

00. FUNCTIONS & SETS

sets

$$A = \{x \mid \text{properties of } x\}$$

- $A \subseteq B$: A is a subset of B
- $A \not\subseteq B$: A is not a subset of B
- $A = B \iff A \subseteq B \wedge B \subseteq A$
- operations on sets**
 - union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
 - intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
 - difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$

common notations on sets:

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ where $\mathbb{N} = \mathbb{Z}^+$
- \emptyset : empty set

closed interval (inclusive):

$$[a, b] = \{x \mid a \leq x \leq b\}$$

open interval (exclusive):

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, \infty) = \{x \mid a < x\}$$

functions

- existence:** $\forall a \in A, f(a) \in B$
- uniqueness:** $\forall a \in A$ has only one image in B .
- for $f: A \rightarrow B$
 - domain: A , codomain: B
 - range: $\{f(x) \mid x \in A\}$
- for this mod:
 - $A, B \subseteq \mathbb{R}$
 - if A is not stated, the domain of f is the largest possible set for which f is defined
 - if B is not stated, $B = \mathbb{R}$

graphs of functions

The graph of f is the set

$$G(f) := \{(x, f(x)) \mid x \in A\}$$

- if $A, B \subseteq \mathbb{R}$ then $G(f) \subseteq A \times B \subseteq \mathbb{R} \times \mathbb{R}$
- each element is a point on the Cartesian plane \mathbb{R}^2

algebra of functions

function	domain
$(f+g)(x) := f(x) + g(x)$	$A \cap B$
$(f-g)(x) := f(x) - g(x)$	$A \cap B$
$(fg)(x) := f(x)g(x)$	$A \cap B$
$(f/g)(x) := f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

types of functions

- rational function:** $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials and $Q(x) \neq 0$
 - every polynomial is a rational function ($Q(x) = 1$)
- algebraic function:** constructed from polynomials using algebraic operations
- a function f is **increasing** on a set I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$.
- a function f is **decreasing** on a set I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$.
- even/odd:
 - even function:** $\forall x, f(-x) = f(x)$

- symmetric about the y -axis
- odd function:** $\forall x, f(-x) = -f(x)$
 - symmetric about the origin O
- any function defined on \mathbb{R} can be decomposed *uniquely* into the sum of an even function and an odd function
- power function:** x^n
 - x^n is $\begin{cases} \text{an odd function,} & \text{if } n \text{ is odd} \\ \text{an even function,} & \text{if } n \text{ is even} \end{cases}$

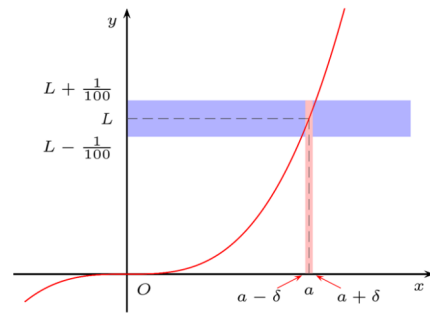
01. LIMITS

precise definition of limits

Let f be a function defined on an open interval containing a , except possibly at a .

The limit of $f(x)$ (as x approaches a) equals L if,

$$\text{for every } \epsilon > 0 \text{ there is } \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$



informally,

- $0 < |x - a| < \delta \Rightarrow x$ is close to but not equal to a .
- $0 < |f(x) - L| < \epsilon \Rightarrow f(x)$ is arbitrarily close to L .

limit laws

you cannot apply any laws on limits UNLESS you have shown that the limit exists!

- Let $c \in \mathbb{R}$. $\lim_{x \rightarrow a} c = c$
- $\lim_{x \rightarrow a} x = a$

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant.

- $\lim_{x \rightarrow a} (cf(x)) = cL = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} f(x) = 0$

direct substitution property

Let f be a polynomial or rational function.

If a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

If $f(x) = g(x)$ for all x near a except possibly at a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

If a is not in the domain (e.g. 0 denominator), don't apply directly - convert to an equivalent function and then sub in

inequalities on limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

lemma

if $f(x) \leq g(x)$ for all x near a (except possibly at a), then $L \leq M$.

lemma

If $f(x) \geq 0$ for all x , then $L \geq 0$.

one-sided limits

- limit laws also hold for one-sided limits

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

$$f(x) \rightarrow L \Leftarrow x \rightarrow a \Leftrightarrow \begin{cases} x \rightarrow a^+ \Rightarrow f(x) \rightarrow L \\ x \rightarrow a^- \Rightarrow f(x) \rightarrow L \end{cases}$$

definition of one-sided limits

$$\text{LH Limit: } \lim_{x \rightarrow a^-} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$

