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Assignment-2

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Q1. n^{th} energy eigenfunction of quantum harmonic oscillator is

$$\langle x'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\left(\frac{1}{x_0^{n+1/2}}\right)\left(x' - x_0^2\frac{d}{dx'}\right)^n e^{-1/2\left(\frac{x'}{x_0}\right)^2}$$

Here, x_0 = length scale of oscillator, $\sqrt{\frac{\hbar}{m\omega}}$.

$$\boxed{\begin{aligned} a|n'\rangle &= \sqrt{n'}\,|n'-1\rangle \\ a^\dagger|n'\rangle &= \sqrt{n'+1}\,|n'+1\rangle \end{aligned}}$$

Now, $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$

$$\Rightarrow x^2 = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2(a+a^\dagger)(a+a^\dagger) = \frac{\hbar}{2m\omega}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)$$

Also, $p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$

$$\Rightarrow p^2 = \left(i\sqrt{\frac{m\hbar\omega}{2}}\right)^2(a^\dagger - a)(a^\dagger - a)$$

$$= -\frac{m\hbar\omega}{2}\left(a^\dagger a^\dagger - a^\dagger a - aa^\dagger + aa\right)$$

Now the expectation values, $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle$ can be calculated.

We can write $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$$

For this, let us calculate $\langle p \rangle$, $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle p^2 \rangle$.

$$\langle x \rangle = \langle n' | x | n' \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left\{ \underset{0}{\langle n' | a | n' \rangle} + \underset{0}{\langle n' | a^+ | n' \rangle} \right\} = 0 \quad \text{— (1)}$$

$$\langle x^2 \rangle = \langle n' | x^2 | n' \rangle = \frac{\hbar}{2m\omega} \left\{ \langle n' | a a | n' \rangle + \langle n' | a a^+ | n' \rangle + \langle n' | a^+ a | n' \rangle + \langle n' | a^+ a^+ | n' \rangle \right\}$$

$$= \frac{\hbar}{2m\omega} \left\{ \sqrt{n'} \langle n' | a | n'-1 \rangle + \sqrt{n'+1}\sqrt{n'+1} \langle n' | n' \rangle + \sqrt{n'}\sqrt{n'} \langle n' | n' \rangle + \sqrt{n'+1} \left(\langle n' | a^+ | n'+1 \rangle \right) \right\}$$

$$= \frac{\hbar}{2m\omega} \left\{ \sqrt{n'}\sqrt{n'-1} \underset{0}{\langle n' | n'-2 \rangle} + \left(\sqrt{n'+1}\right)^2 \overset{1}{\langle n' | n' \rangle} + \left(\sqrt{n'}\right)^2 \overset{1}{\langle n' | n' \rangle} + \sqrt{n'+1}\sqrt{n'+2} \underset{0}{\left(\langle n' | n'+2 \rangle \right)} \right\}$$

$$= \frac{\hbar}{2m\omega} \left[(n'+1) + n' \right] = \frac{\hbar}{2m\omega} (2n'+1) \quad \text{— (2)}$$

$$\langle p \rangle = \langle n' | p | n' \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left[\langle n' | a^+ | n' \rangle - \langle n' | a | n' \rangle \right]$$

$$= i\sqrt{\frac{\hbar\omega m}{2}} \left[\sqrt{n'+1} \underset{0}{\langle n' | n'+1 \rangle} - \sqrt{n'} \underset{0}{\langle n' | n'-1 \rangle} \right]$$

$$= 0 \quad \text{— (3)}$$

$$\langle p^2 \rangle = \langle n' | p^2 | n' \rangle = \frac{-m\hbar\omega}{2} \left\{ \langle n' | a^+ a^+ | n' \rangle + \langle n' | a a | n' \rangle - \langle n' | a^+ a | n' \rangle - \langle n' | a a^+ | n' \rangle \right\}$$

$$= \frac{-m\hbar\omega}{2} \left\{ \sqrt{n'+1}\sqrt{n'+2} \overset{0}{\langle n' | n'+2 \rangle} + \sqrt{n'}\sqrt{n'} \overset{1}{\langle n' | n' \rangle} - \sqrt{n'+1}\sqrt{n'+1} \underset{1}{\langle n' | n' \rangle} + \sqrt{n'}\sqrt{n'-1} \underset{0}{\langle n' | n'-1 \rangle} \right\}$$

$$= -\frac{m\hbar\omega}{2}\left[-2n'-1\right] = \frac{m\hbar\omega}{2}\left[2n'+1\right] \quad (4)$$

