# Comments on "A Linear Time Algorithm for the Optimal Discrete IRS Beamforming"

Dogan Kutay Pekcan
Ender Ayanoglu, Fellow, IEEE

Abstract—Comments on [?] are provided. Updated necessary and sufficient conditions for its Lemma 1 are given. Consequently, an updated Algorithm 1 is provided with full specification. Simulation results with improved performance over the implementation of Algorithm 1 are provided.

Index Terms—Intelligent reflective surface (IRS), reconfigurable intelligent surface (RIS), discrete beamforming for IRS/RIS.

## I. Introduction

Reference [?] presented an algorithm to solve the problem of finding the values $\theta_1, \theta_2, \ldots, \theta_N$ to maximize $|h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}|$ where $\theta_n \in \Phi_K$ and $\Phi_K = \{\omega, 2\omega, \ldots, K\omega\}$ with $\omega = \frac{2\pi}{K}$ and $j = \sqrt{-1}$. The set $\Phi_K$ can equivalently be described as $\{0, \omega, 2\omega, \ldots, (K-1)\omega\}$. In [?], the values $h_n \in \mathbb{C}$, $n = 1, 2, \ldots, N$ are the channel coefficients and $\theta_n$ are the phase values added to the corresponding $h_n$ by an intelligent reflective surface (IRS), also known as reconfigurable intelligent surface (RIS).

## II. Two Statements from [?]

Towards achieving its goal, [?] introduced the following lemma.

Lemma 1: For an optimal solution $(\theta_1^*, \ldots, \theta_n^*)$ to problem (8), each $\theta_n^*$ must satisfy

$$\theta_n^* = \arg\min_{\theta_n \in \Phi_K} |(\theta_n + \alpha_n - \underline{/\mu}) \mod 2\pi| \qquad (11)$$

where $\underline{/\mu}$ stands for the phase of $\mu$ in (10)[1].

In [?], problem (8) is defined as

$$\underset{\boldsymbol{\theta}}{\text{maximize}} \ f(\boldsymbol{\theta}) \qquad (8a)$$

$$\text{subject to } \theta_n \in \Phi_K \quad \text{for} \quad n = 1, 2, \ldots, N \qquad (8b)$$

where

$$f(\boldsymbol{\theta}) = \frac{1}{\beta_0^2} \left| \beta_0 e^{j\alpha_0} + \sum_{n=1}^{N} \beta_n e^{j(\alpha_n + \theta_n)} \right|^2, \qquad (7b)$$

$h_n = \beta_n e^{j\alpha_n}$ for $n = 0, 1, \ldots, N$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$. Also, $g$ is defined as

$$g = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n^*} \qquad (9)$$

and $\mu$ as

$$\mu = \frac{g}{|g|}. \qquad (10)$$

Lemma 1 does not hold. This can be seen by numerical examples. We give one such example in Table ??. In this table, we look at the simple case of $K = 2$, $N = 2$. According to Lemma 1 in [?], the condition in (11) should satisfy (8) for this simple case. We draw values of $h_n$ according to the first paragraph of Sec. IV in [?]. We list these values in rows 2–4 of Table ??. We define

$$g_0(\theta_1, \theta_2) = h_0 + \sum_{n=1}^{2} h_n e^{j\theta_n} \qquad (19)$$

and list the values of $g_0(\theta_1, \theta_2)$ for all possible $\theta_1, \theta_2 \in \{0, \pi\}$. There are four such values and they are listed in rows 5–8 of Table ??. The set of values for $\theta_1$ and $\theta_2$ that maximize $|g_0|$, or equivalently, that achieve $g$ in (9), are $\theta_1 = \theta_2 = \pi$ as in row 8 of Table ??. Note that this operation results in $\underline{/\mu} = 2.3719$ radians as shown in column 5 of row 8 of Table ??.