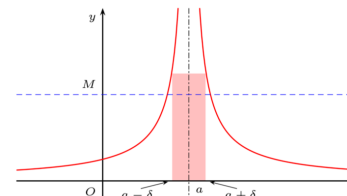
$$\text{RH Limit: } \lim_{x \rightarrow a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$

definition of infinite limits ($\lim f(x) = \infty$)

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for every $M > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > M$



negative infinite limit:

$$0 < |x - a| < \delta \Rightarrow f(x) < M$$

- ∞ is NOT a number \Rightarrow an infinite limit does NOT exist

limits to infinity ($\lim_{x \rightarrow \infty}$)

Suppose f is defined on $[M, \infty)$ for some $M \in \mathbb{R}$:

$$\lim_{x \rightarrow \infty} f(x) = L:$$

For every $\epsilon > 0$, there exists N such that $x > N \Rightarrow |f(x) - L| < \epsilon$

$$\lim_{x \rightarrow \infty} f(x) = \infty:$$

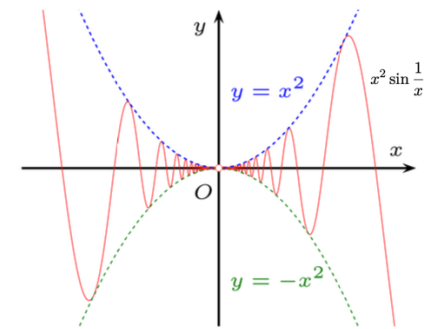
For every $M > 0$, there exists N such that $x > N \Rightarrow f(x) > M$

squeeze theorem

Suppose $f(x)$ is bounded by $g(x)$ and $h(x)$ where

- $g(x) \leq f(x) \leq h(x)$ for all x near a (except at a), and
- $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$.

Then $\lim_{x \rightarrow a} f(x) = L$.



02. CONTINUOUS FUNCTIONS

definition of continuity

a function f is **continuous at** $a \iff$

f is continuous from the left and from the right at a .

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$$

a function f is **continuous at an interval** if it is continuous at every number in the interval.

f is continuous on **open interval** (a, b)

$\iff f$ is continuous at every $x \in (a, b)$

f is continuous on **closed interval** $[a, b]$

$$\iff \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$$

precise definition of continuity

a function f is **continuous** at a number a if

for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

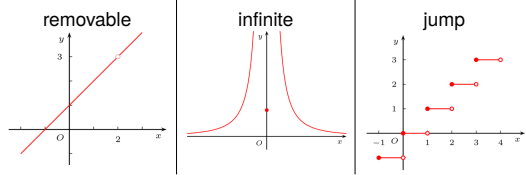
- aka $\lim_{x \rightarrow a} f(x) = f(a)$

continuity test

f is continuous at a ⇔

- 1. f is defined at a (a is in the domain of f)
- 2. lim_{x→a} f(x) exists
- 3. lim_{x→a} f(x) = f(a)

examples of discontinuity



properties of continuous functions

let f and g be functions continuous at a. let c be a constant.

- 1. cf is continuous at a
- 2. f + g is continuous at a
- 3. f - g is continuous at a
- 4. fg is continuous at a
- 5. f/g is continuous at a, provided g(a) ≠ 0

other properties

- a polynomial is continuous everywhere
- a rational function is continuous on its domain
 - if P(x) and Q(x) are polynomials, P(x)/Q(x) is continuous whenever Q(x) ≠ 0.
- f(x) = c is continuous on ℝ for all c ∈ ℝ.
- f(x) = x is continuous on ℝ.

trigonometric functions

- f(x) = sin x and g(x) = cos x are continuous everywhere
- tan x, sec x are continuous whenever cos x ≠ 0
 - domain: ℝ \ {±π/2, ±3π/2, ±5π/2, ...}
- cot x, csc x are continuous whenever sin x ≠ 0
 - domain: ℝ \ {0, ±π, ±2π, ...}

composite of continuous functions

if f is continuous at b and lim_{x→a} g(x) = b, then

lim_{x→a} f(g(x)) = f(lim_{x→a} g(x)) = f(b)

if g is continuous at a and f is continuous at g(a), then f ∘ g is continuous at a.

lim_{x→a} (f ∘ g)(x) = (f ∘ g)(a)

substitution theorem

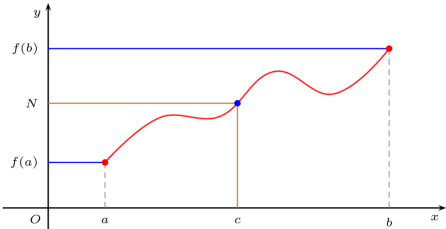
Suppose y = f(x) such that lim_{x→a} f(x) = b. If

- 1. g is continuous at b, OR
- 2. lim_{y→b} g(y) exists and f is one-to-one.
 - ∀x near a, except at a, f(x) ≠ b and lim_{y→b} g(y) exists

Then lim_{x→a} g(f(x)) = lim_{y→b} g(y)

intermediate value theorem

Let f be a function continuous on [a, b] with f(a) ≠ f(b).
Let N be a number between f(a) and f(b).
Then there exists c ∈ (a, b) such that f(c) = N.



