New Solutions of Nonlocal NLS, mKdV and Hirota Equations

Avinash Khare

Physics Department, Savitribai Phule Pune University
Pune 411007, India

Avadh Saxena

Theoretical Division and Center for Nonlinear Studies, Los Alamos
National Laboratory, Los Alamos, New Mexico 87545, USA

Abstract

In this paper, we provide several novel solutions of the Ablowitz-Musslimani as well Yang's versions of the nonlocal nonlinear Schrödinger (NLS) equation, nonlocal modified Korteweg-de Vries (mKdV) as well as nonlocal Hirota equations. In each case we compare and contrast the corresponding solutions of the relevant local equation. Further, we provide new solutions of the local NLS, local mKdV and local Hirota equations which are not the solutions of the corresponding nonlocal equations.

1 Introduction

After the seminal papers of Ablowitz and Musslimani (AM) [1, 2] about the nonlocal NLS equation and its integrability, in recent years nonlocal mKdV [3, 4] as well as well as nonlocal Hirota equations [5, 6] have also been introduced. Moreover, Yang [7] has also introduced another variant of the nonlocal NLS equation and shown its integrability. During the last few years it has been realized that optics and photonics can provide an ideal ground for testing some of the consequences of the AM variant of the nonlocal NLS [8]. In general, beyond nonlinear optics and photonic waveguides, nonlocal, nonlinear equations arise and find applications in a number of other physical contexts [9] spanning condensed matter physics, high energy physics, hydrodynamics, electromagnetics, elasticity, etc.

Several years ago, we obtained [10, 11] a large number of solutions of the AM version of the nonlocal NLS. We might add here that some aspects of the nonlocal mKdV equation [12, 13, 14, 15] and nonlocal Hirota equation [16] including few exact solutions have also been reported in the literature. Further, recently we have also obtained several solutions of the nonlocal mKdV, nonlocal Hirota and Yang variant of the nonlocal NLS [17]. Additionally, in the same paper [17] we showed that all these nonlocal equations admit superposed solutions which can be re-expressed as a sum of two kinks, or kink-anti-kink or two pulse solutions.

The feeling at that time was that perhaps one has exhausted the various possible solutions of the nonlocal equations. Of course, being nonlinear equations, one can never be sure about it. Recently, we discovered new solutions of the symmetric ϕ^4 equation [18]. Subsequently, we realized that even the above mentioned nonlocal equations will admit similar novel solutions. The purpose of this paper is to present new solutions of the above nonlocal equations. In each case we also enquire if the corresponding local equation admits such a solution as well, and if yes, under what conditions. Finally, for completeness, in three Appendices we present those solutions of the local NLS, mKdV and Hirota equations which are not the solutions of the corresponding nonlocal equations.

The plan of the paper is the following. In Sec. II we present new solutions of the nonlocal AM variant of the NLS equation. In each case we compare and contrast with the solutions of the (local) NLS equation (in case they are admitted). In Sec. III we present novel solutions of the Yang variant of the nonlocal NLS. In Appendix A we present those solutions of the local NLS equation which are not the solutions of either the AM or the Yang nonlocal variant of the NLS equation. In Sec. IV we present new solutions of nonlocal mKdV equation and compare and contrast them with the corresponding solutions of the local mKdV. In Appendix B we present solutions of the local mKdV which are not the solutions of the nonlocal mKdV. In Sec. V we present new solutions of nonlocal Hirota equation and compare and contrast them with the corresponding solutions of the local Hirota equation. In Appendix C we present solutions of the local Hirota equation which are not the solutions of the nonlocal Hirota equation.

2 New Solutions of Ablowitz-Musslimani Variant of Nonlocal NLS Model

The Ablowitz-Musslimani (AM) variant of the nonlocal NLS is given by

$$i\psi_t(x,t) + \psi_{xx}(x,t) + g\psi^2(x,t)\psi^*(-x,t) = 0\,, \tag{1}$$

where for comparison with local NLS, we have set the coupling constant to be g instead of $2g$. We now show that apart from the several solutions already obtained, this equation admits several new solutions, both real and complex which we list one by one. We also compare and contrast the solutions admitted by the corresponding local NLS equation

$$i\psi_t(x,t) + \psi_{xx}(x,t) + g|\psi|^2\psi(x,t) = 0\,. \tag{2}$$

It is worth reminding that while the local NLS Eq. (2) admits the plane wave solution

$$\psi(x,t) = Ae^{i(kx-\omega t)}\,, \tag{3}$$

where

$$\omega = k^2 - gA^2\,, \tag{4}$$

the AM variant nonlocal NLS Eq. (1) admits a rather unusual plane wave solution

$$\psi(x,t) = Ae^{(kx-i\omega t)}\,, \tag{5}$$

provided

$$\omega = -k^2 - gA^2\,. \tag{6}$$

As a result, while corresponding to any stationary soliton solution of local NLS Eq. (2) there is always a moving soliton solution with velocity $2k$ and ω being replaced by $\omega - k^2$. However, the nonlocal NLS Eq. (1) only admits stationary soliton solutions. As an illustration, local NLS Eq. (1) admits the solution

$$\psi(x,t) = A\text{sech}(\beta(x-vt))e^{i(kx-\omega t)}\,, \tag{7}$$

provided

$$v = 2k\,, \quad \omega = k^2 - \beta^2\,, \quad A^2 = 2\beta^2\,. \tag{8}$$

On the other hand, nonlocal NLS Eq. (1) only admits the solution

$$\psi(x,t) = A\text{sech}(\beta x)e^{-i\omega t}\,, \tag{9}$$

provided

$$\omega = -\beta^2\,, \quad A^2 = 2\beta^2\,. \tag{10}$$

We now present 11 new (3 real and 8 complex but PT-invariant) solutions of Eq. (1) and compare these with those admitted by the corresponding local NLS Eq. (2). As we show in the next section, the corresponding Yang's nonlocal NLS variant in contrast admits only 3 new solutions. It turns out that there are solutions admitted by the local NLS Eq. (2), but not by the nonlocal Eq. (1) (or Yang's nonlocal variant), which we present in Appendix A.

Solution I

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{A\sqrt{m}\mathrm{sn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}\,, \quad A, D > 0\,, \tag{11}$$

is an exact solution of the nonlocal Eq. (1) provided $g > 0$.

$$D = 1\,, \quad 2gA^2 = \beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{12}$$

Note, here m is the modulus of Jacobi elliptic functions [19].

In contrast the local NLS Eq. (2) admits the solution (11) provided

$$D = 1\,, \quad 2gA^2 = -\beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{13}$$

Thus in contrast to the nonlocal case, in the local case the solution is only valid for $g < 0$.

Solution II

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{A\sqrt{m}\mathrm{cn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}\,, \quad A, D > 0\,, \tag{14}$$

is an exact periodic pulse solution of nonlocal Eq. (1) provided

$$D^2 = 1-m > 0\,, \quad 2gA^2 = -m\beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{15}$$

Remarkably the local NLS Eq. (2) also admits the solution (14) provided relations (15) are satisfied.

Solution III

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\left[F - \frac{A\mathrm{dn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}\right]\,, \quad D, F, A > 0\,, \tag{16}$$

is an exact periodic pulse solution of the nonlocal Eq. (1) provided

$$\begin{aligned} &D^2 = \sqrt{1-m} > 0\,, \quad gA^2 = -2(1-\sqrt{1-m})^2\beta^2\,, \\ &F = A/2\,, \quad \omega = -[(2-m)+6\sqrt{1-m}]\frac{\beta^2}{2}\,. \end{aligned} \tag{17}$$

Remarkably the local NLS Eq. (2) also admits the solution (16) provided the relations (17) are satisfied.

Complex PT-invariant Periodic and Hyperbolic Pulse and Kink Solutions

We now show that one has several periodic and hyperbolic complex PT-invariant pulse as well as kink solutions which are distinct from the well known periodic and hyperbolic complex PT-invariant pulse and kink solutions. We discuss these one by one. It is worth noting that, in contrast, the local NLS Eq. (2) does not admit any of the complex PT-invariant periodic solutions as given below (see Solutions IV to XI).

Solution IV

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{\sqrt{m}[A\mathrm{cn}(\beta x,m) + iB\mathrm{sn}(\beta x,m)]}{D+\mathrm{dn}(\beta x,m)}\,,\quad A,D>0\,, \tag{18}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue 1 of the nonlocal Eq. (1) provided

$$2gA^2 = (D^2-1)\beta^2\,,\quad 2gB^2 = (D^2-1+m)\beta^2\,,\quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{19}$$

Clearly such a solution exists if either

$$D^2 > 1\,,\quad g > 0\,, \tag{20}$$

or

$$0 < D^2 < 1-m\,,\quad g < 0\,. \tag{21}$$

Note that for both these cases $\omega < 0$.

Solution V

In the limit $m = 1$, the solution IV goes over to the complex PT-invariant hyperbolic solution with PT-eigenvalue 1, i.e.

$$\psi(x,t) = e^{i\omega t}\frac{[A\mathrm{sech}(\beta x) + iB\tanh(\beta x)]}{D+\mathrm{sech}(\beta x)}\,,\quad A,D>0\,, \tag{22}$$

provided

$$2gA^2 = (D^2-1)\beta^2\,,\quad 2gB^2 = D^2\beta^2\,,\quad \omega = -\frac{\beta^2}{2}\,. \tag{23}$$

Thus this solution only exists if $g > 0, D^2 > 1$.

