}{d\tau} = g_{33}\frac{dx^3}{d\tau} = K_2$

$$\Rightarrow K_2 = -r^2\sin^2\theta\,\dot{\phi}$$

$$\Rightarrow K_1 = c^2(1-2\Phi)\dot{t} \text{ and } K_2 = -r^2\sin^2\theta\,\dot{\phi}$$

(c) $K^\mu = \frac{dx^\mu}{d\lambda}$

$$\Rightarrow K^\mu = (1, 1, 0, 0)$$

(d) Geodesic eq^n is given by $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$

$$\Rightarrow \frac{dK^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}K^\alpha K^\beta = 0.$$

$\alpha = 0, 1 ; \mu = 1$

$$\frac{dK^y}{d\lambda} + \Gamma^y_{xx}(K^x)^2 + \Gamma^y_{tt}(K^t)^2 + 2\Gamma^y_{xt}K^x K^t = 0$$

$$\Gamma^{y}_{xt} = \frac{1}{2} g^{yy}\left[\partial_t g_{yx} + \partial_x g_{ty} - \partial_y g_{xt}\right] = 0$$

$$\Gamma^{y}_{xx} = \frac{1}{2} g^{yy}\left[\partial_x g_{yx} + \partial_x g_{yx} - \partial_y g_{xx}\right] = -\frac{1}{2} g^{yy} \frac{\partial g_{xx}}{\partial y}$$

For weak field approximation, $\phi << 1$

$$\Rightarrow \frac{GM}{c^2 r} << 1$$

$$\Rightarrow g_{yy} = -1$$

So, $\Gamma^{y}_{xx} = \frac{1}{2} \frac{\partial g_{xx}}{\partial y} = \frac{1}{2} \frac{\partial g_{xx}}{\partial r} \cdot \frac{\partial r}{\partial y}$

$r = \sqrt{x^2 + y^2}$

$$\frac{dr}{dy} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\Gamma^{y}_{xx} = -\frac{1}{2} \frac{y}{r} \frac{\partial}{\partial r}(1 - 2\phi) = -\frac{1}{2} \frac{y}{r} \frac{\partial}{\partial r}\left[1 + \frac{2GM}{c^2 r}\right]$$

$$+ \frac{y}{r} \frac{GM}{c^2 r^2}$$

$$= \frac{GMy}{c^2 (x^2+y^2)^{3/2}}$$

Now $\Gamma^{y}_{tt} = \frac{1}{2} g^{yy}\left[\partial_t g_{yt} + \partial_t g_{yt} - \partial_y g_{tt}\right]$

$$= \frac{-1}{2} g^{yy} \frac{\partial g_{tt}}{\partial y} = \frac{1}{2} \frac{\partial g_{tt}}{\partial r} \frac{\partial r}{\partial y}$$

$$= \frac{1}{2} \frac{y}{r} \frac{\partial}{\partial r}(1+2\Phi) = \frac{1}{2} \frac{y}{r} \frac{\partial}{\partial r}\left[1 - 2\frac{GM}{c^2 r}\right]$$

$$= \frac{1}{2} \frac{y}{r}\left(\frac{2GM}{c^2 r^2}\right) = \frac{GMy}{c^2 (x^2+y^2)^{3/2}}$$

Also $\underline{K^x = K^t}$

$$\frac{dK^y}{d\lambda} = \frac{-2GMy}{c^2(x^2+y^2)^{3/2}} = 0$$

$$\frac{dK^y}{d\lambda} = \frac{-GMy}{c^2(x^2+y^2)^{3/2}}$$

(e) $\frac{dK^y}{d\lambda} = \frac{2GMy}{c^2(x^2+y^2)^{3/2}}$

$\frac{dx}{d\lambda} = 1 \quad \therefore dx = d\lambda$

$$\therefore \Delta K^y = \int_{-\infty}^{\infty} \frac{2GMy}{c^2(x^2+y^2)^{3/2}} dx$$

Put $x = b\tan\theta$, $y = b$

$$\Delta K^y = \frac{2GM}{c^2} \int_{-\pi/2}^{\pi/2} \frac{b \cdot b\sec^2\theta \, d\theta}{(b^2\tan^2\theta + b^2)^{3/2}}$$

$$= \frac{2GM}{c^2}\int_{-\pi/2}^{\pi/2} \frac{b^2 \sec^2\theta}{b^3 \sec^3\theta}\, d\theta$$

$$= \frac{2GM}{c^2 b}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{2GM}{c^2 b}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2GM}{c^2 b}[1+1] = -\frac{4GM}{c^2 b}$$

$$\Delta k^3 = \frac{4 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{9 \times 10^{16}\, b}$$

$$= \frac{5.9288 \times 10^{3}}{b}$$

$$= \frac{5.9288 \times 10^{3}}{R_{sun}}\left(\frac{R_{sun}}{b}\right)$$

$$= \frac{5.9288 \times 10^{3}}{6.9634 \times 10^{3}}\left(\frac{R_{sun}}{b}\right)$$

$$= 8.514\left(\frac{R_{sun}}{b}\right) \text{ arcseconds.}$$

Q2 (a)