PSEUDOCOVERING AND DIGITAL COVERING SPACES

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ABSTRACT. The notions of a local (k_0, k_1)-isomorphism and a weakly local (k_0, k_1)-isomorphism play crucial roles in developing a digital (k_0, k_1)-covering space and a pseudo-(k_0, k_1)-covering space, respectively. In relation to the study of pseudo-(k_0, k_1)-covering spaces, since there are some works to be refined and improved in the literature, the recent paper [11] improved and corrected some mistakes occurred in the literature. One of the important things is that the notion of a pseudo-(k_0, k_1)-covering map in [7, 10] was revised to be more broadened in [11]. Thus this new version is proved to be equivalent to a weakly local (k_0, k_1)-isomorphic surjection [11]. The present paper contains some works in [11] and we only deals with k-connected digital images (X, k).

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1. **Introduction**

The notion of a pseudo-(k_0, k_1)-covering space was initially introduced in 2012 [7]. Indeed, it was intended to make a digital (k_0, k_1)-covering space in [2, 3, 4, 6] more generalized and broader. Hence it was defined by using three conditions among which two of them, i.e., the conditions (1) and (2) for a pseudo-(k_0, k_1)-covering space (see Definition 4), are equal to those for a digital (k_0, k_1)-covering space(see Definition 5). Meanwhile, the other condition (3) for a pseudo-(k_0, k_1)-covering space is different from the condition (3) for a digital (k_0, k_1)-covering space (see Definitions 4 and 5 in the present paper). To be specific, the former was defined by using the notion of a weakly local (WL-, for brevity)(k_0, k_1)-isomorphism and the latter was characterized by using the concept of a local (k_0, k_1)-isomorphism. Thus these two conditions (3) are quite different from each other. However, when combining the two conditions (1) and (2) with each of the conditions (3), a pseudo-(k_0, k_1)-covering space implies to a digital (k_0, k_1)-covering space (see Theorem 3.3 of [13]). Probably, in [7], there seems to be a gap between the author's intension for establishing a pseudo-(k_0, k_1)-covering space and the mathematical presentation of it. Hence the recent paper [11] revised the original version of the condition (1) for a pseudo-(k_0, k_1)-covering space (see Definition 4 in the present paper) to finally make a distinction between a digital (k_0, k_1)-covering and a revised version of the original version of a pseudo-(k_0, k_1)-covering map in Definition 4. In detail, we will shortly see the revised version of Definition 4 via Definition 4.1 of [11].

The present paper only deals with k-connected digital images unless otherwise stated and often uses the notion := to introduce some terms.

The four papers [7, 10, 13, 14] studied various properties of "pseudocovering spaces". Since the two of them [7, 10] have some errors and the others [13, 14] also have some mistakes relating to the map in (1.1) below, the recent paper [11] corrected and improved them, which makes them so clear. More precisely, with the original version of a pseudo-(k_0, k_1)-covering map (see Definition 4 in the present paper), the map p in (1.1) below is not a pseudo-$(2, k)$-covering map (see Proposition 3.2 of [13]). Since there are some errors in the proof of Proposition 3.2 of [13], we note that the paper [11] corrected it.

$$\left\{\begin{array}{l} p : (\mathbb{Z}^+, 2) \to SC_k^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}} \text{ defined by} \\ p(t) = x_{t(mod\, l)}, \text{ where } \mathbb{Z}^+ := [0, \infty)_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, t \geq 0\}. \end{array}\right\} \qquad (1.1)$$

Hence the recent paper [11] fully explained the process of a non-pseudo-$(2, k)$-covering map of p in (1.1).

Next, we also note that there are some mistakes on the identity of (4.2) in Proposition 4.4 and Corollary 4.5 of [10]. The paper [11] pointed out these defects and corrected them and verified that a WL-(k_0, k_1)-surjection is not equivalent to a pseudo-(k_0, k_1)-covering map followed by Definition 4 in the present paper.

To sum up, the recent paper [11] did corrections and improvements, as follows:
(1) Corrections of the map p of (4.1) in the proof of Remark 4.3(2) of [10].
(2) Corrections of the identity of (4.2) of Proposition 4.4 and Corollary 4.5 of [10].
(3) Revision of the notion of a pseudo-(k_0, k_1)-covering space of Definition 4.
(4) Correction of the proof of Proposition 3.2 of [13] and related works in [14].
(5) Improvement of the proof of Theorem 3.3 of [14].
In addition, we confirm that the example in (1.1) now becomes an example for the revised version of a pseudo-$(2, k)$-covering space in [11].

