# Comments on "A Linear Time Algorithm for the Optimal Discrete IRS Beamforming"

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Abstract—Comments on [?] are provided. Updated necessary and sufficient conditions for its Lemma 1 are given. Consequently, an updated Algorithm 1 is provided with full specification. Simulation results with improved performance over the implementation of Algorithm 1 are provided.

Index Terms—Intelligent reflective surface (IRS), reconfigurable intelligent surface (RIS), discrete beamforming for IRS/RIS.

## I. Introduction

Reference [?] presented an algorithm to solve the problem of finding the values $\theta_1, \theta_2, \ldots, \theta_N$ to maximize $|h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}|$ where $\theta_n \in \Phi_K$ and $\Phi_K = \{\omega, 2\omega, \ldots, K\omega\}$ with $\omega = \frac{2\pi}{K}$ and $j = \sqrt{-1}$. The set $\Phi_K$ can equivalently be described as $\{0, \omega, 2\omega, \ldots, (K-1)\omega\}$. In [?], the values $h_n \in \mathbb{C}$, $n = 1, 2, \ldots, N$ are the channel coefficients and $\theta_n$ are the phase values added to the corresponding $h_n$ by an intelligent reflective surface (IRS), also known as reconfigurable intelligent surface (RIS).

## II. Two Statements from [?]

Towards achieving its goal, [?] introduced the following lemma.

Lemma 1: For an optimal solution $(\theta_1^*, \ldots, \theta_n^*)$ to problem (8), each $\theta_n^*$ must satisfy

$$\theta_n^* = \arg\min_{\theta_n \in \Phi_K} |(\theta_n + \alpha_n - \angle\mu) \mod 2\pi| \tag{11}$$

where $\angle\mu$ stands for the phase of $\mu$ in (10)[1].

In [?], problem (8) is defined as

$$\underset{\boldsymbol{\theta}}{\text{maximize}} \ f(\boldsymbol{\theta}) \tag{8a}$$

$$\text{subject to } \theta_n \in \Phi_K \quad \text{for} \quad n = 1, 2, \ldots, N \tag{8b}$$

where

$$f(\boldsymbol{\theta}) = \frac{1}{\beta_0^2} \left| \beta_0 e^{j\alpha_0} + \sum_{n=1}^{N} \beta_n e^{j(\alpha_n + \theta_n)} \right|^2, \tag{7b}$$

$h_n = \beta_n e^{j\alpha_n}$ for $n = 0, 1, \ldots, N$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$. Also, $g$ is defined as

$$g = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n^*} \tag{9}$$

and $\mu$ as

$$\mu = \frac{g}{|g|}. \tag{10}$$

Lemma 1 does not hold. This can be seen by numerical examples. We give one such example in Table ??. In this table, we look at the simple case of $K = 2$, $N = 2$. According to Lemma 1 in [?], the condition in (11) should satisfy (8) for this simple case. We draw values of $h_n$ according to the first paragraph of Sec. IV in [?]. We list these values in rows 2–4 of Table ??. We define

$$g_0(\theta_1, \theta_2) = h_0 + \sum_{n=1}^{2} h_n e^{j\theta_n} \tag{19}$$

and list the values of $g_0(\theta_1, \theta_2)$ for all possible $\theta_1, \theta_2 \in \{0, \pi\}$. There are four such values and they are listed in rows 5–8 of Table ??. The set of values for $\theta_1$ and $\theta_2$ that maximize $|g_0|$, or equivalently, that achieve $g$ in (9), are $\theta_1 = \theta_2 = \pi$ as in row 8 of Table ??. Note that this operation results in $\angle\mu = 2.3719$ radians as shown in column 5 of row 8 of Table ??.

At this point, we would like to emphasize that [?] uses a particular convention for the phases of complex numbers. They are defined to be in $[0, 2\pi)$, see the text that follows (2) in [?]. We use the same convention in generating Table ??, see its column 5, as well as in generating Table ??. With this convention, we list $\theta_n + \alpha_n - \angle\mu$ and $(\theta_n + \alpha_n - \angle\mu) \mod 2\pi$ for possibilities of $\theta_n = 0$ and $\theta_n = \pi$ and $n = 1, 2$ in rows 1–8 of Table ??.[2] It can be seen from rows 1–4 of Table ?? that the method results in $\theta_1 = \pi$ as the potential $\theta_1^*$, which we know from the discussion in the previous paragraph to be correct. When we carry out the calculation $(\theta_2 + \alpha_2 - \angle\mu) \mod 2\pi$ in rows 5–8 of Table ??, we find that the method suggests $\theta_2 = 0$ should be $\theta_2^*$. However, we know from the exhaustive search in rows 5–8 of Table ?? that $\theta_2^* = \pi$. Thus, Lemma 1 is not correct.

It is possible to come up with a correct lemma similar to Lemma 1. We specify this lemma below.

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This work is partially supported by NSF grant 2030029.



[1] To prevent confusion, we will use the same equation numbers (7)–(13) in [?]. Our own equation numbers, not available in [?], will begin at (19) and will be incremented from that number on. Similarly, we will introduce Lemma 2 and Algorithm 2 in lieu of Lemma 1 and Algorithm 1 in [?]. Note that a lemma or an algorithm with number 2 does not exist in [?].

