

# PSEUDOCOVERING AND DIGITAL COVERING SPACES

SANG-EON HAN

ABSTRACT. The notions of a local  $(k_0, k_1)$ -isomorphism and a weakly local  $(k_0, k_1)$ -isomorphism play crucial roles in developing a digital  $(k_0, k_1)$ -covering space and a pseudo- $(k_0, k_1)$ -covering space, respectively. In relation to the study of pseudo- $(k_0, k_1)$ -covering spaces, since there are some works to be refined and improved in the literature, the recent paper [11] improved and corrected some mistakes occurred in the literature. One of the important things is that the notion of a pseudo- $(k_0, k_1)$ -covering map in [7, 10] was revised to be more broadened in [11]. Thus this new version is proved to be equivalent to a weakly local  $(k_0, k_1)$ -isomorphic surjection [11]. The present paper contains some works in [11] and we only deals with  $k$ -connected digital images  $(X, k)$ .

---

AMS Classification: 54C08, 05C40, 68U05

Keywords: (digital) covering space, pseudocovering space, (weakly) local  $(k_0, k_1)$ -isomorphism, digital lifting property.

## 1. Introduction

The notion of a pseudo- $(k_0, k_1)$ -covering space was initially introduced in 2012 [7]. Indeed, it was intended to make a digital  $(k_0, k_1)$ -covering space in [2, 3, 4, 6] more generalized and broader. Hence it was defined by using three conditions among which two of them, i.e., the conditions (1) and (2) for a pseudo- $(k_0, k_1)$ -covering space (see Definition 4), are equal to those for a digital  $(k_0, k_1)$ -covering space (see Definition 5). Meanwhile, the other condition (3) for a pseudo- $(k_0, k_1)$ -covering space is different from the condition (3) for a digital  $(k_0, k_1)$ -covering space (see Definitions 4 and 5 in the present paper). To be specific, the former was defined by using the notion of a weakly local  $(WL-, \text{ for brevity}) (k_0, k_1)$ -isomorphism and the latter was characterized by using the concept of a local  $(k_0, k_1)$ -isomorphism. Thus these two conditions (3) are quite different from each other. However, when combining the two conditions (1) and (2) with each of the conditions (3), a pseudo- $(k_0, k_1)$ -covering space implies to a digital  $(k_0, k_1)$ -covering space (see Theorem 3.3 of [13]). Probably, in [7], there seems to be a gap between the author's intension for establishing a pseudo- $(k_0, k_1)$ -covering space and the mathematical presentation of it. Hence the recent paper [11] revised the original version of the condition (1) for a pseudo- $(k_0, k_1)$ -covering space (see Definition 4 in the present paper) to finally make a distinction between a digital  $(k_0, k_1)$ -covering and a revised version of the original version of a pseudo- $(k_0, k_1)$ -covering map in Definition 4. In detail, we will shortly see the revised version of Definition 4 via Definition 4.1 of [11].

The present paper only deals with  $k$ -connected digital images unless otherwise stated and often uses the notion  $:=$  to introduce some terms.

The four papers [7, 10, 13, 14] studied various properties of “pseudocovering spaces”. Since the two of them [7, 10] have some errors and the others [13, 14] also have some mistakes relating to the map in (1.1) below, the recent paper [11] corrected and improved them, which makes them so clear. More precisely, with the original version of a pseudo- $(k_0, k_1)$ -covering map (see Definition 4 in the present paper), the map  $p$  in (1.1) below is not a pseudo- $(2, k)$ -covering map (see Proposition 3.2 of [13]). Since there are some errors in the proof of Proposition 3.2 of [13], we note that the paper [11] corrected it.

$$\left\{ \begin{array}{l} p : (\mathbb{Z}^+, 2) \rightarrow SC_k^{m,l} := (x_i)_{i \in [0, l-1]_{\mathbb{Z}}} \text{ defined by} \\ p(t) = x_{t \pmod{l}}, \text{ where } \mathbb{Z}^+ := [0, \infty)_{\mathbb{Z}} := \{t \in \mathbb{Z} \mid t \geq 0\}. \end{array} \right\} \quad (1.1)$$

Hence the recent paper [11] fully explained the process of a non-pseudo- $(2, k)$ -covering map of  $p$  in (1.1).

Next, we also note that there are some mistakes on the identity of (4.2) in Proposition 4.4 and Corollary 4.5 of [10]. The paper [11] pointed out these defects and corrected them and verified that a  $WL$ -( $k_0, k_1$ )-surjection is not equivalent to a pseudo- $(k_0, k_1)$ -covering map followed by Definition 4 in the present paper.

