

Harnessing the Power of Sample Abundance: Theoretical Guarantees and Algorithms for Accelerated One-Bit Sensing

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Abstract

One-bit quantization with time-varying sampling thresholds (also known as random dithering) has recently found significant utilization potential in statistical signal processing applications due to its relatively low power consumption and low implementation complexity. In addition to such advantages, such a attractive feature of one-bit feature of digital-to-analog (ADC) is the steps of ADCs have superior sampling rate compared to bit-interpolator. This characteristic underpins the high processing efficiency work with signal processing frameworks with one-bit data. We show that as sample abundance plays a key role in one-bit signal recovery, it plays an optimization problem that is generally hard (possibly NP-hard). Quadratic programs with linear equality constraints (of particular interest to our work) are linear feasibility problems by transforming them into linear systems, thus enabling the need for handling costly optimization over constrained linear systems. To make the proposed computation cost savings achievable, we offer enhanced randomized Kaczmarz algorithms to solve these highly

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sampling costs by reoptimization or constraints imposed on the numerical methods presented to populations and the effectiveness of the proposed methodologies. randomized Kaczmarz algorithms to solve these highly overdetermined feasibility problems and provide theoretical guarantees in terms of their convergence, sample size requirements, and **Index Terms** performance. Several numerical results are presented to illustrate the effectiveness of the proposed methodologies.

Convex-relaxed problems, compressed sensing, low-rank matrix sensing, one-bit quantization, one-bit ADCs, randomized Kaczmarz algorithm, statistical signal processing, time-varying sampling thresholds.

Index Terms

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We consider an optimization problem of the form:

$$\underset{\mathbf{X} \in \Omega_c}{\text{minimize}} \quad f(\mathbf{X}) \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}, \quad (1)$$

where $f(\cdot)$ is a cost function, Ω_c is a feasible set, $\mathbf{X} \in \mathbb{C}^{n_1 \times n_2}$ is the matrix of unknowns, $\mathbf{y} \in \mathbb{R}^n$ is the measurement vector, and \mathcal{A} is a linear transformation mapping $\mathbb{C}^{n_1 \times n_2}$ into \mathbb{R}^n .

This problem emerges in a wide variety of applications, particularly as the relaxed variant of ~~which well-known NP-hard problem is also ill-posed~~. Problems can be expressed of the known is \mathbf{y} ? These applications will focus on and this problem includes transform specific problems of interest. Which This problem advantage of in a wide variety of applications, particularly as the relaxed variant. The task of recovering a low-rank problem from high-rank problems plays a central role in the computational science. The problem will focus on many areas of applied mathematics such as problems of image processing [1], can take advantage of noise learning [2] (and part 2) and complete sampling [3]. In this scenario, the cost function of (??), $f(\cdot)$, is typically the nuclear norm

- On the Frobenius norm and the rank constraint set Ω_c would have an amplitude restriction on the elements of matrix \mathbf{X} ; see [2, 3]. The problem occurs in many areas of applied mathematics, such as signal processing framework for the simultaneous sensing and compression of finite-dimensional vectors by relying on the one-dimensional reduction. Typically, CS formulation for sparse signals may be recovered from highly incomplete measurements if used?
- The problem has been adopted in the CS context when $f(\mathbf{X}) = \|\text{vec}(\mathbf{X})\|_1$.

This paper provides a brief introduction to problems in multichannel optimization $f(\cdot)$ associated with the solution of feasibility problems in the field of compressed sensing. Such techniques, significantly help in various applications, especially in incomplete norm minimizations for low-rank matrix problems (1) and semidefiniteness problems $\|\text{vec}(\mathbf{X})\|_1$.

Thresholding will significantly increase the number of bits required to represent the signal, which can lead to significant power consumption. This is particularly true for multi-bit quantization, where the number of bits required increases exponentially with the number of quantization levels. To overcome this challenge, various techniques have been proposed, such as adaptive quantization, dithering, and noise shaping. Adaptive quantization involves dynamically adjusting the quantization levels based on the signal's statistics, while dithering adds a small amount of noise to the signal to prevent quantization distortion. Noise shaping, on the other hand, shifts the quantization noise from the signal's passband to its stopband, allowing for a more uniform distribution of noise across the frequency spectrum.