Solution VI

It is straightforward to check that

$$\psi(x,t) = e^{i\omega t}\frac{\sqrt{m}[A\mathrm{sn}(\beta x,m) + iB\mathrm{cn}(\beta x,m)]}{D+\mathrm{dn}(\beta x,m)}\,,\quad A,D>0\,, \tag{24}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue -1 of nonlocal Eq. (1) provided

$$2gA^2 = (D^2 - 1 + m)\beta^2\,, \quad 2gB^2 = (D^2 - 1)\beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{25}$$

Thus such a solution exists if either

$$D^2 > 1\,, \quad g > 0\,, \tag{26}$$

or

$$0 < D^2 < 1 - m\,, \quad g < 0\,. \tag{27}$$

Note that for both these cases $\omega < 0$.

Solution VII

In the limit $m = 1$, the solution VI goes over to the complex PT-invariant hyperbolic solution with PT-eigenvalue -1, i.e.

$$\psi(x,t) = e^{i\omega t}\frac{[A\tanh(\beta x) + iB\mathrm{sech}(\beta x)]}{D + \mathrm{sech}(\beta x)}\,, \quad A, D > 0\,, \tag{28}$$

provided

$$2gA^2 = D^2\beta^2\,, \quad 2gB^2 = (D^2 - 1)\beta^2\,, \quad \omega = -\frac{\beta^2}{2}\,. \tag{29}$$

Thus this solution only exists if $g > 0$ and $D^2 > 1$.

Solution VIII

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{[A\mathrm{dn}(\beta x, m) + iB\sqrt{m}\mathrm{sn}(\beta x, m)]}{D + \mathrm{cn}(\beta x, m)}\,, \quad A > 0, D > 1\,, \tag{30}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue 1 of the nonlocal Eq. (1) provided

$$2gA^2 = (D^2 - 1)\beta^2\,, \quad 2mgB^2 = (mD^2 - m + 1)\beta^2\,, \quad \omega = -(2m-1)\frac{\beta^2}{2}\,. \tag{31}$$

Thus this solution exists only if $g > 0$.

Solution IX

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{[A\sqrt{m}\mathrm{sn}(\beta x, m) + iB\mathrm{dn}(\beta x, m)]}{D + \mathrm{cn}(\beta x, m)}\,, \quad A, B > 0, D > 1\,, \tag{32}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue -1 of the nonlocal Eq. (1) provided

$$2mgA^2 = (1-m+mD^2)\beta^2\,, \;\; 2gB^2 = (D^2-1)\beta^2\,, \;\; \omega = -(2m-1)\frac{\beta^2}{2}\,. \tag{33}$$

Thus this solution exists only if $g > 0$.

Solution X

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{[A + iB\sin(\beta x)]}{D+\cos(\beta x)}\,, \quad A, B > 0, D > 1\,, \tag{34}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue 1 of the nonlocal Eq. (1) provided

$$2gA^2 = (D^2-1)\beta^2\,, \;\; 2gB^2 = \beta^2\,, \;\; \omega = \frac{\beta^2}{2}\,. \tag{35}$$

Thus this solution exists only if $g, \omega > 0$.

Solution XI

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{[A\sin(\beta x) + iB]}{D+\cos(\beta x)}\,, \quad A, B > 0, D > 1\,, \tag{36}$$

is an exact complex PT-invariant periodic solution with PT-eigenvalue -1 of the nonlocal Eq. (1) provided

$$2gA^2 = \beta^2\,, \;\; 2gB^2 = (D^2-1)\beta^2\,, \;\; \omega = \frac{\beta^2}{2}\,. \tag{37}$$

Thus this solution exists only if $g, \omega > 0$. As mentioned above, 6 solutions of (local) NLS Eq. (2) which are not the solutions of either the AM or Yang variant of nonlocal NLS Eq. (1), are presented in Appendix A.

3 New Solutions of Yang's Nonlocal NLS Equation

The Yang variant of the nonlocal NLS is given by

$$i\psi_t(x,t) + \psi_{xx}(x,t) + \frac{g}{2}[|\psi(x,t)|^2 + |\psi(-x,t)|^2]\psi(x,t) = 0\,, \tag{38}$$

where for comparison with local NLS, we have set the coupling constant to be $g/2$ instead of g.

It is worth reminding that similar to the local NLS, the Yang variant of the nonlocal NLS equation also admits the plane wave solution (3) where the dispersion relation is again given by Eq. (4). However, while corresponding to any stationary soliton solution of local NLS Eq. (2) there is always a moving soliton solution with velocity $2k$ and ω being replaced by $\omega - k^2$ (e.g. see Eq. (7)), the corresponding Yang (as well as AM nonlocal variant) equation only admit the stationary soliton solution.

We now show that apart from the several solutions already obtained [17], this equation admits three new solutions.

Solution I

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{A\sqrt{m}\mathrm{sn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}\,, \quad A, D > 0\,, \tag{39}$$

is an exact solution of the nonlocal Eq. (38) provided $g > 0$

$$D = 1\,, \quad 2gA^2 = -\beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{40}$$

Thus while local NLS and Yang's nonlocal variant admit solution (39) for $g < 0$, the AM nonlocal variant admits this solution in case $g > 0$.

Solution II

It is easy to check that

$$\psi(x,t) = e^{i\omega t}\frac{A\sqrt{m}\mathrm{cn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}\,, \quad A, D > 0\,, \tag{41}$$

is an exact periodic pulse solution of the nonlocal Eq. (38) provided

$$D^2 = 1-m > 0\,, \quad 2gA^2 = -m\beta^2\,, \quad \omega = -(2-m)\frac{\beta^2}{2}\,. \tag{42}$$

Remarkably the local NLS Eq. (2) as well as AM nonlocal NLS variants also admit the solution (41) provided the relations (42) are satisfied.

Solution III

It is easy to check that

$$\psi(x,t) = e^{i\omega t}[F - \frac{A\mathrm{dn}(\beta x,m)}{D+\mathrm{dn}(\beta x,m)}]\,, \quad D, F, A > 0\,, \tag{43}$$

is an exact periodic pulse solution of the nonlocal Eq. (38) provided

$$D^2 = \sqrt{1-m} > 0\,, \quad gA^2 = -2(1-\sqrt{1-m})^2\beta^2\,,$$
$$F = A/2\,, \quad \omega = -[(2-m) + 6\sqrt{1-m}]\frac{\beta^2}{2}\,. \tag{44}$$

The local NLS Eq. (2) as well as the AM nonlocal NLS variant also admit the solution (43) provided the relations (44) are satisfied.

4 New Solutions of a nonlocal mKdV Equation

Recently a nonlocal mKdV equation has been introduced [3]

$$u_t(x,t) + u_{xxx}(x,t) + 6gu(x,t)u(-x,-t)u_x(x,t) = 0\,. \tag{45}$$

Here $g = +1(-1)$ corresponds attractive (repulsive) mKdV. We now show that apart from the several solutions obtained recently [17], there are 26 new solutions of the nonlocal mKdV Eq. (45). We also enquire if the corresponding local mKdV equation

$$u_t(x,t) + u_{xxx}(x,t) + 6gu^2(x,t)u_x(x,t) = 0\,, \tag{46}$$

also admits such solutions and if yes under what conditions. For completeness in Appendix B we mention those solutions which are solutions of the local mKdV Eq. (46) but not of the nonlocal Eq. (45).

Solution I

Inspired by the solution of NLS as obtained long time ago by Zakharav and Shabat [20], we now show that the nonlocal mKdV Eq. (45) admits a similar solution. In particular, it is straightforward to check that

$$u(x,t) = \sqrt{n}[B\tanh(\xi) + iA]e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{47}$$

is an exact PT-invariant solution of the nonlocal mKdV Eq. (45) with PT-eigenvalue -1 provided

$$A^2 + B^2 = 1\,, \quad g = +1\,, \quad \beta = \sqrt{n}B\,, \tag{48}$$

$$\omega = -(6n + k^2)k\,, \quad v = 4nB^2 - 3k^2 - 6\sqrt{n}Ak - 6n\,. \tag{49}$$

It is worth pointing out that in contrast the (local) mKdV equation (46) does not admit the solution (47).

Solution II

Yet another PT-invariant solution with PT-eigenvalue +1 of the nonlocal mKdV Eq. (45) is

$$u(x,t) = \sqrt{n}[A + iB\tanh(\xi)]e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{50}$$

provided

$$A^2 + B^2 = 1\,, \quad g = -1\,, \quad \beta = \sqrt{n}B\,, \tag{51}$$

$$\omega = -(6n + k^2)k\,, \quad v = 4nB^2 - 3k^2 + 6\sqrt{n}Ak - 6n\,. \tag{52}$$

It is interesting that while solution I with PT-eigenvalue -1 is admitted in case $g = 1$, the solution II with PT-eigenvalue $+1$ is admitted only if $g = -1$. Note that in contrast the (local) mKdV equation (46) does not admit the solution (50).

Solution III

Remarkably, unlike the local mKdV Eq. (46), the nonlocal mKdV Eq. (45) admits the plane wave solution

$$u(x,t) = Ae^{i(kx-\omega t)}\,, \tag{53}$$

provided

$$\omega = k(6gA^2 - k^2)\,. \tag{54}$$

It turns out that unlike the local mKdV Eq. (46), the nonlocal mKdV Eq. (45) admits several real solutions multiplied by the plane wave as given by Eq. (53). We now present 12 such new solutions.

Solution IV

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic kink solution

$$u(x,t) = A\sqrt{m}\,\mathrm{sn}(\xi,m)e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{55}$$

provided

$$g = 1\,, \quad A = \beta\,, \quad \omega = -k[k^2 + 3(1+m)A^2]\,, \quad v = -[3k^2 + (1+m)A^2]\,. \tag{56}$$

In the limit $\omega = k = 0$ we get back the well known real periodic kink solution $u(x,t) = A\sqrt{m}\,\mathrm{sn}(\xi,m)$ [10, 11].