03. DERIVATIVES

definition of derivatives

- f is differentiable at a if f'(a) exists
- f'(a) is the slope of y = f(x) at x = a
 - f'(a) = dy/dx |_{x=a}
 - dy/dx := lim_{x→0} Δy/Δx (derivative of y with respect to x)
- f'(x) = y' = dy/dx = df/dx = d/dx f(x) = D_x f(x) = ...

the derivative of a function f

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

the derivative of a function f at a number a is

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

tangent line

the tangent line to y = f(x) at (a, f(a)) is the line passing through (a, f(a)) with slope f'(a):

y = f'(a)(x - a) + f(a)

differentiable functions

- f is differentiable at a if
 - f'(a) := lim_{x→0} (f(a+h)-f(a))/h exists.
- f is differentiable on (a, b) if
 - f is differentiable at every c ∈ (a, b)

differentiability & continuity

- differentiability ⇒ continuity
 - if f is differentiable at a, then f is continuous at a.
- continuity ⇏ differentiability

differentiability

- every polynomial and rational function is differentiable on its domain
 - the domain of f' may be smaller than the domain of f.
- trigonometric functions are differentiable on the domain

differentiation

differentiation of trigonometric functions

lim_{θ→0} sin θ / θ = 1 | lim_{θ→0} (1 - cos θ) / θ = 0

chain rule

If g is differentiable at a and f is differentiable at b = g(a), then F = f ∘ g is differentiable at a and F'(a) = (f ∘ g)'(a) = f'(b)g'(a) = f'(g(a))g'(a)

If z = f(y) and y = g(x), then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$
$$\frac{dz}{dx} |_{x=a} = \frac{dz}{dy} |_{y=b} \frac{dy}{dx} |_{x=a}$$

generalised chain rule

h is differentiable at a; g is differentiable at B = h(a); f is differentiable at c = g(b).

$$(f \circ (g \circ h))' = f' \circ (g \circ h) \cdot (g \circ h)' = f'(c)g'(b)h'(a)$$

Leibniz notation:

If y = h(x), z = g(y), w = f(z),

$$\frac{dw}{dx} = \frac{dw}{dz} \frac{dz}{dy} \frac{dy}{dx}$$

implicit differentiation

- assumes that dy/dx exists

second derivative

$$f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}$$
$$f' = D(f) \Rightarrow f'' := D^2(f)$$

higher derivatives

$$f^{(0)} := f$$

For any positive integer n, f^{(n)} := (f^{(n-1)})'

if y = f(x), then f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x)

- for f(x) = 1/x, f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}
- for f(x) = x^m, f^{(n)}(x) = \begin{cases} \frac{m! x^{m-n}}{(m-n)!} & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}

04. APPLICATIONS OF DIFFERENTIATION

extreme values of functions

Let f be a function with domain D.

- local max/min ⇏ global max/min
- global max/min ⇏ local max/min

global (absolute) max/min

- aka extreme values

f has a global **maximum** at c ∈ D
⇔ f(c) ≥ f(x) for all x ∈ D

f has a global **minimum** at c ∈ D
⇔ f(c) ≤ f(x) for all x ∈ D

local (relative) max/min

- aka "turning points"
- "all x near c" = for all x in an open interval containing c

f has a local **maximum** at c ∈ D
⇔ f(c) ≥ f(x) for all x near c

f has a local **minimum** at c ∈ D
⇔ f(c) ≤ f(x) for all x near c

extreme value theorem

existence

if f is continuous on a finite closed interval [a, b], then f attains extreme values on [a, b].

value

the extreme value occurs at either critical numbers or the endpoints (x = a, x = b).

critical numbers

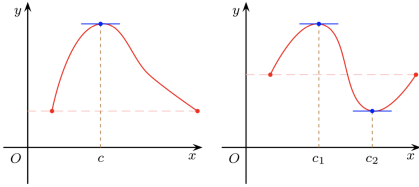
c ∈ D is a **critical number** of f if f'(c) = 0, or f'(c) does not exist.

fermat's theorem

- If f has a local maximum or minimum at c, then
1. c is a critical number.
 2. If f'(c) exists, then f'(c) = 0.

