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Q.1. $$f(ax) \leftrightarrow \frac{F(s/a)}{|a|}$$

can be written as: $$\mathcal{F}\{f(ax)\}(s) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Consider the case of $a > 0$,

Multiplying the integral

$$\mathcal{F}\{f(ax)\}(s) = \int_{-\infty}^{\infty} f(a) e^{-i2\pi sx} dx \quad \text{—— (1)}$$

by $\frac{a}{|a|} = 1$ and the exponent by $\frac{a}{a} = 1$ gives:

$$\mathcal{F}\{f(ax)\}(s) = \frac{1}{|a|} \int_{-\infty}^{\infty} f(ax) e^{-i2\pi (s/a)ax} a\, dx$$

We now substitute $u = ax \Rightarrow du = a\,dx$:

$$\mathcal{F}\{f(ax)\}(s) = \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{-i2\pi (\frac{s}{a}) u} du$$

$$= \frac{1}{|a|} F\left(\frac{s}{a}\right) \quad \text{—— (2)}$$

Case $a<0$:

Multiplying equation ① by $|a|/|a| = 1$ and the exponent by $-|a|/a = 1$ and using the fact that $a = -|a|$, we get:

$$F\{f(ax)\}(s) = \frac{1}{|a|}\int_{x=-\infty}^{x=\infty} f(-|a| \quad)e^{-i2\pi(s/a)(-|a|x)}|a|dx$$

We now substitute $u = -|a|x$ and $du = -|a|dx$:

$$F\{f(ax)\}(s) = \frac{-1}{|a|}\int_{u=\infty}^{u=-\infty} f(u)e^{-i2\pi(s/a)u}du$$

$$= \frac{1}{|a|}\int_{u=-\infty}^{u=\infty} f(u)e^{-i2\pi(s/a)u}du$$

$$= \frac{1}{|a|}F\left(\frac{s}{a}\right) \quad ③$$

Hence from both ② and ③, we can say that for both cases $a<0$ and $a>0$,

$$F\{f(ax)\}(s) = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

Which can also be compacted to the form: $f(ax) \leftrightarrow \frac{F(s/a)}{|a|}$

Hence proved

Q. 2. We know $f(x-a) \leftrightarrow \int_{-\infty}^{\infty} f(x-a)\, e^{-i2\pi s x}\, dx = F(s)$

Let $x-a=u$ and $dx = du$

$\Rightarrow f(u) \leftrightarrow \int_{-\infty}^{\infty} f(u)\, e^{-i2\pi s(u+a)}\, du$

$$= e^{-i2\pi a} \int_{-\infty}^{\infty} f(u)\, e^{-i2\pi s}\, du$$

$\Rightarrow f(x-a) \leftrightarrow e^{-i2\pi a} F(s)$

Hence proved.

Q. 3. (a) Given that $f(x) = \begin{cases} 0 & \forall\, x < 0 \\ e^{-x} & \forall\, x > 0 \end{cases}$

FT is defined as

$$F\{f(x)\}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\, dx$$

$$= \int_{0}^{\infty} e^{-x} e^{-i2\pi s x}\, dx$$

$$= \int_{0}^{\infty} e^{-(i2\pi s x + 1)x}\, dx$$

$$= \left[\frac{e^{-(i2\pi s + 1)x}}{-(i2\pi s + 1)}\right]_0^{\infty} = \frac{e^{-\infty} - 1}{-(i2\pi s + 1)}$$

$$\Rightarrow \quad F(s) = \frac{1}{1 + 2\pi i s}$$

(b) Given $f(x) = e^{-a|x|}$ $(a > 0)$

FT:

$$F\{f(x)\}(s) = F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\, dx$$

$$= \int_{-\infty}^{0} e^{ax} e^{-i2\pi s x}\, dx + \int_{0}^{\infty} e^{-ax} e^{-i2\pi s x}\, dx$$

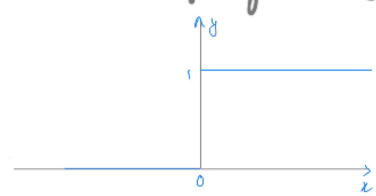
$$= \left[\frac{e^{(a-i2\pi s)x}}{a-i2\pi s}\right]\Bigg|_{-\infty}^{0} + \left[\frac{e^{-(a+i2\pi s)x}}{-(i2\pi s+a)}\right]\Bigg|_{0}^{\infty}$$

$$= \lim_{b\to\infty}\left[\frac{e^{(a-i2\pi s)x}}{a-i2\pi s}\Bigg|_{-b}^{0} + \frac{e^{-(a+i2\pi s)x}}{-(i2\pi s+a)}\Bigg|_{0}^{b}\right]$$

$$= \frac{1}{a-i2\pi s} + \frac{1}{i2\pi s+a} = \frac{\cancel{i2\pi s}+a+a-\cancel{i2\pi s}}{a^2+4\pi^2 s^2}$$

$$\Rightarrow F(s) = \frac{2a}{a^2+4\pi^2 s^2}$$

unit step function:



(c) The unit step function is given by:

$$f(x) = \begin{cases} 1 & \forall\ x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

FT:

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi sx}\, dx = \int_{0}^{\infty} e^{-i2\pi sx}\, dx$$

$$= \frac{e^{-i2\pi sx}}{-i2\pi s}\Bigg|_{0}^{\infty} = \frac{1}{i2\pi s}$$

$$\Rightarrow F(s) = \frac{1}{i2\pi s}$$

(d) $f(x) = \cos\omega_0 x$

We know that we can write $\cos\omega_0 x = \dfrac{e^{i\omega_0 x} + e^{-i\omega_0 x}}{2}$

FT:

$$\mathcal{F}\{f(x)\}(s) = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i\omega_0 x} + e^{-i\omega_0 x}\right) e^{-i2\pi s x}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{i(\omega_0 - 2\pi s)x} + e^{-i(\omega_0 + 2\pi s)x}\right] dx$$

$$= \pi\,\delta(2\pi s - \omega_0) + \pi\,\delta(2\pi s + \omega_0)$$

$$\left[\because \int_{-\infty}^{\infty} e^{-i\omega x}\, dx = 2\pi\delta(\omega)\right]$$

(e) given $f(x) = \delta(x)$

FT:

$$\mathcal{F}\{f(x)\}(s) = \int_{-\infty}^{\infty} \delta(x)\, e^{-i2\pi s x}\, dx$$

$$= e^{i2\pi s(0)}$$

$$\Rightarrow F(s) = 1$$

using the property $\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$

(f) Given $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$$F\{f(x)\}(s) = F(s) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i2\pi s x} \cdot e^{-x^2/2\sigma^2} \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp -\left[i2\pi s x + \frac{x^2}{2\sigma^2} \right] dx$$

Completing the square in the exponential, we can write the above as:

$$F(s) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\left(\frac{x}{\sqrt{2}\sigma} + i\sqrt{2}\pi s\sigma \right)^2 + 2\pi^2 s^2 \sigma^2 \right] dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{2\pi^2 s^2 \sigma^2} \int_{-\infty}^{\infty} \exp\left(\frac{x}{\sqrt{2}\sigma} + i\sqrt{2}\pi s\sigma \right)^2 dx$$

$$= \frac{e^{2\pi^2 s^2 \sigma^2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{(x + i2\pi s\sigma^2)^2}{2\sigma^2} \right] dx$$

Replacing $x + i2\pi s\sigma^2 = u$ and $dx = du$,

$$= \frac{e^{2\pi^2 s^2 \sigma^2}}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} \exp\left(\frac{u^2}{2\sigma^2} \right) du$$

$$\Rightarrow \quad F(s) = \frac{2\pi^2 s^2 \sigma^2}{\sqrt{2\pi}\,\sigma} f(x)$$

This shows that the FT of a Gaussian is also a Gaussian.

Q.4. $F\{f(x)\cos(2\pi\nu x)\}(s) = \int_{-\infty}^{\infty} f(x)\cos 2\pi\nu x \, e^{2i\pi s x}\, dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\left(e^{i2\pi\nu x} + e^{-i2\pi\nu x}\right) e^{i2\pi x s}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\left(e^{i2\pi x(\nu+s)} + e^{i2\pi x(s-\nu)}\right) dx$$

$$= \frac{1}{2}\left[\underbrace{\int_{-\infty}^{\infty} f(x)\, e^{i2\pi x(\nu+s)}\, dx}_{F(\nu+s)} + \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{i2\pi x(s-\nu)}\, dx}_{F(s-\nu)}\right]$$

$$\Rightarrow F\{f(x)\cos 2\pi\nu x\}(s) = \frac{1}{2}\left[F(\nu+s) + F(s-\nu)\right]$$

This can be compacted to the following form:

$$f(x)\cos(2\pi\nu x) \leftrightarrow \frac{1}{2}\left[F(\nu+s) + F(s-\nu)\right]$$

Hence proved.