2. **Preliminaries**

In relation to the study of some properties of a pseudo-(k_0, k_1)-covering space, to make the paper self-contained, we will refer to some notions. Naively, a digital image (X, k) can be considered to be a set $X \subset \mathbb{Z}^n$ with one of the k-adjacency of $\mathbb{Z}^n$ from (2.1) below (or a digital k-graph on $\mathbb{Z}^n$ [5]). Indeed, the papers [12, 15] considered $(X, k), X \subset \mathbb{Z}^n, n \in \{1, 2, 3\}$, with 2-adjacency on $\mathbb{Z}$, $4, 8$-adjacency on $\mathbb{Z}^2$, and $6, 18, 26$-adjacency on $\mathbb{Z}^3$. As the generalization of the low dimensional cases, the digital k-adjacency relations (or digital k-connectivity) for $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, were initially established in [8] (see also [2, 3, 4]), as follows:
For a natural number t, $1 \leq t \leq n$, the distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide. Indeed, the numbers of t and n of $k(t, n)$ above is very important. For instance, on $\mathbb{Z}^2$, two types of digital k-adjacencies exist such as $k(1, 2) = 4$ and $k(2, 2) = 8$. Meanwhile, on $\mathbb{Z}^4$, four kinds of digital k-adjacencies exist such as $k(1, 4) = 8$, $k(2, 4) = 32$, $k(3, 4) = 64$, $k(4, 4) = 80$. Then, even though the 8-adjacency are used on both $\mathbb{Z}^2$ and $\mathbb{Z}^4$, using $k(2, 2) = 8$

and $k(1,4) = 8$, we can make a distinction between them efficiently. According to this statement, the well-presented $k(t,n)$-adjacency relations (or digital k-connectivities) of $\mathbb{Z}^n, n \in \mathbb{N}$, are formulated [8] (see also [4]) as follows:

$$k := k(t,n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n := \frac{n!}{(n-i)!\ i!}. \tag{2.1}$$

Based on the k-adjacency relations of $\mathbb{Z}^n$ in (2.1), $n \in \mathbb{N}$, we will call the pair (X,k) a digital image on $\mathbb{Z}^n$, $X \subset \mathbb{Z}^n$.

A simple closed k-curve (or simple k-cycle) with l elements in $\mathbb{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [4, 12], $l(\geq 4) \in \mathbb{N}$, is defined to be the set $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\,i-j\,| = \pm 1 (mod\, l)$. Then, the number l of $SC_k^{n,l}$ depends on both the dimension n of $\mathbb{Z}^n$ and the k-adjacency (see many types of $SC_k^{n,l}$ in (5) on the page of 6 of [9]).
For a digital image (X,k) and $x \in X$, we follow the notation

$$N_k(x,1) := \{x' \in X \,|\, x \text{ is } k\text{-adjacent to } x'\} \cup \{x\}, \tag{2.2}$$

which is called a digital k-neighborhood of x in (X,k) [2, 3, 4, 6]. Indeed, this notion has been effectively used in studying both pseudocovering spaces and digital covering spaces. For every point x of a digital image (X,k), an $N_k(x,1)$ always exists in (X,k), the digital continuity of [15] can be represented by the following form.

Proposition 2.1. [4, 6] *Let (X,k_0) and (Y,k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0,k_1)-continuous if and only if for every point $x \in X$, $f(N_{k_0}(x,1))$ is a subset of $N_{k_1}(f(x),1)$.*

Owing to a digital k-graph theoretical feature of a digital image (X,k), we have often used a (k_0,k_1)-*isomorphism* in [5] instead of a (k_0,k_1)-*homeomorphism* in [1], as follows:

Definition 1. [1] *(see also* [5]*) For two digital images (X,k_0) in $\mathbb{Z}^{n_0}$ and (Y,k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0,k_1)-isomorphism if h is a (k_0,k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1,k_0)-continuous. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-isomorphism.*

Based on this approach, we can develop the notion of a "radius 2-(k_0,k_1)-isomorphism" or a radius 2-(k_0,k_1)-covering map [2, 3] to establish the so-called " digital homotopy lifting theorem" which is essential in studying digital homotopy theory.

3. Remarks on the earlier verion of a pseudo-(k_0, k_1)-covering space in [7, 10]

Since the notions of a digital (k_0, k_1)-covering map and a pseudo-(k_0, k_1)-covering map are so related to the notion of a (weakly) local (k_0, k_1)-isomorphism, we first need to recall it, as follows:

Definition 2. [2, 4, 10] *For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, consider a map $h : (X, k_0) \to (Y, k_1)$. Then the map h is said to be a local (k_0, k_1)-isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$ i.e., the restriction map $h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to N_{k_1}(h(x), 1))$ is a (k_0, k_1)-isomorphism. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.*

The paper [7] defined the following notion which is weaker than a local (k_0, k_1)-isomorphism.

Definition 3. [7] *For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a weakly local (WL-, for brevity) (k_0, k_1)-isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1)) \subset (Y, k_1)$, i.e., the restriction map $h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to h(N_{k_0}(x, 1))$ is a (k_0, k_1)-isomorphism. In particular, if $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a weakly local k_0-isomorphism (or a WL-k_0-isomorphism).*

Using this notion, the paper [7] defined the notion of a pseudo-(k_0, k_1)-covering space, as follows:

Definition 4. [7] *Let (E, k_0) and (B, k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection such that for any $b \in B$,*
(1) for some index set M, $p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i \in M} N_{k_0}(e_i, 1)$ with $e_i \in p^{-1}(b) := p^{-1}(\{b\})$;
(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1)$ is an empty set; and
(3) the restriction of p to $N_{k_0}(e_i, 1)$ from $N_{k_0}(e_i, 1)$ to $N_{k_1}(b, 1)$ is a WL-(k_0, k_1)-isomorphism for all $i \in M$.
Then the map p is called a pseudo-(k_0, k_1)-covering map, (E, p, B) is said to be a pseudo-(k_0, k_1)-covering and (E, k_0) is called a pseudo-(k_0, k_1)-covering space over (B, k_1).