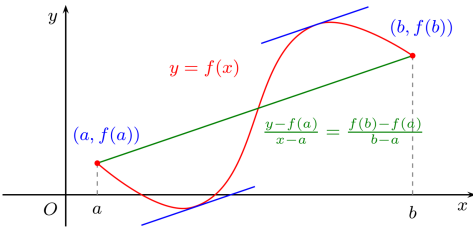
Rolle's Theorem

Let f be a function such that f is continuous on [a, b], f is differentiable on (a, b), and f(a) = f(b).
Then there is a number c ∈ (a, b) such that f'(c) = 0.



mean value theorem

Let f be a function such that f is continuous on [a, b] and f is differentiable on (a, b).
Then there exists c ∈ (a, b) such that f'(c) = \frac{f(b) - f(a)}{b - a}



- generalisation of Rolle's theorem when f(a) = f(b).

ordinary differential equations

Let f and g be continuous on [a, b].
If f'(x) = g'(x) for all x ∈ (a, b), then f(x) = g(x) + C on [a, b] for a constant C.

increasing/decreasing test

Let f be continuous on [a, b] and differentiable on (a, b).

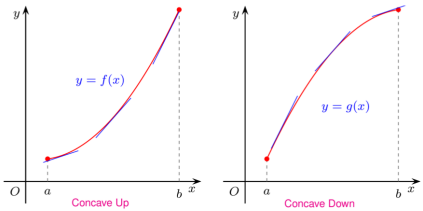
- f'(x) > 0 for any x ∈ (a, b) ⇒ f is increasing.
 - f is increasing ⇒ f'(x) ≥ 0 on (a, b)
- f'(x) < 0 for any x ∈ (a, b) ⇒ f is decreasing.
 - f is decreasing ⇒ f'(x) ≤ 0 on (a, b)
- f'(x) = 0 ⇒ f could be increasing OR decreasing.

first derivative test

Let f be continuous and c be a critical number of f . Suppose f is differentiable near c (except possibly at c). At c , if f' changes from:

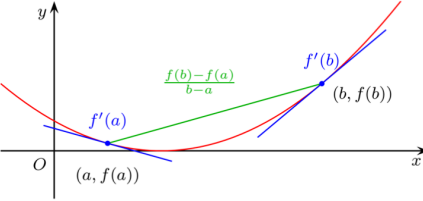
- (+) to (-) $\Rightarrow f$ has a local **maximum** at c
- (-) to (+) $\Rightarrow f$ has a local **minimum** at c
- no change in sign $\Rightarrow f$ has neither local max/min at c .

concavity



f is **concave up** on an open interval $I \Leftrightarrow f'$ is increasing
 \Leftrightarrow for $a < b \in I$, $f'(a) < f'(b)$
 $\Leftrightarrow f(x) > f'(y)(x - y) + f(y)$ for any $x \neq y \in I$

f is **concave down** on an open interval $I \Leftrightarrow f'$ is decreasing
 \Leftrightarrow for $a < b \in I$, $f'(a) > f'(b)$
 $\Leftrightarrow f(x) < f'(y)(x - y) + f(y)$ for any $x \neq y \in I$



concavity test

- $f'' > 0$ on $I \Rightarrow f$ is concave up on I
- $f'' < 0$ on $I \Rightarrow f$ is concave down on I

second derivative test

- If $f'(c) = 0$ and $f''(c)$ exists,
- $f''(c) < 0 \Rightarrow f$ has a **local maximum** at c .
 - $f''(c) > 0 \Rightarrow f$ has a **local minimum** at c .
 - $f''(c) = 0 \Rightarrow$ inconclusive

inflection point

- A point P on the curve $y = f(x)$ is an inflection point if
 - f is continuous at P , and
 - the concavity of the curve changes at P .
- if c is an inflection point and f is twice differentiable at c , then $f''(c) = 0$.

Taylor's Theorem

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n,$$

where $R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{(n+1)}$ for c between x and a

Taylor Series

As $R - n \rightarrow 0$ as $n \rightarrow \infty$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

L'Hopital's Rule

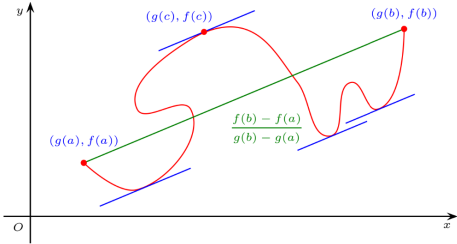
Let f and g be functions such that

- $(\frac{0}{0}) \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, OR
- $(\frac{\infty}{\infty}) \lim_{x \rightarrow a} |f(x)| = \lim_{x \rightarrow a} |g(x)| = \infty$,
- f and g are differentiable near a (except at a),
- $g'(x) \neq 0$ near a (except at a).

