$|I_\alpha| = m_\alpha$ *for any* $\alpha \in \mathbb{N}_d$. *Let* u *be an integer with* $\sum_{\alpha=1}^{d'} m_\alpha < u < \sum_{\alpha=1}^{d'+1} m_\alpha$, *we impose an additional condition below:*

$$\{\pi_1(1), \pi_1(2), \cdots, \pi_1(m_1), \pi_2(1), \cdots, \pi_{d'+1}(u')\} = \{1, 2. \cdots, u\},$$

where $0 < u' < m_{d'+1}$. *Let* $(n_1, n_2, \cdots, n_{d+v})$ *and* $(p_1, p_2, \cdots, p_{d'})$ *be the* q*-ary representations of* n *and* p*, respectively. Let*

$$f(x) = \sum_{\alpha=1}^{d} \sum_{\beta=1}^{m_\alpha - 1} a_{\alpha,\beta} x_{\pi_\alpha(\beta)} x_{\pi_\alpha(\beta+1)} + \sum_{\alpha=1}^{d} \sum_{\beta=1}^{m_\alpha} \sum_{k=1}^{v} b_{\alpha,\beta,k} x_{\pi_\alpha(\beta)} x_{m-v+k} + \sum_{l=1}^{q-1} \sum_{s=1}^{m} c_{s,l} x_s^l + c_0, \tag{1}$$

$$f_n^p(x) = f(x) + \sum_{\alpha=1}^{d} n_\alpha x_{\pi_\alpha(1)} + \sum_{k=1}^{v} n_{k+d} x_{m-v+k} + c \sum_{\alpha=1}^{d'} p_\alpha x_{\pi_\alpha(m_\alpha)}, \tag{2}$$

where $a_{\alpha,\beta}, c \in \mathbb{Z}_q^*$ *are co-prime with* q *and* $b_{\alpha,\beta,k}, c_{s,l}, c_0 \in \mathbb{Z}_q$. *Then* $\{\mathcal{F}^0, \mathcal{F}^1, \cdots, \mathcal{F}^{q^{d'}-1}\}$ *generates a* $(q^{d'}, q^{v+d}, L)$*-MOCS with* $L = a_m q^{m-1} + \sum_{k=1}^{v-1} a_k q^{m-v+k-1} + q^u$, $a_k \in \mathbb{Z}_q$ *and* $a_m \in \mathbb{Z}_q^*$, *where* $\mathcal{F}^p = \{\mathbf{f}_0^p, \mathbf{f}_1^p, \cdots, \mathbf{f}_{q^{v+d}-1}^p\}$.

Proof. Since for sequences $\mathbf{f}_n^p$ and $\mathbf{f}_n^{p'}$, $R_{\mathbf{f}_n^{p'}, \mathbf{f}_n^p}(-\tau) = R^*_{\mathbf{f}_n^p, \mathbf{f}_n^{p'}}(\tau)$, then it suffice to prove that for $0 \le p, p' \le q^{d'} - 1$ and $0 < \tau \le L - 1$,

$$R_{\mathcal{F}^p, \mathcal{F}^{p'}}(\tau) = \sum_{n=0}^{q^{v+d}-1} \sum_{i=0}^{L-1-\tau} \xi^{f_{n,i}^p - f_{n,i+\tau}^{p'}} = \sum_{i=0}^{L-1-\tau} \sum_{n=0}^{q^{v+d}-1} \xi^{f_{n,i}^p - f_{n,i+\tau}^{p'}} = 0,$$

where $f_{n,i}^p$ and $f_{n,j}^{p'}$ are the $(i+1)$-th and the $(j+1)$-th element of sequence $\mathbf{f}_n^p$ and $\mathbf{f}_n^{p'}$, respectively. For simplicity, we assume $a_k \neq 0$ for any $k \in \mathbb{N}_{v-1}$. Throughout this paper, for a given integer i, we set $j = i + \tau$ and let $(i_1, i_2, \cdots, i_m)$ and $(j_1, j_2, \cdots, j_m)$ be the q-ary representations of i and j, respectively. Let $(p_1, p_2, \cdots, p_{d'})$ and $(p'_1, p'_2, \cdots, p'_{d'})$ are the q-ary representations of p and p', respectively.

