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Assignment - 1

Q1. Car - garage paradox.

⇒ Initial assumptions and given data:

According to given data, the car and the garage have equal lengths at rest.

I am assuming that the garage is open on both ends. (to avoid any collisions)

The Paradox:

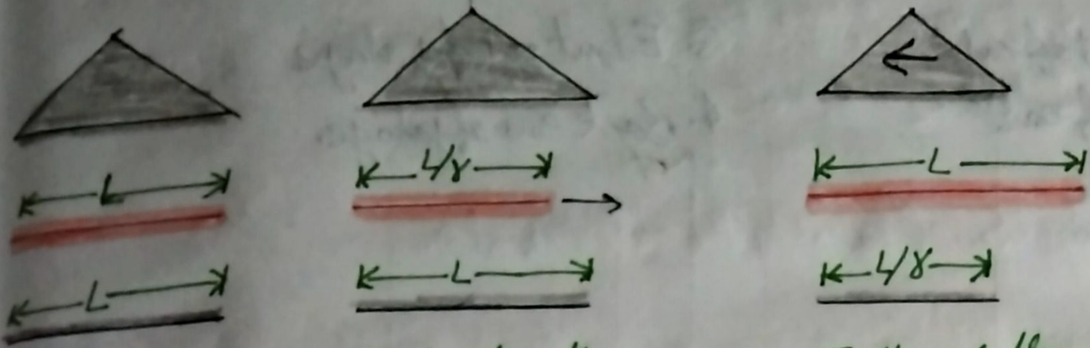
The given situation involves the "simultaneity" of observers in two different frames for the same event and the Lorentz contractions of frames for them.

What's happening?

Mechanic says: When we move the car at a high velocity through the garage, due to its speed, it goes through Lorentz contraction and becomes significantly shorter, i.e, for that instant in time, it can be kept in the garage.

Driver says: According to him/her, the car is stationary and the garage is moving with high velocity. It is therefore the garage which is length contracted, too small to contain the car.

Overview of the situation:



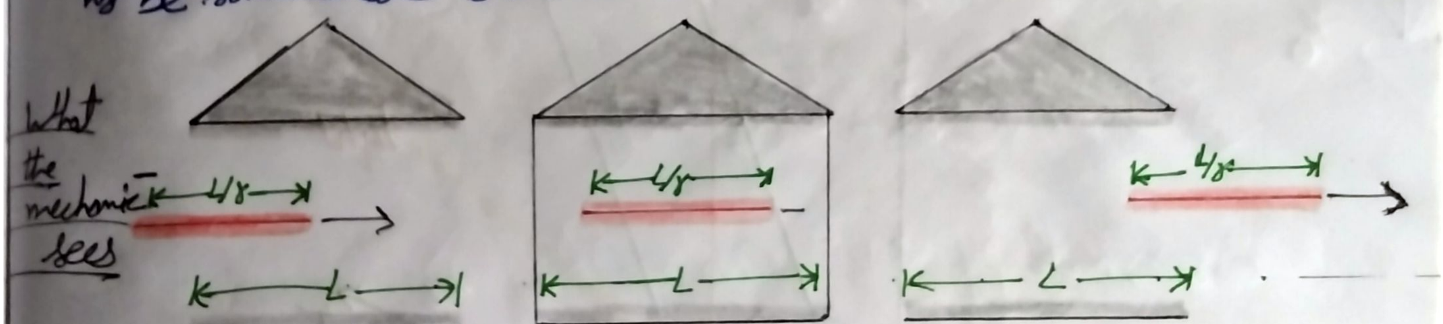
At rest: car and garage of equal length

In the garage frame: Car goes through length contraction and fits in garage

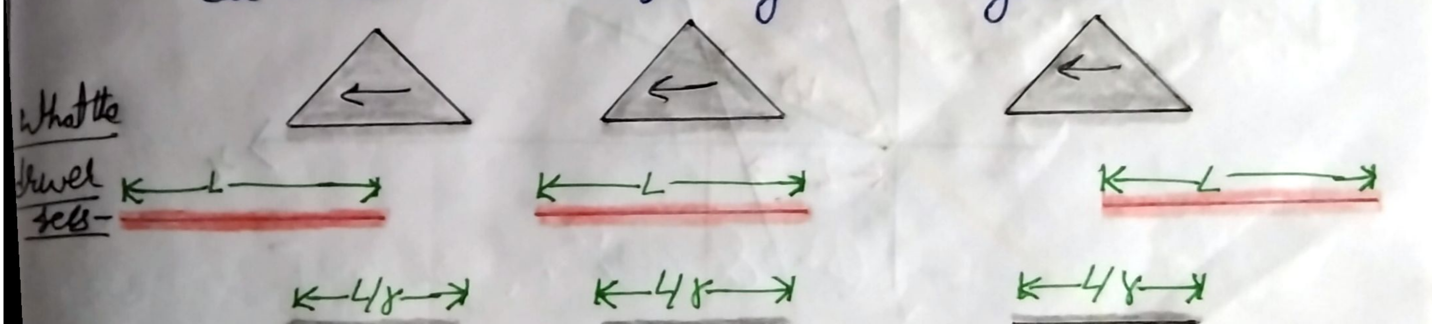
In the car frame: The garage becomes contracted and becomes too small for the car.

How to explain what's really happening?

The solution to the apparent paradox is in the relativity of the "simultaneity".
What one observer considers to be two simultaneous events may not be simultaneous to the other



Car enders | Car fits easily | Car goes out.



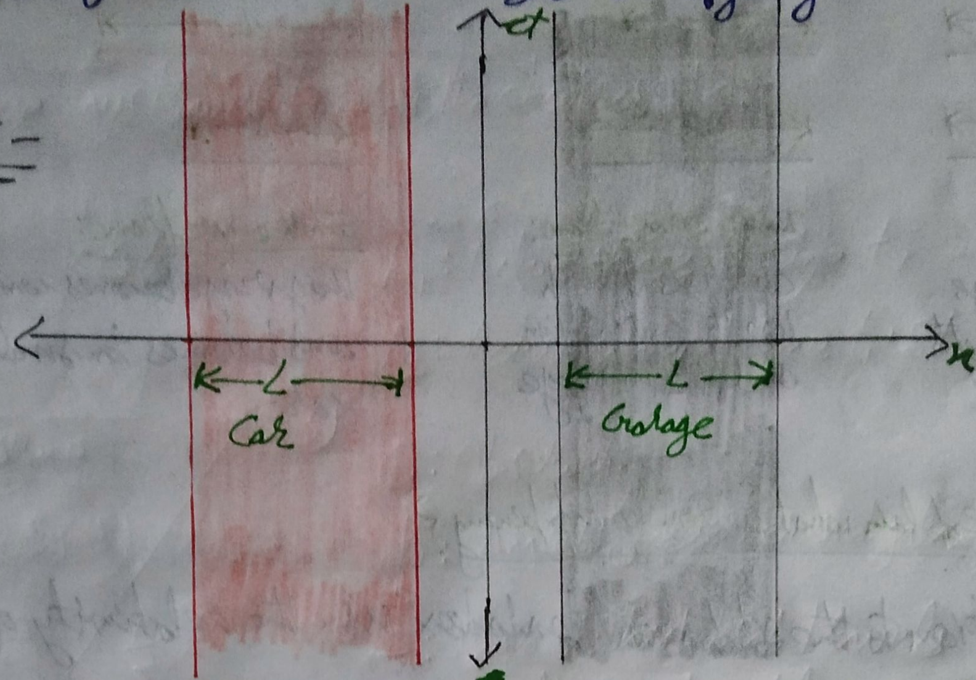
Garage passes over car | Car doesn't fit | Garage passes over car.