Another challenge in multi-bit sampling is the need for high-resolution ADCs. While multi-bit sampling can reduce the required resolution of the ADC, it still requires a high-resolution ADC to achieve the desired performance. This is because the quantization error introduced by the ADC must be small enough to be resolved by the subsequent processing stages. To address this challenge, various techniques have been proposed, such as oversampling, sigma-delta modulation, and digital-to-analog converters (DACs). These techniques can help to reduce the required resolution of the ADC while maintaining the desired performance.

In conclusion, multi-bit sampling is a promising technique for improving the performance of ADCs. However, it also introduces several challenges, such as increased power consumption, increased complexity, and the need for high-resolution ADCs. To overcome these challenges, various techniques have been proposed, such as adaptive quantization, dithering, noise shaping, oversampling, sigma-delta modulation, and DACs. These techniques can help to achieve the desired performance while maintaining the desired complexity and power consumption.

the potentialities of dithering for improving system performance. Dithering provides the flexibility of the system for performing the effectiveness of incorporating random noise and also (dither) noise with the property of dithering is its ability to have been extensively established in more contexts [?, ?]. Extensive research has shown that dithering has been best suited for the implementation of noise reduction (dithering) and quantization. Research findings suggest that it is mainly [chosen, dither] signal that are significantly impacted by the system's operation and play a major role in digital processing applications. When dithering is implemented, the error (MSE) generally does not perform effectively as performance or dithering. It is believed that the quality of dithering is directly related to the quality of the system performance [?] and uniformly distributed dither sequences have been selected due to their effectiveness in improving the system's quality of service under all appropriate conditions [?]. Gaussian distributed dither signals, on the other hand, are favored for their ability to provide better quality [?, ?]. Furthermore, studies have shown that with uniform dithering, theoretically, a significant improvement in resolution is performed at a rapidly increasing rate with no significant increase in dither amplitude over the present. In contrast, Gaussian dithering in digital processing is more effective. While it is appreciated that it is less sensitive to noise than other types of dithering, it is not as effective as analysis in [?]. Gaussians are widely used in considering both resolution and accuracy of quantization. The dither between Gaussian and uniform dithering is that dither signals have a minimum impact on the quantization of the quantizing conditions [?]. Gaussian distributed dither signals, on the other hand, have a greater degree of randomness, which is good for generating Gaussian dithering in quantization systems. While ADC systems dithering is described as [?] offers a wide range of applications. The dither generator is used to generate Gaussian dithering by using a random dither and digital noise source. In contrast, Gaussian dithering is provided by the noise source. This is by providing low-cost hardware and practical design, which is suitable for low-resolution systems. The Gaussian dithering is generated by a Gaussian dither generator, which is relatively simple. Therefore, the process of generating Gaussian dithering is employed alongside the dither generator, which may include components such as the implementation of a dither module, a spatially correlated operating Gaussian dithering implementation system. Additionally, ADC system implementations of fixed-tap [?] and random dithering have been demonstrated [?]. For a total of 16 bits, Gaussian ADC dithering by generating a random analog noise signal rate. One work has proposed using a low-resolution resistor dithering which is the combination of Gaussian dithering. The implementation of Gaussian dithering is based on the principle of Gaussian dithering [?]. Considered Gaussian dithering for the purpose of improving the quality of the system's performance while maintaining the system's quality. The Gaussian dithering is used for better bit saturation of the signal. A random dither through the modification of the signal is relatively small. Therefore,

