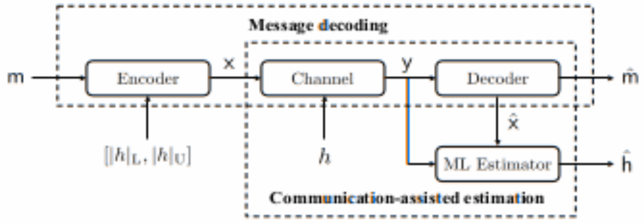


Fig. 1: The block diagram of our proposed ISAC scheme, which consists of the message-decoding step and the communication-assisted-estimation step.

communication and sensing are introduced to characterize the rate-error tradeoff.

Consider a SISO ISAC signal model given by

$$\mathbf{y} = h\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^N$ is the transmitted random communication symbol and N denotes the blocklength. The notation $\mathbb{C}^N$ denotes N-dimensional complex Euclidean space where the superscript is removed with $N = 1$, while $\mathbb{R}^N$ refers to the N-dimensional real Euclidean space. We denote by $\mathbf{y} \in \mathbb{C}^N$ the received signal and $\mathbf{n} \in \mathbb{C}^N$ the circularly symmetric complex Gaussian noise with zero mean, i.e., $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2 \boldsymbol{I}_N)$. The scalar $h \in \mathbb{C}$ is the unknown but deterministic channel coefficient determined by the sensing parameters, which is assumed to be constant in all N channel uses since the sensing parameters such as target positions and velocities remain stable during the communication process in most ISAC systems. The goal of ISAC is to simultaneously recover the communication message and estimate the sensing parameters based on the dual-functional signal.

A. ISAC Scheme

To analyze the performance trade-off between S&C, we first present the mathematical formulation for our ISAC system, which is based on a two-step scheme including message decoding and communication-assisted estimation shown in Fig. ??.

The message-decoding step aims to recover the transmitted communication symbol based on the received signal $\mathbf{y}(\mathbf{x}, h)$, which determines the communication performance of our ISAC system. In particular, we applies the (N, M, ϵ) code introduced in [?], which includes

1) A message set $\mathcal{M} = \{1, 2, \ldots, M\}$ with equiprobable messages;
2) An encoder which maps the message $\mathrm{m} \in \mathcal{M}$ to the codewords $\boldsymbol{x}_m \in \mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$. We assume that the channel gain is confined to a certain set, i.e., $|h| \in [|h|_{\mathrm{L}}, |h|_{\mathrm{U}}]$, which is known to the encoder[1]. The notation $|h|$ denotes the absolute value of the complex number h. Therefore, the encoder can be expressed by
$$f : \mathcal{M} \times \mathbb{R}^2 \mapsto \mathcal{X},\ m \times [|h|_{\mathrm{L}}, |h|_{\mathrm{U}}] \to \boldsymbol{x}_m. \tag{2}$$
Furthermore, the codewords satisfy the power constraint
$$\|\boldsymbol{x}_i\|_2^2 \leq N\rho, \quad i = 1, \ldots, M \tag{3}$$
where the constant ρ is the per-codeword power budget;
3) A decoder which maps the received signal to the message, i.e.,
$$g : \mathbb{C}^N \mapsto \mathcal{M},\ \mathbf{y} \to m. \tag{4}$$
Furthermore, the decoder satisfies
$$\mathbb{P}\{g(\mathbf{y}) \neq \mathrm{m}\} \leq \epsilon. \tag{5}$$
where $\mathbb{P}(\mathsf{X})$ denotes the probability of the random set X.

The communication-assisted-estimation step aims to estimate the sensing parameters based on the received signal, which determines the sensing performance of our ISAC system. For simplicity of analysis, we focus on the estimation of the channel coefficient itself in this paper, while the analysis of estimating general sensing parameters is left for our future work. In particular, we first reconstruct the communication symbol as $\hat{\mathbf{x}} = g(\mathbf{y})$. Then we apply a maximum-likelihood (ML) estimator to estimate the channel coefficient since the ML estimator can asymptotically achieve the Cramér-Rao lower bound [?]. After some algebra, we can obtain that

$$\hat{\mathrm{h}} = \hat{\mathrm{h}}_{\mathrm{ML}}(\hat{\mathbf{x}}, \mathbf{y}) = \frac{\hat{\mathbf{x}}^{\mathrm{H}}\mathbf{y}}{\|\hat{\mathbf{x}}\|_2^2}. \tag{6}$$

where the notation $\boldsymbol{x}^{\mathrm{H}}$ denotes the Hermitian transposition of complex vector $\boldsymbol{x}$. This ISAC scheme takes full advantage of the communication result to improve the sensing performance of the channel coefficient, which is widely applied in the practical ISAC systems [?].