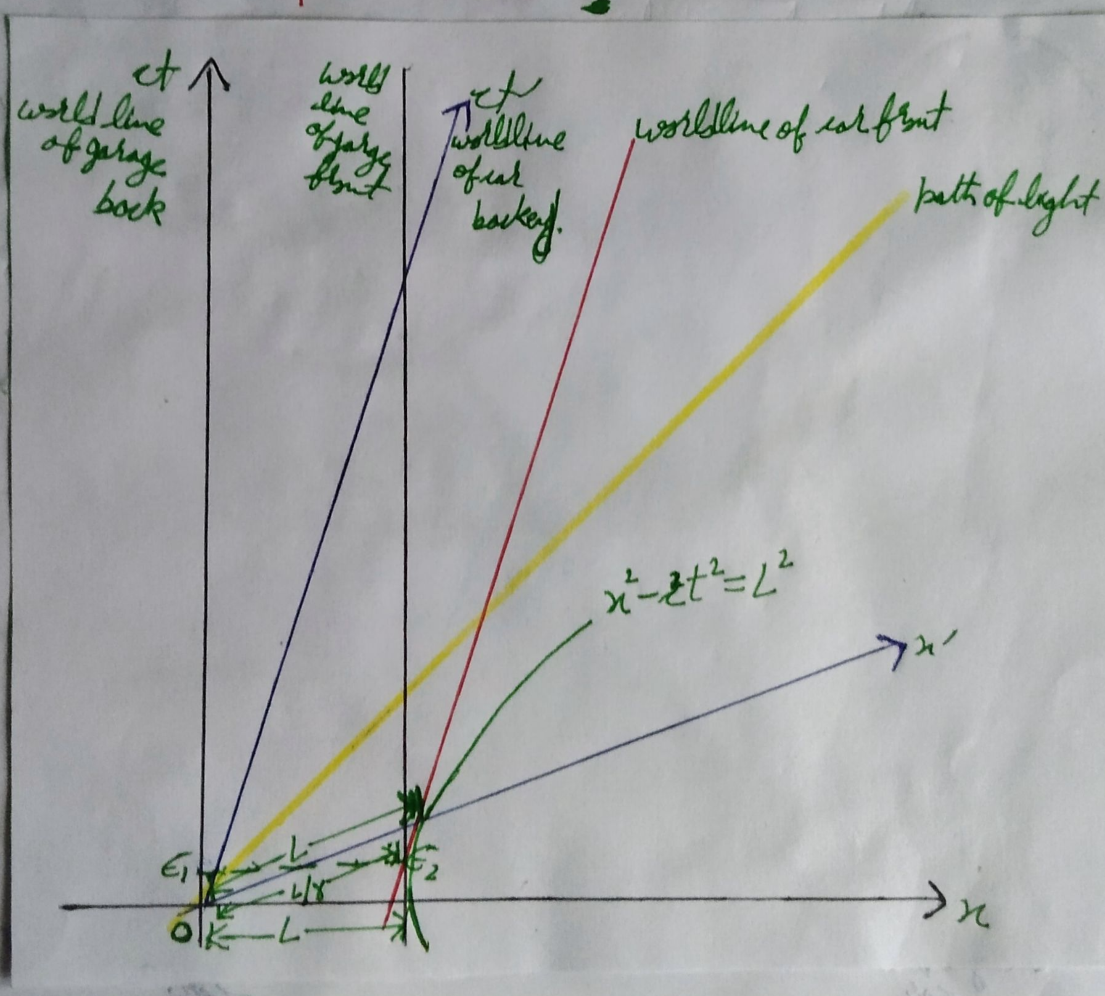
Spacetime Diagrams

To take an attempt to explain what's happening we consider four worldlines —

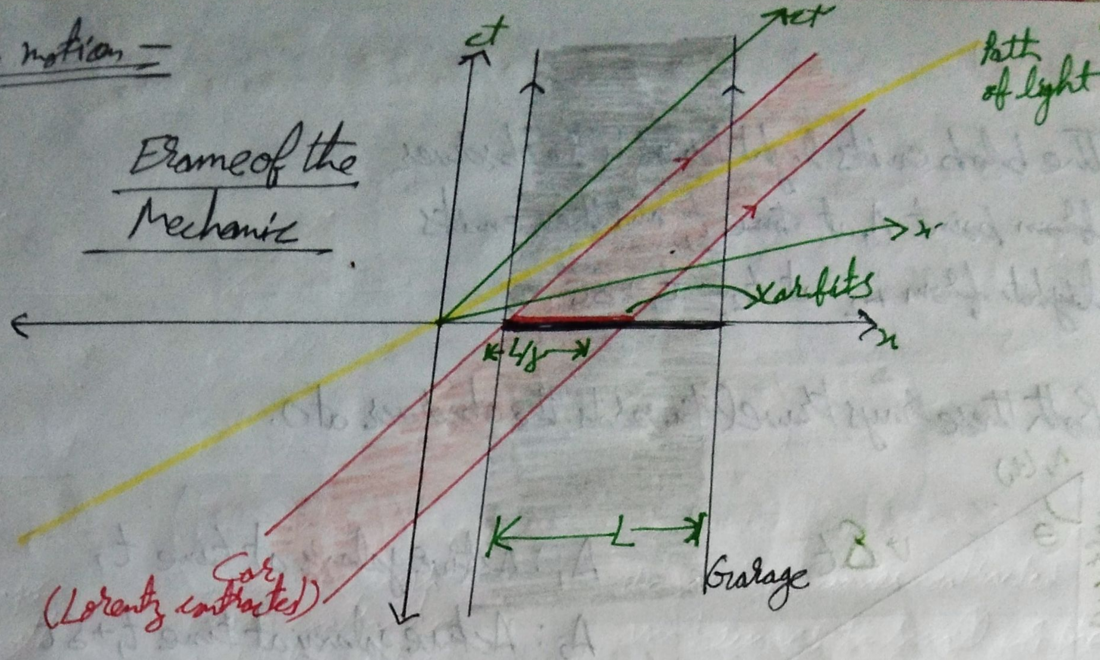
1. Front end of car
2. Back end of car
3. Front end of garage
4. Back end of garage.

