7/9/21

Assignment - 2

Q1. State and prove Kelvin's vorticity theorem.

⇒ As we know, vorticity ω satisfies general vorticity equation.

i.e $$\frac{\partial \omega}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega})$$ (For an incompressible or barotropic fluid)

"Kelvin's vorticity theorem states that conservation of circulation for an ideal fluid acted upon by conservative forces."

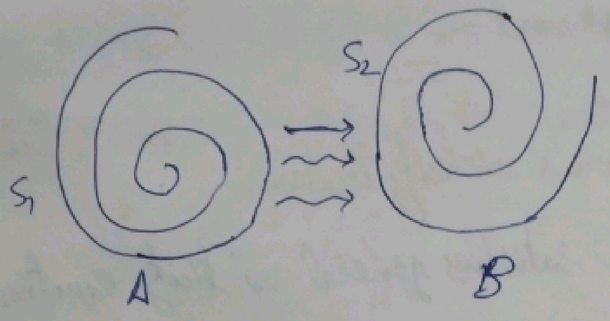
The theorem talks about preserving the vortex flux only for ideal fluids as viscosity is zero.

This happens as ideal fluids have no resistance to the shear stress b/w its layers, Hence it is impossible to change the rotation state of the fluid particles.

For non ideal fluids, this is not true. In this case, a phenomenon called vortex shedding takes place due to friction b/w fluid layers as viscosity ≠ 0.

Mathematically, $$\frac{D}{Dt} \boxed{\int_S \vec{\omega}\, d\vec{S} = 0}$$ Circulation

Hence, $$\frac{D}{Dt}(\Gamma) = 0$$ (Conservation of circulation)



"Vortex flux remains constant"

$t_1 \longrightarrow t_2 = t + \delta t$

Consider a surface S_1 inside a fluid at time t_1.

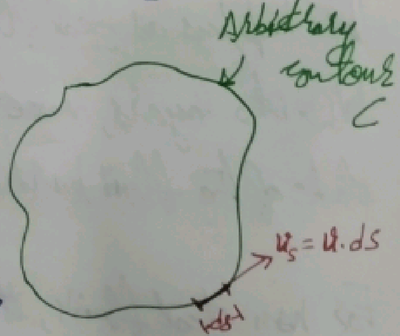
The flux of vorticity linked with this surface $= \int_{S_1} \omega \, dS$

After time dt, the element moves to S_2.

Hence, vorticity linked with S_2 = $\int_{S_2} \omega \, dS$

According to Kelvin's theorem,

$$\int_{S_1} \omega \, dS = \int_{S_2} \omega \, dS$$



Proof - Mathematically, we can write

$$\frac{d}{dt}\int_S \vec{\omega}\cdot d\vec{S} = \frac{d}{dt}\int_S (\vec{\nabla}\times\vec{V})\, d\vec{S} = \frac{d}{dt}\underbrace{\oint_C \vec{V}\cdot dl}_{\text{velocity around closed contour } C}$$

$$\therefore \frac{d}{dt}(\Gamma) = \frac{d}{dt}\oint_C \vec{V}\cdot dl = \oint_C \left[\frac{d\vec{V}}{dt}\cdot dl + V\frac{d(dl)}{dt}\right.$$

Euler momentum conservation equation,

$$\frac{dV}{dt} = F - \frac{1}{\rho}\nabla P$$

$$\Rightarrow \frac{d}{dt}(\Gamma) = \oint_c \left[\left(F - \frac{1}{\rho}\nabla P\right) dl + V \cdot \frac{d}{dt}\left(\frac{dr}{dt}\right)\right]$$

$$= \oint_c \left[\left(F - \frac{1}{\rho}\nabla P\right) dl + V \cdot dV\right]$$

If F is a conservative force, then $F = -\nabla \Omega$ (Ω is some potential quantity)

$$= \oint_c \left[\left(-\nabla\Omega - \frac{1}{\rho}\nabla P\right) dl + \frac{1}{2} d(V^2)\right]$$

$$= \oint_c \left[-\nabla(dl\,\Omega) - \frac{1}{\rho}\nabla P\, dl + \frac{1}{2} d(V^2)\right]$$

($\because \nabla dl = d$)

$$= \oint_c \left[-d\,\Omega - \frac{1}{\rho} dP + \frac{1}{2} d(V^2)\right]$$

Now, $dP = 0$ as the given fluid is barotropic.

$$= \oint_c \left[-d\,\Omega + \frac{1}{2} d(V^2)\right]$$

$$= 0$$

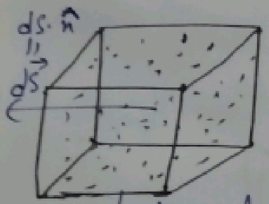
This is due to the fact that Ω and V do not change around a closed path.