Case 1: If $i_{\pi_\alpha(1)} \neq j_{\pi_\alpha(1)}$ for some $\alpha \in \mathbb{N}_d$ or $i_{m-v+k} \neq j_{m-v+k}$ for some $k \in \mathbb{N}_v$. Then

$$R_{\mathcal{F}^p, \mathcal{F}^{p'}}(\tau) = \sum_{i=0}^{L-1-\tau} \xi^{f_i - f_j} \prod_{\alpha=1}^{d} \left(\sum_{n_\alpha=0}^{q-1} \xi^{n_\alpha \left(i_{\pi_\alpha(1)} - j_{\pi_\alpha(1)}\right)} \right) \prod_{\alpha=1}^{d'} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}} A = 0.$$

where $A = \prod_{k=1}^{v} \left(\sum_{n_{d+k}=0}^{q-1} \xi^{n_{d+k}(i_{m-v+k} - j_{m-v+k})} \right)$.

Case 2: If $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \mathbb{N}_d$, $i_{m-v+k} = j_{m-v+k}$ for all $k \in \mathbb{N}_v$, and $i_m = j_m = 0$. Then

$$R_{\mathcal{F}^p, \mathcal{F}^{p'}}(\tau) = q^{d+v} \sum_{i=0}^{L-1-\tau} \xi^{f_i - f_j} \prod_{\alpha=1}^{d'} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}}.$$

$i \leq q^m - 1 - \tau,\ i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}\}$. Thus we obtain that

$$\begin{aligned}
R_{\mathcal{F}^p,\mathcal{F}^{p'}}(\tau) &= \sum_{i=0}^{q^m-1-\tau} \sum_{n=0}^{q^d-1} \xi^{f^p_{n,i} - f^{p'}_{n,j}} \\
&= \sum_{i=0}^{q^m-1-\tau} \xi^{f_i - f_j} \prod_{\alpha=1}^{d} \left(\sum_{n_\alpha=0}^{q-1} \xi^{n_\alpha \left(i_{\pi_\alpha(1)} - j_{\pi_\alpha(1)}\right)} \right) \prod_{\alpha=1}^{d} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}} \\
&= \sum_{i \in S_1(\tau)} \xi^{f_i - f_j} \prod_{\alpha=1}^{d} \left(\sum_{n_\alpha=0}^{q-1} \xi^{n_\alpha \left(i_{\pi_\alpha(1)} - j_{\pi_\alpha(1)}\right)} \right) \prod_{\alpha=1}^{d} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}} \\
&\quad + \sum_{i \in S_2(\tau)} \xi^{f_i - f_j} \prod_{\alpha=1}^{d} \left(\sum_{n_\alpha=0}^{q-1} \xi^{n_\alpha \left(i_{\pi_\alpha(1)} - j_{\pi_\alpha(1)}\right)} \right) \prod_{\alpha=1}^{d} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}} \\
&= q^d \sum_{i \in S_2(\tau)} \xi^{f_i - f_j} \prod_{\alpha=1}^{d} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}},
\end{aligned}$$

where $(p_{k,1}, p_{k,2}, \cdots, p_{k,d})$ is the q-ary representation of p_k for any $k \in \{1, 2\}$. For any $i \in S_2(\tau)$, according to the Case 2 of first part in Theorem **??**, we have

$$f_{i^{(t)}} - f_i - f_{j^{(t)}} + f_j = t a_{\alpha_1, \beta_1 - 1} \left(i_{\pi_{\alpha_1}(\beta_1)} - j_{\pi_{\alpha_1}(\beta_1)} \right)$$

and

$$(\xi^{f_i - f_j} + \xi^{f_{i^{(1)}} - f_{j^{(1)}}} + \xi^{f_{i^{(2)}} - f_{j^{(2)}}} + \cdots + \xi^{f_{i^{(q-1)}} - f_{j^{(q-1)}}}) \prod_{\alpha=1}^{d} \xi^{p_\alpha i_{\pi_\alpha(m_\alpha)} - p'_\alpha j_{\pi_\alpha(m_\alpha)}} = 0.$$