At this point, we would like to emphasize that [?] uses a particular convention for the phases of complex numbers. They are defined to be in $[0, 2\pi)$, see the text that follows (2) in [?]. We use the same convention in generating Table ??, see its column 5, as well as in generating Table ??. With this convention, we list $\theta_n + \alpha_n - \underline{/\mu}$ and $(\theta_n + \alpha_n - \underline{/\mu}) \mod 2\pi$ for possibilities of $\theta_n = 0$ and $\theta_n = \pi$ and $n = 1, 2$ in rows 1–8 of Table ??.[2] It can be seen from rows 1–4 of Table ?? that the method results in $\theta_1 = \pi$ as the potential $\theta_1^*$, which we know from the discussion in the previous paragraph to be correct. When we carry out the calculation $(\theta_2 + \alpha_2 - \underline{/\mu}) \mod 2\pi$ in rows 5–8 of Table ??, we find that the method suggests $\theta_2 = 0$ should be $\theta_2^*$. However, we know from the exhaustive search in rows 5–8 of Table ?? that $\theta_2^* = \pi$. Thus, Lemma 1 is not correct.

It is possible to come up with a correct lemma similar to Lemma 1. We specify this lemma below.

The authors are with the Center for Pervasive Communications and Computing (CPCC), Department of Electrical Engineering and Computer Science, University of California, Irvine.

This work is partially supported by NSF grant 2030029.



[1]To prevent confusion, we will use the same equation numbers (7)–(13) in [?]. Our own equation numbers, not available in [?], will begin at (19) and will be incremented from that number on. Similarly, we will introduce Lemma 2 and Algorithm 2 in lieu of Lemma 1 and Algorithm 1 in [?]. Note that a lemma or an algorithm with number 2 does not exist in [?].

[2]Note that absolute value signs in (11) are not needed since the argument of the minimum operation in (11) is in $[0, 2\pi)$.

| | Re[·] | Im[·] | $\lvert\cdot\rvert$ | $\angle \in [0, 2\pi)$ (rad.) |
|---|---|---|---|---|
| $h_0$ | $-2.8267 \times 10^{-7}$ | $2.7376 \times 10^{-7}$ | $3.9350 \times 10^{-7}$ | 2.3722 |
| $h_1$ | $1.0958 \times 10^{-10}$ | $-1.0501 \times 10^{-11}$ | $1.1008 \times 10^{-10}$ | 6.1876 |
| $h_2$ | $-1.2238 \times 10^{-11}$ | $-2.6605 \times 10^{-11}$ | $2.6634 \times 10^{-10}$ | 4.6664 |
| $g_0(\theta_1 = 0, \theta_2 = 0)$ | $-2.8257 \times 10^{-7}$ | $2.7348 \times 10^{-7}$ | $3.9324 \times 10^{-7}$ | 2.3725 |
| $g_0(\theta_1 = 0, \theta_2 = \pi)$ | $-2.8255 \times 10^{-7}$ | $2.7401 \times 10^{-7}$ | $3.9359 \times 10^{-7}$ | 2.3715 |
| $g_0(\theta_1 = \pi, \theta_2 = 0)$ | $-2.8279 \times 10^{-7}$ | $2.7350 \times 10^{-7}$ | $3.9341 \times 10^{-7}$ | 2.3729 |
| $g_0(\theta_1 = \pi, \theta_2 = \pi)$ | $-2.8277 \times 10^{-7}$ | $2.7403 \times 10^{-7}$ | $\mathbf{3.9377 \times 10^{-7}}$ | **2.3719** |

Table 1: Sample calculation for attempting to find optimum $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$ to maximize $|g_0|$ where $g_0(\theta_1, \theta_2, \ldots, \theta_N) = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}$ with $\theta_n \in \Phi_K = \{0, \frac{2\pi}{K}, \ldots, (K-1)\frac{2\pi}{K}\}$, $n = 1, 2, \ldots, N$, for $K = 2$ and $N = 2$. Channel coefficients $h_n$, $n = 0, 1, 2$ are calculated using the technique described in [?]. Rows 5–8 present all values of $g_0$ with all combinations of $\theta_1, \theta_2 \in \Phi_2$, showing that $|g| = \max |g_0(\theta_1, \theta_2)|$ is achieved with $\theta_1^* = \theta_2^* = \pi$.