Hence, $\frac{d}{dt}(\Gamma) = 0$ or $\frac{\Delta}{\Delta t}\int_c \omega \cdot dS = 0$

Thus, any motion that started from a state of rest at some initial time will remain irrotational for all subsequent times. In this case, circulation will vanish.

Q2. Obtain the conservative form for momentum and energy conservation eqution and thereby get the expression of their respective flux.

$\Rightarrow$ Conservation of momentum in fluid element –



δV (Vol)
ρ (density)
$\vec{v}$ (vel)

Newton's second law: $\underbrace{\rho \delta V}_{m} \underbrace{\frac{Dv}{Dt}}_{a} = \sum \vec{F}$ —— (1)

δF_{body} $\quad$ $\delta F_{surface}$

$d\vec{F}_{body} = \rho \delta V \vec{F} \longrightarrow$ force per unit mass

$d\vec{F}_{surface} \propto d\vec{S}$

$d\vec{F}_{surface} = -P_{ij}(d\vec{S})\hat{n}$

$(F_{surface})_i = -\oint P_{ij}\, dS_j = -\int \frac{\partial P_{ij}}{\partial x_j} \delta V$

$\rightarrow$ Using Gauss's theorem

Eq (1) becomes

$$\rho \frac{D v_i}{D t} = \rho F_i - \frac{\partial P_{ij}}{\partial x_j}$$ for a particular fluid element.

We assume isotropic pressure acting on all fluid elements hence,

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{F} - \vec{\nabla} P$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = F_{body} - \frac{\vec{\nabla} P}{\rho}$$

Conservation of energy in fluid element -

If we have a ~~liquid~~ fluid in thermodynamic equilibrium then,

$$dQ = dU + PdV \quad \text{— (5)}$$ (only for small fluid element)

fluid element of mass δm

$$dQ = \delta m \, dq$$

$$dU = \delta m \, d\mathcal{E}$$

where q and $\mathcal{E}$ are U and Q per unit mass respectively.

$$dV = \frac{1}{\rho}\delta m$$

Dividing (5) get

$$\frac{dq}{dt} = \frac{d\mathcal{E}}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

$$P\frac{d}{dt}\left(\frac{1}{\rho}\right) = -\frac{P}{\rho^2}\underbrace{\frac{d\rho}{dt}}_{\text{from mass conservation eq}^n} = \frac{P}{\rho}\nabla v$$

$$\frac{dq}{dt} = \frac{d\mathcal{E}}{dt} + \frac{P}{\rho}\nabla\cdot v$$

$$\rho\frac{d\mathcal{E}}{dt} + P\nabla v = \rho\frac{dq}{dt} = -\mathcal{L}$$ = Rate of heat gain per unit volume.

"If there are heat flows within a fluid, then some region may gain heat."

The heat flux F inside a continuous substance can be assumed to be proportional to the temp gradient.

i.e, $F = -k\nabla T$ → Thermal conductivity

Also, $\oint F\cdot ds = \int \nabla\cdot F\, dV$ (Heat loss rate from a volume of fluid).

$$\mathcal{L} = \nabla\cdot F = -\nabla(k\nabla T)$$

Thus, the heat loss rate from a volume of fluid is equal to the heat flux integrated over the bounding surface.

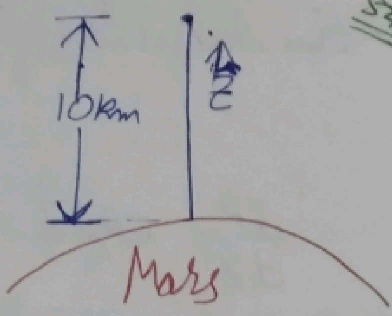
Q3. Given - $R_{mars} = 3.38 \times 10^{8}$ cm

$M_{mars} = 6.42 \times 10^{23}$ kg

$\rho_{surface} = 0.02$ kg/m^3

$P_{surface} = 6$ milibars

$\mu = 43.34$



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To solve the following problem, we need to make some initial assumptions for the sake of simplicity. We assume the atmosphere to be

1) In steady state $\frac{\partial}{\partial t} = 0$

2) Incompressible

3) Isothermal (T = constant, PV = constant)

4) $g = \frac{GM}{R^2}$ = constant.