According to the above discussion, we know that the ideal correlation property is available for any $\tau > 0$. Now, we need to prove that for any $0 \leq p \neq p' \leq q^d - 1$ and $\tau = 0$,

$$R_{\mathcal{F}^p,\mathcal{F}^{p'}}(0) = \sum_{n=0}^{q^d-1} R_{\mathbf{f}^p_n, \mathbf{f}^{p'}_n}(0) = \sum_{n=0}^{q^d-1} \sum_{i=0}^{q^m-1} \xi^{\sum_{\alpha=1}^{d} (p_\alpha \oplus p'_\alpha) i_{\pi_\alpha(m_\alpha)}} = 0.$$

Put $\mathbf{d} = \sum_{\alpha=1}^{d} (p_\alpha \oplus p'_\alpha) \mathbf{x}_{\pi_\alpha(m_\alpha)}$. Due to each $\mathbf{x}_{\pi_\alpha(m_\alpha)}$ is a balanced sequence, the linear combination of $\mathbf{x}_{\pi_1(m_1)}, \mathbf{x}_{\pi_2(m_2)}, \cdots, \mathbf{x}_{\pi_d(m_d)}$ is balanced, i.e., $\mathbf{d}$ is balanced. Then we have

$$R_{\mathcal{F}^p,\mathcal{F}^{p'}}(0) = \sum_{n=0}^{q^d-1} \sum_{i=0}^{q^m-1} \xi^{\sum_{\alpha=1}^{d} (p_\alpha \oplus p'_\alpha) i_{\pi_\alpha(m_\alpha)}} = 0,$$

which completes the proof.

□

With the help of the above Theorem **??**, the following $(q^{v+d}, q^d, q^m, q^{m-v})$-ZCCSs can be obtained easily.

Remark 4.3. *According to Lemma* **??**, *we know the ZCCS constructed from Theorem* **??** *is optimal since* $M/N = q^{v+d}/q^{d} = L/Z$ *is available. In particular, when* $v = 0$, *the Theorem* **??** *changes into Theorem* **??**.

Example 4.4. *Let* $a_{1,1} = b = 1$, $q = 4$, $m = 3$, $v = 1$, $d = 1$, $m_1 = 2$, $(\pi_1(1), \pi_1(2)) = (2, 1)$, $h_0 = 1$, $(h_{1,1}, h_{2,1}, h_{3,1}) = (1, 2, 2)$, $(h_{1,2}, h_{2,2}, h_{3,2}) = (3, 1, 0)$ *and* $(h_{1,3}, h_{2,3}, h_{3,3}) = (2, 1, 3)$ *in Theorem* **??**. *Then* $\{\mathcal{F}^0, \mathcal{F}^1, \cdots, \mathcal{F}^{15}\}$ *forms a quaternary* $(16, 4, 64, 16)$*-ZCCS, where* $\mathcal{F}^3$ *and* $\mathcal{F}^{10}$ *are given by*