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$
provided that the RHS limit exists or is $\pm \infty$

Cauchy's Mean Value Theorem

Let f, g be continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) \neq 0$ for any $x \in (a, b)$. Consider a curve defined by $t \mapsto (g(t), f(t))$. Then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$


05. INTEGRALS

definite integral

Let f be a continuous function on $[a, b]$ divided into n intervals.

Riemann sum

$$[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)] \Delta x = \sum_{i=1}^n f(x_i^*) \Delta x$$

the lengths of subintervals are not necessarily equal

- $\max\{|x_i - x_{i-1}| : i = 1, \dots, n\} \rightarrow 0$

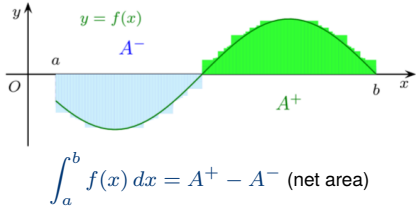
definite integral of f from a to b :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

where $\Delta x = \frac{b-a}{n}$

- f is **integrable** from a to b if $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists.
- continuous functions are integrable.
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- $\int_a^a f(x) dx = 0$

geometric meaning



properties

- let f and g be continuous functions.
- $\int_a^b c dx = (b - a)c$
 - $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
 - $\int_a^c f(x) dx = \int_b^c f(x) dx \pm \int_a^b f(x) dx$
 - suppose $f(x) \geq 0$ on $[a, b]$. Then $\int_a^b f(x) dx \geq 0$.
 - suppose $f(x) \geq g(x)$ on $[a, b]$.
 - Then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.
 - suppose $m \leq f(x) \leq M$ on $[a, b]$.
 - Then $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$.

fundamental theorem of calculus

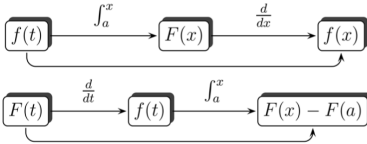
- for $g(x) = \int_a^x f(t) dt$ ($a \leq x \leq b$),
- g is continuous on $[a, b]$
 - g is differentiable on (a, b)
 - $g'(x) = f(x)$ on (a, b) or $\frac{d}{dx} \int_a^x f(t) dt = f(x)$



if F is continuous on $[a, b]$, and $F' = f$ on (a, b) ,

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_{x=a}^{x=b}$$

$$\int_a^x \frac{d}{dx} F(t) dt = F(x) - F(a)$$



indefinite integral

- **indefinite integral** of f , $\int f(x) dx = F(x) + c$
- **antiderivative** (of a continuous function f): a continuous function F such that $F' = f$.
 - antiderivatives of f are functions of form $F + c$
 - indefinite integral is a family of antiderivatives
- properties of indefinite integral
 - $\int (af(x) \pm bg(x)) dx = a \int f(x) dx \pm b \int g(x) dx$

integration by parts

$$u dv = uv - \int v du$$

substitution rule (I)

let $u = g(x)$ be a differentiable function.

indefinite integral

if f and g' are continuous,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

definite integral

if g' are continuous on $[a, b]$, and f is continuous on the range of $u = g(x)$,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

substitution rule (II)

let f and g' be continuous functions, and $x = g(t)$ is a one-to-one differentiable function.

$$\int f(x) dx = \int f(g(t))g'(t) dt$$

improper integral

for discontinuous integrands

if f is continuous on $[a, b)$ and discontinuous at b ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if f is continuous on $(a, b]$ and discontinuous at a ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

- $\int_a^b f(x) dx$ is the limit of integrals.
 - **converges** if the limit exists
 - **diverges** if the limit does not exist

discontinuity in the interior of the interval

suppose f has discontinuity at $c \in (a, b)$. then

$$\int_a^b f(x) dx = \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

over infinite intervals

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

if $\int_a^t f(x) dx$ exists for every $t \geq a$, then the **improper integral** of f from a to ∞ is

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

if $\int_t^b f(x) dx$ exists for every $t \leq b$, then the **improper integral** of f from $-\infty$ to b is

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

- NOTE: $\int_{-\infty}^{\infty} f(x) dx \neq \lim_{a \rightarrow \infty} \int_{-a}^a f(x) dx$