Here, we can use the hydrostatic eqn.

$$\vec{\nabla} P = -\rho g \hat{z}$$

$$\frac{dP}{dz} = -\rho g \hat{z}$$

$$\int \frac{dP}{dz} = -\int \rho g \hat{z}$$

$$\int_{P_{surface}}^{P_z} dP = -\rho g \int_0^z dz$$

$$P_z - P_{surface} = -\rho g z$$

$$\Rightarrow P_z = P_{surface} - \rho g z$$

$$P_z = 6 - (2 \times 10^{-5}) \times 374.82 z$$

$$\begin{aligned} g &= \frac{GM}{R^2} \\ &= \frac{6.67 \times 10^{-11} \times 6.42 \times 10^{23}}{(3.38 \times 10^{6})^2} \\ &= 3.74 \text{ m/s}^2 \\ &= 374.82 \text{ cm/s}^2 \end{aligned}$$

From ideal gas law, $P = \frac{nRT}{V} = \frac{\rho K_B T}{\mu m}$

$$\frac{dP}{dz} = -\rho g \hat{z}$$

$$\Rightarrow \frac{K_B T}{\mu m} \frac{d\rho}{dz} = -\rho g$$

$$\frac{K_B T}{\mu m} \int_{\rho_{surface}}^{\rho_z} \frac{1}{\rho} d\rho = -g \int_0^z dz$$

$$\rho_z = \rho_{surface} e^{\frac{-mgz\mu}{K_B T}}$$

Temp of martian atmosphere $= 210.37K$

$$\rho_z = \rho_{surface} e^{\left(\frac{-3.74 \times 43.34 \times 1.67 \times 10^{-27}}{1.38 \times 10^{-23} \times 210.37} z\right)}$$

$$\rho_z = \rho_{surface} e^{(-9.32 \times 10^{-5} z)}$$

$$\rho_z = 0.02\, e^{(-9.32 \times 10^{-5} z)} \text{ kg/m}^3$$

Q3. (Variation of pressure)

$$\int \frac{dP}{dz} = -\int \rho_s e^{-9.32\times10^{-5}\times Z} g$$

as $\rho_Z = \rho_{surface} e^{-9.32\times10^{-5}\times Z}$

$$\int_{P_s}^{P_Z} dP = -\rho_s g \int_0^Z e^{-9.32\times10^{-5} Z} dZ$$

$$P_Z = P_s + \frac{\rho_s g}{9.32\times10^{-5}}\left[\left(e^{-9.32\times10^{-5}\times Z}\right) - 1\right]$$

$$= 600 + \frac{0.02\times3.74}{9.32\times10^{-5}}\left[e^{-9.32\times10^{-5}\times Z} - 1\right]$$

$$P_Z = 600 + 802.57\left[e^{-9.32\times10^{-5}\times Z} - 1\right]$$

where P_Z is the value of pressure at a given height above Martian surface

Q4. Show that for a steady flow, the quantity

$$B = \frac{1}{2}v^2 + \int \frac{dP}{\rho} + \phi$$

is constant along a streamline.

$\Rightarrow$ As we know the body force acting on a fluid is usually conservative

So, $F = -\nabla\phi$ —— (1)

From Euler's eq^n, $\frac{\partial V}{\partial t} + (V\cdot\nabla)V = \frac{-1}{\rho}\nabla P + F$ —— (2)

$\Rightarrow (V\cdot\nabla)V = \frac{1}{2}(\nabla)(V\cdot V) - V\times(\nabla\times V)$ —— (3)

Putting the value from (3) in (2), we have

$$\nabla\left(\frac{1}{2}V^2\right) - V\times(\nabla\times V) = -\frac{1}{\rho}\nabla P - \nabla\phi$$

Now we take a line integral along a streamline, 'dl' is a line element of the streamline

$$\int\left[\left(\nabla\frac{1}{2}V^2\right) - V\times(\nabla\times V) + \frac{1}{\rho}\nabla P + \nabla\phi\right]dl = 0$$

Since dl and V are in the same direction, $(V\times(\nabla\times V))\,dl = 0$

$\Rightarrow \int\left(\nabla\frac{1}{2}V^2\right) + \int\frac{1}{\rho}\nabla P + \nabla\phi = \text{constant}$

$\Rightarrow$ The remaining term is equal to some constant along a streamline

Integral $\int \frac{1}{\rho} \, dP$ has to be evaluated from a refence point on the streamline to the point where all the other quantities are considered.