Solution V

In the limit $m = 1$, the solution IV goes over to the complex hyperbolic kink solution

$$u(x,t) = A\tanh(\xi)e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{57}$$

provided

$$g = 1\,, \quad A = \beta\,, \quad \omega = -k[k^2 + 6A^2]\,, \quad v = -[3k^2 + 2A^2]\,. \tag{58}$$

In the limit $\omega = k = 0$ we get back the well known real hyperbolic kink solution $u(x,t) = A\tanh(\xi)$ [10, 11].

Solution VI

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic pulse solution

$$u(x,t) = A\mathrm{dn}(\xi, m)e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{59}$$

provided

$$g = 1\,, \quad A = \beta\,, \quad \omega = -k[k^2 - 3(2-m)A^2]\,, \quad v = -[3k^2 - (2-m)A^2]\,. \tag{60}$$

In the limit $\omega = k = 0$ we get back the well known real periodic pulse solution $u(x,t) = A\mathrm{dn}(\xi, m)$ [10, 11].

Solution VII

It is straightforward to check that the nonlocal mKdV Eq. (45) admits another complex periodic pulse solution

$$u(x,t) = A\sqrt{m}\mathrm{cn}(\xi, m)e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{61}$$

provided

$$g = 1\,, \quad A = \beta\,, \quad \omega = -k[k^2 - 3(2m-1)A^2]\,, \quad v = -[3k^2 - (2m-1)A^2]\,. \tag{62}$$

In the limit $\omega = k = 0$ we get back the well known real periodic pulse solution $u(x,t) = A\sqrt{m}\mathrm{cn}(\xi, m)$ [10, 11].

Solution VIII

In the limit $m = 1$, the solutions VI and VII go over to the complex hyperbolic pulse solution

$$u(x,t) = A\mathrm{sech}(\xi)e^{i(kx-\omega t)}\,, \quad \xi = \beta(x - vt)\,, \tag{63}$$

provided

$$g = 1\,, \quad A = \beta\,, \quad \omega = -k[k^2 - 3A^2]\,, \quad v = -[3k^2 - A^2]\,. \tag{64}$$

In the limit $\omega = k = 0$ we get back the well known real hyperbolic pulse solution $u(x,t) = A\mathrm{sech}(\xi)$ [10, 11].

Solution IX

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic solution

$$u(x,t) = \frac{A\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)} e^{i(kx-\omega t)}\,, \quad \xi = \beta(x-vt)\,, \tag{65}$$

provided

$$g = -1\,,\ 0 < m < 1\,,\ A = \beta\,,\ \omega = -k[k^2+3(1+m)A^2]\,,\ v = -[3k^2+(1+m)A^2]\,. \tag{66}$$

In the limit $\omega = k = 0$ we get back the well known real periodic solution $u(x,t) = A\sqrt{m}\mathrm{cn}(\xi,m)/\mathrm{dn}(\xi,m)$ [10, 11].

Solution X

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic solution

$$u(x,t) = \frac{A\sqrt{m(1-m)}\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)} e^{i(kx-\omega t)}\,, \quad \xi = \beta(x-vt)\,, \tag{67}$$

provided

$$g = -1\,,\ 0 < m < 1\,,\ A = \beta\,,\ \omega = -k[k^2-3(2m-1)A^2]\,,\ v = -[3k^2-(2m-1)A^2]\,. \tag{68}$$

In the limit $\omega = k = 0$ we get back the well known real periodic solution $u(x,t) = A\sqrt{m}\mathrm{sn}(\xi,m)/\mathrm{dn}(\xi,m)$ [10, 11].

Solution XI

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic solution

$$u(x,t) = \frac{A\sqrt{1-m}}{\mathrm{dn}(\xi,m)} e^{i(kx-\omega t)}\,, \quad \xi = \beta(x-vt)\,, \tag{69}$$

provided

$$g = 1\,,\ 0 < m < 1\,,\ A = \beta\,,\ \omega = -k[k^2-3(2-m)A^2]\,,\ v = -[3k^2-(2-m)A^2]\,. \tag{70}$$

In the limit $\omega = k = 0$ we get back the well known real periodic solution $u(x,t) = A\sqrt{1-m}/\mathrm{dn}(\xi,m)$ [10, 11].

Solution XII

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic superposed solution

$$u(x,t) = [A\mathrm{dn}(\xi,m) + \sqrt{m}B\mathrm{cn}(\xi,m)]e^{i(kx-\omega t)}\,, \quad \xi = \beta(x-vt)\,, \tag{71}$$

provided

$$
\begin{aligned}
&g=1\,, \quad 0<m<1\,, \quad B=\pm A\,, \quad 2A=\beta\,,\\
&\omega=-k[k^2-6(1+m)A^2]\,, \quad v=-[3k^2-2(1+m)A^2]\,.
\end{aligned} \tag{72}
$$

In the limit $\omega=k=0$ we get back the well known real periodic superposed solution $u(x,t)=A\mathrm{dn}(\xi,m)+B\sqrt{m}\mathrm{cn}(\xi,m)$ [10, 11].

Solution XIII

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic superposed solution

$$
u(x,t)=\left[A\mathrm{dn}(\xi,m)+\frac{B\sqrt{1-m}}{\mathrm{dn}(\xi,m)}\right]e^{i(kx-\omega t)}\,, \quad \xi=\beta(x-vt)\,, \tag{73}
$$

provided

$$
\begin{aligned}
&g=1\,, \quad B=\pm A\,, \quad A=\beta\,, \quad v=-3k^2+[2-m\mp 6\sqrt{1-m}]A^2\,,\\
&0<m<1\,, \quad \omega=-k^3+3k[2-m\pm 6\sqrt{1-m}A^2]\,.
\end{aligned} \tag{74}
$$

In the limit $\omega=k=0$ we get back the well known real periodic superposed solution $u(x,t)=A\mathrm{dn}(\xi,m)+B\sqrt{1-m}/\mathrm{dn}(\xi,m)$ [10, 11].

Solution XIV

It is easy to check that the nonlocal mKdV Eq. (45) admits the complex periodic superposed solution

$$
u(x,t)=\left[\frac{A\sqrt{1-m}}{\mathrm{dn}(\xi,m)}+\frac{iB\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}\right]e^{i(kx-\omega t)}\,, \quad \xi=\beta(x-vt)\,, \tag{75}
$$

provided

$$
\begin{aligned}
&g=1\,, \quad B=\pm A\,, \quad 2A=\beta\,, \quad \omega=-k[k^2+6(2m-1)A^2]\,,\\
&0<m<1\,, \quad v=-[3k^2+2(2m-1)A^2]\,.
\end{aligned} \tag{76}
$$

In the limit $\omega=k=0$ we obtain the complex periodic superposed solution

$$
u(x,t)=\frac{A\sqrt{1-m}}{\mathrm{dn}(\xi,m)}+\frac{iB\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}\,, \tag{77}
$$

provided

$$
g=1\,, \quad 0<m<1\,, \quad B=\pm A\,, \quad 2A=\beta\,, \quad v=-2(2m-1)A^2\,. \tag{78}
$$

Solution XV

It is not difficult to check that the nonlocal mKdV Eq. (45) admits the complex periodic superposed solution

$$u(x,t) = \left[\frac{A\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)} + \frac{iB\sqrt{1-m}}{\mathrm{dn}(\xi,m)}\right] e^{i(kx-\omega t)}\,, \quad \xi = \beta(x-vt)\,, \tag{79}$$

provided

$$\begin{aligned} &g=-1\,, \quad B=\pm A\,, \quad 2A=\beta\,, \quad \omega=-k[k^2+6(2m-1)A^2]\,, \\ &0<m<1\,, \quad v=-[3k^2+2(2m-1)A^2]\,. \end{aligned} \tag{80}$$

In the limit $\omega = k = 0$ we obtain the complex periodic superposed solution

$$u(x,t) = \frac{A\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)} + \frac{iB\sqrt{1-m}}{\mathrm{dn}(\xi,m)}\,, \tag{81}$$

provided

$$g=-1\,, \quad 0<m<1\,, \quad B=\pm A\,, \quad 2A=\beta\,, \quad v=-2(2m-1)A^2\,. \tag{82}$$

Solution XVI

Remarkably, unlike local mKdV Eq. (46), the nonlocal mKdV Eq. (45) admits yet another plane wave type solution

$$u(x,t) = Ae^{kx-\omega t}\,, \tag{83}$$

provided

$$\omega = k(k^2+6gA^2)\,. \tag{84}$$

Solution XVII

It is straightforward to check that the nonlocal mKdV Eq. (45) will admit solutions similar to solutions IV to XV except the factor of $e^{i(kx-\omega t)}$ is replaced by the factor of $e^{kx-\omega t}$ provided one replaces k^3 by $-k^3$ in the expression for ω and replace k^2 by $-k^2$ in the expression for v. As an illustration, it is easy to check that the nonlocal mKdV Eq. (45) admits the solution

$$u(x,t) = A\sqrt{m}\mathrm{sn}(\xi,m)e^{(kx-\omega t)}\,, \quad \xi=\beta(x-vt)\,, \tag{85}$$

provided

$$g=1\,, \quad A=\beta\,, \quad \omega=k[k^2-3(1+m)A^2]\,, \quad v=[3k^2-(1+m)A^2]\,. \tag{86}$$

In the limit $\omega = k = 0$ we get back the well known real periodic kink solution $u(x,t) = A\sqrt{m}\mathrm{sn}(\xi,m)$ [10, 11].

Solution XVIII

In order to obtain the other solutions of the nonlocal mKdV Eq. (45), we define a new variable $\xi = \beta(x - vt)$, in terms of which the nonlocal mKdV Eq. (45) takes the form

$$\beta^2 u_{\xi\xi\xi}(\xi) = v u_\xi(\xi) - 6g u(\xi) u(-\xi) u_\xi(\xi) \,. \tag{87}$$

It is then clear that those solutions of the (local) mKdV equation (46) for which $u(-\xi) = \pm u(\xi)$ are clearly also the solution of the nonlocal mKdV Eq. (45) with same or opposite sign of g. We now present several new solutions of nonlocal mKdV.