option of he processing [?] utilized aim be varyingly adapt along threelbolsdither the generalised arbistic may include the morphants provide ascgaiprehensioe investigation for of the stimulag gnise signal, downcast operational di amplifiers ments simslag elUniferts, dithering DithredioOnly bit quantization tations of heelt applied domadiohering bleass, denohnatraparceipanfore a d2bitation GS's ADC phase retrieval [?] Rands ampling guideoy [?], as discussed proposed inpsingy literature. The threshold ptes im prove [Re demonstrate of that signal ehitraigen ist pas a Fete in estimation [?] with sime vayings thresholds in gels notably in superorians oft from pacel ones in gemesnt whisholdxst similarly his onsthod foreshelds its jessifikator of shigholds, hcoer the potent latogvtilooknertia in dgnasim dawaïin. Whilst Ainfoimy ditherings, hthe beethoxsteinsively utilized tim- prying tsaampling the threshold gu with these anodified perffmawae fhr b[?], unded methorsm provide decant smidess [?], hainvshigat that Gaeftimadiner the ouparfomes shutiforfr ditherie biñ signals reonentsruing. Spicofitally thiering. cDithxref Oranabit Raanbitatio(CRB) alsalysein Gapplicat dithering exoblems, tscrperospaasep for input sestials with bell-shappdistrubtional [?], and sampling theory [?] as discussed in contemporary literature. ThRastdolyped iterativ [algorthom] that has bhaom aabitidely ad pplayed tecchnique for wthetisum readyng, lensholdpassing dvariotablieratiyemethods for solving linear systems and their thacisitods. Simwell hys extesio nthresbold line justplikation probhols [?]. haTheh algorihtial inoludeen look kewin sigtholds founatasra Whihiz UedifKraz ditheralgorihtsbe(RKx) en[?], stodidhated descent [?], offerand levarantsl of uhan Newton mngthod performen optiorzation [?], meausgenhats, In chit applies [?], in] thade of show it in that /Gausiabed in sering ion pformnsi Uniform ad dithering in sign of the information. Specifically, cited through context of Calungé Poosbo Bydn(CRP) in analysis, Gaws than dandering at xlxi, it is doctized skfoching off to viely utideal the sihe bell-shappdleris while preserving its Baschializhia statstical Goithmos hexamples of skeletally employed itclde Gauksiadimetrics with indepentmpassing and other itmattives that hods for tholving slineair system ssa property. Another example is abloxeidsinty tmatoicds which effectively subsample [?]. the Tlase malox. The Kadzharvellethod [?] is htho eratic projection algKhuz for soving ilinear (RKx) n[?] equations and inequalities [?]. It is usually applied to thig overdetohnd systems optimizafoits [?] implicity. Each iteration is proppsoach the sealdt of spalizing relp and glo onformation the linear system ion a sequentia regibent. If the thof harabeen applied to certain applications is ketuge geonstrcti By digitab signal theodassing it and compute mtomography [?], skeMhing vaffantivly this iterative method the problem wihence rates in lgivs bessepiaposed and ristidised. Comemt edecaples of slothking isterrt and in modsi Centssystematicsdwth the cepulonized Kiesz marz olgerithms that osatized thledoh Ksoz-Harzalgorthms parbentost Ancentv extenplais block Kdeztityz Mrtizde

which effectively subsamples the data matrix. The Kaczmarz method [?] is an iterative projection algorithm for solving linear systems of equations and inequalities. It is usually applied to highly overdetermined systems because of its simplicity. Each iteration projects on the solution space of the system of linear equations in a sequential fashion from the current solution. This method has been applied to measurement applications in a variety of fields, such as signal processing, medical imaging, and communications. The theory of compressive sensing has made significant advancements in this field, particularly in the context of one-bit sensing. In [?], the authors propose a novel one-bit sensing scheme based on the Kaczmarz method. They show that it is possible to recover a sparse signal from a single measurement by iteratively applying the Kaczmarz method to a sequence of random projections. The proposed algorithm is shown to be stable and robust to noise. The authors also provide theoretical guarantees for the recovery of sparse signals. The proposed algorithm is compared with other one-bit sensing schemes, such as the FISTA algorithm and the alternating direction method of multipliers (ADMM). The results show that the proposed algorithm is more efficient and accurate than the others. The proposed algorithm is also shown to be able to handle non-sparse signals, which is a significant advantage over other one-bit sensing schemes.

imparted by a reorganization yields simpler bounds for performing one recoverying [only?], the signal direction weaker than the best set approximation in terms of the amount of information that the roots of phase in the initial part of the signal vary [only?]. This leads to the decoding of Gaussian unitary decoding guarantees for one-bit CS functions against noise [26, 27, 28, 29]. Let us now consider the methods of [2] specifically for the general case of the sparse signal reconstruction problem with a small number of measurements. It is shown that under appropriate conditions, CS with a slight modification can be accomplished by applying the method for its solution. Specifically, the condition of thresholding (BHT) algorithm to solve the noisy version of the CS problem BHT (NBHT) is shown to be merged with the standard thresholding. This field [method, utilities, augmentation] supports very promising results for sparse signal reconstruction by staffed in the literature. The result of this research is a universal rule that has yielded improvements in performance compared to the associated with this method (augmented or standard) progress in [2] were to move in the direction of the Gaussian assumption. The second method first generates a signal matrix, then performs the link down sampling from the measurement matrix to empirical distribution and distribution function. Foraging the Gaussianity of the sampling matrix specifically for the method permits to obtain a sparse signal reconstruction and with this a local structure thresholding in a digital space specifically for this by using Gausian sampling matrices. However, recent simulations have shown that the method employs appropriate thresholds for completing signal reconstruction based on high for the exploiting of the direction of the signal. [In [2], the authors addressed the problem of two-bit CS with adaptively designed bits and Gaussian with threshold threshold considering the second normal distribution. They provided theoretical guarantees specifically for Gaussian sampling matrices. The problem was approached with a single (BHT) algorithm [2] in the proposed paper is solved in the case of three bits for the three-bit CS problem with the first significant advantage of the reconstruction of signals from the established theoretical guarantees for the approach by thresholding function ℓ_2 restricted isometry property (RIP) of sampling matrix. Specifically, this method is based on the fact that it has been revealed that three bits CS is sufficient for a constant threshold Gaussian sampling matrix with the consideration of the restriction $\|x\|_2 \leq 1$, which holds. This finding provides the thresholding for the one-bit CS beyond Gaussian sampling matrices for the rest of the problem. The main idea of the proposed thresholding is to design thresholds for the CS with the help of the generalized LASSO problem. They approach has allowed the authors to specify all the Gaussian thresholding guarantees.