Based on the notion of a pseudo-(k_0, k_1)-covering space, the paper [7] referred to the map $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l}$ as in (1.1) for a pseudo-$(2, k)$-covering map. Indeed, the paper [7] made a mistake to take this map as a pseudo-$(2, k)$-covering map (see Remark 3.1 below). By contary to the condition (1) of Definition 4, the map p is not a pseudo-$(2,k)$-covering map, as follows:

Remark 3.1. *(Proposition 3.2 of* [13]*) The map* $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l} := (c_i)_{i \in [0,l-1]_{\mathbb{Z}}}, l \geq 4$*, in (1.1) is not a pseudo-*$(2,k)$*-covering map.*

Since the proof of this assertion in [13] is incorrect, the paper [11] corrected the errors.

To compare between a digital covering space and a pseudocovering space, we need to recall the notion of a digital covering space as follows:

Definition 5. [3, 4, 6] *Let* (E, k_0) *and* (B, k_1) *be digital images in* $\mathbb{Z}^{n_0}$ *and* $\mathbb{Z}^{n_1}$*, respectively. Let* $p : E \to B$ *be a surjection such that for any* $b \in B$*, the conditions (1) and (2) are equal to those of Definition 4; and the condition (3) is the following:*
The restriction of p *to* $N_{k_0}(e_i, 1)$ *from* $N_{k_0}(e_i, 1)$ *to* $N_{k_1}(p(e_i), 1)$ *is a* (k_0, k_1)*-isomorphism for all* $i \in M$*.*
Then the map p *is called a digital* (k_0, k_1)*-covering map,* (E, p, B) *is said to be a digital* (k_0, k_1)*-covering and* (E, k_0) *is called a digital* (k_0, k_1)*-covering space over* (B, k_1)*.*

Based on Definitions 4 and 5, the following is obtained.

Theorem 3.2. *(see Corollary 4 of* [9]*) In Definition 4, as a special case, assume that* (E, k_0) *and* (B, k_1) *are* k_0*- and* k_1*-connected, respectively. Then a digital* (k_0, k_1)*-covering map is equivalent to a local* (k_0, k_1)*-isomorphism.*

In relation to the study between a WL-(k_0, k_1)-isomorphic surjection and a pseudo-(k_0, k_1)-covering map, there are the following two incorrect statements in [10] (see the identity (3.8) of Proposition 3.4 and Corollary 3.5 below) which were corrected in the paper [11], as follows:

Proposition 3.3. [11] *(Correction of the identity of (4.2) in Proposition 4.4 of* [10]*) Let* $p : (E, k_0) \to (B, k_1)$ *be a* WL*-*(k_0, k_1)*-isomorphic surjection. Then, for any* $b \in B$ *with* $e_i \in p^{-1}(\{b\})$*, for some index set* M *we obtain*

$$p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i \in M} N_{k_0}(e_i, 1) \text{ with } e_i \in p^{-1}(\{b\}). \tag{3.8}$$

This statement of (3.8) was corrected as follows (see Remark 3.10 of [11]*).*

$$\bigcup_{i \in M} N_{k_0}(e_i, 1) \subset p^{-1}(N_{k_1}(b, 1)) \text{ with } e_i \in p^{-1}(\{b\}). \tag{3.9}$$

Corollary 3.4. *(Correction of Corollary 4.5 of* [10]*) (1) A WL-local (k_0, k_1)-isomorphic surjection is equivalent to a pseudo-(k_0, k_1)-covering map of Definition 4.*
This statement was corrected in [11] *as follows:*
(2) While a pseudo-(k_0, k_1)-covering map of Definition 4 implies a WL-local (k_0, k_1)-isomorphic surjection, the converse does not hold.
(3) However, with the revised version of a pseudo-(k_0, k_1)-covering map in Definition 4.1 of [11]*, a WL-local (k_0, k_1)-isomorphic surjection implies a new version of a pseudo-(k_0, k_1)-covering map in Definition 4.1 of* [11].

4. **Summary**

The paper [11] revised the condition (1) of the original version of a pseudo-(k_0, k_1)-space. Based on this revision, it turns out that while a digital covering space implies a revised version of a pseudo-covering space in [11], the converse does not hold. Besides, we note that a WL-(k_0, k_1)-isomorphic surjection is equivalent to a revised version of a pseudo-(k_0, k_1)-map. Finally, since some suitable corrections on some mistakes and errors on the study of a digital covering, a pseudocovering, and a WL-(k_0, k_1)-isomorphism were made in [11], we can find them shortly.

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