FYS-STK4155 Project 1

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I. THEORY

II. METHOD

III. RESULTS

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REFERENCES

Appendix A: Mean values and variances calculations

The main regression method used in this report is the ordinary least squares method. This appensix shows the calculations for some of the equations used to produce the results shiwn in this report.

We have assumed that our data can be described by the continous function $f(\boldsymbol{x})$, and an error term $\boldsymbol{\epsilon}\ N(0, \sigma^2)$. If we approximate the function with the solution derived from a model $\tilde{\boldsymbol{y}} = X\boldsymbol{\beta}$ the data can be described with $\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$. The expectation value

$$
\begin{aligned}
\mathbb{E}(\boldsymbol{y}) &= \mathbb{E}(X\boldsymbol{\beta} + \boldsymbol{\epsilon}) \\
&= \mathbb{E}(X\boldsymbol{\beta}) + \mathbb{E}(\boldsymbol{\epsilon}) && \text{where the expected value } \boldsymbol{\epsilon} = 0 \\
\mathbb{E}(y_i) &= \sum_{j=0}^{P-1} X_{i,j}\beta_j && \text{for the each element} \\
&= X_{i,*}\beta_i && \text{where } _* \text{ replace the sum over index } i
\end{aligned}
$$

The variance for the element y_i can be found by

$$
\begin{aligned}
\mathbb{V}(y_i) &= \mathbb{E}\big[(y_i - \mathbb{E}(y_i))^2\big] \\
&= \mathbb{E}(y_i^2) - (\mathbb{E}(y_i)^2) \\
&= \mathbb{E}((X_{i,*}\beta_i + \epsilon_i)^2) - (X_{i,*}\beta_i)^2 \\
&= \mathbb{E}((X_{i,*}\beta_i)^2 + 2\epsilon_i X_{i,*}\beta_i + \epsilon^2) - (X_{i,*}\beta_i)^2 \\
&= \mathbb{E}((X_{i,*}\beta_i)^2) + \mathbb{E}(2\epsilon_i X_{i,*}\beta_i) + \mathbb{E}(\epsilon^2) - (X_{i,*}\beta_i)^2 \\
&= (X_{i,*}\beta_i)^2 + \mathbb{E}(\epsilon^2) - (X_{i,*}\beta_i)^2 \\
&= \mathbb{E}(\epsilon^2) = \sigma^2
\end{aligned}
$$

The expression for the optimal parameter

$$
\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}
$$

We find the expected value of $\hat{\boldsymbol{\beta}}$

$$
\begin{aligned}
\mathbb{E}(\hat{\boldsymbol{\beta}}) &= \mathbb{E}((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}) \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\mathbb{E}(\boldsymbol{y}) && \text{using that } \boldsymbol{X} \text{ is a non-stochastic variable} \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\beta} && \text{using } \mathbb{E}(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta} \\
&= \boldsymbol{\beta}
\end{aligned}
$$

we can find the variance by

$$
\begin{aligned}
\mathbb{V}(\hat{\boldsymbol{\beta}}) &= \mathbb{E}\big[(\hat{\boldsymbol{\beta}} - \mathbb{E}(\hat{\boldsymbol{\beta}}))^2\big] \\
&= \mathbb{E}(\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T) - \mathbb{E}(\hat{\boldsymbol{\beta}})^2 \\
&= \mathbb{E}(((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y})((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y})^T) - \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \\
&= \mathbb{E}((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}) - \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\mathbb{E}(\boldsymbol{y}\boldsymbol{y}^T)\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1} - \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{\beta}\boldsymbol{\beta}^T\boldsymbol{X}^T + \sigma^2)\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1} - \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \\
&= \boldsymbol{\beta}\boldsymbol{\beta}^T + \sigma^2((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}) - \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \\
&= \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}
\end{aligned}
$$

Appendix B: Bias-variance trade-off