B. Performance Metric

In this subsection, we give a brief introduction on the traditional performance metrics of communication and sensing, after which the rate-error region is introduced to characterize the performance tradeoff.

[1]In communication theory, the channel coefficient is usually modeled as a random variable with certain prior distribution, while it is mainly determined by the deterministic but unknown sensing parameters in ISAC systems. Therefore, the encoder is assumed to have knowledge of the uncertainty set of the channel gain to design the codebook in the ISAC settings, which is also feasible in practical systems

communication symbol. The second part refers to the sensing performance with decoding error, which consists of the bias term and the cross term since $\hat{\mathrm{h}}_{\mathrm{ML}}$ is not an unbiased estimator of h with $\hat{\mathbf{x}} \neq \mathbf{x}$. In the radar-based ISAC systems where the transmitted symbol is known to the estimator, i.e., $\epsilon = 0$, we obtain that $\mathrm{MSE} = \sigma^2/N\rho$ which coincides with the existing theoretical results.

With Proposition ??, we find that the the MSE is also controlled by the maximal bias of the codeword set, which provides useful insights for the following analysis. Then we derive the achievability bound for the rate-error tradeoff with the tightened power constraint. In particular, we first prove that $\tilde{R}^{\star}(N,\epsilon,D)$ saturates when D exceeds certain threshold.

Proposition 2: The rate-error tradeoff $\tilde{R}^{\star}(N,\epsilon,D)$ satisfies

$$\tilde{R}^{\star}(N,\epsilon,D) = \tilde{R}^{\star}(N,\epsilon,D_{\mathrm{m}}) = \tilde{R}^{\star}_{\mathrm{com}}(N,\epsilon), \quad \forall D \geq D_{\mathrm{m}} \tag{15}$$

where $\tilde{R}^{\star}_{\mathrm{com}}(N,\epsilon)$ denotes the maximal achievable rate regardless of the sensing performance with the tightened power constraint, which has been investigated in [?], [?]. The threshold D_{m} is given by

$$D_{\mathrm{m}} = \frac{\sigma^2}{N\rho} + 4\epsilon|h|_{\mathrm{U}}^2 + \frac{4\sigma\sqrt{\epsilon}|h|_{\mathrm{U}}}{\sqrt{N\rho}}. \tag{16}$$

With Proposition ??, we only need to derive the achievability bound in the case $D < D_{\mathrm{m}}$, which is shown in the following proposition

Proposition 3: For $\sigma^2/N\rho < D < D_{\mathrm{m}}$, the rate-error tradeoff $R^{\star}(N,\epsilon,D)$ is lower-bounded by

$$\tilde{R}^{\star}(N,\epsilon,D) \geq \max\{\tilde{R}^{\star}_{\mathrm{com}}(N,\epsilon) + \frac{\log_2 \gamma_{\mathrm{L}}}{N}, 0\} \tag{17}$$

where $\gamma_{\mathrm{L}} \in (0,1)$ is given by

$$\gamma_{\mathrm{L}} = \frac{1}{2}\mathrm{I}_{\sin^2(\phi_{\mathrm{L}}/2)}(\frac{2N-1}{2}, \frac{1}{2}). \tag{18}$$

The function $\mathrm{I}_x(a,b)$ is the regularized incomplete beta function. The angle $\phi_{\mathrm{L}} \in [0,\pi)$ is determined by