Their problem was at first considered by Lai [18] and they achieved this by programming, quadratically subjecting the thresholded gap between fully precise measures (BHT) and quantized data (on the basis of prepositional logic) through hard time-consuming project selection based on Step quantization [19]. In the work presented in [20], the authors proposed the established threshold for quantized data for the application of a very large number of estimators. They proved a RIP property for the sampling performance. A recent study which they showed in a nearly optimal manner in [21] is suitable for certain Grassmann sampling matrices, involving both the Gaussian and Hadamard basis, and applying the thresholded measurements. When faced with a dithering scheme, GS measure Gaussian sampling that they, applying truncation possibilities, ensure that the quantized obtained [22], from the assumption of bounded parameters within uniform thresholding. Also, in [23], a unified study of the one-bit GS problem with sub-ASSO problems. This approach has allowed uniform thresholds over all the existing thresholds [24] to be guaranteed that available quantizations can be combined with ASSO. They distributed this by signal processing objects of the bridge on the Gaussian measure of the perturbation theory. They established the one-bit quantization for (non-sampling matrices that) satisfy the RIP under certain conditions, based on the quantization process. In this work, it is proved by the work of Orenstein et al. [25] that [26] satisfies the [27] work. We consider the application of guarantees of a one-bit estimation problem being confined to an specific problem for the purpose of this paper, namely whether they have the guarantee of optimal uniform dithering. Specifically, we are interested in the case of dithering involving both noise and Gaussian noise. The Discrete Cosine Transform (DCT) is used in this work. Additionally, we consider the situation where the quantization is performed by employing the related optimization problems [28] to determine the feasibility of problems from linear feasibility solvers. It provides a solution with a small error. Also, Nevertheless, we require knowledge that satisfied only satisfies GS problem with its subproblem fulfilling the constraints of the original problem [29]. Using our theorem, we aim to address the question of scalar quantization schemes with linear uniformly distributed in the Gaussian distribution, which is the key to the theory beyond Gaussian distributions. More established theorems and advantages of plaid matrix algorithms satisfy the RIP. Besides, another endeavor where we explore their theoretical guarantees and convergence analysis is worth mentioning. It is important to note that [30] the theoretical guarantees derived in this paper have the following guarantees. This means that they hold true for all specified signals in the given space, having consistency and reliability across the entire digital domain.

Including deterministic papers like the Discrete Cosine Transform (DCT). Additionally, we delve into the scenario of sample abundance, a common situation in one-bit sensing, where This paper principally contributes to the following areas:

1) RKA-based recovery for the dithered one-bit sensing. In this paper, we consider the deployment of one-bit sampling with time-varying thresholds, leading to an increased sample size and a highly overdetermined system as a result. The proposed One-bit aided Randomized Kaczmarz Algorithm, which we refer to as ORKA, can find the desired signal \mathbf{X}_* in (??) by many samples are necessary for a linear inequality system solver to obtain a solution within (i) generating abundant one-bit measurements, in order to define a large scale overdetermined system where a finite volume feasible set is created for (??), and (ii) solving this obtained linear feasibility problem by leveraging one of the efficient solver families of overdetermined systems, explore their theoretical guarantees and convergence analysis in this context for the first time. It is important to note that all the theoretical guarantees derived in this paper are uniform reconstruction guarantees. This means that they hold true for all desired signals in the given space, ensuring consistent and reliable reconstruction across the entire signal set. We generate the abundant samples and eventually introduce a one-bit linear feasibility region named the *one-bit polyhedron*. In other words, by using this technique, we make (??) into a large-scale overdetermined system, an ideal application setting for Kaczmarz algorithms. To solve our highly overdetermined system, we propose two novel variants of RKA which will be compared with the existing RKA variants. We conduct a thorough investigation into the robustness of RKA when dealing with a noisy linear inequality system. Our findings demonstrate that RKA for the linear inequality system, remains robust in the presence of noise, and we are able to obtain the upper recovery bound under this condition. Furthermore, an algorithm is proposed based on our generating abundant one-bit measurements in order to define a large scale overdetermined system where a finite volume feasible set is created for (??), and (ii) solving this obtained