It is not difficult to check that

$$u(\xi) = \frac{A\sqrt{m}\,\mathrm{sn}(\xi, m)}{D + \mathrm{dn}(\xi, m)} \,, \quad D > 0 \,, \tag{88}$$

is an exact periodic kink solution of the nonlocal mKdV Eq. (45) provided

$$g = 1 \,, \quad D = 1 \,, \quad 4A^2 = \beta^2 \,, \quad v = -(2 - m)\frac{\beta^2}{2} \,. \tag{89}$$

Note that Eq. (88) is also a solution of the local mKdV Eq. (46) except $g = -1$ in that case (but otherwise same relations as given by Eq. (89).

Solution XIX

It is straightforward to check that

$$u(\xi) = \frac{A\sqrt{m}\,\mathrm{cn}(\xi, m)}{D + \mathrm{dn}(\xi, m)} \,, \quad D > 0 \,, \tag{90}$$

is an exact periodic pulse solution of the nonlocal mKdV Eq. (45) provided

$$g = -1 \,, \quad D^2 = 1 - m > 0 \,, \quad 4A^2 = m\beta^2 \,, \quad v = -(2 - m)\frac{\beta^2}{2} \,. \tag{91}$$

Note that Eq. (90) is also a solution of the local mKdV Eq. (46) provided the relations (91) are satisfied.

Solution XX

It is easy to check that

$$u(\xi) = F + \frac{A}{D + \mathrm{dn}(\xi, m)} \,, \quad A, D > 0 \,, \tag{92}$$

is an exact solution of the nonlocal mKdV Eq. (45) provided

$$\begin{aligned} &6gF^2 - v = [6D^2 - (2 - m)]\beta^2 \,, \quad 2gFA = D[2 - m - 2D^2]\beta^2 \,, \\ &gA^2 = (D^2 - 1)(D^2 + m - 1)\beta^2 \,. \end{aligned} \tag{93}$$

Thus if $g > 0$ then either $D^2 > 1$ or $0 < D^2 < 1 - m$. On the other hand if $g < 0$ then $1 - m < D < 1$.

Note that Eq. (92) is also a solution of the local mKdV Eq. (46) provided the relations (93) are satisfied.

We thus have 4 possibilities:

1. $g = 1\,,\quad F > 0\,,\quad D^2 < 1 - m\,,\quad 6F^2 - v < (4 - 5m)\beta^2$.

2. $g = 1\,,\quad F < 0\,,\quad D^2 > 1\,,\quad 6F^2 - v > (4 + m)\beta^2$.

3. $g = -1\,,\quad F > 0\,,\quad (2 - m)/2 < D^2 < 1\,,\quad -(4 + m)\beta^2 < v + 6F^2 < -(2 - m)\beta^2$.

4. $g = -1\,,\quad F < 0\,,\quad 1 - m < D^2 < (2 - m)/2\,,\quad -(2 - m)\beta^2 < v + 6F^2 < (5m - 4)\beta^2$.

Observe that the solution (92) can be re-expressed as

$$u(\xi) = \frac{FD + A + F\mathrm{dn}(\xi, m)}{D + \mathrm{dn}(\xi, m)}\,, \tag{94}$$

which oscillates between $u = \frac{A+FD+F}{D+1}$ and $\frac{A+FD+F\sqrt{1-m}}{D+\sqrt{1-m}}$.

There are two special cases in which the solution XX takes a somewhat simpler form.

Case a: $F = 0$

In case $F = 0$, the solution (92) takes the form

$$u(\xi) = \frac{A}{D + \mathrm{dn}(\xi, m)}\,,\quad A, D > 0\,, \tag{95}$$

which holds good provided

$$g = -1\,,\quad v = -2(2 - m)\beta^2\,,\quad D^2 = \frac{2 - m}{2}\,,\quad 4A^2 = m^2\beta^2\,. \tag{96}$$

Case b: $A = -FD$

In case $A = -FD$, the solution (92) takes the form

$$u(\xi) = \frac{F\mathrm{dn}(\xi, m)}{D + \mathrm{dn}(\xi, m)}\,,\quad F, D > 0\,, \tag{97}$$

which holds good provided

$$g = -1\,,\quad v = -2(2 - m)\beta^2\,,\quad D^2 = \frac{2(1 - m)}{2 - m}\,,\quad F^2 = \frac{m^2}{2(2 - m)}\beta^2\,. \tag{98}$$

Solution XXI

In the $m = 1$ limit, the solution (92) goes over to the hyperbolic pulse solution

$$u(\xi) = F + \frac{A}{D + \mathrm{sech}(\xi)}\,, \quad A, D > 0\,, \tag{99}$$

provided

$$v = \frac{2D^2 + 1}{2(D^2 - 1)}\beta^2\,, \quad gA^2 = D^2(D^2 - 1)\beta^2\,, \quad gF^2 = \frac{(2D^2 - 1)^2}{4(D^2 - 1)}\beta^2\,. \tag{100}$$

Thus $g > (<)\ 0$ depending on if $D > (<)\ 1$. We thus have three possibilities:

1. $g = 1\,, \quad F > 0\,, \quad D^2 > 1\,, \quad 6F^2 - v > 5\beta^2$.

3. $g = -1\,, \quad F > 0\,, \quad 0 < D^2 < 1/2\,, \quad -2\beta^2 < v + 6F^2 < \beta^2$.

4. $g = -1\,, \quad F < 0\,, \quad 1/2 < D^2 < 1\,, \quad -5\beta^2 < v + 6F^2 < -2\beta^2$.

Note that Eq. (99) is also a solution of the local mKdV Eq. (46) provided the relations (100) are satisfied.

It is easy to check that this solution does not exist in case $FD = -A$. However, it does exist in the other limit of $F = 0$. In case $F = 0$, the solution (99) takes the form

$$u(\xi) = \frac{A}{D + \mathrm{sech}(\xi)}\,, \quad A, D > 0\,, \tag{101}$$

which holds good provided

$$g = -1\,, \quad v = -2\beta^2\,, \quad D^2 = \frac{1}{2}\,, \quad A^2 = \frac{\beta^2}{4}\,. \tag{102}$$

Solution XXII

It is easy to check that

$$u(\xi) = F - \frac{A}{D + \mathrm{cn}(\xi, m)}\,, \quad A > 0, D > 1\,, \tag{103}$$

is an exact solution of the nonlocal mKdV Eq. (45) provided

$$g = 1\,, \quad 6F^2 - v = [6mD^2 - (2m - 1)]\beta^2\,,$$
$$2FA = D[2mD^2 - (2m - 1)]\beta^2\,, \quad A^2 = (D^2 - 1)(mD^2 + 1 - m)\beta^2\,. \tag{104}$$

Note that Eq. (103) is also a solution of the local mKdV Eq. (46) provided the relations (104) are satisfied.

Thus this solution only exists if $g > 0, D^2 > 1\,, \quad 6F^2 - v > (4m + 1)\beta^2$. Note that the solution (103) can be re-expressed as

$$u(\xi) = \frac{FD - A + F\mathrm{cn}(\xi, m)}{D + \mathrm{cn}(\xi, m)}\,, \quad A > 0, D > 1\,. \tag{105}$$

Unlike the solution XX, this solution does not exist in case $F = 0$ since here $D > 1$. However, it does exist in case $FD = A$. In case $A = FD$, the solution (103) takes the form

$$u(\xi) = \frac{F\mathrm{cn}(\xi, m)}{D + \mathrm{cn}(\xi, m)}\,, \quad F, D > 0\,, \tag{106}$$

which holds good provided

$$g = 1\,, \;\; v = 2(1-2m)\beta^2 > 0\,, \;\; D^2 = \frac{2(1-m)}{(1-2m)}\,, \;\; 2A^2 = \frac{1}{(1-2m)}\beta^2\,. \tag{107}$$

Note that this solution exists only if $m < 1/2$.

Solution XXIII

In the limit $m = 0$, the solution XXII goes over to the trigonometric solution

$$u(\xi) = F - \frac{A}{D + \cos(\xi)}\,, \quad A > 0, D > 1\,, \tag{108}$$

provided

$$g = 1\,, \;\; 6F^2 - v = \beta^2\,, \;\; 2FA = D\beta^2\,, A^2 = (D^2 - 1)\beta^2\,. \tag{109}$$

Thus for this solution $g > 0\,, \;\; F > 0$. Note that Eq. (108) is also a solution of the local mKdV Eq. (46) provided the relations (109) are satisfied.

The solution (108) takes a simpler form in case $FD = A$. In case $A = FD$, the solution (108) takes the form

$$u(\xi) = \frac{F\cos(\xi)}{D + \cos(\xi)}\,, \quad F, D > 0\,, \tag{110}$$

which holds good provided

$$g = 1\,, \;\; v = 2\beta^2 > 0\,, \;\; D^2 = 2\,, \;\; F^2 = \frac{\beta^2}{2}\,. \tag{111}$$

Solution XXIV

It is easy to check that

$$u(\xi) = F - \frac{A}{D + \xi^2}\,, \quad D > 0\,, \tag{112}$$

is an exact solution of the nonlocal mKdV Eq. (1) provided

$$g = 1\,, \;\; v = 3/2D > 0\,, \;\; A^2 = 4D\,, \;\; F = \frac{1}{2\sqrt{D}}\,. \tag{113}$$

Thus this solution can also be re-expressed as

$$u(\xi) = \frac{\xi^2 - 3D}{2\sqrt{D}(D + \xi^2)}\,, \quad D > 0\,. \tag{114}$$

Note that Eq. (112) is also a solution of the local mKdV Eq. (46) provided the relations (113) are satisfied. Note that unlike all other solutions discussed so far, this is a solution with a power law tail.