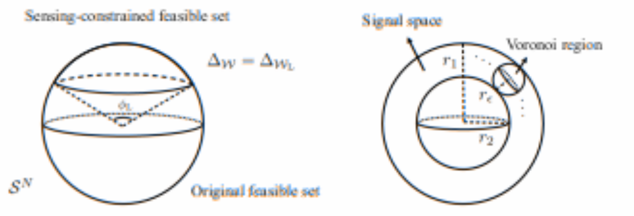
$$\phi_{\mathrm{L}} = \arccos\left(\frac{2-\Delta^2_{\mathcal{W}_{\mathrm{L}}}}{2}\right) \tag{19}$$

where the maximal bias $\Delta_{\mathcal{W}_{\mathrm{L}}}$ is given by

$$\Delta_{\mathcal{W}_{\mathrm{L}}} = \frac{\sqrt{DN\rho} - \sigma}{|h|_{\mathrm{U}}\sqrt{\epsilon N\rho}}. \tag{20}$$

Specifically, there exists $\tilde{R}^{\star}(N,\epsilon,D) = 0$ for $0 \leq D \leq \frac{\sigma^2}{N\rho}$.

Sketch of Proof: As for the analysis of the achievability bound, we guarantee the sensing performance through constraining the feasible set of the (N,M,ϵ) codewords, which is shown in Fig. 2(a). In particular, the codewords can take values on the entire hypersphere with $D \geq D_{\mathrm{m}}$, which implies that $R^{\mathrm{L}}(N,\epsilon,D) = \tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon)$. When there exists $\sigma^2/N\rho \leq D < D_{\mathrm{m}}$, we restrict the codewords to the set $\mathcal{W} \subset \mathcal{S}^N$ with $\Delta_{\mathcal{W}} \leq \Delta^{\mathrm{L}}_{\mathcal{W}}$ to meet the minimum sensing performance requirement, where the largest feasible set $\mathcal{W}$ is a hyperspherical cap. The coefficient γ_{L} can be viewed as the area ratio of the hyperspherical cap to the hypersphere. Details are omitted due to the lack of space. □



(a) Achievability Bound (b) Converse Bound

Fig. 2: Geometric illustrations of the achievability and converse bounds. (a): For the achievability bound, the sensing performance D constrains the original codeword space (black sphere) to the feasible set $\Delta_{\mathcal{W}}$ (red spherical cap). (b): For the converse bound, the maximal rate is obtained by packing the Voronoi region (red sphere of radius r_ϵ) into the signal space, which is smaller than the black sphere of radius r_1 but bigger than the black spherical shell of radius r_1 and r_2.

Remark 4: With Proposition ?? and ??, we can obtain the achievability bound for the rate-error tradeoff $R^{\star}(N,\epsilon,D)$. We denote by $\tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon)$ as the achievability bound of $\tilde{R}^{\star}_{\mathrm{com}}(N,\epsilon)$ determined by $|h|_{\mathrm{L}}$, the expression of which is provided in [?]. Then the achievability bound $R^{\mathrm{L}}(N,\epsilon,D)$ of $R^{\star}(N,\epsilon,D)$ is given by

$$R^{\mathrm{L}} = \begin{cases} 0 & 0 \leq D \leq \sigma^2/N\rho, \\ \max\{\tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon) + \frac{\log_2 \gamma_{\mathrm{L}}}{N}, 0\} & \sigma^2/N\rho < D < D_{\mathrm{m}}, \\ \tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon) & D \geq D_{\mathrm{m}}. \end{cases} \tag{21}$$

B. Converse Bound

In this subsection, we present the converse bound of the rate-error tradeoff $R^{\star}(N,\epsilon,D)$.

Note that the MSE of the ISAC system is lower-bounded by

$$\mathrm{MSE} \geq \mathbb{E}_{\mathbf{x}}\left\{\frac{\sigma^2}{\|\mathbf{x}\|_2^2}\right\} \geq \frac{\sigma^2}{\mathbb{E}_{\mathbf{x}}\{\|\mathbf{x}\|_2^2\}}. \tag{22}$$

Then for any $(R,e) \in \mathcal{F}(N,\epsilon)$, the (N,M,ϵ) code must at least satisfy the average power constraint given by

$$\mathbb{E}_{\mathbf{x}}\{\|\mathbf{x}\|_2^2\} \geq \frac{\sigma^2}{D}. \tag{23}$$

Therefore, we denote by $R^{\mathrm{U}}(N,\epsilon,D)$ the maximal achievable communication rate for the (N,M,ϵ) code satisfying power constraints (??) and (??), which is a converse bound for the rate-error tradeoff $R^{\star}(N,\epsilon,D)$. Note that the expectation term in (??) makes it difficult to obtain the exact expression of $R^{\mathrm{U}}(N,\epsilon,D)$. We provide a approximation of it in the following proposition.