[2] Note that absolute value signs in (11) are not needed since the argument of the minimum operation in (11) is in $[0, 2\pi)$.

hand, Proposition 1 is compatible with Lemma 2. To see this, assume $\mu$ satisfies (12). Then,

$$\angle\mu \in \Big(\alpha_n + \Big(k - \frac{1}{2}\Big)\omega, \alpha_n + \Big(k + \frac{1}{2}\Big)\omega\Big). \tag{25}$$

Since $\omega = \frac{2\pi}{K}$,

$$\alpha_n - \angle\mu \in \Big((-2k-1)\frac{\pi}{K}, (-2k+1)\frac{\pi}{K}\Big) \tag{26}$$

considering the reversal of order due to the substraction of $\angle\mu$. Now, let $\theta_n = k\omega = 2k\frac{\pi}{K}$. Then

$$\theta_n + \alpha_n - \angle\mu \in \Big(-\frac{\pi}{K}, \frac{\pi}{K}\Big) \tag{27}$$

and thus $\cos(\theta_n + \alpha_n - \angle\mu)$ is the largest among all other possibilities for $\theta_n$ because the slice $(-\frac{\pi}{K}, \frac{\pi}{K})$ corresponds to the largest values of the cosine function among all slices corresponding to different values of $\theta_k \in \Phi_K$ for $k = 1, 2, \ldots, K$.

## III. New Algorithm

We now specify Algorithm 2 to replace Algorithm 1 in [?]. In doing so, not only do we incorporate Lemma 2 instead of Lemma 1 but also we eliminate the many uncertainties present in Algorithm 1 of [?].

**Algorithm 2** Update for Algorithm 1

Initialization: Compute $s_{nk} = e^{j(\alpha_n + (k-0.5)\omega)}$ for $n = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, K$.
Eliminate duplicates among $s_{nk}$ and sort to get $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_L < 2\pi$.
Let, for $l = 1, 2, \ldots, L$, $\mathcal{N}(\lambda_l) = \{n | s_{nk} = \lambda_l\}$.
Set $\angle\mu = 0$. For $n = 1, 2, \ldots, N$, calculate $\theta_n = \arg\max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu)$.
Set $g_1 = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}$, `absgmax` $= 0$.
**for** $l = 2, 3, \ldots, L$ **do**
    For each $n \in \mathcal{N}(\lambda_l)$, let $(\theta_n + \omega \leftarrow \theta_n)$ mod $\Phi_K$.
    Let
$$g_l = g_{l-1} + \sum_{n \in \mathcal{N}(\lambda_l)} h_n \big(e^{j\theta_n} - e^{j(\theta_n - \omega) \bmod \Phi_K}\big)$$
    **if** $|g_l| >$ `absgmax` **then**
        Let `absgmax` $= |g_l|$
        Store $\theta_n$ for $n = 1, 2, \ldots, N$
    **end if**
**end for**
Read out $\theta_n^*$ as the stored $\theta_n$, $n = 1, 2, \ldots, N$.

## IV. Results and Remarks

Because its description is based on Lemma 1, which does not provide an equivalency condition for finding $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$, the performance of Algorithm 1 will in general not achieve the optimum result for SNR Boost [?].

We have implemented Algorithm 1 to the best of our interpretation. We have also implemented Algorithm 2. We present the CDF results for SNR Boost [?] in Fig. ?? for $K = 2$ and $N = 16$, 64, and 256, using the average of 1,000 realizations of the channel. Clearly, Algorithm 1 is not optimal. Algorithm 2 performs better than Algorithm 2 although the gains decrease with $N$. Plots for $K = 4$ show smaller gains as compared to $K = 2$, but still, Algorithm 2 always performs better than Algorithm 1 for the same $K$ and $N$.

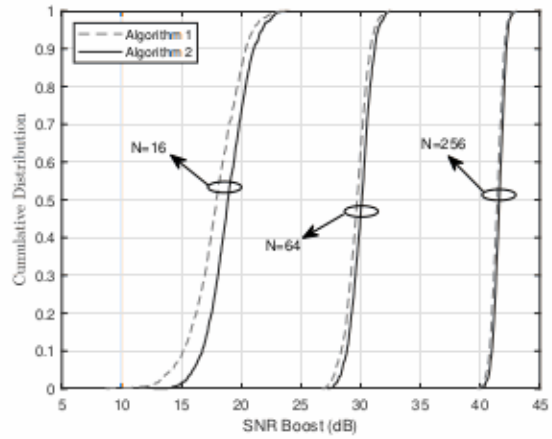


Fig. 1: CDF plots for SNR Boost with Algorithm 1 and Algorithm 2, $K = 2$.

We note that it is possible to convert the maximization of $\cos(\theta_n + \alpha_n - \angle\mu)$ to the minimization of a simple expression. For example, minimization of $f_1(x) = \pi - |(x \bmod 2\pi) - \pi|$ is the same as maximization of $\cos(x)$ within the context of Lemma 2. However, this is different than minimization of $|x \bmod 2\pi|$ proposed in Lemma 1 of [?]. The reason can be seen by plotting these functions against $x$. While $f_1(x)$ and $\cos(x)$, in addition to being periodic with period $2\pi$, have even symmetry around odd multiples of $\pi$, $|x \bmod 2\pi|$ (or equivalently, $(x \bmod 2\pi)$) does not have this symmetry.