To sum up, the recent paper [11] did corrections and improvements, as follows:

- (1) Corrections of the map  $p$  of (4.1) in the proof of Remark 4.3(2) of [10].
- (2) Corrections of the identity of (4.2) of Proposition 4.4 and Corollary 4.5 of [10].
- (3) Revision of the notion of a pseudo- $(k_0, k_1)$ -covering space of Definition 4.
- (4) Correction of the proof of Proposition 3.2 of [13] and related works in [14].
- (5) Improvement of the proof of Theorem 3.3 of [14].

In addition, we confirm that the example in (1.1) now becomes an example for the revised version of a pseudo- $(2, k)$ -covering space in [11].

## 2. Preliminaries

In relation to the study of some properties of a pseudo- $(k_0, k_1)$ -covering space, to make the paper self-contained, we will refer to some notions. Naively, a digital image  $(X, k)$  can be considered to be a set  $X \subset \mathbb{Z}^n$  with one of the  $k$ -adjacency of  $\mathbb{Z}^n$  from (2.1) below (or a digital  $k$ -graph on  $\mathbb{Z}^n$  [5]). Indeed, the papers [12, 15] considered  $(X, k)$ ,  $X \subset \mathbb{Z}^n$ ,  $n \in \{1, 2, 3\}$ , with 2-adjacency on  $\mathbb{Z}$ , 4, 8-adjacency on  $\mathbb{Z}^2$ , and 6, 18, 26-adjacency on  $\mathbb{Z}^3$ . As the generalization of the low dimensional cases, the digital  $k$ -adjacency relations (or digital  $k$ -connectivity) for  $X \subset \mathbb{Z}^n$ ,  $n \in \mathbb{N}$ , were initially established in [8] (see also [2, 3, 4]), as follows:

For a natural number  $t$ ,  $1 \leq t \leq n$ , the distinct points  $p = (p_1, p_2, \dots, p_n)$  and  $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$  are  $k(t, n)$ -adjacent if at most  $t$  of their coordinates differ by  $\pm 1$  and the others coincide. Indeed, the numbers of  $t$  and  $n$  of  $k(t, n)$  above is very important. For instance, on  $\mathbb{Z}^2$ , two types of digital  $k$ -adjacencies exist such as  $k(1, 2) = 4$  and  $k(2, 2) = 8$ . Meanwhile, on  $\mathbb{Z}^4$ , four kinds of digital  $k$ -adjacencies exist such as  $k(1, 4) = 8$ ,  $k(2, 4) = 32$ ,  $k(3, 4) = 64$ ,  $k(4, 4) = 80$ . Then, even though the 8-adjacency are used on both  $\mathbb{Z}^2$  and  $\mathbb{Z}^4$ , using  $k(2, 2) = 8$

and  $k(1, 4) = 8$ , we can make a distinction between them efficiently. According to this statement, the well-presented  $k(t, n)$ -adjacency relations (or digital  $k$ -connectivities) of  $\mathbb{Z}^n, n \in \mathbb{N}$ , are formulated [8] (see also [4]) as follows:

$$k := k(t, n) = \sum_{i=1}^t 2^i C_i^n, \text{ where } C_i^n := \frac{n!}{(n-i)! i!}. \quad (2.1)$$

Based on the  $k$ -adjacency relations of  $\mathbb{Z}^n$  in (2.1),  $n \in \mathbb{N}$ , we will call the pair  $(X, k)$  a digital image on  $\mathbb{Z}^n, X \subset \mathbb{Z}^n$ .

A simple closed  $k$ -curve (or simple  $k$ -cycle) with  $l$  elements in  $\mathbb{Z}^n, n \geq 2$ , denoted by  $SC_k^{n,l}$  [4, 12],  $l(\geq 4) \in \mathbb{N}$ , is defined to be the set  $(x_i)_{i \in [0, l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^n$  such that  $x_i$  and  $x_j$  are  $k$ -adjacent if and only if  $|i - j| = \pm 1 \pmod{l}$ . Then, the number  $l$  of  $SC_k^{n,l}$  depends on both the dimension  $n$  of  $\mathbb{Z}^n$  and the  $k$ -adjacency (see many types of  $SC_k^{n,l}$  in (5) on the page of 6 of [9]).