Using (1) and (2), we have

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}(2n'+1) \quad (5)$$

and from (3) and (4), we have

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2}(2n'+1) \quad (6)$$

Using (5) and (6), we have

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar}{2m\omega} \times \frac{m\hbar\omega}{2}(2n'+1)^2$$

$$\Rightarrow \langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{\,}\left(n'+\frac{1}{2}\right)^2$$

Now, to calculate the energy eigenfunction for $n = 4$.

$$\underline{\underline{\langle x|4\rangle}} = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^4 \cdot 4!}}\left(\frac{m\omega}{\hbar}\right)^{5/4}\left(x' - \frac{\hbar}{m\omega}\frac{d}{dx'}\right)^4 e^{-\frac{x^2 m\omega}{2\hbar}}$$

$$= \frac{e^{-\frac{m\omega x^2}{2\hbar}}}{384}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} H_4\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \quad (7)$$

Here $H_4(z)$ is the Hermite polynomial of order 4.

$$H_4(z) = (-1)^4 e^{(z^2)}\frac{d^4}{dx^4}\left(e^{(-z^2)}\right)$$

The solution is given by $H_4(z) = 16z^4 - 48z^2 + 12$.

Substituting z we have

$$H_4\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) = 16\left(\frac{m^2\omega^2 x^4}{\hbar^2}\right) - 48\left(\frac{m\omega x^2}{\hbar}\right) + 12$$

Using this value in (7), we have

$$\langle x|4\rangle = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{e^{-\frac{m\omega x^2}{2\hbar}}}{384}\left[\frac{16m^2\omega^2x^4}{\hbar^2} - \frac{48m\omega x^2}{\hbar} + 12\right]$$

Q2. $$\langle x\rangle = \int_{-\infty}^{\infty}\langle\alpha|x'\rangle x'\langle x'|\alpha\rangle = \int_{-\infty}^{\infty}|\langle x'|\alpha\rangle|^2 x'dx'$$

$$= \frac{1}{d\sqrt{\pi}}\int_{-\infty}^{\infty} x' e^{\left(\frac{-x'^2}{d^2}\right)}dx' = 0 \quad (\because \text{odd function}). \quad \text{—(1)}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty}|\langle x'|\alpha\rangle|^2 x'^2 dx' = \int_{-\infty}^{\infty}\frac{d^2}{2}e^{(-x'^2/d^2)}dx' = \frac{d^2}{2} \quad \text{—(2)}$$

$$\langle p\rangle = \frac{i\hbar}{d^3\sqrt{\pi}}\int_{-\infty}^{\infty} x\cdot e^{\left(\frac{-x^2}{d^2}\right)}dx = 0 \quad \text{—(3)}$$

$$\langle p^2\rangle = \frac{1}{d\sqrt{\pi}}\int_{-\infty}^{\infty} e^{(-x^2/2d^2)}\left(-\hbar^2\frac{\partial^2}{\partial x^2}e^{-x^2/2d^2}\right)\cdot dx$$

$$= \frac{-\hbar^2}{d\sqrt{\pi}}\left[\frac{1}{d^4}\times\frac{d^2}{2}\sqrt{\pi d^2} - \sqrt{\pi d^2}\right] \text{ after integrating}$$

$$= \frac{\hbar^2}{2d^2} \text{ on further solving.} \quad \text{—(4)}$$

From (1), (2), (3), and (4), we have

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \left[\langle x^2\rangle - \langle x\rangle^2\right]\left[\langle p^2\rangle - \langle p\rangle^2\right] = \frac{d^2}{2}\cdot\frac{\hbar^2}{2d^2}$$

$$= \frac{\hbar^2}{4}$$

This is the minimum uncertainty wavepacket.