| | |
|---|---|
| $(\theta_1 = 0) + \alpha_1 - \angle\mu$ | 3.8158 |
| mod $((\theta_1 = 0) + \alpha_1 - \angle\mu, 2\pi)$ | 3.8158 |
| $(\theta_1 = \pi) + \alpha_1 - \angle\mu$ | 6.9574 |
| mod $((\theta_1 = \pi) + \alpha_1 - \angle\mu, 2\pi)$ | 0.67417 |
| $(\theta_2 = 0) + \alpha_2 - \angle\mu$ | 2.2945 |
| mod $((\theta_2 = 0) + \alpha_2 - \angle\mu, 2\pi)$ | 2.2945 |
| $(\theta_2 = \pi) + \alpha_2 - \angle\mu$ | 5.4361 |
| mod $((\theta_2 = \pi) + \alpha_2 - \angle\mu, 2\pi)$ | 5.4361 |
| $\cos((\theta_1 = 0) + \alpha_1 - \angle\mu)$ | -0.7812 |
| $\cos((\theta_1 = \pi) + \alpha_1 - \angle\mu)$ | 0.7812 |
| $\cos((\theta_2 = 0) + \alpha_2 - \angle\mu)$ | -0.6672 |
| $\cos((\theta_2 = \pi) + \alpha_2 - \angle\mu)$ | 0.6672 |

Table 2: Continuation of the sample calculation for attempting to find optimum $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$ to maximize $|g_0|$. Rows 1–8 present the calculation of $\min_{\theta_n \in \Phi_K}$ mod $(\theta_n + \alpha_n - \angle\mu, 2\pi)$ for $n = 1, 2, \ldots, N$, as specified in [?] to attempt to find the optimum values of $\theta_n$. This calculation results in values $\theta_1 = 0$ and $\theta_2 = \pi$, which are not $\theta_1^*, \theta_2^*$. Rows 9-12 present the calculation of $\max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu)$ to find $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$ as discussed in this comment. This technique finds the optimum values of $\theta_n$, $n = 1, 2, \ldots, N$.

Lemma 2: For an optimal solution $(\theta_1^*, \theta_2^*, \ldots, \theta_n^*)$, it is necessary and sufficient that each $\theta_n^*$ satisfy

$$\theta_n^* = \arg \max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu) \tag{20}$$

where $\angle\mu$ stands for the phase of $\mu$ in (10).

Proof: We can rewrite (9) as

$$\begin{aligned} |g| &= \beta_0 e^{j(\alpha_0 - \angle\mu)} + \sum_{n=1}^{N} \beta_n e^{j(\alpha_n + \theta_n - \angle\mu)} \qquad (21) \\ &= \beta_0 \cos(\alpha_0 - \angle\mu) + j\beta_0 \sin(\alpha_0 - \angle\mu) \\ &\quad + \sum_{n=1}^{N} \beta_n \cos(\theta_n + \alpha_n - \angle\mu) \\ &\quad + j\sum_{n=1}^{N} \beta_n \sin(\theta_n + \alpha_n - \angle\mu). \qquad (22) \end{aligned}$$

Because $|g|$ is real-valued, the second and fourth terms in (??) sum to zero, and

$$|g| = \beta_0 \cos(\alpha_0 - \angle\mu) + \sum_{n=1}^{N} \beta_n \cos(\theta_n + \alpha_n - \angle\mu) \tag{23}$$

from which (??) follows as a necessary and sufficient condition for Lemma 2 to hold. ∎

Rows 9–12 of Table ?? illustrate that this method finds $\theta_1^*$ and $\theta_2^*$. More extensive calculations can be carried out to show that an exhaustive search as in rows 5–8 of Table ?? confirms that Lemma 2 holds for a wide set of $K$ and $N$ values as well as a wide set of channel coefficients $h_0, h_1, \ldots, h_N$.