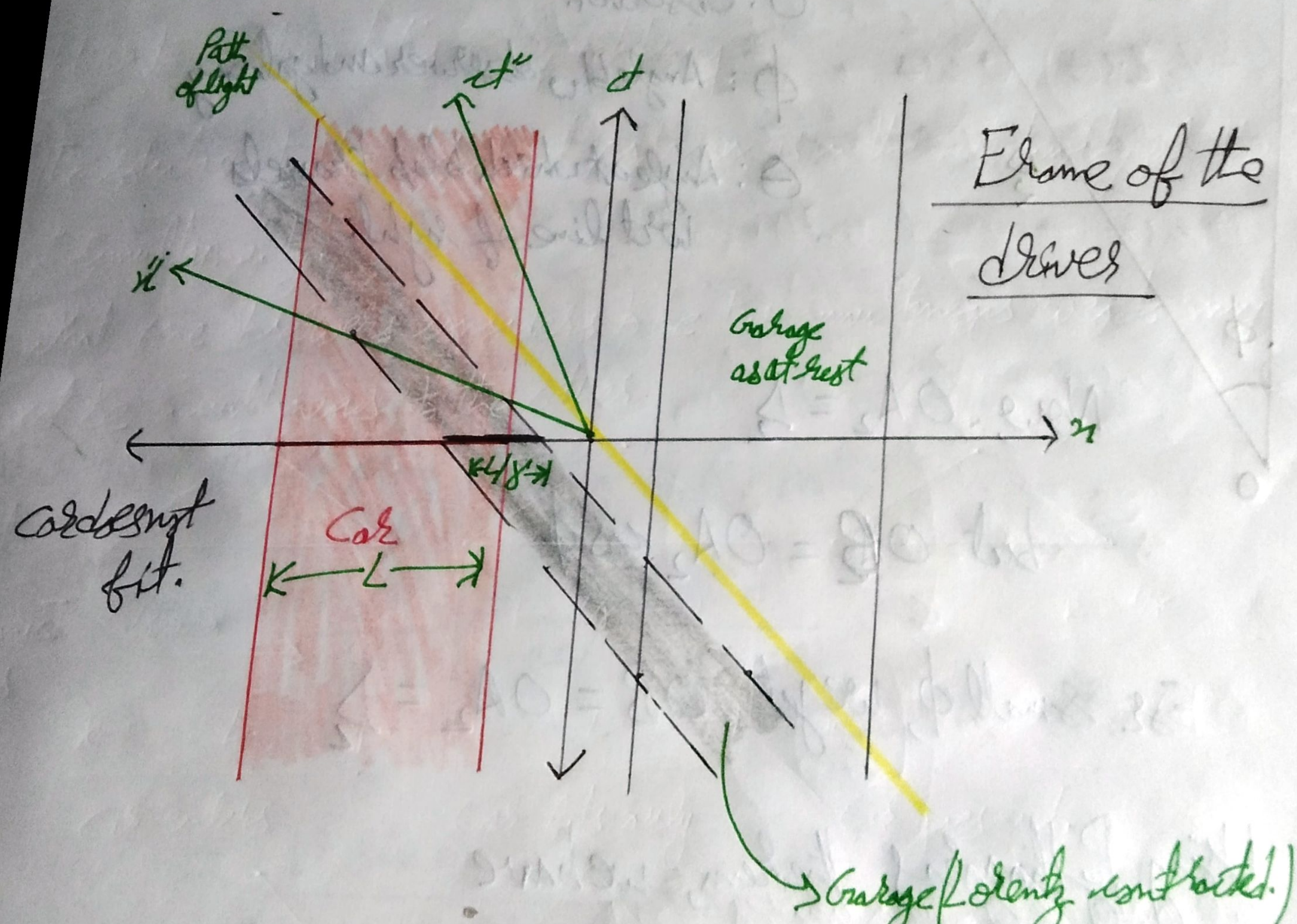
At rest —





In motion —

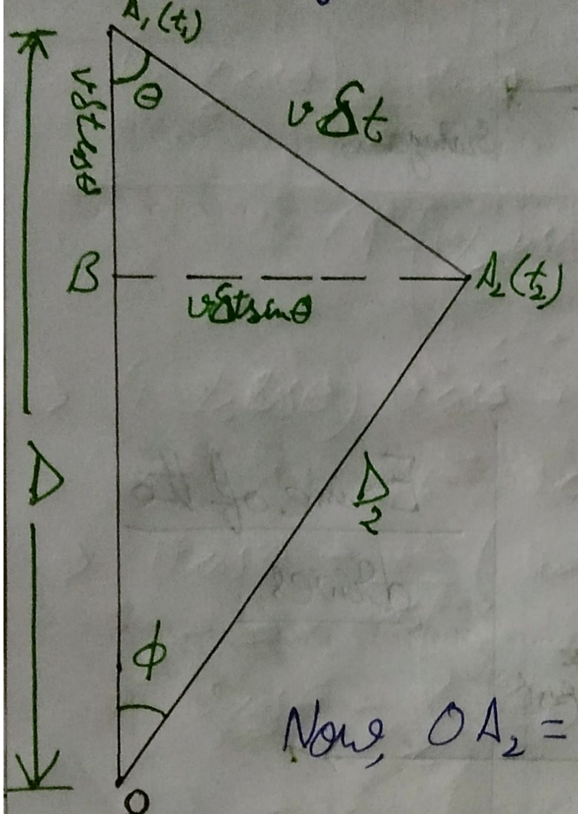

Frame of the Mechanic
ct
ct'
Path of light
x'
x
x car fits
L/γ
L
Garage
Car (Lorentz contracted)


Path of light
ct''
ct
Frame of the driver
x''
Garage as at rest
x
L/γ'
Car doesn't fit.
Car
L
Garage (Lorentz contracted.)

Q2. (a)

The blob emits light towards the observer from point A_1 at time t_1 and then emits light from A_2 at $t_2 = t_1 + \delta t$.

Both these rays travel towards the observer at O.



A_1 : Active galaxy at time t_1

A_2 : Active galaxy at time $t_1 + \delta t$

O : Observer.

ϕ : Angle b/w observer and galaxy

θ : Angle at which blob travels wrt line of sight.

Now, $OA_2 = D_2$

but $OB = OA_2 \cos\phi$

For small ϕ, we get $OB = OA_2 = D_2$

Using Pythagoras theorem, we have

$$OA_2 = \sqrt{OB^2 + A_2B^2}$$

$$= \sqrt{(D - v\delta t\cos\theta)^2 + (v\delta t\sin\theta)^2}$$

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$$OA_2 = \sqrt{\Delta^2 - 2v\Delta\delta t\cos\theta + v^2\delta t^2(\cos^2\theta + \sin^2\theta)}$$

$$OA_2 = \Delta\sqrt{1 + \frac{v^2\delta t^2}{\Delta^2} - \frac{2v\delta t\cos\theta}{\Delta}}$$

$$OA_2 = \Delta\sqrt{1 - \frac{2v\delta t\cos\theta}{\Delta}} \quad \left(\because \delta t \Rightarrow \text{small} \Rightarrow \delta t^2 \longrightarrow 0\right)$$

$$OA_2 = \Delta\left(1 - \frac{v\delta t\cos\theta}{\Delta} - \frac{1}{8}\frac{4v^2\delta t^2\cos^2\theta}{\Delta^2} - \cdots\right)$$

Neglecting higher order terms, we get

$$OA_2 = \Delta - v\delta t\cos\theta = \Delta_2$$

$$\Rightarrow \Delta - \Delta_2 = v\delta t\cos\theta.$$

Now, the observer recieves two light signals separated by time δt —

$$\delta t' = t_1 + \frac{\Delta}{c} - t_2 - \frac{\Delta_2}{c} = \delta t + \frac{\Delta - \Delta_2}{c}$$

$$= \delta t - \frac{v}{c}\delta t\cos\theta.$$

Now, the ~~app~~ velocity along the sky, (A_2B) path,

is $$v_{app} = \frac{A_2B}{\delta t'} = \frac{v\delta t\sin\theta}{\delta t\left(1 - \frac{v}{c}\cos\theta\right)}$$

$$= \frac{v\sin\theta}{\left(1 - \frac{v}{c}\cos\theta\right)}$$

(b) To find the value of θ, at which apparent speed is max,

we take $\frac{d\,v_{app}}{d\theta} = 0$

$$\Rightarrow \frac{d}{d\theta}\left(\frac{v\sin\theta}{1-\frac{v}{c}\cos\theta}\right) = 0$$

$$\Rightarrow \frac{\left(1-\frac{v}{c}\cos\theta\right)\frac{d}{d\theta}v\sin\theta - v\sin\theta\left(\frac{d}{d\theta}\left(1-\frac{v}{c}\cos\theta\right)\right)}{\left(1-\frac{v}{c}\cos\theta\right)^2} = 0$$

$$\Rightarrow \left(1-\frac{v}{c}\cos\theta\right)v\cos\theta + \frac{v^2}{c}\sin^2\theta = 0$$

$$\Rightarrow v\cos\theta - \frac{v^2}{c}\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow v\cos\theta - \frac{v^2}{c} = 0$$

$$\Rightarrow \cos\theta = \frac{v}{c}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{v}{c}\right)$$

This value of θ maximises v_{app}.

(c) Using the value of $\theta = \cos^{-1}(v/c)$, we have P2

$$v_{app} = \frac{v \sin\theta}{1 - v/c \cos\theta}$$

$$= \frac{v\sqrt{1-\cos^2\theta}}{1 - v/c \cos\theta}$$

$$= \frac{v\sqrt{1 - v^2/c^2}}{1 - \frac{v^2}{c^2}} = \frac{v}{\sqrt{1-v^2/c^2}} = v\gamma$$

but, $\gamma \gg 1$

$\Rightarrow$ v_{app} can be greater than the speed of light.