In Heisenberg's picture, kets do not evolve, instead the operators evolve with time.

Using Heisenberg's eq^n of motion for a free particle, $x_i(t)$

$$x_i(t) = x_i(t=0) + \frac{p_i}{m}(t=0)\, t$$

$$p_i(t) = p_i(t=0) \quad (\because \text{momentum of a free particle doesn't change})$$

$$\langle(\Delta x)^2\rangle_t = \langle x(t)^2\rangle - \langle x(t)\rangle^2$$

$$= \left\langle\left(x(0) + \frac{p(0)}{m}t\right)^2\right\rangle - \left(\langle x(0)\rangle + \frac{t}{m}\langle p(0)\rangle\right)^2$$

$$= \langle x(0)^2\rangle + \frac{t^2}{m^2}\langle p(0)^2\rangle + \frac{t}{m}\left[\langle x(0)\rangle\langle p(0)\rangle + \langle p(0)\rangle\langle x(0)\rangle\right]$$

$$\Rightarrow \langle(\Delta x)^2\rangle_t = \frac{d^2}{2} + \frac{t^2\hbar^2}{2m^2d^2} \quad \text{— (5)}$$

and $\langle(\Delta p)^2\rangle_t = \langle(\Delta p)^2\rangle_{t=0} = \dfrac{\hbar^2}{2d^2}$

$$\Rightarrow \langle(\Delta x)^2\rangle_t\langle(\Delta p)^2\rangle_t = \frac{\hbar^2}{4}\left[1 + \frac{\hbar^2t^2}{m^2d^4}\right] = \frac{\hbar^2}{4}(1+\omega^2t^2) \text{ s.t. } \omega = \frac{\hbar}{md^2}$$

In Schrödinger's picture, the operators remain the same but the kets evolve with time.

If we consider a Gaussian wavepacket in momentum space,

$$\langle p|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{\frac{-p^2 d^2}{2\hbar^2}}$$

The ket evolves from $\langle p|\alpha\rangle \rightarrow U\langle p|\alpha\rangle$

$$\Rightarrow U\langle p|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{\left(\frac{-p^2 d^2}{2\hbar^2} - \frac{iHt}{m}\right)}$$

$$\Rightarrow \langle p|\alpha\rangle_t = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{\left(\frac{-p^2 d^2}{2\hbar^2} - \frac{i p^2 t}{2m^2}\right)}$$

$$\langle x\rangle_t = \int_{-\infty}^{\infty} \langle\alpha|p\rangle U^{+} x\, U\langle p|\alpha\rangle\, dp = \int_{-\infty}^{\infty}\langle\alpha|p\rangle_t\, x\,\langle p|\alpha\rangle_t\, dp$$

Now using the value $\boxed{x = i\hbar\frac{\partial}{\partial p}}$

$$\langle x\rangle_t = \frac{i\hbar d}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left[\frac{-p^2 d^2}{\hbar^2}\right]\times\left[\frac{p d^2}{\hbar} + \frac{i p t}{m^2}\right] dp = 0 \quad (\because \text{odd function})$$

$$\langle x^2\rangle_t = \int_{-\infty}^{\infty}\langle\alpha|p\rangle_t\, x^2\,\langle p|\alpha\rangle_t\, dp = -\frac{d\hbar}{\sqrt{\pi}}\left[\frac{-\hbar t^2\sqrt{\pi}}{2d^3 m^2} - \frac{d\sqrt{\pi}\, d^2}{2\hbar}\right]$$

$$= \frac{\hbar^2 t^2}{2d^2 m^2} + \frac{d^2}{2}$$

$\langle p\rangle_t = \langle p\rangle_{t=0} = 0$ ($\because$ momentum does not change for a free particle because there's no external potential term)