Solution XXV

It is straightforward to check that

$$u(\xi) = \frac{[A\sqrt{1-m} + iB\sqrt{m}\mathrm{cn}(\beta\xi, m)]}{\mathrm{dn}(\beta\xi, m)}\,, \tag{115}$$

is an exact solution of the nonlocal mKdV Eq. (1) provided $0 < m < 1$ and

$$g = 1\,, \quad v = -(2m-1)\frac{\beta^2}{2}\,, \quad B = \pm A\,, \quad A^2 = 4\beta^2\,. \tag{116}$$

Note that Eq. (115) is also a solution of the local mKdV Eq. (46) provided the relations (116) are satisfied.

Solution XXVI

It is easy to check that

$$u(\xi) = \frac{[A\sqrt{m}\mathrm{cn}(\beta\xi, m) + iB]}{\mathrm{dn}(\beta\xi, m)}\,, \tag{117}$$

is an exact solution of the nonlocal mKdV Eq. (1) provided $0 < m < 1$ and

$$g = -1\,, \quad v = -(2m-1)\frac{\beta^2}{2}\,, \quad B = \pm A\,, \quad A^2 = 4\beta^2\,. \tag{118}$$

Note that Eq. (117) is also a solution of the local mKdV Eq. (46) provided the relations (118) are satisfied.

In Appendix B we present 14 solutions of local mKdV Eq. (46) which, however, are not the solutions of the nonlocal mKdV Eq. (45).

5 New Solutions of the Nonlocal Hirota Equation

The nonlocal Hirota equation is given by [6]

$$iu_t(x,t) + \alpha[u_{xx}(x,t) + 2gu^2(x,t)u(-x,-t)] + i\beta[u_{xxx}(x,t) + 6gu(x,t)u(-x,-t)u_x(x,t)] = 0\,. \tag{119}$$

We now obtain 19 new solutions of the nonlocal Hirota Eq. (119) and compare and contrast them with the corresponding solutions of the local Hirota equation

$$iu_t(x,t)+\alpha[u_{xx}(x,t)+2g|u|^2(x,t)u(x,t)]+i\beta[u_{xxx}(x,t)+6g|u|^2(x,t)u_x(x,t)]=0\,, \tag{120}$$

Solution I

We now show that like the NLS equation [20] even nonlocal Hirota Eq. (119) admits the PT-invariant solution with PT-eigenvalue -1, i.e.

$$u(x,t)=\sqrt{n}\,[B\tanh(\xi)+iA]e^{i(\omega t-kx)}\,,\quad \xi=\delta(x-vt)\,, \tag{121}$$

provided

$$g=1\,,\quad A^2+B^2=1\,,\quad \delta=\sqrt{n}B\,,\quad \omega=-\alpha(k^2+2n)+\beta k(k^2+6n)\,,$$
$$v=-2\alpha(k-\sqrt{n}A)+\beta(4nB^2-3k^2-6n+6kA\sqrt{n})\,. \tag{122}$$

Note that Eq. (121) is also the solution of the corresponding local Hirota Eq. (120) provided unlike the solution I, $g=-1$ in the local Hirota case while the other relations of Eq. (122) are satisfied.

Solution II

The nonlocal Hirota Eq. (119) also admits a PT-invariant solution with PT-eigenvalue $+1$, i.e.

$$u(x,t)=\sqrt{n}\,[A+iB\tanh(\xi)]e^{i(\omega t-kx)}\,,\quad \xi=\delta(x-vt)\,, \tag{123}$$

provided

$$g=-1\,,\quad A^2+B^2=1\,,\quad \delta=\sqrt{n}B\,,\quad \omega=-\alpha(k^2+2n)+\beta k(k^2+6n)\,,$$
$$v=-2\alpha(k+\sqrt{n}A)+\beta(4nB^2-3k^2-6n-6kA\sqrt{n})\,. \tag{124}$$

Note that Eq. (124) is also the solution of the corresponding local Hirota equation (120) with the same constraints as given by Eq. (124). It is amusing to note that while the solution I holds good in case $g=1$, the solution II only holds good if $g=-1$.

Solution III

It is not difficult to check that both the local (120) as well as the nonlocal Hirota Eq. (119) admit the plane wave solution

$$u(x,t)=Ae^{i(\omega t-kx)}\,, \tag{125}$$

provided

$$\omega = -\alpha(k^2 - 2gA^2) + \beta k(k^2 - 6gA^2)\,. \tag{126}$$

In order to obtain the other solutions of the nonlocal Hirota Eq. (119), we start with the ansatz

$$u(x,t) = e^{i\omega t}\phi(\xi)\,, \quad \xi = \delta(x - vt)\,. \tag{127}$$

On substituting this ansatz in Eq. (119) we obtain

$$\begin{aligned} &\alpha\left[\delta^2\phi_{\xi\xi}(\xi) + 2g\phi^2(\xi)\phi(-\xi) - \frac{\omega\phi}{\alpha}\right] \\ &+i\beta\delta\left[\delta^2\phi_{\xi\xi\xi}(\xi) + 6g\phi(\xi)\phi(-\xi)\phi_\xi(\xi) - \frac{v\phi_\xi(\xi)}{\beta}\right] = 0\,. \end{aligned} \tag{128}$$

The two cases in which we can obtain the solutions of the Eq. (128) are when $\phi(-\xi) = \pm\phi(\xi)$. This is because in that case we can rewrite Eq. (128) as

$$\begin{aligned} &\alpha\left[\delta^2\phi_{\xi\xi}(\xi) \pm 2g\phi^3(\xi) - \frac{\omega\phi}{\alpha}\right] \\ &+i\beta\delta\frac{d}{d\xi}\left[\delta^2\phi_{\xi\xi}(\xi) \pm 2g\phi^3(\xi) - \frac{v\phi(\xi)}{\beta}\right] = 0\,, \end{aligned} \tag{129}$$

so that as long as

$$\frac{\omega}{\alpha} = \frac{v}{\beta}\,, \tag{130}$$

then all solutions of equation

$$\delta^2\phi_{\xi\xi}(\xi) = \frac{\omega}{\alpha}\phi \mp 2g\phi^3(\xi)\,, \tag{131}$$

automatically solve Eq. (129) as long as the relation (130) is valid. We now present several solutions of Eq. (131) and hence nonlocal Hirota Eq. (119) satisfying relation (130).

Solution IV

It is easy to check that

$$u(x,t) = e^{i\omega t}\frac{A\sqrt{m}\,\mathrm{cn}(\delta\xi, m)}{D + \mathrm{dn}(\delta\xi, m)}\,, \quad A, D > 0\,, \tag{132}$$

is an exact periodic pulse solution of the nonlocal Hirota Eq. (119) provided relation (130) is satisfied and further

$$D^2 = 1 - m > 0\,, \quad 4gA^2 = -m\delta^2\,, \quad \omega = -(2 - m)\frac{\alpha\delta^2}{2}\,. \tag{133}$$

Note that this solution exists only if $g < 0, \omega < 0$. It is worth pointing out that Eq. (132) is also the solution of the corresponding local Hirota equation (120) provided Eq. (133) is satisfied.

Solution V

It is easy to check that

$$u(x,t) = e^{i\omega t}\left[F - \frac{A\mathrm{dn}(\delta\xi, m)}{D + \mathrm{dn}(\delta\xi, m)}\right]\,, \quad D, F, A > 0\,, \tag{134}$$

is an exact periodic pulse solution of the nonlocal Hirota Eq. (119) provided relation (130) is satisfied and further

$$\begin{aligned} &D^2 = \sqrt{1-m} > 0\,, \quad gA^2 = -(1-\sqrt{1-m})^2\delta^2\,, \\ &F = A/2\,, \quad \omega = -[(2-m) + 6\sqrt{1-m}]\frac{\delta^2}{2}\,. \end{aligned} \tag{135}$$

Note that this solution exists only if $g < 0, \omega < 0$. It is worth pointing out that Eq. (134) is also the solution of the corresponding local Hirota Eq. (120) provided the relations (135) are satisfied.

Solution VI

A novel complex (but which is not PT-invariant) solution of the nonlocal Hirota Eq. (119) is

$$u(x,t) = e^{i\omega t}\left[\frac{A\sqrt{1-m}}{\mathrm{dn}(\delta x, m)} + \frac{iB\sqrt{m}\mathrm{cn}(\delta x, m)}{\mathrm{dn}(\delta x, m)}\right]\,, \tag{136}$$

provided relation (130) is satisfied and if further

$$B = \pm A\,, \quad 0 < m < 1\,, \quad 4gA^2 = -\delta^2\,, \quad \omega = -(2m-1)\frac{\delta^2}{2}\,. \tag{137}$$

Note that Eq. (136) is also the solution of the corresponding local Hirota Eq. (120) provided the relations (137) are satisfied.

Solution VII

Another novel complex (but which is not PT-invariant) solution of the nonlocal Hirota Eq. (119) is

$$u(x,t) = e^{i\omega t}\left[\frac{A\sqrt{m}\mathrm{cn}(\delta x, m)}{\mathrm{dn}(\delta x, m)} + \frac{iB\sqrt{1-m}}{\mathrm{dn}(\delta x, m)}\right]\,, \tag{138}$$

provided relation (130) is satisfied and if further

$$B = \pm A\,, \quad 0 < m < 1\,, \quad 4gA^2 = \delta^2\,, \quad \omega = -(2m-1)\frac{\delta^2}{2}\,. \tag{139}$$

Note that Eq. (138) is also the solution of the corresponding local Hirota Eq. (120) provided the relations (139) are satisfied.