For a digital image  $(X, k)$  and  $x \in X$ , we follow the notation

$$N_k(x, 1) := \{x' \in X \mid x \text{ is } k\text{-adjacent to } x'\} \cup \{x\}, \quad (2.2)$$

which is called a digital  $k$ -neighborhood of  $x$  in  $(X, k)$  [2, 3, 4, 6]. Indeed, this notion has been effectively used in studying both pseudocovering spaces and digital covering spaces. For every point  $x$  of a digital image  $(X, k)$ , an  $N_k(x, 1)$  always exists in  $(X, k)$ , the digital continuity of [15] can be represented by the following form.

**Proposition 2.1.** [4, 6] *Let  $(X, k_0)$  and  $(Y, k_1)$  be digital images in  $\mathbb{Z}^{n_0}$  and  $\mathbb{Z}^{n_1}$ , respectively. A function  $f : X \rightarrow Y$  is  $(k_0, k_1)$ -continuous if and only if for every point  $x \in X$ ,  $f(N_{k_0}(x, 1))$  is a subset of  $N_{k_1}(f(x), 1)$ .*

Owing to a digital  $k$ -graph theoretical feature of a digital image  $(X, k)$ , we have often used a  $(k_0, k_1)$ -isomorphism in [5] instead of a  $(k_0, k_1)$ -homeomorphism in [1], as follows:

**Definition 1.** [1] (see also [5]) *For two digital images  $(X, k_0)$  in  $\mathbb{Z}^{n_0}$  and  $(Y, k_1)$  in  $\mathbb{Z}^{n_1}$ , a map  $h : X \rightarrow Y$  is called a  $(k_0, k_1)$ -isomorphism if  $h$  is a  $(k_0, k_1)$ -continuous bijection and further,  $h^{-1} : Y \rightarrow X$  is  $(k_1, k_0)$ -continuous. If  $n_0 = n_1$  and  $k_0 = k_1$ , then we call it a  $k_0$ -isomorphism.*

Based on this approach, we can develop the notion of a “radius 2- $(k_0, k_1)$ -isomorphism” or a radius 2- $(k_0, k_1)$ -covering map [2, 3] to establish the so-called “digital homotopy lifting theorem” which is essential in studying digital homotopy theory.

### 3. Remarks on the earlier verion of a pseudo- $(k_0, k_1)$ -covering space in [7, 10]

Since the notions of a digital  $(k_0, k_1)$ -covering map and a pseudo- $(k_0, k_1)$ -covering map are so related to the notion of a (weakly) local  $(k_0, k_1)$ -isomorphism, we first need to recall it, as follows:

**Definition 2.** [2, 4, 10] *For two digital images  $(X, k_0)$  in  $\mathbb{Z}^{n_0}$  and  $(Y, k_1)$  in  $\mathbb{Z}^{n_1}$ , consider a map  $h : (X, k_0) \rightarrow (Y, k_1)$ . Then the map  $h$  is said to be a local  $(k_0, k_1)$ -isomorphism if for every  $x \in X$ ,  $h$  maps  $N_{k_0}(x, 1)$   $(k_0, k_1)$ -isomorphically onto  $N_{k_1}(h(x), 1)$  i.e., the restriction map  $h|_{N_{k_0}(x, 1)} : N_{k_0}(x, 1) \rightarrow N_{k_1}(h(x), 1)$  is a  $(k_0, k_1)$ -isomorphism. If  $n_0 = n_1$  and  $k_0 = k_1$ , then the map  $h$  is called a local  $k_0$ -isomorphism.*

The paper [7] defined the following notion which is weaker than a local  $(k_0, k_1)$ -isomorphism.

**Definition 3.** [7] *For two digital images  $(X, k_0)$  in  $\mathbb{Z}^{n_0}$  and  $(Y, k_1)$  in  $\mathbb{Z}^{n_1}$ , a map  $h : X \rightarrow Y$  is called a weakly local (WL-, for brevity)  $(k_0, k_1)$ -isomorphism if for every  $x \in X$ ,  $h$  maps  $N_{k_0}(x, 1)$   $(k_0, k_1)$ -isomorphically onto  $h(N_{k_0}(x, 1)) \subset (Y, k_1)$ , i.e., the restriction map  $h|_{N_{k_0}(x, 1)} : N_{k_0}(x, 1) \rightarrow h(N_{k_0}(x, 1))$  is a  $(k_0, k_1)$ -isomorphism. In particular, if  $n_0 = n_1$  and  $k_0 = k_1$ , then the map  $h$  is called a weakly local  $k_0$ -isomorphism (or a WL- $k_0$ -isomorphism).*