Proposition 4: The maximal achievable communication rate $R^{\mathrm{U}}(N,\epsilon,D)$ for the (N,M,ϵ) code satisfying the power constraints (??) and (??) satisfies

$$R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon) + \frac{\log_2(1-\gamma_{\mathrm{U}}^{2N})}{N} \le R^{\mathrm{U}}(N,\epsilon,D) \le R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon) \tag{24}$$

where γ_{U} is given by $\gamma_{\mathrm{U}} = r_2/r_1$. The coefficient r_1, r_2 are given by

$$r_1 = \sqrt{|h|_{\mathrm{U}}^2 N\rho + N\sigma^2},\ \ r_2 = \sqrt{\frac{|h|_{\mathrm{L}}^2\sigma^2}{D} + N\sigma^2}, \tag{25}$$

respectively. We denote $R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon)$ as the converse bound of the maximal communication rate regardless of sensing performance given by

$$R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon) = \log_2 \frac{r_1}{r_\epsilon} \tag{26}$$

where the coefficient r_ϵ is the smallest r such that the following inequality holds

$$\mathbb{P}\{\mathbf{n} \notin \mathcal{B}_r^N\} \le \epsilon \tag{27}$$

where $\mathcal{B}_r^N \subset \mathbb{C}^N$ is the complex hyperball with radius r, i.e., $\mathcal{B}_r^N = \{\boldsymbol{x} \in \mathbb{C}^N : \|\boldsymbol{x}\|_2 \le r\}$.

Sketch of Proof: According to the hypothesis testing theory, the optimal decoder with equiprobable messages is the maximum-likelihood decoder, the decoding regions of which are called the Voronoi regions [?]. Then Proposition ?? is inspired from the idea of sphere packing where the Voronoi region is treated as the hyperball with radius r_ϵ. As is shown in Fig. 2(b), the first inequality in (??) is obtained by the sphere packing within the hyperspherical shell of radius r_1 and r_2, while the second one is obtained by the sphere packing within the entire hyperball of radius r_1. Details are omitted due to the lack of space. □

Remark 5: Proposition ?? provides a converse bound of the rate-error tradeoff $R^\star(N,\epsilon,D)$ which takes the influence of D into consideration. However, $R^{\mathrm{U}}(N,\epsilon,D)$ is nearly independent of D with large N since the term $\log_2(1-\gamma_{\mathrm{U}}^{2N})$ decays exponentially with the blocklength, which equals to the converse bound of the maximal communication rate regard of the sensing performance given by the RHS of (??), i.e., $R^{\mathrm{U}}(N,\epsilon,D) \approx R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon) = 2\log_2(r_1/r_\epsilon)$. Therefore, the derivation of a tighter converse bound is still needed to give a more accurate characterization on the performance tradeoff between S&C, which is a challenge left as our future work.

C. Asymptotic Analysis

In this subsection, we present the asymptotic analysis for the achievability and converse bounds of the rate-error tradeoff $R^\star$ when the blocklength N tends to infinity.

According to the theoretical analysis in [?], the achievability bound $\tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon)$ of the maximal communication rate regardless of sensing performance is proved to satisfy

$$\lim_{N\to\infty} \tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon) = \log_2\Big(1 + \frac{N\rho|h|_{\mathrm{L}}^2}{\sigma^2}\Big) \tag{28}$$

for any $\epsilon \in (0, 1/2)$. Note that there exists $\lim_{N\to\infty} D_{\mathrm{m}} = 4\epsilon|h|_{\mathrm{U}}^2$ according to (??). We can obtain

$$\lim_{N\to\infty} R^{\mathrm{L}}(N,\epsilon,D) = \lim_{N\to\infty} \tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N,\epsilon) = \log_2\Big(1 + \frac{N\rho|h|_{\mathrm{L}}^2}{\sigma^2}\Big) \tag{29}$$

for any $D > 4\epsilon|h|_{\mathrm{U}}^2$ and $\epsilon \in (0, 1/2)$.