Solution VIII

It is easy to check that

$$u(x,t) = e^{i\omega t}\frac{A\sqrt{m}\mathrm{sn}(\delta\xi, m)}{D + \mathrm{dn}(\delta\xi, m)}\,, \quad A, D > 0\,, \tag{140}$$

is an exact periodic kink solution of the nonlocal Hirota Eq. (119) provided relation (130) is satisfied and if further

$$D = 1\,, \quad 4gA^2 = \delta^2\,, \quad \omega = -(2-m)\frac{\delta^2}{2}\,. \tag{141}$$

Note that Eq. (140) is also the solution of the corresponding local Hirota Eq. (120) provided unlike the solution V, $g < 0$ while the other relations of Eq. (141) are satisfied.

Solution IX

We now show that the nonlocal Hirota Eq. (119) also admits solutions with a more general ansatz of the form

$$u(x,t) = Ae^{i(\omega t - kx)}\phi(\xi)\,, \quad \xi = \delta(x - vt)\,. \tag{142}$$

In this case the relation (130) between ω and v is no more valid. We now present 12 such solutions.

It is easy to check that

$$u(x,t) = Ae^{i(\omega t - kx)}\sqrt{m}\mathrm{sn}(\xi, m)\,, \quad \xi = \delta(x - vt)\,, \tag{143}$$

is an exact periodic kink solution of Eq. (119) provided

$$\begin{aligned} &g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 + (1+m)A^2)\,, \\ &\omega = -\alpha[k^2 + (1+m)A^2] + \beta k[k^2 + 3(1+m)A^2]\,. \end{aligned} \tag{144}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution X

In the limit $m = 1$, the solution VIII goes over to the hyperbolic moving kink solution

$$u(x,t) = Ae^{i(\omega t - kx)}\tanh(\xi)\,, \quad \xi = \delta(x - vt)\,, \tag{145}$$

provided

$$g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 + 2A^2)\,,$$
$$\omega = -\alpha[k^2 + 2A^2] + \beta k[k^2 + 6A^2]\,. \tag{146}$$

As expected, in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XI

It is easy to check that

$$u(x,t) = Ae^{i(\omega t - kx)}\mathrm{dn}(\xi, m)\,, \quad \xi = \delta(x - vt)\,, \tag{147}$$

is an exact periodic pulse solution of Eq. (119) provided

$$g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 - (2-m)A^2)\,,$$
$$\omega = -\alpha[k^2 - (2-m)A^2] + \beta k[k^2 - 3(2-m)A^2]\,. \tag{148}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XII

Yet another exact periodic pulse solution of nonlocal Hirota Eq. (119) is

$$u(x,t) = Ae^{i(\omega t - kx)}\sqrt{m}\mathrm{cn}(\xi, m)\,, \quad \xi = \delta(x - vt)\,, \tag{149}$$

provided

$$g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 - (2m-1)A^2)\,,$$
$$\omega = -\alpha[k^2 - (2m-1)A^2] + \beta k[k^2 - 3(2m-1)A^2]\,. \tag{150}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XIII

In the limit $m = 1$, the solutions X and XI go over to the hyperbolic moving pulse solution

$$u(x,t) = Ae^{i(\omega t - kx)}\mathrm{sech}(\xi)\,, \quad \xi = \delta(x - vt)\,, \tag{151}$$

provided

$$g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 - A^2)\,,$$
$$\omega = -\alpha[k^2 - A^2] + \beta k[k^2 - 3A^2]\,. \tag{152}$$

As expected, in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XIV

It is easy to check that

$$u(x,t) = e^{i(\omega t - kx)} \frac{A\sqrt{1-m}}{\mathrm{dn}(\xi,m)}\,, \quad \xi = \delta(x - vt)\,, \tag{153}$$

is an exact periodic pulse solution of Eq. (119) provided $0 < m < 1$ and further

$$\begin{aligned} &g = 1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 - (2-m)A^2)\,, \\ &\omega = -\alpha[k^2 - (2-m)A^2] + \beta k[k^2 - 3(2-m)A^2]\,. \end{aligned} \tag{154}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XV

Yet another exact periodic pulse solution of nonlocal Hirota Eq. (119) is

$$u(x,t) = e^{i(\omega t - kx)} \frac{A\sqrt{m(1-m)}\mathrm{sn}(\xi,m)}{\mathrm{dn}(\xi,m)}\,, \quad \xi = \delta(x - vt)\,, \tag{155}$$

provided $0 < m < 1$ and further

$$\begin{aligned} &g = -1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 - (2m-1)A^2)\,, \\ &\omega = -\alpha[k^2 - (2m-1)A^2] + \beta k[k^2 - 3(2m-1)A^2]\,. \end{aligned} \tag{156}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XVI

It is easy to check that

$$u(x,t) = e^{i(\omega t - kx)} \frac{A\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}\,, \quad \xi = \delta(x - vt)\,, \tag{157}$$

is an exact periodic kink solution of Eq. (119) provided $0 < m < 1$ and further

$$\begin{aligned} &g = -1\,, \quad A = \delta\,, \quad v = -2\alpha k - \beta(3k^2 + (1+m)A^2)\,, \\ &\omega = -\alpha[k^2 + (1+m)A^2] + \beta k[k^2 + 3(1+m)A^2]\,. \end{aligned} \tag{158}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XVII

Remarkably, the nonlocal Hirota Eq. (119) also admits the superposed solution

$$u(x,t) = e^{i(\omega t-kx)}[A\mathrm{dn}(\xi,m) + B\sqrt{m}\mathrm{cn}(\xi,m)]\,, \quad \xi = \delta(x-vt)\,, \tag{159}$$

provided $0 < m < 1$ and further

$$\begin{aligned} &g = 1\,, \quad 2A = \delta\,, \quad B = \pm A\,, \quad v = -2\alpha k - \beta[3k^2 - 2(1+m)A^2]\,, \\ &\omega = -\alpha[k^2 - 2(1+m)A^2] - \beta k[k^2 - 6(1+m)A^2]\,. \end{aligned} \tag{160}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XVIII

Remarkably, the nonlocal Hirota Eq. (119) also admits the superposed solution

$$u(x,t) = e^{i(\omega t-kx)}\left[A\mathrm{dn}(\xi,m) + \frac{B\sqrt{1-m}}{\mathrm{dn}(\xi,m)}\right]\,, \quad \xi = \delta(x-vt)\,, \tag{161}$$

provided $0 < m < 1$ and further

$$\begin{aligned} &g = 1\,, \quad A = \delta\,, \quad B = \pm A\,, \\ &v = -2\alpha k - \beta[3k^2 - (2-mm)A^2 - 6\sqrt{1-m}A^2]\,, \\ &\omega = -\alpha[k^2 - (2-m)A^2 - 6\sqrt{1-m}A^2] \\ &-\beta k[k^2 - 3(2-m)A^2 - 18\sqrt{1-m}A^2]\,. \end{aligned} \tag{162}$$

Note that in the limit $k = 0$ we recover the solution obtained earlier [17] satisfying relation (130).

Solution XIX

Remarkably, the nonlocal Hirota Eq. (119) also admits the complex superposed solution

$$u(x,t) = e^{i(\omega t-kx)}\left[\frac{A\sqrt{1-m}}{\mathrm{dn}(\xi,m)} + \frac{iB\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)}\right]\,, \quad \xi = \delta(x-vt)\,, \tag{163}$$

provided $0 < m < 1$ and further

$$\begin{aligned} &g = 1\,, \quad 2A = \delta\,, \quad B = \pm A\,, \quad v = -2\alpha k - \beta[3k^2 + 2(2m-1)A^2]\,, \\ &\omega = -\alpha[k^2 + 2(2m-1)A^2] - k\beta[k^2 + 6(2m-1)A^2]\,. \end{aligned} \tag{164}$$

Note that in the limit $k = 0$ we recover the solution V satisfying relation (130).

Solution XX

The nonlocal Hirota Eq. (119) also admits another complex superposed solution

$$u(x,t) = e^{i(\omega t - kx)} \left[\frac{A\sqrt{m}\mathrm{cn}(\xi,m)}{\mathrm{dn}(\xi,m)} + \frac{iB\sqrt{1-m}}{\mathrm{dn}(\xi,m)} \right], \quad \xi = \delta(x - vt)\,, \tag{165}$$

provided $0 < m < 1$ and further

$$\begin{aligned} &g = -1\,, \quad 2A = \delta\,, \quad B = \pm A\,, \quad v = -2\alpha k - \beta[3k^2 + 2(2m-1)A^2]\,, \\ &\omega = -\alpha[k^2 + 2(2m-1)A^2] - k\beta[k^2 + 6(2m-1)A^2]\,. \end{aligned} \tag{166}$$

Note that in the limit $k = 0$ we recover the solution VI satisfying relation (130).

In Appendix C we present 5 solutions of the local Hirota Eq. (120) which however are not the solutions of the nonlocal Hirota Eq. (119).

6 Conclusion and Open Problems

In this paper we have obtained several new solutions of Ablowitz-Musslimani as well as Yang nonlocal variants of the NLS equations, nonlocal mKdV equation and nonlocal Hirota equation. Further, we have compared and contrasted with the solutions of the corresponding local NLS, local mKdV and local Hirota equations, respectively. In particular, we found that unlike the local mKdV equation, the nonlocal mKdV equation admits not only a plane wave solution of the form $e^{i(kx-\omega t)}$, but also several solutions multiplied by the same factor. Besides, we found that unlike the local NLS, the Ablowitz-Musslimani nonlocal variant of the NLS equation admits several complex PT-invariant solutions. For completeness, in 3 appendices we have also mentioned those solutions of the local NLS, mKdV and Hirota equations which are, however, not the solutions of the corresponding nonlocal NLS, nonlocal mKdV and nonlocal Hirota equations.