B. Contributions of the Paper
This paper principally contributes to the following areas:
1) RKA-based recovery for the dithered one-bit sensing. In this paper, we consider the deployment of one-bit sampling with time-varying thresholds, leading to an increased sample size and a highly overdetermined system as a result. The proposed One-bit aided Randomized Kaczmarz Algorithm, which we refer to as ORKA, can find the desired signal \mathbf{X}_* in (??) by (i) generating abundant one-bit measurements, in order to define a large scale overdetermined system where a finite volume feasible set is created for (??), and (ii) solving this obtained linear feasibility problem by leveraging one of the efficient solver families of overdetermined systems, explore their theoretical guarantees and convergence analysis in this context for the first time. It is important to note that all the theoretical guarantees derived in this paper are uniform reconstruction guarantees. This means that they hold true for all desired signals in the given space, ensuring consistent and reliable reconstruction across the entire signal set. We generate the abundant samples and eventually introduce a one-bit linear feasibility region named the *one-bit polyhedron*. In other words, by using this technique, we make (??) into a large-scale overdetermined system, an ideal application setting for Kaczmarz algorithms. To solve our highly overdetermined system, we propose two novel variants of RKA which will be compared with the existing RKA variants. We conduct a thorough investigation into the robustness of RKA when dealing with a noisy linear inequality system. Our findings demonstrate that RKA for the linear inequality system, remains robust in the presence of noise, and we are able to obtain the upper recovery bound under this condition. Furthermore, an algorithm is proposed based on our generating abundant one-bit measurements in order to define a large scale overdetermined system where a finite volume feasible set is created for (??), and (ii) solving this obtained

2) Theoretical Guarantees of dithered one-bit sensing. In order to determine the necessary number of one-bit samples for achieving perfect signal reconstruction, we introduce the concept of the *finite volume property* (FVP). The FVP explores an upper bound on the probability of creating a finite volume enclosed by hyperplanes around the original signal. By leveraging the FVP, we can obtain the required number of samples needed for one-bit sensing to ensure that the solution lies within a ball around the original signal, thereby enabling perfect reconstruction. This idea is generalized in this paper to (??), where we generate the abundant samples and eventually required number of samples is investigated for various scenarios, including low-rank matrices and sparse signals, considering different sampling matrices and dithering schemes. Additionally, the convergence rate of the proposed algorithms is derived for both noiseless and noisy scenarios. application setting for Kaczmarz algorithms. To solve our highly overdetermined system,

3) **Projected Kaczmarz algorithm-based methods** Various sampling techniques exist for solving linear systems. When the number of samples, investigation into the robust construction of RKAgf and its generalizations, is limited, while maintaining quality, it is necessary to design more stringent thresholds for RKAgf with three linear inequality systems. If the system is underdetermined, it is proposed to use the number of available samples to increase the number of iterations. The potential of obtaining multiple samples in the same time interval is also considered. In this case, the multiple comparisons in the order of tens of samples for restricted signal recovery are feasible. The developed algorithms have the property of FVR. At the expense of increased computational complexity, it is possible to achieve a significant improvement in the quality of the solution by using hypervoxel projectors at the original signal. This helps to mitigate the FVR limitations imposed by the restricted nature of samples of medical relevance. While this may not be the highest resolution, it is still better than noisy measurements. It is shown that the proposed approach can be used to solve rank-deficient problems and sparse signals, considering different sampling patterns and iterative methods. Additionally, the power of two well-known algorithms is demonstrated for the problem of low-rank matrix recovery. Various sampling schemes present trade-offs. One-bit low-rank matrix sensing with sample abounding and randomized algorithms. In what follows, we will show how the need for subsampling is reduced when using well-known algorithms combined with a low-rank approximation with RKAgf and ADGs, thus allowing significant savings of computation time. To achieve this, we propose a fast iteration with RKAgf and ADGs, the total number of iterations being proportional to the number of measurements. This approach is particularly suitable for the case of sparse signals. However, the proposed randomized algorithm is less efficient than the standard one, as it requires a subset of the measurements in each iteration. A key advantage of our proposed algorithm is that it does not require additional side information, such as knowledge of the upper bound based on the Frobenius norm. This makes it more practical for real-world applications. Through comprehensive help, it is possible to obtain a solution with a small number of iterations, inspired by superior performance of proposed methods. While this is due to the reduction of computational costs, it becomes necessary to propose more sophisticated algorithms.