$$\begin{bmatrix} \mathbf{f}_0^3 \\ \mathbf{f}_1^3 \\ \mathbf{f}_2^3 \\ \mathbf{f}_3^3 \end{bmatrix} = \begin{bmatrix} 1212133132323311121213313232331112121331323233111212133132323311 \\ 1212200210102200121220021010220012122002101022001212200210102200 \\ 1212311332321133121231133232113312123113323211331212311332321133 \\ 1212022010100022121202201010002212120220101000221212022010100022 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{f}_0^{10} \\ \mathbf{f}_1^{10} \\ \mathbf{f}_2^{10} \\ \mathbf{f}_3^{10} \end{bmatrix} = \begin{bmatrix} 1133121231133232331130301331101011331212311332323311303013311010 \\ 1133232313312121331101013113030311332323133121213311010131130303 \\ 1133303031131010331112121331323211333030311310103311121213313232 \\ 1133010113310303331123233113212111330101133103033311232331132121 \end{bmatrix}$$

The sum of aperiodic auto-correlation of sequences $\mathcal{F}^3$ *is presented in Figure* **??** *and the sum of aperiodic cross-correlation of sequences* $\mathcal{F}^3$ *and* $\mathcal{F}^{10}$ *is presented in Figure* **??**.

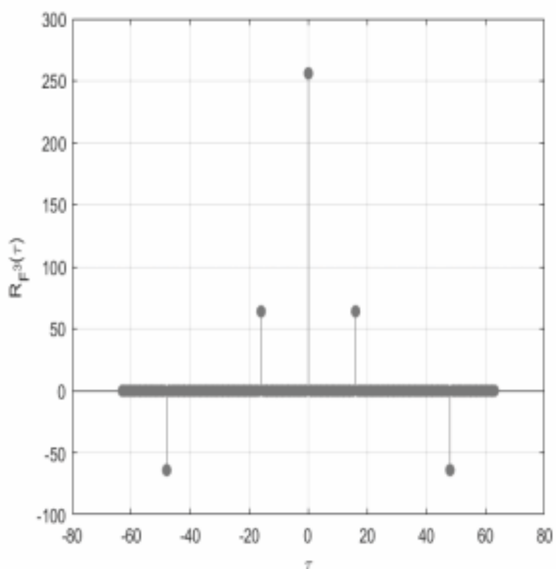

Figure 1: Auto-correlation of $\mathcal{F}^3$

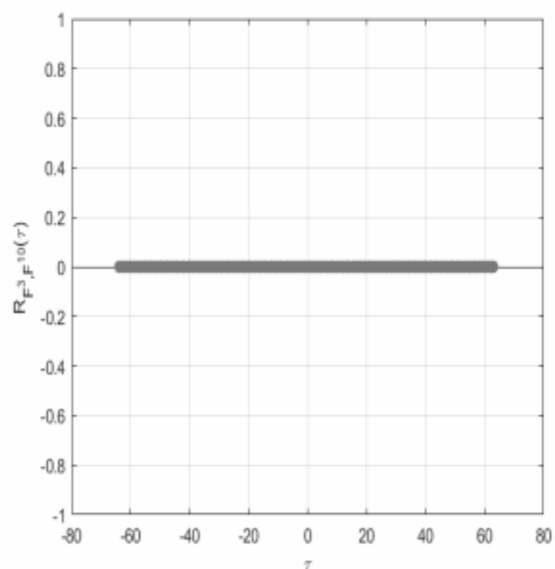

Figure 2: Cross-correlation of F^3 and $\mathcal{F}^{10}$

Table 2: Summary of Existing ZCCSs

Source	Based on	Parameters	Conditions	Optimal	Remark
[?]	GBF	$(2^{k+1}, 2^{k+1}, 3\cdot 2^m, 2^{m+1})$	$0 < k \le m$	✓	Direct
[?]	GBF	$(2^{k+2}, 2^{k+2}, 2^m\cdot 2L, 2^{m+1}L)$	$0 < k \le m$	✓	Direct
[?]	GBF	$(2^{k+2}, 2^{k+2}, 3\times 2^m, 2^{m+1}L)$	$m \ge 0, k > 0$	✓	Direct
[?]	GBF	$(2^{k}, 2^{k}, 2^m, 2^{m+1})$	$v \le m, 0 \le m-v$	✓	Direct
[?]	GBF	$(2^{k+v}, 2^k, 2^m, 2^{m-v})$	π is a permutation of	✓	Direct
[?]	GBF	$(2^n, 2^n, 2^{m-1}+2, 2^{m-2}+2^{\pi(m-3)}+1)$	π is a permutation of $\mathbb{N}_{m-2}$, $m \ge 3$	✓	Direct
[?]	GBF	$(2^{n+1}, 2^{n+1}, 2^{m-1}+2, 2^{m-2}+2^{\pi(m-3)+1})$	π is a permutation of $\mathbb{N}_{m-2}$, $m \ge 2$, $q \ge 2$,	✓	Direct
[?]	GBF	$(2^{n+p}, 2^n, 2^m, 2^{m-p})$	$p \le m$	✓	Direct