There are several open questions. For example, there are many discrete nonlocal nonlinear equations like nonlocal Ablowitz-Ladik equation, nonlocal discrete NLS (DNLS) equation or discrete saturable nonlocal DNLS equation and the obvious question is if these nonlocal equations also admit similar new novel solutions. Secondly, whether the corresponding coupled nonlocal equations like nonlocal coupled NLS and coupled nonlocal mKdV also admit similar new solutions. We hope to address some of these issues in near future.

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8 Appendix A: NLS Solutions Which are Not Solutions of the Nonlocal NLS Equation

We now present six new solutions of the local NLS Eq. (2) which though are not the solutions of the Ablowitz-Musslimani nonlocal Eq. (1).

Solution IA

It is easy to check that

$$\psi = e^{i\omega t}\frac{A\mathrm{dn}(\beta x, m)}{D+\mathrm{sn}(\beta x, m)}\,, \quad A > 0, D > 1\,, \tag{167}$$

is an exact periodic pulse solution of the local NLS Eq. (1) provided $0 < m < 1$ and further

$$mD^2 = 1\,, \quad 4gmA^2 = (1-m)\beta^2\,, \quad \omega = (1+m)\frac{\beta^2}{2}\,. \tag{168}$$

Solution IIA

It is easy to check that

$$\psi = e^{i\omega t}\frac{[A\mathrm{dn}(\beta x, m) + B\sqrt{m}\mathrm{cn}(\beta x, m)]}{D+\mathrm{sn}(\beta x, m)}\,, \quad A, B > 0, D > 1\,, \tag{169}$$

is an exact superposed periodic solution of the local NLS Eq. (1) provided

$$4gA^2 = (D^2-1)\beta^2\,, \quad 4mgB^2 = (mD^2-1)\beta^2\,, \quad \omega = (1+m)\frac{\beta^2}{2}\,. \tag{170}$$

Thus, this solution exists only if either $D^2 > 1/m, g > 0$ or $D^2 < 1, g < 0$.

Solution IIIA

It is easy to check that

$$\psi = e^{i\omega t}\left[F + \frac{A\mathrm{sn}(\beta x, m)}{D+\mathrm{sn}(\beta x, m)}\right]\,, \quad D > 1, F, A > 0\,, \tag{171}$$

is an exact periodic pulse solution of the local NLS Eq. (1) provided

$$A = \frac{2F}{\sqrt{m}}\,, \quad gA^2 = -\frac{(1-\sqrt{m})^2\beta^2}{\sqrt{m}}\,, \quad D^2 = \frac{1}{\sqrt{m}}\,,$$
$$0 < m < 1\,, \quad \omega = -\left[\frac{3(3+m)\sqrt{m}}{2} + (2m-1)\right]\beta^2\,. \tag{172}$$

Solution IVA

It is well known that the local NLS Eq. (1) admits both Adn and Bsn/dn as exact solutions [17] but we now show that remarkably, even their superposition is an exact solution of Eq. (1). In particular, it is easy to check that

$$\psi = e^{i\omega t}\left[A\mathrm{dn}(\beta x, m) + \frac{B\sqrt{m(1-m)}\mathrm{sn}(\beta x, m)}{\mathrm{dn}(\beta x, m)}\right]\,, \tag{173}$$

is an exact solution of Eq. (1) provided

$$0 < m < 1\,, \quad B = \pm A\,, \quad 4gA^2 = \beta^2\,, \quad \omega = (1+m)\beta^2\,. \tag{174}$$

Note, this solution is not an eigenstate of parity, i.e. while the first term is even under $x \to -x$, the second term is odd under parity.

Solution VA

It is not so well known that the local NLS Eq. (2) admits the PT-invariant solution with PT-eigenvalue -1, i.e.

$$\psi = \sqrt{n}[B\tanh(\xi) + iA]e^{i(kx-\omega t+\theta_0)}\,, \quad \xi = \beta(x - vt + t_0)\,, \tag{175}$$

provided

$$g = -1\,, \quad A^2+B^2 = 1\,, \quad \beta = \sqrt{n/2}B\,, \quad \omega = k^2+n\,, \quad v = 2k+\sqrt{2n}A\,. \tag{176}$$

Solution VIA

The local NLS Eq. (2) also admits the PT-invariant solution with PT-eigenvalue $+1$, i.e. solution

$$\psi = \sqrt{n}[A + iB\tanh(\xi)]e^{i(kx-\omega t+\theta_0)}\,, \quad \xi = \beta(x - vt + t_0)\,, \tag{177}$$

provided

$$g = -1\,, \quad A^2+B^2 = 1\,, \quad \beta = \sqrt{n/2}B\,, \quad \omega = k^2+n\,, \quad v = 2k-\sqrt{2n}A\,. \tag{178}$$

Notice that the solutions VA and VIA hold good under the same conditions except for the relation for the velocity v.

9 Appendix B: mKdV Solutions Which are Not Solutions of the Nonlocal mKdV Equation

We now present 14 new solutions of the (local) mKdV equation (46), neither of which is a solution of the nonlocal mKdV Eq. (45).

Solution IB

It is easy to check that

$$u(\xi) = F - \frac{A}{D + \mathrm{sn}(\xi, m)}\,, \quad D > 1\,, \quad \xi = \beta(x - vt)\,, \tag{179}$$

is an exact solution of mKdV Eq. (120) provided

$$\begin{aligned} v - 6gF^2 &= [6mD^2 - (1+m)]\beta^2\,, \\ 2gFA &= D[(1+m) - 2mD^2]\beta^2\,, \quad gA^2 = -(D^2-1)(mD^2-1)\beta^2\,. \end{aligned} \tag{180}$$

Since $D > 1$, it implies that

$$\begin{aligned} &1 < D^2 < 1/m\,, \quad g = 1\,, \\ &D^2 > 1/m\,, \quad g = -1\,. \end{aligned} \tag{181}$$

This solution can be re-expressed as

$$u(\xi) = \frac{FD - A + F\mathrm{sn}(\xi, m)}{D + \mathrm{sn}(\xi, m)}\,, \quad D > 1\,, \quad \xi = \beta(x - vt)\,, \tag{182}$$

There are two special cases in which the Solution IB takes a simpler form which we now discuss one by one.

Case a: $F = 0$

In case $F = 0$, the solution IB takes the simpler form

$$u(\xi) = -\frac{A}{D + \mathrm{sn}(\xi, m)}\,, \quad D > 1\,, \quad \xi = \beta(x - vt)\,, \tag{183}$$

provided

$$g = -1\,, \quad v = 2(1+m)\beta^2\,, \quad D^2 = \frac{(1+m)}{2m}\,, \quad 4A^2 = (1-m)^2\beta^2\,. \tag{184}$$

Case b: $FD = A$

In case $FD = A$, the solution IB takes the simpler form

$$u(\xi) = \frac{F\mathrm{sn}(\xi, m)}{D + \mathrm{sn}(\xi, m)}\,, \quad D > 1\,, \quad \xi = \beta(x - vt)\,, \tag{185}$$

provided

$$g = 1 \,, \quad v = 2(1+m)\beta^2 \,, \quad D^2 = \frac{2}{(1+m)} \,, \quad F^2 = \frac{(1-m)^2\beta^2}{2(1+m)} \,. \tag{186}$$

Solution IIB

In the limit $m = 1$, the solution IB goes over to the hyperbolic kink solution

$$u(\xi) = F - \frac{A}{D + \tanh(\xi)} \,, \quad D > 1 \,, \tag{187}$$

provided

$$g = -1 \,, \quad v + 6F^2 = 2[3D^2 - 1]\beta^2 \,, \quad FA = D(D^2-1)\beta^2 \,, A^2 = (D^2-1)^2\beta^2 \,. \tag{188}$$

Solution IIIB

It is easy to check that

$$u(\xi) = \frac{A\mathrm{dn}(\beta x, m)}{D + \mathrm{sn}(\beta\xi, m)} \,, \quad D > 1 \,, \quad \xi = \beta(x - vt) \,, \tag{189}$$

is an exact solution of mKdV Eq. (120) provided

$$g = 1 \,, \quad v = \frac{(1+m)\beta^2}{2} \,, \quad D = m^{-1/2} \,, \quad A^2 = \frac{(1-m)\beta^2}{4m} \,. \tag{190}$$

Solution IVB

It is easy to check that

$$u(\xi) = \frac{A\mathrm{dn}(\beta x, m) + B\sqrt{m}\mathrm{cn}(\beta x, m)}{D + \mathrm{sn}(\beta\xi, m)} \,, \quad D > 1 \,, \quad \xi = \beta(x - vt) \,, \tag{191}$$

is an exact solution of mKdV Eq. (120) provided

$$g = 1 \,, \quad v = \frac{(1+m)\beta^2}{2} \,, \quad 4A^2 = (D^2-1)\beta^2 \,, \quad 4mB^2 = (mD^2-1)\beta^2 \,. \tag{192}$$

Thus this solution exists only if $mD^2 > 1$.

Solution VB

Remarkably, mKdV Eq. (46) also admits a complex PT-invariant pulse solution with PT-eigenvalue +1

$$u(\xi) = \frac{[A\sqrt{m}\mathrm{cn}(\beta\xi, m) + iB\sqrt{m}\mathrm{sn}(\beta\xi, m)]}{D + \mathrm{dn}(\beta\xi, m)} \,, \quad D > 0 \,, \tag{193}$$

provided

$$4gA^2 = (D^2 - 1)\beta^2\,, \quad 4gB^2 = (D^2 - 1 + m)\beta^2\,, \quad v = -\frac{(2-m)\beta^2}{2}\,. \quad (194)$$

Thus there are two possibilities.

1. $g = 1\,, \quad mD^2 > 1$.

2. $g = -1\,, \quad D^2 < 1 - m$. Thus in this case $4A^2 = (1 - D^2)\beta^2\,, \;\; 4B^2 = (1 - m - D^2)\beta^2$.