Using this notion, the paper [7] defined the notion of a pseudo- $(k_0, k_1)$ -covering space, as follows:

**Definition 4.** [7] *Let  $(E, k_0)$  and  $(B, k_1)$  be digital images in  $\mathbb{Z}^{n_0}$  and  $\mathbb{Z}^{n_1}$ , respectively. Let  $p : E \rightarrow B$  be a surjection such that for any  $b \in B$ ,*

*(1) for some index set  $M$ ,  $p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i \in M} N_{k_0}(e_i, 1)$  with  $e_i \in$*

*$p^{-1}(b) := p^{-1}(\{b\})$ ;*

*(2) if  $i, j \in M$  and  $i \neq j$ , then  $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1)$  is an empty set; and*

*(3) the restriction of  $p$  to  $N_{k_0}(e_i, 1)$  from  $N_{k_0}(e_i, 1)$  to  $N_{k_1}(b, 1)$  is a WL- $(k_0, k_1)$ -isomorphism for all  $i \in M$ .*

*Then the map  $p$  is called a pseudo- $(k_0, k_1)$ -covering map,  $(E, p, B)$  is said to be a pseudo- $(k_0, k_1)$ -covering and  $(E, k_0)$  is called a pseudo- $(k_0, k_1)$ -covering space over  $(B, k_1)$ .*

Based on the notion of a pseudo- $(k_0, k_1)$ -covering space, the paper [7] referred to the map  $p : (\mathbb{Z}^+, 2) \rightarrow SC_k^{n, l}$  as in (1.1) for a pseudo- $(2, k)$ -covering map. Indeed, the paper [7] made a mistake to take this map as a pseudo- $(2, k)$ -covering map (see Remark 3.1 below). By contary

to the condition (1) of Definition 4, the map  $p$  is not a pseudo- $(2, k)$ -covering map, as follows:

**Remark 3.1.** (*Proposition 3.2 of [13]*) The map  $p : (\mathbb{Z}^+, 2) \rightarrow SC_k^{n,l} := (c_i)_{i \in [0, l-1]_{\mathbb{Z}}}, l \geq 4$ , in (1.1) is not a pseudo- $(2, k)$ -covering map.

Since the proof of this assertion in [13] is incorrect, the paper [11] corrected the errors.

To compare between a digital covering space and a pseudocovering space, we need to recall the notion of a digital covering space as follows:

**Definition 5.** [3, 4, 6] Let  $(E, k_0)$  and  $(B, k_1)$  be digital images in  $\mathbb{Z}^{n_0}$  and  $\mathbb{Z}^{n_1}$ , respectively. Let  $p : E \rightarrow B$  be a surjection such that for any  $b \in B$ , the conditions (1) and (2) are equal to those of Definition 4; and the condition (3) is the following:

The restriction of  $p$  to  $N_{k_0}(e_i, 1)$  from  $N_{k_0}(e_i, 1)$  to  $N_{k_1}(p(e_i), 1)$  is a  $(k_0, k_1)$ -isomorphism for all  $i \in M$ .

Then the map  $p$  is called a digital  $(k_0, k_1)$ -covering map,  $(E, p, B)$  is said to be a digital  $(k_0, k_1)$ -covering and  $(E, k_0)$  is called a digital  $(k_0, k_1)$ -covering space over  $(B, k_1)$ .

Based on Definitions 4 and 5, the following is obtained.

**Theorem 3.2.** (see Corollary 4 of [9]) In Definition 4, as a special case, assume that  $(E, k_0)$  and  $(B, k_1)$  are  $k_0$ - and  $k_1$ -connected, respectively. Then a digital  $(k_0, k_1)$ -covering map is equivalent to a local  $(k_0, k_1)$ -isomorphism.