[?]	GBF	$(2^{k+p+1}, 2^{k+1}, 2^m, 2^{m-p})$	$k+p \le m$	✓	Direct
[?]	GBF	$(2^{k+1}, 2^{k+1}, 3(2^{m-1}+2^{m-3}), 2(2^{m-1}+2^{m-3}))$	$m \ge 5, k > 0$	✓	Direct
[?]	GBF	$(R2^{k+1}, 2^{k+1}, R(2^{m-1}+2^{m-3}), 2^{m-1}+2^{m-3})$	$m \ge 5$, $k \ge 0$, and R is even	✓	Direct
[?]	BH Matrix	(MP, M, MP, M)	M, P are the order of BH matrix	✓	Indirect
[?]	Optimal ZCP, BH Matrix	$(MP, M, M^{N+1}P, M^{N+1})$	M, P are the order of BH matrix, $N > 0$	✓	Indirect
[?]	Hadamard product	$(2^{n+1}, 2^{n+1}, N, Z)$	$N \ge 3$, N is odd, $Z \ge \lceil \frac{N}{2} + 1 \rceil$	✓	Direct
[?]	ZCP	$(2^m, 2^m, L, Z)$	$Z \ge \lceil \frac{L}{2} \rceil$	✓	Direct
[?]	PBF	$(\prod_{i=1}^{l} p_i 2^{n+1}, 2^{n+1}, 2^m \prod_{i=1}^{l} p_i, 2^m)$	$\forall p_i$ is a prime, $m > 0$	✓	Direct
[?]	PBF	$(p2^{k+1}, 2^{k+1}, p2^m, 2^m)$	p is a prime number,	✓	Direct
[?]	MVF	$(\prod_{i=1}^{k} p_i^2, \prod_{i=1}^{k} p_i, \prod_{i=1}^{k} p_i^{m_i}, \prod_{i=1}^{k} p_i^{m_i-1})$	p_i is a prime number,	✓	Direct
[?]	EBF	$(q^{v+1}, q, q^m, q^{m-v})$	$q \ge 2$, $v \le m$	✓	Direct
Theorem ??	EBF	$(q^{v+d}, q^d, q^m, q^{m-v})$	$v < m$, $d \le m \le v$, $q \ge 2$ v is a positive integer	✓	Direct

6 Conclusion

In this paper, we mainly present a construction of optimal ZCCSs and a construction of MOCSs with flexible lengths based on EBFs. According to the arbitrariness of q, the proposed MOCSs cover the result in [?] and have non-power-of-two lengths when $q = 2$. Moreover, the resulting MOCSs can be obtained directly from EBFs without using tedious sequence operations. The proposed MOCSs with flexible lengths find many applications in wireless communication due to its good correlation properties. The proposed ZCCSs are optimal with respect to the theoretical upper bound and we can obtain a new class of ZCCSs of arbitrary lengths with large zero correlation zone width.

Declarations

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Ethics approval and consent to participate Not applicable.

Consent for publication Not applicable.