(d) Given, $v_{app} \simeq 10c$.

for θ_{max}, $v_{app} = \frac{v}{\sqrt{1 - v^2/c^2}}$

$$v_{app}^2 = \frac{v^2}{1 - v^2/c^2}$$

$$\Rightarrow v_{app}^2 = \frac{c^2 v^2 + v^2 v_{app}^2}{c^2}$$

$$\Rightarrow v_{app}^2 = \left(1 + \frac{v_{app}^2}{c^2}\right) v^2$$

$$\Rightarrow v^2 = \frac{v_{app}^2}{\left(1 + \frac{v_{app}^2}{c^2}\right)}$$

$$v = \frac{v_{app}}{\sqrt{1 + \frac{v_{app}^2}{c^2}}}$$

Using this and $\theta = \cos^{-1}(v/c)$, we have

$$\theta = \cos^{-1}\left(\frac{\frac{v_{app}}{\sqrt{1+ v_{app}^2/c^2}}}{c}\right)$$

$$= \cos^{-1}\left(\frac{v_{app}}{\sqrt{c^2 + v_{app}^2}}\right)$$

Now, taking $v_{app} \simeq 10c$, we get

$$\theta \simeq \cos^{-1}\left(\frac{10c}{\sqrt{c^2+100c^2}}\right)$$

$$\Rightarrow \theta \simeq \cos^{-1}\left(\frac{10c}{\sqrt{101}\cdot c}\right)$$

$$\Rightarrow \theta \simeq \cos^{-1}\left(\frac{10}{\sqrt{101}}\right)$$

$$\Rightarrow \theta \simeq \cos^{-1}(0.9950372)$$

$$\Rightarrow \theta \simeq 5.71 \text{ degrees}$$

Q3. (a) $\tanh \chi = \frac{v}{c}$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \text{ and } \beta = \frac{v}{c}$$

$$\Rightarrow \tanh \chi = \beta.$$

$$\gamma = (1-\beta^2)^{-1/2} = (1 - \tanh^2 \chi)^{-1/2} = (\operatorname{sech}^2 \chi)^{-1/2}$$

$$\Rightarrow \gamma = \cosh \chi$$

$$\Rightarrow \gamma\beta = \tanh\chi \cdot \cosh\chi = \sinh\chi$$

$$\Rightarrow \gamma = \cosh\chi \quad ; \gamma\beta = \sinh\chi$$

(b) For a frame moving along x-dir^n, Lorentz boost is given by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

For boost v, rapidity $= \alpha$; w, rapidity $= \beta$

Then, $\gamma_v = \cosh\alpha$; $\gamma_v\beta_v = \sinh \alpha$

and $\gamma_w = \cosh\beta$, $\gamma_w\beta_w = \sinh\beta$.

$\Rightarrow$ for boost v, Lorentz transformation matrix is $\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}$

Similarly for w, $\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}$

For successive boosts v and w, the Lorentz transformation will be

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}}_{v} \underbrace{\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}}_{w} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\alpha\cdot\cosh\beta + \sinh\alpha\sinh\beta & -\cosh\alpha\sinh\beta - \sinh\alpha\cosh\beta \\ -\cosh\alpha\sinh\beta - \sinh\alpha\cosh\beta & \sinh\alpha\sinh\beta + \cosh\alpha\cosh\beta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh(\alpha+\beta) & -\sinh(\alpha+\beta) \\ -\sinh(\alpha+\beta & \cosh(\alpha+\beta) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Clearly, we can see that rapidity associated with the boosts v and w is additive and is $\alpha+\beta$.

(c) As all stars recede from the previous at a velocity

$$\beta c = \frac{9c}{10}$$

$$\beta = 9/10$$

or $\tan\chi = 9/10$.

Using the principle of additivity of rapidity, after N boosts, the matrix looks like

$$\begin{pmatrix} \cosh[(N-1)\chi] & -\sinh[(N-1)\chi] \\ -\sinh[(N-1)\chi] & \cosh[(N-1)\chi] \end{pmatrix}$$

i.e, for the Nth star wrt first,

$$\gamma = \cosh[(N-1)\chi] \quad \text{and} \quad \gamma\beta = [\sinh\{(N-1)\chi\}]$$

$$\beta = \frac{\gamma\beta}{\gamma} = \frac{\sinh[(N-1)\chi]}{\cosh[(N-1)\chi]} = \tanh[(N-1)\chi]$$

$$\Rightarrow \frac{v}{c} = \tanh(N-1)\chi$$

$$v = c\tanh\left((N-1)\tanh^{-1}\beta\right) = c\tanh\left((N-1)\frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right)\right)$$

But, $\beta = 9/10$ (given)

$$\Rightarrow v = c\tanh\left(\left(\frac{N-1}{2}\right)\log\left(\frac{19}{1}\right)\right)$$

$$\Rightarrow v = c\tanh\left(\frac{(N-1)}{2}\log 19\right)$$

This is the required expression of velocity of Nth star wrt first star.

as $\tanh(\theta) < 1$ for any value of θ,

$$v_{1N} < c$$

Despite adding the velocities many times, we can't exceed the limit ~~c~~ c.

Q4. The phase of a wave can be represented as

$$\phi = \omega t - \mathbf{k}\cdot\mathbf{r}$$

$$= \frac{\omega t \times c}{c} - \mathbf{k}\cdot\mathbf{r}$$

$$= \left(\frac{\omega}{c}, -\mathbf{k}\right)\cdot\left(ct, \mathbf{r}\right) = k_\mu x^\mu$$

As phase is a scalar, it is invariant.

The transformation is

$$\begin{pmatrix} \omega'/c \\ k_x' \\ k_y' \\ k_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

$$\Rightarrow \omega'/c = \gamma\left(\omega/c - \beta k_x\right) \Rightarrow \omega' = \gamma\left(\omega - v k_x\right)$$

$$k_x' = \gamma\left(k_x - \beta\frac{\omega}{c}\right)$$

$$k_y' = k_y$$

$$k_z' = k_z$$

Q5(a) Given: Frame 1 moves with 3velocity β_1

$\Rightarrow$ wrt stationary frame, it moves with $\beta_1 c$.

Frame 2 moves with 3velocity β_2

$\Rightarrow$ wrt stationary frame, it moves with $\beta_2 c$

Or, more simply, stationary frame moves wrt frame 1 with velocity $-\beta_1 c$

frame 2 moves with respect to frame (stationary) with velocity $\beta_2 c$.

For frame 1, this becomes simply a case of classic velocity addition where β is the relative velocity b/w frame 1 and 2.

As $\bar{u} = \bar{v} \oplus \bar{u}'$, we substitute $u = \beta$

$v = -\beta_1$

and $u' = \beta_2$

$$u_{\parallel} = \frac{\bar{u}' + \bar{v}}{1 + \frac{\bar{u}_{\parallel}' \bar{v}}{c^2}}$$ (when velocities are along one direction)

and

$$u_{\perp} = \frac{\sqrt{1 - \frac{v^2}{c^2}} \; u_{\perp}'}{1 + \frac{\bar{v} \cdot \bar{u}_{\parallel}'}{c^2}}$$

Now, to obtain the velocity addition formula for $\beta = -\beta_1 \oplus \beta_2$, For β_1 and β_2,

$$\vec{\beta}_1^{\parallel} = \frac{\bar{\beta}_2 \cdot \bar{\beta}_1}{\beta_2^2} \bar{\beta}_2$$

and $$\bar{\beta}_1^{\perp} = \frac{-\bar{\beta}_2 \times (\bar{\beta}_2 \times \bar{\beta}_1)}{\beta_2^2} = \frac{(\bar{\beta}_2 \cdot \bar{\beta}_1)\bar{\beta}_2 - (\bar{\beta}_2 \cdot \bar{\beta}_2)\bar{\beta}_1}{\beta_2^2}.$$