06. INVERSE FUNCTIONS & INTEGRATION

one to one functions

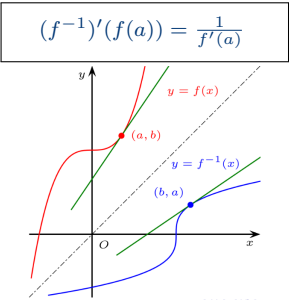
let f be a function with domain D .
 f is **one-to-one** if, for any $a, b \in D$,
 $a \neq b \Rightarrow f(a) \neq f(b)$
OR $f(a) = f(b) \Rightarrow a = b$

inverse function

- let f be a one-to-one function with domain A and range B .
- its **inverse function** f^{-1} is the function with
 - domain B and range A , and
 - $f^{-1}(y) = x \iff y = f(x)$ for any $x \in A, y \in B$
 - $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$
 - $(f^{-1})^{-1} = f$
 - NOTE: $(f(x))^{-1}$ is the reciprocal of the value of $f(x)$

properties

- let f be a *one-to-one continuous* function on an open interval I .
- the inverse function f^{-1} is also continuous.
 - if f is differentiable at $a \in I$, and $f'(a) \neq 0$, then
 - f^{-1} is differentiable at $b = f(a)$
 - $(f^{-1})'(b) = \frac{1}{f'(a)}$



techniques of integration

integration of rational functions

- for $f = \frac{A(x)}{B(x)}$,
- manipulate such that $\deg A(x) < \deg B(x)$, then decompose into partial fractions
 - common rational functions:

$$\begin{aligned} \int \frac{1}{(x+a)^k} dx &= \begin{cases} \ln|x+a| + K, & \text{if } k = 1 \\ \frac{(x+a)^{1-k}}{1-k} + K, & \text{if } k \geq 2 \end{cases} \\ \int \frac{u}{(u^2+d^2)^r} du &= \begin{cases} \frac{1}{2} \ln(u^2+d^2), & \text{if } r = 1 \\ \frac{(u^2+d^2)^{1-r}}{2(1-r)}, & \text{if } r \geq 2 \end{cases} \\ \int \frac{1}{(u^2+d^2)^r} du &= \frac{1}{d^{2r-1}} \int \frac{1}{(t^2+1)^r} dt \end{aligned}$$

partial fractions

- for each linear factor $(x+a)^k$:
 - $\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \dots + \frac{A_k}{(x+a)^k}$
- for each quadratic factor $(x^2+bx+c)^r$:
 - $\frac{B_1x+C_1}{x^2+bx+c} + \dots + \frac{B_rx+C_r}{(x^2+bx+c)^r}$

common trigonometric substitutions

- $\sqrt{a^2-x^2}, \quad x = a \sin t, \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sqrt{x^2-a^2}, \quad x = a \sec t, \quad t \in [0, -\frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}]$
- $a^2+x^2, \quad x = a \tan t, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

universal trigonometric substitution

any rational expression in $\sin x$ and $\cos x$ can be integrated using the substitution $t = \tan \frac{x}{2}, \quad x \in (-\pi, \pi)$.

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

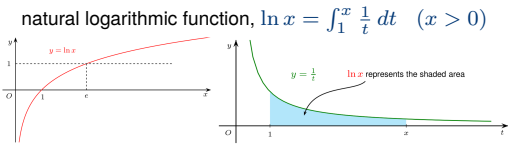
derivatives of trigonometric functions

function	derivative	function	derivative
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$\frac{-1}{1+x^2}$

trigonometric identities

- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
- $\sec^{-1} x + \csc^{-1} x = \begin{cases} \frac{\pi}{2}, & \text{if } x \geq 1 \\ \frac{5\pi}{2}, & \text{if } x \leq -1 \end{cases}$

natural logarithmic function



- $\ln x < 0$ for $0 < x < 1$; $\ln x > 0$ for $x > 1$; $\ln 1 = 0$
- $\ln x$ is increasing on \mathbb{R}^+ ($\frac{d}{dx} \ln x > 0$)

logarithmic differentiation I

aka take \ln on both sides and implicitly differentiate

for $y = f_1(x)f_2(x) \cdots f_n(x)$ (product of nonzero functions),

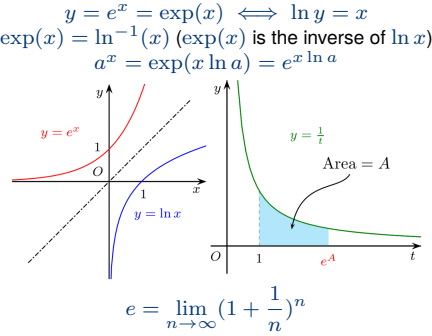
$$\begin{aligned} \ln|y| &= \ln|f_1(x)| + \ln|f_2(x)| + \dots + \ln|f_n(x)| \\ \frac{dy}{dx} &= \left[\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \dots + \frac{f'_n(x)}{f_n(x)} \right] y \\ &= \left[\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \dots + \frac{f'_n(x)}{f_n(x)} \right] f_1(x)f_2(x) \cdots f_n(x) \end{aligned}$$