Q. 5. $f(x) = e^{-x}$; $g(x) = \sin x$.

Convolution of $f(x)$ and $g(x)$ is defined as:

$$f*g = \int_0^x f(t)\, g(x-t)\, dt$$

$\Rightarrow$ In the given case, $f*g = \int_0^x e^{-t} \sin(x-t)\, dt$

$$= \frac{1}{2i}\int_0^x e^{-t}\left(e^{i(x-t)} - e^{-i(x-t)}\right) dt$$

$$= \frac{1}{2i}\int_0^x \left[e^{i(x-t)-t} - e^{-[i(x-t)+t]}\right] dt$$

$$= \frac{1}{2i}\left[\frac{e^{i(x-t)-t}}{-i-1} + \frac{e^{-[i(x-t)+t]}}{1-i}\right]_0^x$$

$$= \frac{1}{2i}\left[\frac{e^{-x}}{-(1+i)} + \frac{e^{-x}}{1-i} - \frac{e^{ix}}{-(1+i)} - \frac{e^{-ix}}{(1-i)}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-x} - ie^{-x} - e^{ix} + ie^{ix}}{-2} + \frac{e^{-x} + ie^{-x} - e^{ix} + ie^{-ix}}{2}\right]$$

$$= \frac{1}{4i}\left[\underline{i}e^{-x} + \cancel{e^{ix}} - \underline{ie}^{ix} - \cancel{e^{-x}} + \cancel{e^{-x}} + \underline{ie}^{-x} - \cancel{e^{ix}} + ie^{-ix}\right]$$

$$\frac{\cancel{2i}\, e^{-x}}{2\cancel{4i}} - \frac{i\left(e^{ix} - e^{-ix}\right)}{4i}$$

$$\Rightarrow \frac{e^{-x}}{2} - \frac{i}{2}\sin x = f*g$$

Q.6. Again, convolution is defined as:

$$f(x) * g(x) = \int_0^x f(x') \, g(x - x') \, dx'$$

$$\therefore \; f(x) * \delta(x) = \int_0^x f(x') \, \delta(x - x') \, dx' \quad \text{——} \; (1)$$

Using the property of $\delta(x)$ function that

$\int f(x) \, \delta(x-a) \, dx = f(a)$, equation (1) becomes:

$$f(x) * \delta(x) = f(x)$$

Hence proved

Q.7. $f(x) = \begin{cases} 1 & \forall \; |x| < 1/2 \\ e^{-x} & |x| \geq 1/2 \end{cases}$

FT:

$$F\{f(x)\}(s) = \int_{-\infty}^{-1/2} e^{x} e^{-i2\pi s x} \, dx + \int_{-1/2}^{1/2} e^{-i2\pi s x} \, dx + \int_{1/2}^{\infty} e^{-x} e^{-i2\pi s x} \, dx$$

$$= \left. \frac{e^{(1-i2\pi s)x}}{(1-i2\pi s)} \right|_{-\infty}^{-1/2} + \left. \frac{e^{-i2\pi s x}}{-i2\pi s} \right|_{-1/2}^{1/2}$$

$$+ \left. \frac{e^{-(1+i2\pi s)x}}{-(1+i2\pi s)} \right|_{1/2}^{\infty}$$

$$= \frac{e^{-(1-i2\pi s)/2}}{(1-i2\pi s)} + \frac{e^{-i\pi s}}{-i2\pi s} - \frac{e^{i\pi s}}{-i2\pi s}$$

$$+ \frac{e^{-(1+i2\pi s)/2}}{(1+i2\pi s)}$$

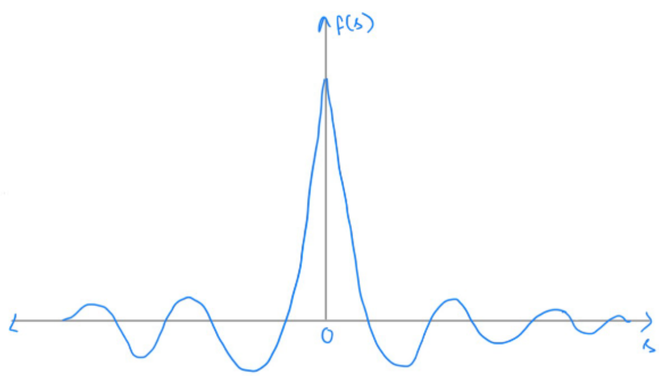
$$= \frac{\sin \pi s}{\pi s} + e^{-1/2}\left[\frac{e^{-i\pi s}}{1-i2\pi s} + \frac{e^{i\pi s}}{1+i2\pi s}\right]$$

$$= \frac{\sin \pi s}{\pi s} + e^{-1/2}\left[\frac{e^{-i\pi s}(1+i2\pi s) + (1-i2\pi s)e^{i\pi s}}{(1-i2\pi s)(1+i2\pi s)}\right]$$