Considering these as boosts, we can calculate $\beta_{\parallel}$ and $\beta_{\perp}$

$$\bar{\beta}_{\parallel} = \frac{-\bar{\beta}_1^{\parallel} + \bar{\beta}_2}{1 - \bar{\beta}_2 \cdot \bar{\beta}_1}$$

and $$\bar{\beta}_{\perp} = \frac{-\sqrt{1-\beta_2^2}\,\bar{\beta}_1^{\perp}}{1 - \beta_2 \cdot \beta_1}$$

Substituting $\bar{\beta}_1^{\parallel}$ and $\bar{\beta}_1^{\perp}$ into $\bar{\beta}_{\parallel} + \bar{\beta}_{\perp}$, we have

$$\bar{\beta}_{\parallel} + \bar{\beta}_{\perp} = \frac{-\bar{\beta}_1^{\parallel} + \bar{\beta}_2}{1 - \bar{\beta}_2 \cdot \bar{\beta}_1} + \frac{-\sqrt{1-\beta_2^2}\,\bar{\beta}_1^{\perp}}{1 - \bar{\beta}_2 \cdot \bar{\beta}_1}$$

$$= \frac{-\frac{\bar{\beta}_2 \cdot \bar{\beta}_1}{\beta_2^2} \cdot \bar{\beta}_2 + \bar{\beta}_2}{1 - \bar{\beta}_1 \cdot \bar{\beta}_2} - \frac{\sqrt{1-\beta_2^2}\left(\frac{(\bar{\beta}_2 \cdot \bar{\beta}_1)\bar{\beta}_2 - (\bar{\beta}_2 \cdot \bar{\beta}_2)\bar{\beta}_1}{\beta_2^2}\right)}{1 - \bar{\beta}_1 \cdot \bar{\beta}_2}$$

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$$\beta = \frac{1}{1-\bar{\beta}_1\cdot\bar{\beta}_2}\left[\bar{\beta}_2\left\{1-\frac{\bar{\beta}_2\cdot\bar{\beta}_1}{\beta_2^2}\left(1-\sqrt{1-\beta_2^2}\right)\right\}-\bar{\beta}_1\sqrt{1-\beta_2^2}\right]$$

Now we need to find β^2

$$\beta^2=\frac{1}{(1-\bar{\beta}_1\cdot\bar{\beta}_2)^2}\Bigg[\underbrace{\bar{\beta}_2\cdot\bar{\beta}_2\left\{1-\frac{\beta_2\cdot\beta_1}{\beta_2^2}\left(1-\sqrt{1-\beta_2^2}\right)\right\}^2}_{1}+\underbrace{(\beta_1\beta_1)(1-\beta_2^2)}_{2}$$
$$\underbrace{-2(\beta_2\cdot\beta_1)\left\{1-\frac{\beta_2\beta_1}{\beta_2^2}\left(1-\sqrt{1-\beta_2^2}\right)\right\}\left\{\sqrt{1-\beta_2^2}\right\}}_{3}\Bigg]$$

Calculating each separately -

(1) $$\bar{\beta}_2\cdot\bar{\beta}_2\left\{1-\frac{\beta_2\beta_1}{\beta_2^2}\left(1-\sqrt{1-\beta_2^2}\right)\right\}^2 = \cancel{\bar{\beta}_2\cdot\bar{\beta}_2\left(1-\frac{\beta_2\beta_1}{\beta_2^2}\right)\left(1-\sqrt{1-\beta_2^2}\right)^2}$$

$$=\bar{\beta}_2\cdot\bar{\beta}_2+\Bigg\{\left(-2(\bar{\beta}_2\cdot\bar{\beta}_1)\right)-2(\bar{\beta}_2\cdot\bar{\beta}_1)\sqrt{1-\beta_2^2}+\frac{2(\bar{\beta}_2\cdot\bar{\beta}_1)^2}{\beta_2^2}$$
$$\left(-2\frac{(\beta_2\cdot\beta_1)^2}{\beta_2^2}\sqrt{1-\beta_2^2}\right)-(\bar{\beta}_2\cdot\bar{\beta}_1)^2\Bigg\}$$

(2) $$(\beta_1\cdot\beta_1)(1-\beta_2^2) = (\beta_1\cdot\beta_1)-(\beta_1\beta_1)(\beta_2\beta_2)$$

(3) $$-2(\beta_2\cdot\beta_1)\left\{1-\frac{\beta_2\beta_1}{\beta_2^2}\left(1-\sqrt{1-\beta_2^2}\right)\left(\sqrt{1-\beta_2^2}\right)\right\} = -2(\beta_2\cdot\beta_1)\sqrt{1-\beta^2}+$$
$$2\left\{\left(\frac{(\beta_2\cdot\beta_1)^2}{\beta_2^2}\sqrt{1-\beta_2^2}\right)-\frac{(\beta_2\beta_1)^2}{\beta_2^2}+(\beta_2\beta_1)^2\right\}$$

The terms underlined or marked are same, just with opposite signs and get cancelled out.

Then, after cancelling out the terms we have,

$$= (\bar{\beta}_2 \cdot \bar{\beta}_2) + \left\{ -2(\bar{\beta}_2 \cdot \bar{\beta}_1) - (\bar{\beta}_1 \cdot \bar{\beta}_1)^2 \right\} + 2(\beta_2 \cdot \beta_1)^2$$

$$+ (\beta_1 \beta_1) - (\beta_1 \beta_1)(\beta_2 \beta_2)$$

$$= \left[(\beta_1 \cdot \beta_2)^2 - 2(\beta_1 \beta_2) + \beta_2 \beta_2 \right] + \left[(\beta_1 \cdot \beta_2)^2 - (\beta_1 \beta_1)(\beta_2 \beta_2) \right]$$

$$= \underline{(\beta_1 - \beta_2)^2 - |\beta_1 \times \beta_2|^2}$$

Using this value, we can write β^2 as

$$\beta^2 = \frac{(\beta_1 - \beta_2)^2 - |\beta_1 \times \beta_2|^2}{(1 - \bar{\beta}_1 \cdot \bar{\beta}_2)^2}$$

<u>Hence proved</u>

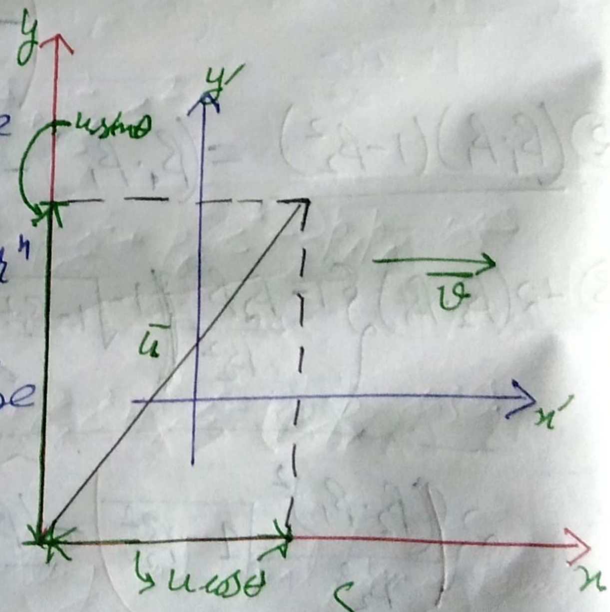
<u>(b)</u> From the S' frame, S frame is travelling with a velocity '$-v$'

In the S frame the bullet has the coordinates

$(t, ut\cos\theta, ut\sin\theta)^*$ "ignoring Z dir"

Let coordinates of the S' frame be

(t', x', y')



Since S' frame is moving along +ive x axis, so, the Lorentz factor, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