logarithmic differentiation II

for $y = f(x)^{g(x)} (f(x) > 0)$,

$$\begin{aligned} \ln y &= g(x) \ln f(x) \Rightarrow \frac{dy}{dx} = y \frac{d}{dx} [g(x) \ln f(x)] \\ \lim_{x \rightarrow a} (f(x)^{g(x)}) &= \lim_{x \rightarrow a} \exp(g(x) \ln f(x)) \\ &= \exp \left(\lim_{x \rightarrow a} g(x) \ln f(x) \right) \end{aligned}$$

exponential function



- $\ln(e^x) = x$ for $x \in \mathbb{R}$ and $e^{\ln y} = y$ for $y \in \mathbb{R}^+$
- common equations
 - $\lim_{x \rightarrow \infty} e^x = \infty, \quad \lim_{x \rightarrow -\infty} e^x = 0$
 - $\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}^+$
 - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

properties

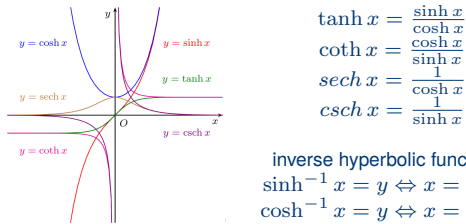
- $a^u a^v = a^{u+v}$
- $a^{-u} = \frac{1}{a^u}$
- $(a^u)^v = a^{uv}$
- $(a^x)' = a^x \ln a$
- $\frac{d}{dx} x^r = r x^{r-1}$
- $\int x^r dx = \begin{cases} \frac{x^{r+1}}{r+1} + C & \text{if } r \neq -1, \\ \ln x + C & \text{if } r = -1, \end{cases}$
 - if r is irrational, then x^r is only defined for $x \geq 0$.
- $\lim_{x \rightarrow \infty} e^x = \infty, \quad \lim_{x \rightarrow -\infty} e^x = 0$
- $\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}^+$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

hyperbolic trigonometric functions

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, & (\sinh x)' &= \cosh x \\ \cosh x &= \frac{e^x + e^{-x}}{2}, & (\cosh x)' &= \sinh x \end{aligned}$$

- $\cosh^2 x - \sinh^2 x = 1$
- parametrization represents a **hyperbola** -

let $\begin{cases} x = \cosh t, \\ y = \sinh t. \end{cases}$ Then $x^2 - y^2 = 1$



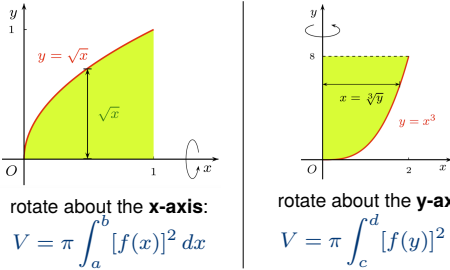
- properties
 - $\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), x \in \mathbb{R}$
 - $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), x \geq 1$
 - $\tanh^{-1} x = \frac{1}{2} \ln(\frac{1+x}{1-x}), \quad -1 < x < 1$
 - $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$
 - $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx} \tanh^{-1} x = \text{sech } x$

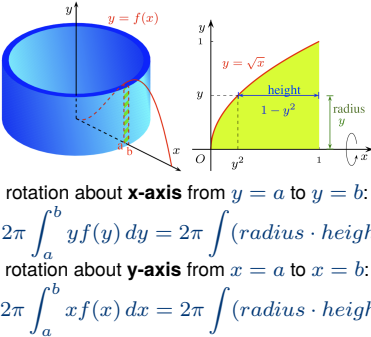
07. APPLICATIONS OF INTEGRALS

volume

disk/washer method



method of cylindrical shells



arc length

- a function f is **smooth** if f' is continuous.
- arc length,

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

parametric curves:

$$\text{arc length} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

surface area of revolution

Let f be a smooth function such that $f(x) \geq 0$ on $[a, b]$. Then the area of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$ about the x -axis is

$$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

08. ORDINARY DIFFERENTIAL EQUATIONS

$$\begin{aligned} \frac{dy}{dx} &= f(x) \Rightarrow y = \int f(x) dx \\ \frac{dy}{dx} &= f(y) \Rightarrow x = \int \frac{1}{f(y)} dy \end{aligned}$$