Reference [?] attempts to decide a range of $\mu$ for which $\theta_n^* = k\omega$ must hold, making use of Lemma 1. Towards that end, it first defines a sequence of complex numbers with respect to each $n = 1, 2, \ldots, N$ as

$$s_{nk} = e^{j(\alpha_n + (k-0.5)\omega)}, \text{ for } k = 1, 2, \ldots, K. \tag{12}$$

Then, [?] defines, for any two points $a$ and $b$ on the unit circle $C$, arc$(a : b)$ to be the unit circular arc with $a$ as the initial end and $b$ as the terminal end in the counterclockwise direction; in particular, it defines arc$(a : b)$ as an open arc with the two endpoints $a$ and $b$ excluded. With this definition, [?] states the following proposition follows from Lemma 1.

Proposition 1: A sufficient condition for $\theta_n^* = k\omega$ is

$$\mu \in \text{arc}(s_{nk} : s_{n,k+1}). \tag{13}$$

Reference [?] states that "letting $\theta_n = k\omega$ is guaranteed to minimize the gap $|(\theta_n + \alpha_n - \angle\mu) \mod 2\pi|$ whenever $\mu$ lies in its associated arc, and thus $k\omega$ must be optimal according to Lemma 1."

Now, let $K = 2$ and thus $\omega = \frac{2\pi}{K} = \pi$, and the two possibilities for $\theta$ are $\theta^1 = \pi$ and $\theta^2 = 2\pi$, or equivalently $\theta^2 = 0$. According to (12), we have

$$s_{n1} = e^{j(\alpha_n + \frac{\pi}{2})}, \quad s_{n2} = e^{j(\alpha_n + \frac{3\pi}{2})}. \tag{24}$$

According to Proposition 1, if $\mu \in \text{arc}(s_{n1} : s_{n2})$ then $\theta_n^* = \omega = \pi$ should hold. Assume $\mu$ is in arc$(s_{n1}, s_{n2})$. Then, it can be observed that $\alpha_n - \angle\mu \in (\frac{\pi}{2}, \frac{3\pi}{2})$, paying attention to the change of order due to the subtraction of $\angle\mu$. In particular, let $\mu$ be such that $\alpha_n - \angle\mu \in (\frac{\pi}{2}, \pi)$. When this is the case, note that $(\theta^1 + \alpha_n - \angle\mu) \in (\frac{3\pi}{2}, 2\pi)$ while $(\theta^2 + \alpha_n - \angle\mu) \in (\frac{\pi}{2}, \pi)$. Thus, $|(\theta^2 + \alpha_n - \angle\mu) \mod 2\pi| < |(\theta^1 + \alpha_n - \angle\mu) \mod 2\pi|$, and according to Lemma 1, $\theta_n^* = \theta^2 = 0$, in contradiction with Proposition 1. On the other hand, Proposition 1 is compatible with Lemma 2. To see this, assume $\mu$ satisfies (12). Then,

$$\angle\mu \in \left(\alpha_n + \left(k - \frac{1}{2}\right)\omega, \alpha_n + \left(k + \frac{1}{2}\right)\omega\right). \tag{25}$$

Since $\omega = \frac{2\pi}{K}$,

$$\alpha_n - \angle\mu \in \left((-2k-1)\frac{\pi}{K}, (-2k+1)\frac{\pi}{K}\right) \tag{26}$$

considering the reversal of order due to the substraction of $\angle\mu$. Now, let $\theta_n = k\omega = 2k\frac{\pi}{K}$. Then

$$\theta_n + \alpha_n - \angle\mu \in \left(-\frac{\pi}{K}, \frac{\pi}{K}\right) \tag{27}$$

and thus $\cos(\theta_n + \alpha_n - \angle\mu)$ is the largest among all other possibilities for $\theta_n$ because the slice $(-\frac{\pi}{K}, \frac{\pi}{K})$ corresponds to the largest values of the cosine function among all slices corresponding to different values of $\theta_k \in \Phi_K$ for $k = 1, 2, \ldots, K$.