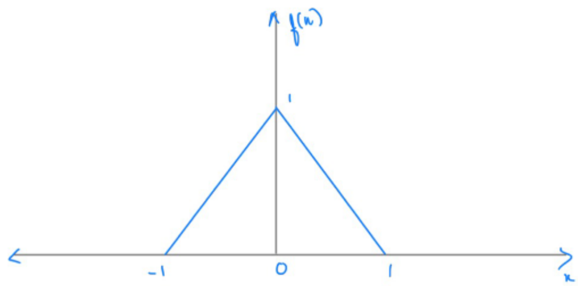
$$= \frac{\sin \pi s}{\pi s} + e^{-1/2}\left[\frac{e^{-i\pi s} + e^{i\pi s} - i2\pi s(e^{i\pi s} - e^{-i\pi s})}{1+4\pi^2 s^2}\right]$$

$$\Rightarrow F(s) = \frac{\sin \pi s}{\pi s} + \frac{e^{-1/2}}{1+4\pi^2 s^2}\left(2\cos \pi s + 4\pi s \sin \pi s\right)$$

The graph of this fourier transform is given by:



Q4.



Graph of the given function

The FT of given $f(x) = \begin{cases} 1-|x| & \forall \ |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$ is given by:

$$F\{f(x)\}(s) = F(s) = \int_{-1}^{0} (1+x)\, e^{-i2\pi s x}\, dx + \int_{0}^{1} (1-x)\, e^{-i2\pi s x}\, dx$$

$$= \left[\frac{e^{-i2\pi s x}}{-i2\pi s} - \frac{x e^{-i2\pi s x}}{i2\pi s} - \frac{e^{-i2\pi s x}}{4\pi^2 s^2} \right]_{-1}^{0}$$

$$+ \left[\frac{e^{-i2\pi s x}}{-i2\pi s} + x\frac{e^{-i2\pi s x}}{i2\pi s} + \frac{e^{-i2\pi s x}}{4\pi^2 s^2} \right]_{0}^{1}$$

$$= \left[\cancel{\frac{1}{-i2\pi s}} - \cancel{\frac{1}{4\pi^2 s^2}} + \cancel{\frac{e^{i2\pi s}}{i2\pi s}} - \cancel{\frac{e^{i2\pi s}}{i2\pi s}} + \frac{e^{i2\pi s}}{4\pi^2 s^2} \right]$$

$$+ \left[\cancel{\frac{e^{-i2\pi s}}{-i2\pi s}} + \cancel{\frac{e^{-i2\pi s}}{i2\pi s}} + \frac{e^{-i2\pi s}}{4\pi^2 s^2} + \cancel{\frac{1}{i2\pi s}} + \cancel{\frac{1}{4\pi^2 s^2}} \right]$$

$$\Rightarrow F(s) = \frac{\cos 2\pi s}{2\pi^2 s^2}$$

Q. 9

We know, convolution theorem :

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x') g(x - x') \, dx'$$

Here, $g(x) = u(x)$ is the unit step function given by:

$$u(x) = \begin{cases} 1 & \forall \; x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, $$f * u = \int_{-\infty}^{x} f(x') \, dx'$$

Also,

$$F\{f * u\} = F\{f\} \cdot F\{u\}$$

$\because$ $$F\{u\} = \frac{1}{i(2\pi s)} + \pi \delta(2\pi s)$$

and $F\{f\} = F(s)$,

$\therefore$ $$F\{f * u\} = F(s) \left[\frac{1}{i 2\pi s} + \pi \delta(2\pi s) \right]$$

$$= \frac{F(s)}{i 2\pi s} + \pi F(s) \, \delta(2\pi s)$$

$$= \frac{F(s)}{i2\pi s} + \frac{\pi F(0)\,\delta(s)}{2\pi}$$

$$\Rightarrow F\left\{\int_{-\infty}^{x} f(x')\,dx'\right\} = \frac{1}{2\pi}\left[\frac{F(s)}{is} + \pi F(0)\,\delta(s)\right]$$

Hence proved.

Q.10. If x is replaced by $(-x)$, then,

$$\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx = -\int_{-\infty}^{\infty} f(-x)\, e^{-2\pi i s(-x)}\, dx$$

$$= -\int_{-\infty}^{\infty} f(x)\, e^{i2\pi s x}\, dx$$

Let $x = -y$ & $dx = -dy$.

Then the above becomes

$$\int_{-\infty}^{\infty} e^{-i2\pi s y}\, dy = F(s)$$

Then the corresponding F.T. will just be $F(s)$