separation of variables

dy/dx = f(x)g(y) => 1/g(y) dy = f(x) dx

=> integral 1/g(y) dy = integral f(x) dx

singular solution

- if y = C is a solution to g(y) = 0, then it is a **singular solution** to dy/dx = f(x)g(x).
- singular solution disappears if the equation is 1/g(x) dy/dx = f(x)
- (can ignore singular solutions in this course)

homogenous equations

- Suppose dy/dx = F(x, y) is not separable.
- suppose F(x, y) is **homogenous of degree zero**
 - i.e. F(x, y) = F(tx, ty) for all t in R \ {0}
 - let z = y/x. Then
 - y = xz and dy/dx = x dz/dx + z
 - F(x, y) = F(x/x, y/x) = F(1, z)
 - x dz/dx + z = F(1, z) => separable!

first order linear differential equations

- general equation: dy/dx + p(x)y = q(x)
- find P(x) = integral p(x) dx
 - multiply both sides by integrating factor v(x) = e^{P(x)}:
 - e^{P(x)} dy/dx + e^{P(x)}p(x)y = e^{P(x)}q(x)
 - d/dx (e^{P(x)}y) = e^{P(x)}q(x)
 - integrate with respect to x

e^{P(x)}y = integral e^{P(x)}q(x) dx

y = 1/e^{P(x)} integral e^{P(x)}q(x) dx

note: if the equation is not linear in y but is linear in x, can take the reciprocal and use dx/dy instead.

Bernoulli's equation

dy/dx + p(x)y = q(x)y^n

- if n = 0 or n = 1:
 - the system is linear
- if n != 0, 1:
 - let z = y^{1-n} => dz/dx = (1-n)y^{-n} dy/dx
 - multiply both sides of the equation by (1-n)y^{-n}
 - equation is reduced to a linear equation
 - dz/dx + (1-n)p(x)z = (1-n)q(x)

applications

- compound interest
 - let r be the interest rate (%), A be the money
 - ODE: dA/dt = rA; A(0) = C
 - solve for A(t) = Ce^{rt}
- radiocarbon dating
 - let lambda be the half life, C be % of Carbon left
 - ODE: dC/dt = kC; C(0) = 1; k = -ln 2 / lambda
 - solve C(t) = e^{kt}
- population growth - let M be max. population (carrying capacity), r be the rate of change of population
 - ODE: dP/dt = rP(M - P)

- solve P(t) = M / (1 + (M/P(0) - 1)e^{-rt})
- newton's law of cooling
 - let T_S be the surrounding temperature, r > 0 be the rate of heat loss
 - ODE: dT/dt = -r * (T - T_S)
 - ln |T - T_S| = -rt + C
- draining tank problem (torricelli's law)
 - the rate at which water flows out is proportional to the square root of the water's depth
 - let A be the base area of the tank, R be the rate of flow
 - ODE: A dh/dt = -R

misc

triangle inequality

|a + b| <= |a| + |b| for all a, b in R

binomial theorem

(a + b)^n = sum_{k=0}^n (n choose k) a^{n-k} b^k

= a^n + (n choose 1) a^{n-1} b + ... + (n choose n-1) a b^{n-1} + b^n

where the binomial coefficient is given by

(n choose k) = n! / (k!(n-k)!)

factorisation

a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1})

a^3 - b^3 = (a - b)(a^2 + ab + b^2)

a^3 + b^3 = (a + b)(a^2 - ab + b^2)

misc

- forall x in (0, pi/2), sin x < x < tan x
- sin theta = tan theta / sqrt(tan^2 theta + 1)

differentiation

f(x)	f'(x)
tan x	sec^2 x
csc x	-csc x cot x
sec x	sec x tan x
cot x	-csc^2 x
sin^-1 f(x)	f'(x) / sqrt(1-[f(x)]^2), f(x) < 1
cos^-1 f(x)	-f'(x) / sqrt(1-[f(x)]^2), f(x) < 1
tan^-1 f(x)	f'(x) / (1+[f(x)]^2)
cot^-1 f(x)	-f'(x) / (1+[f(x)]^2)
sec^-1 f(x)	f'(x) / (f(x) sqrt([f(x)]^2 - 1))
csc^-1 f(x)	-f'(x) / (f(x) sqrt([f(x)]^2 - 1))

integration

f(x)	integral f(x)
tan x	ln(sec x), x < pi/2
cot x	ln(sin x), 0 < x < pi
csc x	-ln(csc x + cot x), 0 < x < pi
sec x	ln(sec x + tan x), x < pi/2