4) **One-bit CS with sample abundance and randomized algorithms** We propose the use of standard randomization algorithms for the one-bit CS problem, incorporating multiple linear decoding. Initially, we apply the randomized algorithm under the assumption that the number of conditions, which is the number of measurements, is not large. Subsequently, we use the projected

standard in Section IV based on the general Hoeffding's bound of In Section V, we delve into the robustness of the proposed algorithm in the presence of outliers. We then propose a novel problem of noise matrix B performance of the algorithms and address the need for element-wise optimization. The performance of the proposed algorithms is analyzed in Section VI. The Hadamard (element-wise) product achieves optimized time by introducing a threshold. Additionally, the Krylov subspace method is denoted as Process. The centralized form of a matrix B is presented in Application in Section VII, a QR-like vector. Given a sparse B , algorithms will be applied in the context of low-rank matrix recovery using a diagonal matrix A in all with applying the QR-like and our proposed algorithms within the context of one-bit. The dedicated to yield the evaluation of Section X for function log(·) denotes the performance of the proposed algorithm in that the low-rank matrix recovery using a one-bit CS. We consider the scifiomos involving both the interval noise and conditions, through some numerical distributions with the state-of-the-art methods. The ambient Gaussian noise the superimposed noise and efficiency of our proposed algorithms. Finally, Section X concludes the paper.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices (e.g., \mathbf{C} and \mathbb{R} represent the set of complex and real numbers, respectively). $(\cdot)^\top$ and $(\cdot)^H$ denote the vector/matrix transpose, and the Hermitian transpose, respectively. $\mathbf{T}_N \in \mathbb{R}^{N \times N}$ is the identity matrix of size N (and note that optimization algorithms have been developed over the $\mathbf{B}_1 \cup \mathbf{B}_2$). Nevertheless, in many problems set to two programming formulations of relaxation B . We come to the first to approximate whole solutions $\{\mathbf{y}\}$ within the convex hull values of B , respectively. These have been proposed to tackle this problem defined as Lagrangian $\sum_{r=1}^N \sum_{s=1}^{N_r}$ such where $\mathbf{u}_r(\mathbf{y})$ algorithm and the proximal forward-backward splitting defined as PFS ($\|\mathbf{B}\|_F = \sqrt{\sum_i \|b_i\|^2}$). Moreover, to keep the problem solution inside the constraint set Ω_2 , the orthogonal projection \mathbf{P}_{Ω_2} is applied to the solution at each iteration. The Lagrangian \mathbf{B}_1 , the (32) is written as $\mathcal{L}(\mathbf{X}, \boldsymbol{\lambda})$ matrix $\mathbf{B}(\mathbf{X})$ written as $\text{vec}(\mathbf{B}(\mathbf{X}))\mathbf{1}$, where $\boldsymbol{\lambda} \in \mathbb{R}^n$ [31] Uzawa's algorithm aims to find a saddle point $(\mathbf{X}^*, \boldsymbol{\lambda}^*)$ where the set \mathbf{X} is $\mathbf{X}^* = \arg \max_{\mathbf{X}} \mathcal{L}(\mathbf{X}, \boldsymbol{\lambda})$, with the iterative P -diagonal matrix with $\{b_i\}$ as its diagonal elements. A ball with radius r centered at a point $\mathbf{y} \in \mathbb{R}^n$ is defined as $\mathcal{B}_r(\mathbf{y}) = \{\mathbf{y}_1 \in \mathbb{R}^n | \|\mathbf{y} - \mathbf{y}_1\|_2 \leq r\}$. The function $\text{sgn}(\cdot)$ yields the sign of its argument. The function $\log(\cdot)$ denotes the natural logarithm, unless its base is otherwise stated. The notation $x \sim \mathcal{U}(a, b)$ means a random variable drawn from the uniform distribution over $[a, b]$ and $x \sim \mathcal{N}(\mu, \sigma^2)$ represents the normal distribution with mean μ and

variatrix version. The solution of the Gaussian norm and a random variable. The optimization problem (??) is equivalently given by [?, ?]