In relation to the study between a  $WL$ -( $k_0, k_1$ )-isomorphic surjection and a pseudo- $(k_0, k_1)$ -covering map, there are the following two incorrect statements in [10] (see the identity (3.8) of Proposition 3.4 and Corollary 3.5 below) which were corrected in the paper [11], as follows:

**Proposition 3.3.** [11] (*Correction of the identity of (4.2) in Proposition 4.4 of [10]*) Let  $p : (E, k_0) \rightarrow (B, k_1)$  be a  $WL$ -( $k_0, k_1$ )-isomorphic surjection. Then, for any  $b \in B$  with  $e_i \in p^{-1}(\{b\})$ , for some index set  $M$  we obtain

$$p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i \in M} N_{k_0}(e_i, 1) \text{ with } e_i \in p^{-1}(\{b\}). \quad (3.8)$$

This statement of (3.8) was corrected as follows (see Remark 3.10 of [11]).

$$\bigcup_{i \in M} N_{k_0}(e_i, 1) \subset p^{-1}(N_{k_1}(b, 1)) \text{ with } e_i \in p^{-1}(\{b\}). \quad (3.9)$$

**Corollary 3.4.** (Correction of Corollary 4.5 of [10]) (1) A  $WL$ -local  $(k_0, k_1)$ -isomorphic surjection is equivalent to a pseudo- $(k_0, k_1)$ -covering map of Definition 4.

This statement was corrected in [11] as follows:

(2) While a pseudo- $(k_0, k_1)$ -covering map of Definition 4 implies a  $WL$ -local  $(k_0, k_1)$ -isomorphic surjection, the converse does not hold.

(3) However, with the revised version of a pseudo- $(k_0, k_1)$ -covering map in Definition 4.1 of [11], a  $WL$ -local  $(k_0, k_1)$ -isomorphic surjection implies a new version of a pseudo- $(k_0, k_1)$ -covering map in Definition 4.1 of [11].

#### 4. Summary

The paper [11] revised the condition (1) of the original version of a pseudo- $(k_0, k_1)$ -space. Based on this revision, it turns out that while a digital covering space implies a revised version of a pseudo-covering space in [11], the converse does not hold. Besides, we note that a  $WL$ - $(k_0, k_1)$ -isomorphic surjection is equivalent to a revised version of a pseudo- $(k_0, k_1)$ -map. Finally, since some suitable corrections on some mistakes and errors on the study of a digital covering, a pseudocovering, and a  $WL$ - $(k_0, k_1)$ -isomorphism were made in [11], we can find them shortly.

#### REFERENCES

- [1] L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vis.* **10** (1999) 51–62.
- [2] S.-E. Han, Digital  $(k_0, k_1)$ -covering map and its properties, *Honam Math. J.* **26** (2004) 107–117.
- [3] S.-E. Han, Digital coverings and their applications, *J. Appl. Math. Comput.* **18** (2005) 487–495.
- [4] S.-E. Han, Non-product property of the digital fundamental group, *Inf. Sci.* **171** (2005) 73–91.
- [5] S.-E. Han, On the simplicial complex stemmed from a digital graph, *Honam Math. J.*, **27** (2005) 115–129.
- [6] S.-E. Han, Discrete Homotopy of a Closed  $k$ -Surface, *LNCS 4040*; Springer: Berlin, Germany, 2006; pp. 214–225.
- [7] S.-E. Han, Unique pseudo-lifting property in digital topology, *Filomat* **26**(4) (2012) 739–746.
- [8] S.-E. Han, Estimation of the complexity of a digital image form the viewpoint of fixed point theory. *Appl. Math. Comput.*, **347** (2019) 236–248.
- [9] S.-E. Han, The most refined axiom for a digital covering space, *Mathematics* **8** (1898) (2020) doi:2.3030/math811868.
- [10] S.-E. Han, An equivalent conditions for a pseudo- $(k_0, k_1)$ -covering space, *Filomat*, **36**(15) (2022) 5093–5105.

- [11] S.-E. Han, Remarks on pseudocovering spaces in a digital topological setting, *Filomat* in reivew (2023) 1-8.
- [12] T.Y. Kong; A.Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Science: Amsterdam, The Netherlands, 1996.
- [13] A. Pakdaman, Is there any digital pseudocovering map?, *Caspian Jour. of Math. Sciences* **11**(1) (2022) 210–216.
- [14] A. Pakdaman, Digital semicovering and digital quasicovering maps, *Appl. Gen. Topol.* **24**(1) (2023) 47–57.
- [15] A. Rosenfeld, Continuous functions on digital pictures, *Pattern Recognit. Lett.* **4** (1986) 177–184.

(label1) DEPARTMENT OF MATHEMATICS EDUCATION, INSTITUTE OF PURE AND APPLIED MATHEMATICS, JEONBUK NATIONAL UNIVERSITY, JEONJU-CITY JEONBUK, 54896, REPUBLIC OF KOREA, E-MAIL ADDRESS:SEHAN@JBNU.AC.KR, TEL: 82-63-270-4449.