Solution VIB

Remarkably, mKdV Eq. (46) also admits another complex PT-invariant pulse solution with PT-eigenvalue $+1$

$$u(\xi) = \frac{[A\mathrm{dn}(\beta\xi, m) + iB\sqrt{m}\mathrm{sn}(\beta\xi, m)]}{D + \mathrm{cn}(\beta\xi, m)}\,, \quad D > 1\,, \quad (195)$$

provided

$$g = 1\,, \;\; 4A^2 = (D^2 - 1)\beta^2\,, \;\; 4mB^2 = (mD^2 - m + 1)\beta^2\,, \;\; v = -\frac{(2m-1)\beta^2}{2}\,. \quad (196)$$

Thus this solution only exists if $v < 0$ and $D^2 > 1$.

Solution VIIB

In the limit $m = 1$, the solutions VI and VII go over to the hyperbolic complex PT-invariant pulse solution with PT-eigenvalue $+1$

$$u(\xi) = \frac{[A\mathrm{sech}(\beta\xi) + iB\tanh(\beta\xi)]}{D + \mathrm{sech}(\beta\xi)}\,, \quad D > 0\,, \quad (197)$$

provided

$$g = 1\,, \quad 4A^2 = (D^2 - 1)\beta^2\,, \quad 4B^2 = D^2\beta^2\,, \quad v = -\frac{\beta^2}{2}\,. \quad (198)$$

Thus this solution only exists if $v < 0$ and $D^2 > 1$.

Solution VIIIB

Remarkably, mKdV Eq. (46) also admits a complex PT-invariant kink solution with PT-eigenvalue -1

$$u(\xi) = \frac{[A\sqrt{m}\mathrm{sn}(\beta\xi, m) + iB\sqrt{m}\mathrm{cn}(\beta\xi, m)]}{D + \mathrm{dn}(\beta\xi, m)}\,, \quad D > 0\,, \quad (199)$$

provided

$$4gA^2 = -(D^2 - 1 + m)\beta^2\,, \;\; 4gB^2 = -(D^2 - 1)\beta^2\,, \;\; v = -\frac{(2-m)\beta^2}{2}\,. \quad (200)$$

Thus there are two possibilities.

1. $g=-1\,, \quad D^2>1$.

2. $g=1\,, \quad D^2<1-m$. Thus in this case $4B^2=(1-D^2)\beta^2\,, \quad 4A^2=(1-m-D^2)\beta^2$.

Solution IXB

Remarkably, mKdV Eq. (46) also admits another complex PT-invariant kink solution with PT-eigenvalue -1

$$u(\xi)=\frac{[A\sqrt{m}\mathrm{sn}(\beta\xi,m)+iB\mathrm{dn}(\beta\xi,m)]}{D+\mathrm{cn}(\beta\xi,m)}\,, \quad D>1\,, \tag{201}$$

provided

$$g=-1\,, \; 4mA^2=(mD^2+1-m)\beta^2\,, \; 4B^2=(D^2-1)\beta^2\,, \; v=-\frac{(2m-1)\beta^2}{2}\,. \tag{202}$$

Solution XB

In the limit $m=1$, both the periodic complex kink solutions IX and X go over to the hyperbolic complex kink solution with PT-eigenvalue -1

$$u(\xi)=\frac{[A\tanh(\beta\xi)+iB\mathrm{sech}(\beta\xi)]}{D+\mathrm{sech}(\beta\xi)}\,, \quad D>0\,, \tag{203}$$

provided

$$g=-1\,, \quad 4A^2=D^2\beta^2\,, \quad 4B^2=(D^2-1)\beta^2\,, \quad v=-\frac{\beta^2}{2}\,. \tag{204}$$

Thus this solution only exists if $v<0$ and $D^2>1$.

Solution XIB

It is easy to check that

$$u(\xi)=\frac{A\sin(\beta\xi)+iB}{D+\cos(\beta\xi)}\,, \quad D>1\,, \tag{205}$$

is an exact solution of mKdV Eq. (46) provided

$$g=-1\,, \quad v=\frac{\beta^2}{2}>0\,, \quad 4B^2=-(D^2-1)\beta^2\,, 4A^2=\beta^2\,. \tag{206}$$

Solution XIIB

It is straightforward to check that

$$u(\xi)=\frac{A+iB\sin(\beta\xi)}{D+\cos(\beta\xi)}\,, \quad D>1\,, \tag{207}$$

is an exact solution of mKdV Eq. (46) provided

$$g = 1\,, \quad v = \frac{\beta^2}{2} > 0\,, \quad 4A^2 = -(D^2-1)\beta^2\,, 4B^2 = \beta^2\,. \tag{208}$$

Solution XIIIB
It is not difficult to check that

$$u(\xi) = \frac{\sqrt{1-m}A + B\sqrt{m(1-m)}\mathrm{sn}(\beta\xi, m)}{\mathrm{dn}(\beta\xi, m)}\,, \tag{209}$$

is an exact solution of mKdV Eq. (46) provided

$$g = -1\,, \quad v = \frac{(1+m)\beta^2}{2} > 0\,, \quad 4A^2 = \beta^2\,. \tag{210}$$

Finally, we want to remind about the celebrated Miura transformation [21] which showed a remarkable connection between the solutions of the repulsive mKdV (i.e. Eq. (46) with $g = -1$) and the KdV equation

$$w_t + w_{xxx} - 6ww_x = 0\,. \tag{211}$$

In particular it was shown that if $u(x,t)$ is a solution of the repulsive mKdV equation

$$u_t + u_{xxx} - 6u^2 u_x = 0\,, \tag{212}$$

then $w(x,t) = u^2(x,t) + u_x(x,t)$ is the corresponding solution of the KdV Eq. (211). Out of the 14 solutions given above, we find that the solutions IB, IIIB, XB, XIB, XIIB, XIVB hold good when $g = -1$ while solutions IIB, VIB, IXB hold good for $g = -1$ under certain conditions. Besides, solutions IV, V and VI of Sec. IV also hold good in the case of local mKdV provided $g = -1$. Besides, solutions VII and IX hold good for $g = -1$ in the case of local mKdV under certain conditions. Thus in all these cases, the local KdV Eq. (211) admits solutions related to the corresponding mKdV solutions by the Miura transformation.

10 Appendix C: Solutions of Local Hirota equation Which are Not Solutions of the Nonlocal Hirota Equation

We now present 5 new solutions of (local) Hirota Eq. (120) which are, however, not the solutions of the nonlocal Hirota Eq. (119).

Solution IC

It is easy to check that

$$u(x,t) = e^{i\omega t}\frac{A\text{dn}(\delta\xi, m)}{D + \text{sn}(\delta\xi, m)}\,, \quad D > 1\,, \quad \xi = x - vt\,, \tag{213}$$

is an exact periodic pulse solution of (local) Hirota Eq. (120) provided

$$0 < m < 1\,, \quad mD^2 = 1\,, \quad 4gmA^2 = (1-m)\delta^2\,,$$
$$\frac{\omega}{\alpha} = (1+m)\frac{\delta^2}{2}\,, \quad \omega\beta = v\alpha\,. \tag{214}$$

Solution IIC

It is straightforward to check that

$$u(x,t) = e^{i\omega t}\frac{[A\text{dn}(\delta\xi, m) + \sqrt{m}\text{cn}(\delta\xi, m)]}{D + \text{sn}(\beta x, m)}\,, \quad D > 1\,, \quad \xi = x - vt\,, \tag{215}$$

is an exact superposed periodic solution of Hirota Eq. (120) provided

$$4gA^2 = (D^2-1)\delta^2\,, \quad 4mgB^2 = (mD^2-1)\delta^2\,,$$
$$\frac{\omega}{\alpha} = (1+m)\frac{\delta^2}{2}\,, \quad \omega\beta = v\alpha\,. \tag{216}$$

Thus such a solution exists only if $mD^2 > 1$.

Solution IIIC

In the limit $m = 1$, the solution II does not exist but instead we have a novel pulse solution

$$u(x,t) = e^{i\omega t}\frac{A\text{sech}(\delta\xi)}{D + \tanh(\delta\xi)}\,, \quad D > 1\,, \quad \xi = x - vt\,, \tag{217}$$

provided

$$gA^2 = (D^2-1)\delta^2\,, \quad \frac{\omega}{\alpha} = \delta^2\,, \quad \omega\beta = v\alpha\,. \tag{218}$$

Solution IVC

It is easy to check that

$$u(x,t) = e^{i\omega t}\left[F + \frac{A\text{sn}(\delta\xi, m)}{D + \text{sn}(\delta\xi, m)}\right]\,, \quad A, F > 0\,, \quad D > 1\,, \quad \xi = x - vt\,, \tag{219}$$

is an exact periodic pulse solution of Hirota Eq. (120) provided

$$0 < m < 1\,, \quad gA^2 = -\frac{(1-\sqrt{m})^2\delta^2}{\sqrt{m}}\,, \quad F = A/2\,,$$
$$\frac{\omega}{\alpha} = -[3(3+m)\sqrt{m} + 2(2m-1)]\frac{\delta^2}{2}\,, \quad \omega\beta = v\alpha\,. \tag{220}$$

Solution VC

It is easy to check that

$$u(x,t) = e^{i\omega t}\frac{[A\sqrt{1-m}+B\sqrt{m(1-m)}\mathrm{sn}(\delta\xi,m)]}{\mathrm{dn}(\beta x,m)}\,,\quad \xi = x - vt\,, \tag{221}$$

is an exact superposed periodic solution of Hirota Eq. (120) provided

$$\begin{aligned} &B = \pm A\,,\quad 4gA^2 = \delta^2\,,\quad \omega\beta = v\alpha\,,\\ &\frac{\omega}{\alpha} = (1+m)\frac{\delta^2}{2}\,. \end{aligned} \tag{222}$$

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