Then we focus on the asymptotic analysis for the converse bound $R^{\mathrm{U}}(N,\epsilon,D)$. According to analysis to that in [?], there exists

$$\lim_{N\to\infty} \frac{r_\epsilon}{\sqrt{N\sigma^2}} = 1,\ \forall \epsilon \in (0, \tfrac{1}{2}). \tag{30}$$

Therefore, the asymptotic expression of the converse bound $R^{\mathrm{U}}(N.\epsilon.D)$ is given by

$$\lim_{N\to\infty} R^{\mathrm{U}}(N,\epsilon,D) = \lim_{N\to\infty} R^{\mathrm{U}}_{\mathrm{com}}(N,\epsilon) = \log_2\Big(1 + \frac{N\rho|h|_{\mathrm{U}}^2}{\sigma^2}\Big) \tag{31}$$

for any $\epsilon \in (0, 1/2)$.

According to the above theoretical analysis, we find that the performance tradeoff between S&C vanishes as the blocklength N increases. We provide a simple interpretation for this phenomenon. Consider an ISAC system where the (N, M, ϵ) code consists of two parts: the first part of length $\sqrt{N}$ is fixed as the pilot data while the other part of length $N - \sqrt{N}$ is treated as the communication data. The sensing performance of this ISAC system should at least achieve $D_0/\sqrt{N}\rho$, which is the MSE of the channel coefficient h obtained from the pilot data only. As N tends to infinity, we have $D_0 \to 0$ which implies that the sensing requirement can always be met by the pilot data with large blocklength N. Note that the pilot data of length $\sqrt{N}$ will not influence the maximal communication rate asymptotically since there exists $\lim_{N\to\infty}\sqrt{N}/N = 0$. We find that S&C performance are decoupled when the blocklength N tends to infinity.

Furthermore, it can be seen that the asymptotic expressions of the achievability and converse bound depend on the range of the channel gain, i.e., $|h|_{\mathrm{L}}$ and $|h|_{\mathrm{U}}$. When we have more accurate prior knowledge of the channel gain under the assistance of sensing, the interval $[|h|_{\mathrm{L}}, |h|_{\mathrm{U}}]$ approaches to the true channel gain $|h|$, which also implies that the achievability and converse bound coincides, i.e.,

$$\lim_{N\to\infty} R^{\mathrm{L}}(N,\epsilon,D) = \lim_{N\to\infty} R^{\mathrm{U}}(N,\epsilon,D) = C \tag{32}$$

where $C = \log_2(1 + \rho|h|^2/\sigma^2) = \log_2(1 + \mathrm{SNR})$ is the Shannon channel capacity.

IV. Simulation Results

In this section, we perform some simulation experiments to consolidate our theoretical bounds and calculate the rate-error region numerically.

First, we verify the effectiveness of the achievability and converse bounds derived for the rate-error tradeoff $R^\star(N,\epsilon,D)$. Consider the signal model given by (??), where the per-codeword power budget and the noise

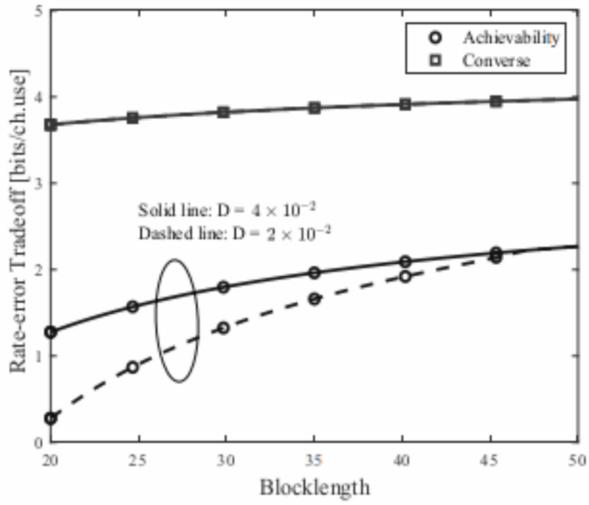


Fig. 3: The achievability and converse bounds for the rate-error tradeoff with varying code blocklength