## III. New Algorithm

We now specify Algorithm 2 to replace Algorithm 1 in [?]. In doing so, not only do we incorporate Lemma 2 instead of Lemma 1 but also we eliminate the many uncertainties present in Algorithm 1 of [?].

---

Algorithm 2 Update for Algorithm 1

---

Initialization: Compute $s_{nk} = e^{j(\alpha_n + (k-0.5)\omega)}$ for $n = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, K$.
Eliminate duplicates among $s_{nk}$ and sort to get $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_L < 2\pi$.
Let, for $l = 1, 2, \ldots, L$, $\mathcal{N}(\lambda_l) = \{n | s_{nk} = \lambda_l\}$.
Set $\angle\mu = 0$. For $n = 1, 2, \ldots, N$, calculate $\theta_n = \arg\max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu)$.
Set $g_1 = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}$, $\texttt{absgmax} = 0$.
for $l = 2, 3, \ldots, L$ do
    For each $n \in \mathcal{N}(\lambda_l)$, let $(\theta_n + \omega \leftarrow \theta_n) \bmod \Phi_K$.
    Let

$$g_l = g_{l-1} + \sum_{n \in \mathcal{N}(\lambda_l)} h_n \left(e^{j\theta_n} - e^{j(\theta_n - \omega) \bmod \Phi_K}\right)$$

    if $|g_l| > \texttt{absgmax}$ then
        Let $\texttt{absgmax} = |g_l|$
        Store $\theta_n$ for $n = 1, 2, \ldots, N$
    end if
end for
Read out $\theta_n^*$ as the stored $\theta_n$, $n = 1, 2, \ldots, N$.

---

## IV. Results and Remarks

Because its description is based on Lemma 1, which does not provide an equivalency condition for finding $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$, the performance of Algorithm 1 will in general not achieve the optimum result for SNR Boost [?].

We have implemented Algorithm 1 to the best of our interpretation. We have also implemented Algorithm 2. We present the CDF results for SNR Boost [?] in Fig. ?? for $K = 2$ and $N = 16$, 64, and 256, using the average of 1,000 realizations of the channel. Clearly, Algorithm 1 is not optimal. Algorithm 2 performs better than Algorithm 2 although the gains decrease with $N$. Plots for $K = 4$ show smaller gains as compared to $K = 2$, but still, Algorithm 2 always performs better than Algorithm 1 for the same $K$ and $N$.

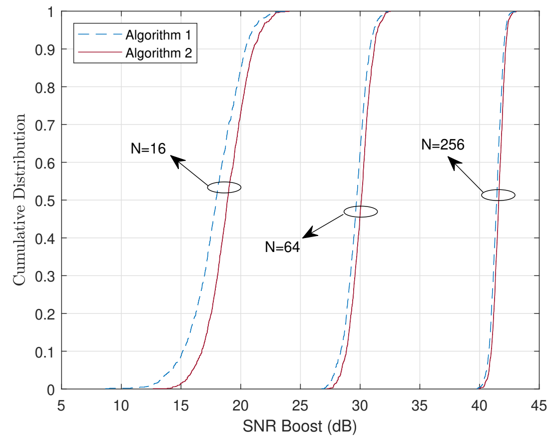


Fig. 1: CDF plots for SNR Boost with Algorithm 1 and Algorithm 2, $K = 2$.

We note that it is possible to convert the maximization of $\cos(\theta_n + \alpha_n - \angle\mu)$ to the minimization of a simple expression. For example, minimization of $f_1(x) = \pi - |(x \bmod 2\pi) - \pi|$ is the same as maximization of $\cos(x)$ within the context of Lemma 2. However, this is different than minimization of $|x \bmod 2\pi|$ proposed in Lemma 1 of [?]. The reason can be seen by plotting these functions against $x$. While $f_1(x)$ and $\cos(x)$, in addition to being periodic with period $2\pi$, have even symmetry around odd multiples of $\pi$, $|x \bmod 2\pi|$ (or equivalently, $(x \bmod 2\pi)$) does not have this symmetry.