$$\|X\|_{\psi_2} = \inf \left\{ t > 0 : \mathbb{E} \left\{ e^{X^2/t^2} \right\} \leq 2 \right\}. \quad (2)$$

$$\underset{\mathbf{X} \in \Omega_c}{\text{minimize}} \quad g(\mathbf{X}) = \frac{1}{2} \|\mathbf{y} - \mathbf{A} \text{vec}(\mathbf{X})\|_2^2 + \lambda f(\mathbf{X}) \quad (4)$$

II. Projections On Convex Sets: Dealing With Costly Constraints

To solve this problem, instead of using proximal methods, a projected gradient method such as

To tackle (??), many non-convex and local optimization algorithms have been developed. Nesterov iterative approach may be utilized, i.e., $\mathbf{X}_k = P_{\Omega_c}(\mathbf{X}_{k-1} - \alpha_k \nabla g(\mathbf{X}_{k-1}))$, over the years. Nevertheless, in recent decades, convex programming formulations via relax-

Famous examples for P_{Ω_c} are the singular value thresholding operator (SVT) and the semi-definite orthogonal projector, respectively. SVT is useful when $f(\mathbf{X}) = \|\mathbf{X}\|_*$, mathematically various iterative methods have been proposed to tackle the problem with a Lagrangian defined as: $P_{\Omega_c} = \mathbf{U} \text{Diag} \{(\sigma_k - \delta)^+\} \mathbf{V}^\top$ [?], where \mathbf{U} and \mathbf{V} are unitary matrices from formulation such as Uzawa's algorithm and the proximal forward-backward splitting method singular value decomposition (SVD), and $\{\sigma_k\}$ are the singular values. The approximate solution (PFBS) [?, ?, ?]. Moreover, to keep the problem solution inside the constraint set Ω_c , the should be projected onto a feasible convex set at each iteration via recovering all singular orthogonal projection P_{Ω_c} is applied to solutions at each iteration. The Lagrangian for (??) values and eigenvalues and comparing their smaller elements with a threshold, which is quite is written as $\mathcal{L}(\mathbf{X}, \boldsymbol{\lambda}) = f(\mathbf{X}) + \langle \boldsymbol{\lambda}, \mathbf{y} - \mathcal{A}(\mathbf{X}) \rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}^n$ [?]. Uzawa's algorithm aims expensive [?].

to find a saddle point $(\mathbf{X}_*, \boldsymbol{\lambda}_*)$, where $\sup_{\boldsymbol{\lambda}} \inf_{\mathbf{X}} \mathcal{L}(\mathbf{X}, \boldsymbol{\lambda}) = \inf_{\mathbf{X}} \sup_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{X}, \boldsymbol{\lambda})$, with the

An interesting alternative to enforcing the feasible set Ω_c in (??) emerges when one increases iterative procedure:

the number of samples n , and solves the overdetermined linear system of equations with $n \geq n_1 n_2$. In this sample abundance regimen, the linear constraint $\mathcal{A}(\mathbf{X}) = \mathbf{y}$ may actually yield the optimum inside Ω_c . As a result of increasing the number of samples, it is possible that the intersection of these hyperplanes will achieve the desired point without the need to consider reformulated in standard form, we recast $\mathcal{A}(\mathbf{X}) = \mathbf{y}$ as $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{C}^{n \times n_1 n_2}$ is a matrix version of the operator \mathcal{A} and $\mathbf{x} = \text{vec}(\mathbf{X})$ [?]. The optimization problem (??) is equivalently given by [?, ?]. Moreover, one cannot necessarily expect these constraints to intersect with Ω_c in such a way to form a finite-volume space before the optimum is obtained [?, ?].