Let us take it as a constant B.

$$\Rightarrow B = \frac{1}{2}v^2 + \int \frac{1}{\rho} \, dP + \phi$$

Hence proved.

Q5. Plot the variation of pressure in terms of P_0 upto 1 AU above the sun where P_0 is the pressure at the inner boundary of the corona.

⇒ To solve the problem, we have to make some assumptions for the solar corona -

① It is spherically symmetric.

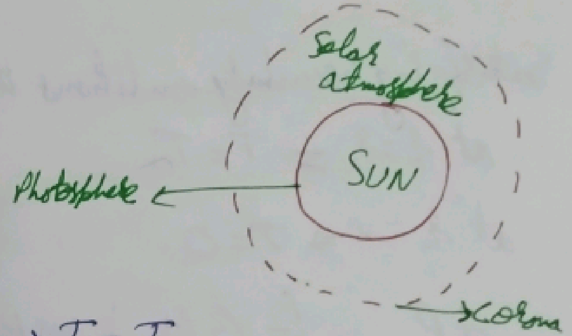
② Thermal conduction is mode of heat transfer hence it is not isothermal.

③ Assume it is hydrostatic balance

④ $m << m_s$

⑤ $\frac{1}{r^2}$ profile of gravity.

⑥ at $r = r_0$ (is base of corona) $\rightarrow T = T_0$



From Hydrostatic equation, as $V=0$, we get $\nabla P = \rho F$ —— ①

Also, $\nabla(K \nabla T) = 0$ from energy conservation eqn. —— ②

Let $F = -g\hat{z}$ (above surface)

① gives ⇒ $$\frac{dP}{dr} = -\frac{GM_s}{r^2}\frac{MP}{K_B T}$$

② gives ⇒ $$\frac{d}{dr}\left(Kr^2 \frac{dT}{dr}\right) = 0$$

From Kinetic theory of plasma, thermal conduction $K \propto T^{5/2}$ (59)

Hence $r^2 T^{5/2} \frac{dT}{dr} = \text{constant}$

$$\int T^{5/2} dT = \int \frac{c}{r^2} dr$$

$$T = T_0 \left(\frac{r_0}{r}\right)^{2/7}$$

Satisfying boundary conditions that

at $r = r_0$ $T = T_0$

at $r = \infty \Rightarrow T = 0$.

$$\frac{dP}{dr} = \left(-\frac{GM_s}{r^2}\right) \frac{mP}{K_B T r_0^{2/7} r^{-2/7}}$$

$$\frac{dP}{dr} = -\left(\frac{GM_s m}{K_B T_0}\right) \frac{P}{r_0^{2/7} r^{12/7}}$$

$$\int_{P_0}^{P} \frac{1}{P} dP = \left(-\frac{GM_s m}{K_B T_0 r_0^{2/7}}\right) \int_{r_0}^{r} r^{-12/7} dr$$

$$[\ln P]_{P_0}^{P} = \left(\frac{-GM_s m}{K_B T_0 r_0^{2/7}}\right) \left[\frac{-7}{5} r^{-5/7}\right]_{r_0}^{r}$$

$$\ln \frac{P}{P_0} = \frac{7}{5} \frac{M_s m}{K_B T_0 r_0^{2/7}} \left[\frac{1}{r^{5/7}} - \frac{1}{r_0^{5/7}}\right]$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{7M_s m}{5K_B T_0 r_0^{2/7}}\left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]$$

$$P = P_0\, e^{\left\{\frac{7M_s m}{5K_B T_0 r_0^{2/7}}\left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]\right\}}$$

$$\Rightarrow P \propto \exp\left(A\left(\frac{r_0}{r}\right)^{5/7} - 1\right)$$

$$\Rightarrow \frac{P}{P_0} = \exp\left[\left(\frac{0.00465}{r}\right)^{5/7} - 1\right]$$ where $r_0 = 0.00465\,AU$

This is the relative pressure above the sun.