The differential time diff in the S' frame is-

$$\Delta t' = \gamma\left(\Delta t - \frac{vx}{c^2}\right) = \gamma\left(\Delta t - \frac{v\Delta t\, u\cos\theta}{c^2}\right)$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma(\Delta t\, u\cos\theta - v\Delta t)$$

$$\Delta y' = \Delta y = u\Delta t\sin\theta$$

Now, we can calculate the velocity components of the bullet in the S' frame. $u_x' = \dfrac{\Delta x'}{\Delta t'} = \dfrac{u\cos\theta - v}{1 - \frac{uv\cos\theta}{c^2}}$

$$u_y' = \frac{\Delta y'}{\Delta t'} = \frac{u\sin\theta}{\gamma\left(1 - \frac{uv\cos\theta}{c^2}\right)}$$

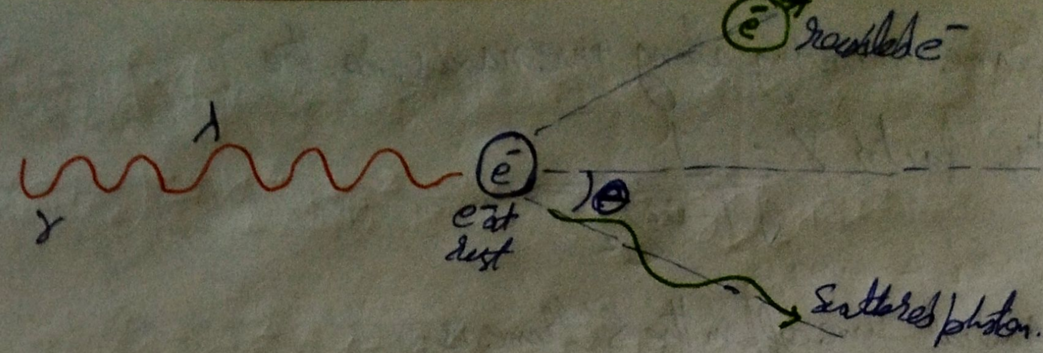
Then $\tan\theta' = \dfrac{u_y'}{u_x'} \Rightarrow \theta' = \tan^{-1}\left(\dfrac{u_y'}{u_x'}\right) = \tan^{-1}\left(\dfrac{u\sin\theta}{\gamma(u\cos\theta - v)}\right)$

$$\theta' = \tan^{-1}\left(\frac{u\sin\theta\sqrt{1-v^2/c^2}}{u\cos\theta - v}\right)$$

If the bullet is assumed to be a photon, then $u = c$

$$\Rightarrow \theta' = \tan^{-1}\left(\frac{c\sin\theta\sqrt{1-v^2/c^2}}{c\cos\theta - v}\right)$$

(L) (58)



Let us derive the expression for change in $\Delta\lambda$,
first we start with the law of conservation of momentum.

$$E_\gamma + E_e = E_{\gamma'} + E_{e'}$$

$$\hbar\omega + m_e c^2 = \hbar\omega' + \sqrt{m_e^2 c^4 + p_e^2 c^2} \quad -①$$

Also, the three momentum components of the four momentum conservation gives the linear momentum conservation rel^n.

$$\bar{p}_\gamma = \bar{p}_{e'} + \bar{p}_{\gamma'}$$

$$\Rightarrow \bar{p}_{e'} = \bar{p}_\gamma - \bar{p}_{\gamma'} = \sqrt{p_\gamma^2 + p_{\gamma'}^2 - 2p_\gamma p_{\gamma'} \cos\theta} \quad -②$$

From ① and ②, we get

$$\hbar\omega + m_e c^2 = \hbar\omega' + \sqrt{m_e^2 c^4 + p_e^2 c^2}$$

$$= \hbar\omega' + \sqrt{m_e^2 c^4 + p_\gamma^2 c^2 + p_{\gamma'}^2 c^2 - 2p_\gamma p_{\gamma'} \cos\theta}$$

$$\hbar(\omega - \omega') + m_e c^2 = \sqrt{m_e^2 c^4 + p_\gamma^2 c^2 + p_{\gamma'}^2 c^2 - 2p_\gamma p_{\gamma'} \cos\theta}$$

Squaring both sides, and cancelling out equal/same terms, we have

$$\hbar(\omega\omega')(1-\cos\theta) = (\omega-\omega')\, m_0 c^2$$

$$\frac{\hbar}{m_0 c^2} = \frac{(\omega-\omega')}{\omega\omega'} \times \frac{1}{1-\cos\theta}$$

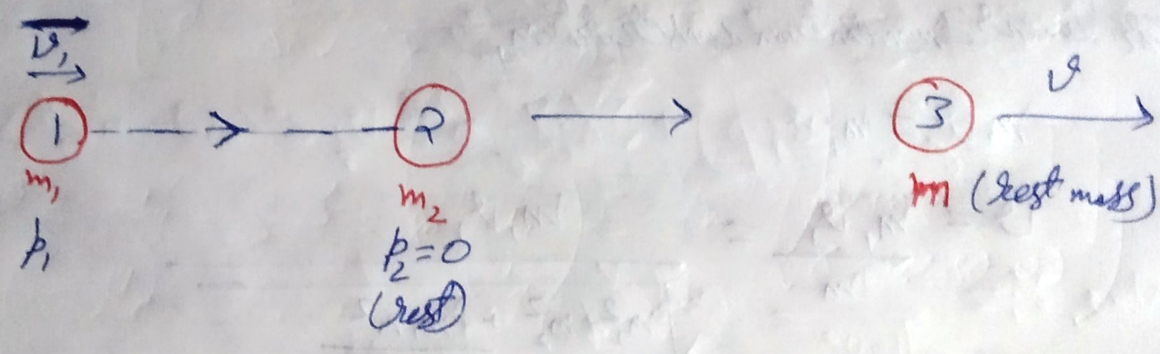
$$\frac{\hbar}{m_0 c^2} = \left(\frac{\lambda'}{2\pi c} - \frac{\lambda}{2\pi c}\right) \times \frac{1}{(1-\cos\theta)}$$

$$\Rightarrow \frac{h}{2\pi c\, m_0 c} = \frac{(\lambda'-\lambda)}{2\pi c} \times \frac{1}{(1-\cos\theta)}$$

$$\Rightarrow \frac{h}{m_0 c} = \frac{\Delta\lambda}{(1-\cos\theta)}$$

or, $$\Delta\lambda = \frac{h(1-\cos\theta)}{m_e c}$$ Ans.

1)



Since this is a particle-particle interaction, we can use the 4-momentum conservation laws.

Initial momentum $= P \equiv \left(\frac{1}{c}(E_1+E_2), \gamma m_1 \vec{v_1}\right)$

Final momentum $= Q \equiv \left(\gamma_v m c, \gamma_v m \vec{v}\right)$

$$P.P = Q.Q$$

$$\frac{1}{c^2}(\epsilon_1+\epsilon_2)^2-\gamma_1^2 m_1^2 v_1^2 = \gamma_0^2 m^2 c^2-\gamma_v^2 m^2 v^2$$

$$\Rightarrow \frac{1}{c^2}(\epsilon_1+\epsilon_2)^2-\gamma_1^2 m_1^2 v_1^2 = m^2 c^2$$

$$m_1^2 c^4-\gamma_1^2 m_1^2 v_1^2 c^2+m_2^2 c^4+\underbrace{2m_2 c^2(T_1+m_1c^2)}_{\text{KE of particle 1.}}-\gamma_1^2 m_1^2 v_1^2 c^2 = m^2 c^4$$