$\langle p^2\rangle_t = \langle p^2\rangle_{t=0} = \frac{\hbar^2}{2d^2}$

$$\langle(\Delta x)^2\rangle_t \langle(\Delta p)^2\rangle_t = \left[\frac{\hbar^2 t^2}{2d^2m^2} + \frac{d^2}{2} - 0\right]\left[\frac{\hbar^2}{2d^2}\right]$$

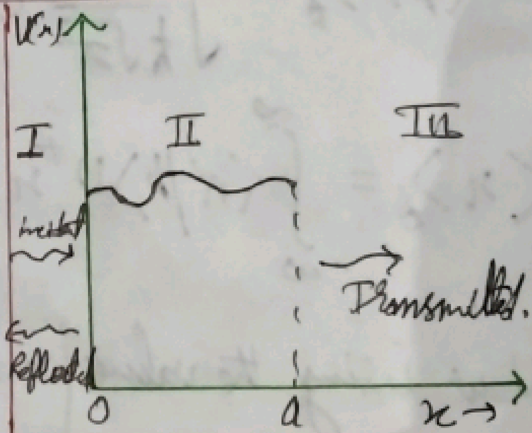
$$= \frac{\hbar^2 t^2}{2d^2m^2} \times \frac{\hbar^2}{2d^2} + \frac{d^2}{2} \times \frac{\hbar^2}{2d^2}$$

$$\langle(\Delta x)^2\rangle_t \langle(\Delta p)^2\rangle_t = \frac{\hbar^2}{4}\left[1 + \frac{\hbar^2}{m^2d^4}t^2\right]$$

Clearly, both the pictures give the same result.

Q3. Let's say wavefunction on the left of the bump is given by $(E>V)$

$$\psi_1(x,t) = \frac{A}{\sqrt{(E-V(x))}}\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty}\sqrt{2m(E-V)}\,dx - \frac{iEt}{\hbar}}$$



A = incident wave amp.
$\frac{iEt}{\hbar}$: time evolution part.

Similarly wavefunction on the right of the bump $(x>a)$ will be given by

$$\psi_3(x,t) = \frac{F}{\sqrt{(E-V(x))^{1/2}}}\, e^{\left[\frac{+i}{\hbar}\int_a^{\infty}\sqrt{2m(E-V(x))}\,dx - \frac{iEt}{\hbar}\right]}$$

where F is the amplitude of transmitted wave.

In the tunneling region, $(0 \leq x \leq a)$, the WKB approximation gives $(V > \epsilon)$

$$\psi_2(x,t) = \frac{P}{\sqrt{(V(x)-\epsilon)^{1/2}}}\, e^{\left[\frac{1}{\hbar}\int_0^a \sqrt{2m(V(x)-\epsilon)}\,dx - i\frac{\epsilon t}{\hbar}\right]}$$

$$+ \frac{Q}{\sqrt{(V(x)-\epsilon)^{1/2}}}\, e^{\left[\frac{-1}{\hbar}\int_0^a \sqrt{2m(V(x)-\epsilon)}\,dx - i\frac{\epsilon t}{\hbar}\right]}$$

P: exponentially increasing amplitude.
Q: exponentially decreasing amplitude.

Essentially, it can be said that P is less than Q.
If the barrier is very high or very wide, then P is very small.
Then the probability of tunneling will be also very small.
Now, at $x=0$, wavefunction must be continuous and hence
$\psi_1(0,t) = \psi_2(0,t)$ and at $x=a$; $\psi_3(a,t) = \psi_2(a,t)$

Thus, we have
$$\frac{|F|}{|A|} \sim e^{\left[\frac{-1}{\hbar}\int_0^a \sqrt{(\epsilon - V(x))2m}\,dx\right]}$$

$\therefore$ Transmission probability, $T = \frac{|F|^2}{|A|^2} \sim e^{-2\gamma}$

where $\gamma = \frac{1}{\hbar}\int_0^a \sqrt{2m(\epsilon - V(x))}\,dx$