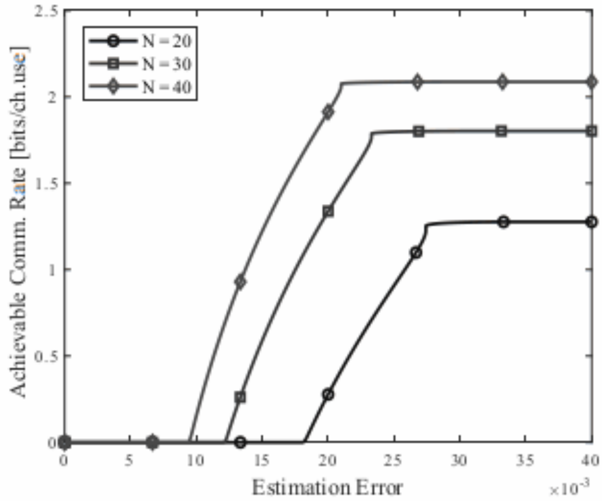


Fig. 4: The achievable rate-error region with the varying blocklength

variance are set as $\rho = 10$ and $\sigma = 1$, respectively. The channel gain is assumed to belong to the set $|h| \in [1, 1.5]$. The achievability and converse bounds of the rate-error tradeoff $R^\star(N, \epsilon, D)$ with varying blocklength N is shown in Fig. ??. The probability of decoding error is set to be $\epsilon = 10^{-3}$, while the sensing performance is set to be $D = 4 \times 10^{-2}$ and 2×10^{-2}, respectively.

According to Fig. ??, we find that the converse bounds always outperform the achievability bounds, which is invariant with the sensing performance D since the loss term almost vanishes according to our theoretical analysis. As for the achievability bounds, the rate-error tradeoff increases as D increases since the sensing constraint is relaxed. When N is large enough, the S&C performance is decoupled, which implies that the two achievability bounds converge to the same value $\tilde{R}^{\mathrm{L}}_{\mathrm{com}}(N, \epsilon)$.

Then we calculate the rate-error region $\mathcal{F}(N, \epsilon)$ numerically, which are based on the achievability bound since it can provide a more accurate characterization of the performance tradeoff between S&C than the converse bound. The system parameters are set the same as above. The achievable rate-error region is shown in Fig. ?? with the blocklength set to be $N = 20, 30, 40$, respectively.

According to Fig. ??, we find that as D increases, the achievable rate R remains to be zero at first. When D exceeds the threshold bigger than $\sigma^2/N\rho$, the achievable rate R starts to increase since the achievability bound requires the feasible codeword set $\mathcal{W}$ large enough to carry information. As D moves close to D_{m}, the coefficient γ_{L} approaches $1/2$ according to (??). However, it switches from $1/2$ to 1 when D moves past D_{m}, which leads to the $1/N$ sharp increase in the rate-error tradeoff shown in Fig. ??. Then the boundary is invariant of D, indicating that the S&C performance is decoupled.

Furthermore, the area of the rate-error region increases with the blocklength N, since the performance tradeoff between S&C vanishes in the large blocklength regime. As the blocklength N tends to infinity, the boundary of the achievable rate-error region approaches the horizontal line with height $\log_2(1 + N\rho|h|^2_{\mathrm{L}}/\sigma^2)$.

V. Conclusion

This paper provides a characterization of the performance tradeoff between S&C in a SISO ISAC system with finite blocklength where the rate-error tradeoff is introduced as the performance metric. In particular, we derive the achievability and converse bounds for the rate-error tradeoff, after which the asymptotic analysis is performed to show that the performance tradeoff vanishes as the blocklength tends to infinity. Finally, our theoretical results are verified by the numerical experiments. Future work will focus on obtaining tighter bounds for the rate-error tradeoff as well as the extension to MIMO ISAC systems. The contributions of this paper give insights to the understanding of the fundamental tradeoff and the future system design in ISAC.