$$m^2 c^4 = m_1^2 c^4+m_2^2 c^4+2m_1 m_2 \gamma_1 c^4$$

$$m^2 = m_1^2+m_2^2+2m_1 m_2 \gamma_1$$

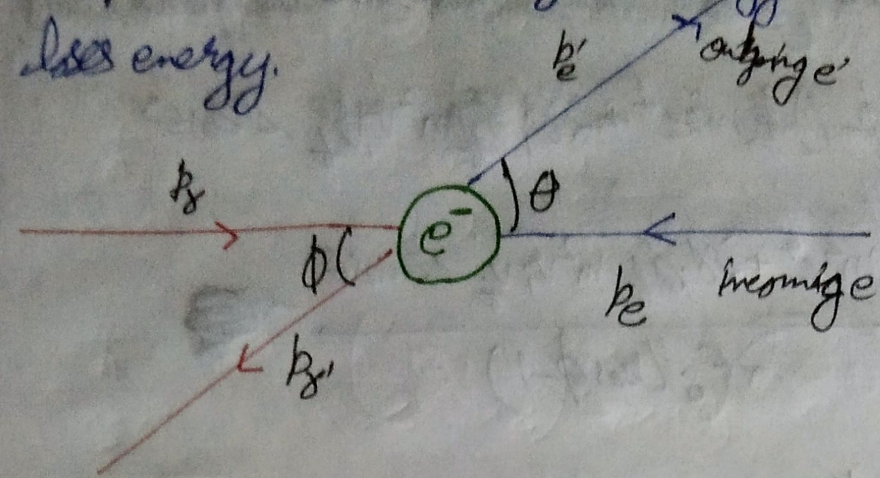
$$m = \sqrt{m_1^2+m_2^2+\frac{2m_1 m_2}{\sqrt{1-\frac{v^2}{c^2}}}}$$

For linear momentum conservation,

$$mv = m_1 v_1$$

$$v = \frac{m_1 v_1}{m} = \frac{m_1 v_1}{\sqrt{m_1^2+m_2^2+\frac{2m_1 m^2}{\sqrt{1-v^2/c^2}}}}$$

(e) In this phenomenon, the photon gains energy and the electron loses energy.



the total energy before and after the collision remains conserved and can be formulated as

$$E_e + E_\gamma = E_e' + E_{\gamma'}$$

$$\sqrt{m^2c^2 + \gamma_i^2 m^2 v_i^2 c^2} + E_\gamma = \sqrt{m^2c^4 + \gamma_f^2 m^2 v_f^2 c^2} + E_{\gamma'}$$

Since $E_e >> mc^2$; m_2c^2 on LHS can be neglected

$$\gamma_i m v_i c + E_\gamma = \sqrt{m^2c^4 + \gamma_f^2 m^2 v_f^2 c^2} + E_{\gamma'}$$

$$E_{\gamma'} - E_\gamma = \gamma_i m v_i c - \sqrt{m^2c^4 + \gamma_f^2 m^2 v_f^2 c^2} \quad \text{—— (1)}$$

From the law of conservation of momentum, we have

$$\overline{p_e} + \overline{p_\gamma} = \overline{p_e'} + \overline{p_{\gamma'}}$$

$$\overline{p_e} - \overline{p_e'} = \overline{p_{\gamma'}} - \overline{p_\gamma}$$

$$\Rightarrow p_e^2 + p_e'^2 - 2p_e p_e' \cos\theta = p_\gamma^2 + p_{\gamma'}^2 - 2p_\gamma p_{\gamma'} \cos\phi$$

$$\Rightarrow \gamma_i^2 m^2 v_i^2 + \gamma_f^2 m^2 v_f^2 - 2\gamma_i \gamma_f m^2 v_i v_f \cos\theta = \left(\frac{h\nu'}{c}\right)^2 + \left(\frac{h\nu}{c}\right)^2 - \left(\frac{2h^2\nu\nu'}{c^2}\right)\cos\phi \quad \text{—— (2)}$$

Taking (1), squaring both sides and cancelling out terms,

$$2E_\gamma E_\gamma(\cos\phi-1) = m^2c^4+2\gamma_i\gamma_f m^2 v_i v_f c^2\cos\theta$$

$$E_{\gamma'} = \frac{m^2c^4+2\gamma_i\gamma_f m^2 v_i v_f c^2\cos\theta}{2E_\gamma(\cos\phi-1)}$$

Subtracting E_γ both sides we have,

$$E_\gamma' - E_\gamma = \frac{m^2c^4+2\gamma_i\gamma_f m^2 v_i v_f c^2\cos\theta - 2E_\gamma^2(\cos\phi-1)}{2E_\gamma(\cos\phi-1)}$$

$$= \frac{E_\gamma^2(1-\cos\phi)+2\gamma_i\gamma_f m^2 v_i v_f c^2\cos\theta - m^2c^4}{2E_\gamma(1-\cos\phi)}$$

For energy transfer to take place,

$$E_\gamma^2(1-\cos\phi)+2\gamma_i\gamma_f m^2 v_i v_f c^2\cos\theta - m^2c^4 > 0.$$

Now, $\gamma_i = \frac{1}{\sqrt{1-\frac{v_i^2}{c^2}}}$ and $\gamma_f = \frac{1}{\sqrt{1-\frac{v_f^2}{c^2}}}$

Putting these in above equation and solving further we have

$$E_\gamma^2(1-\cos\phi)+m^2c^2\left(\frac{2cv_i}{\sqrt{c^2-v_i^2}}\times\frac{cv_f}{\sqrt{c^2-v_f^2}}\cos\theta - c^2\right)>0$$

$$\epsilon_\gamma^2(1-\cos\phi)\, m^2c^4\left(\frac{2v_i v_f \cos\theta}{\sqrt{(c^2-v_i^2)}\sqrt{c^2-v_f^2}}-1\right) > 0$$

To calculate the max. energy the particle can transfer to the photon, we take $\phi = -\pi$ and $\theta = 0$

max energy transfer $\epsilon_{max} = \epsilon_{\gamma'} - \epsilon_\gamma$

$$= \epsilon_\gamma^2(1-\cos(-\pi)) + \left(\frac{2\cos(0)\, v_i v_f}{\sqrt{c^2-v_i^2}\sqrt{c^2-v_f^2}}-1\right) m^2c^4$$

$$= \frac{2\epsilon_\gamma^2}{4\epsilon_\gamma^2} + \frac{m^2c^4}{4\epsilon_\gamma}\left(\frac{2v_i v_f}{\sqrt{c^2-v_i^2}\sqrt{c^2-v_f^2}}-1\right)$$

$$= \frac{1}{2}\epsilon_\gamma + \frac{m^2c^4}{4\epsilon_\gamma}\left(\frac{2v_i v_f}{\sqrt{c^2-v_i^2}\sqrt{c^2-v_f^2}}-1\right)$$

To approximate the energy gained by the photon, we shift to the rest frame of the e^-

In this case, Energy of photon, $\overline{\epsilon_{\gamma'}} = \dfrac{\overline{\epsilon_\gamma}}{1+\frac{\overline{\epsilon_\gamma}}{m_e c^2}(1-\cos\theta')}$

To convert energy to the lab frame, we apply the ~~D~~Doppler formula with Lorentz factor γ_v(~~ ~~) where v is the relativistic velocity of the ~~photon~~ electron.

$$\epsilon_{\gamma'} = \gamma_v \bar{\epsilon}_{\gamma'} \left(1 + \frac{v}{c} \cos\theta_1'\right)$$

Taking $\theta_1' = \pi/2$, we have $\epsilon_{\gamma'} \simeq \gamma_v \bar{E}_\gamma$

Again applying doppler's formula, we have

$$\epsilon_{\gamma'} = \epsilon_\gamma \gamma_v (1 - \beta \cos\theta)$$

~~Agg~~ Again taking $\theta = \pi/2$, we have

$$\epsilon_{\gamma'} \simeq \epsilon_\gamma \cdot \gamma_v \simeq \gamma_v^2 \epsilon_\gamma$$

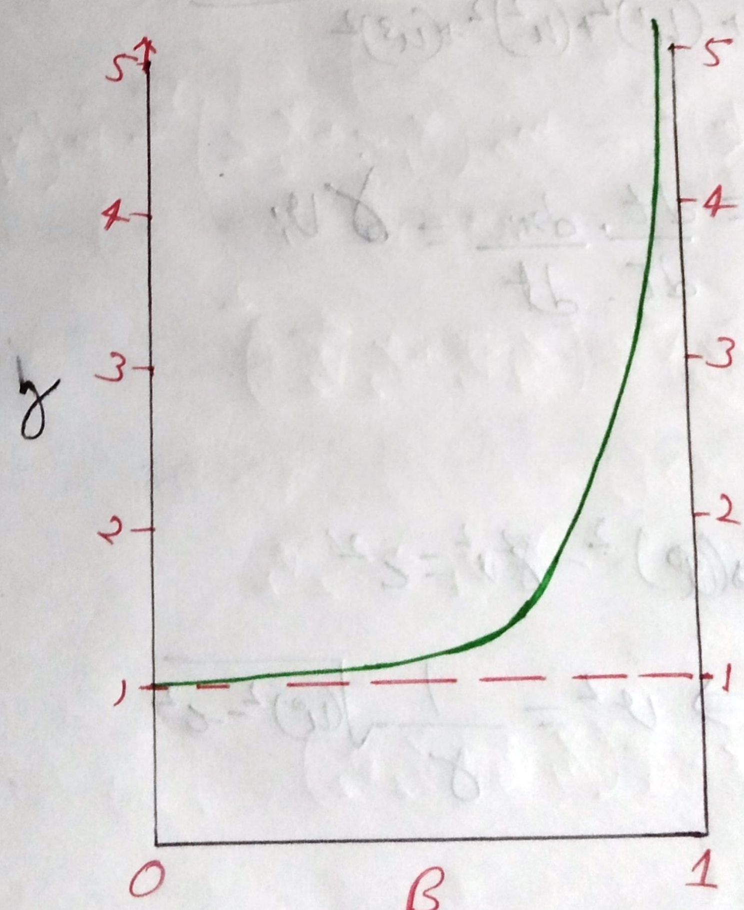
At $\beta = 0$; $\gamma = 1$

At $\beta = 1$; $\gamma = \infty$

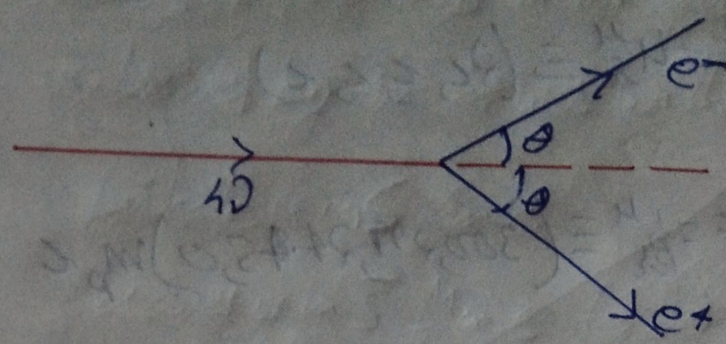
$$\frac{d\gamma}{d\beta} = \frac{\frac{-2\beta}{-2\sqrt{1-\beta^2}}}{1-\beta^2} = \frac{\beta}{(1-\beta^2)^{3/2}}$$

slope at $\beta = 0 \Rightarrow \left.\frac{d\gamma}{d\beta}\right|_{\beta=0} = 0$

at $\beta = 1 \Rightarrow \left.\frac{d\gamma}{d\beta}\right|_{\beta=1} = \infty$



(i)



Initial 4 momentum : $\left[K = \left(\omega/c, \bar{k}\right)\right]$

final 4 momentum : $\left[P = \left(m_0(\gamma_1+\gamma_2)c,\ m_0(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2)\right)\right]$

By law of conservation of 4 momentum

$$K.K = P.P$$

$$\left(\frac{\omega}{c}\right)^2 - |k|^2 = \left(m_0 c(\gamma_1+\gamma_2)\right)^2 - m_0\left(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2\right)^2$$

$$\left(\frac{c|k|}{c}\right)^2 - |k|^2 = m_0\left(\gamma_1 c+\gamma_2 c\right)^2 - m_0\left(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2\right)^2$$

$$\Rightarrow \left(\gamma_1 c+\gamma_2 c\right)^2 = \left(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2\right)^2$$

$$\Rightarrow \gamma_1 c+\gamma_2 c = \gamma_1 v_1 + \gamma_2 v_2$$

or

$$\gamma_1 c+\gamma_2 c = -(\gamma_1 v_1 + \gamma_2 v_2)$$

Case I: $\gamma_1 c + \gamma_2 c = \gamma_1 v + \gamma_2 v$

$$\Rightarrow \gamma_1 (c - v_1) + \gamma_2 (c - v_2) = 0$$

Since $\gamma >> 1$

$\Rightarrow$ this is possible iff $c = v_1 = v_2$

Case II: $\gamma_1 c + \gamma_2 c = -(\gamma_1 v + \gamma_2 v)$

$$\Rightarrow \gamma_1 (c + v_1) + \gamma_2 (c + v_2) = 0$$

$c = -v_1$ and $c = -v_2$

$\Rightarrow$ This process is kinematically forbidden

(9)

Calculating from the frame of rocket A,

velocity of earth frame $= u_1 = c/2$

velocity of rocket B wrt earth frame $= u_2 = c/2$

relative velocity of A wrt B $= v = \dfrac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}} = \dfrac{4c}{5}$

Length of B as seen by A $= L' = L_0 \sqrt{1 - v^2/c^2}$

$$= L_0 \sqrt{1 - \frac{16}{25}} = \frac{3}{5} L_0$$

(k) Particle of rest mass m and four momentum P is observed by an observer with 4 velocity u.

particle is in S frame.

observer is in $\bar{S}$ frame (velocity u)

4 momentum of particle in rest frame

$$p \equiv (E_0/c, p_x, p_y, p_z)$$

When viewed from a frame moving at velocity $u = u_x$ wrt rest frame, 4 momentum will be

$$\vec{\bar{p}} \equiv \left(E/c, \bar{p}_x, \bar{p}_y, \bar{p}_z\right)$$

and 4 velocity, $\vec{\bar{u}} = (c, 0, 0, 0)$

(i) $p \cdot u = \bar{p} \cdot \bar{u}$ (invariant momentum)

$$p \cdot u = \frac{E}{c} \times c - 0$$

$$E = p \cdot u$$

(ii) $\vec{p}^{*}\cdot\vec{p}^{*} = m^2c^2 = p\cdot p \Rightarrow m = \frac{1}{c}\sqrt{p\cdot p}$

(iii) $\vec{p}^{*}\cdot\vec{p}^{*} = \frac{\epsilon^2}{c^2} - \vec{p}^{*}\vec{p}^{*}$

$$|p^{*}|^2 = \frac{\epsilon^2}{c^2} - p^{*}p^{*}$$

But $p^{*}\cdot p^{*} = p\cdot p$ and $\epsilon = p\cdot u$

$$\Rightarrow (p^{*})^2 = \frac{1}{c^2}(p\cdot u)^2 - p\cdot p$$

(iv) $p^{*}\cdot p^{*} = \frac{\epsilon^2}{c^2} - p^2$

$$p^2 = \frac{\epsilon^2}{c^2} - p^{*}\cdot p^{*}$$

$$\gamma^2 m^2 v^2 = \epsilon^2/c^2 - p\cdot p$$

$$\Rightarrow v^2 = \frac{\epsilon^2}{\gamma^2 m^2 c^2} - \frac{p\cdot p}{\gamma^2 m^2}$$

$$v^2 = \frac{\epsilon^2 c^2}{\gamma^2 m^2 c^4} - \frac{p\cdot p\, c^4}{\gamma^2 m^2 c^2} \quad (\because \gamma m c^2 = \epsilon)$$

$$v^2 = \frac{\epsilon^2 c^2}{\epsilon^2} - \frac{c^4\cdot p\cdot p}{\epsilon^2} \Rightarrow v = c^2\sqrt{\left[\frac{1}{c^2} - \frac{p\cdot p}{\epsilon^2}\right]}$$