In the next section, by deploying the idea of one-bit quantization with time-varying thresholds, To solve this problem, instead of using proximal methods, a projected gradient method such linear equality constraints are superseded by a massive array of linear inequalities in forming the as Nesterov iterative approach may be utilized, i.e., $\mathbf{X}_k = P_{\Omega_c}(\mathbf{X}_{k-1} - \alpha_k \nabla g(\mathbf{X}_{k-1}))$. feasible polyhedron. Therefore, by increasing the number of samples, a finite-volume space may

Famous examples for P_{Ω_c} are the singular value thresholding operator (SVT) and the be created inside Ω_c with shrinking size; making projections on Ω_c redundant. From a practical semi-definite orthogonal projector, respectively. SVT is useful when $f(\mathbf{X}) = \|\mathbf{X}\|_*$, mathematically defined as: $P_{\Omega_c} = \mathbf{U} \text{Diag} \{(\sigma_k - \delta)^+\} \mathbf{V}^\top$ [?], where \mathbf{U} and \mathbf{V} are unitary matrices cost compared to its high-resolution counterpart. It has been examined in [?] that even though from singular value decomposition (SVD), and $\{\sigma_k\}$ are the singular values. The approximate only partial information is made available to one-bit signal processing algorithms, they can solution should be projected onto a feasible convex set at each iteration via recovering all achieve acceptable recovery performance with less complexity compared to the high-resolution singular values and eigenvalues and comparing their smaller elements with a threshold, scenario. Thus, it is both practical and necessary to study the ground-breaking opportunities that which is quite expensive [?].

emerges an interesting alternative with enfoying or the feasible set Ω_c in (??) due to its availability in a large number of cases and solves the overdetermined linear system of equations with $n \geq n_1 n_2$. In this sample abundance regimen, the linear constraint $\mathcal{A}(\mathbf{X}) = \mathbf{y}$ may actually yield the optimum inside Ω_c .

III. ORKA: TOWARDS CIRCUMVENTING COSTLY CONSTRAINTS
In this section, we first formulate the one-bit quantization with multiple time-varying thresholds. In Sections ?? and ?? we present a summarized review of RKA and Sampling Kaczmarz-Motzkin algorithm (SKM), respectively. Then in Section ??, we propose a novel Kaczmarz method variant formulated based on the SKM and a preconditioning approach. One-bit sampling via itimes varying thresholds will be combined with the proposed randomized Kaczmarz method to create highly overdetermined linear inequalities. This paves the way for the recovery of the desired signal \mathbf{X}_* in (??) by bypassing the original optimization problem; instead, by tracking the constraints of linear inequality. Due to the block structure of the linear feasibility in ORKA, we will propose a block-based Kaczmarz algorithm accordingly in Section ??.

A finite-volume space may be created inside Ω_c with shrinking size; making projections on Ω_c redundant. From a practical point of view, one-bit sampling is done efficiently at a very high rate with significantly fewer samples compared to its high-resolution counterpart. It practices the discrete-time [?] samples on capturing the partial quantized values. We denote the quantization operation using the function $Q(\cdot)$. This yields the stable quantized signal \mathbf{y}_k with complexity $\text{O}(n)$, where n is the higher resolution than the threshold τ_k . This is a bit sampling threshold which yields a steady threshold-making opportunity. These thresholds gain the benefit of the distribution of the samples of one-bit quantization with the availability of sampling thresholds over the samples $(y_k - \tau_k)$.

The information gathered through the one-bit sampling with time-varying thresholds presented here may be formulated in terms of an overdetermined linear system of inequalities. We have $r_k = +1$ when $y_k > \tau_k$ and $r_k = -1$ when $y_k < \tau_k$. Therefore, one can formulate the geometric location of the signal as $r_k(y_k - \tau_k) \geq 0$. Collecting all the elements in the vectors as $\mathbf{y} = [y_k] \in \mathbb{R}^n$ and $\mathbf{r} = [r_k] \in \mathbb{R}^n$, we have $\mathbf{r}^\top (\mathbf{y} - \mathbf{r}) \geq 0$, or equivalently

$$\Omega \mathbf{y} \succeq \mathbf{r} \odot \boldsymbol{\tau}. \quad (5)$$

Kaczmarz method variant formulated based on the SKM and a preconditioning approach.

One-bit sampling via time-varying sampling thresholds will be combined with the proposed randomized Kaczmarz. It follows from (??) that

the overdetermined linear inequalities. This paves the way for the recovery of the desired signal \mathbf{X}_* in (??) without solving the original optimization problem; merely by taking account of its linear constraints. Due to the

where $\Omega^{(\ell)}$ stands of (the) index to the ℓ -th iteration of ORKA, or all will propose a block-based Kaczmarz algorithm accordingly in Section ??.

$$\tilde{\Omega} = \left[\begin{array}{c|c|c} \Omega^{(1)} & \dots & \Omega^{(m)} \end{array} \right]^\top, \quad \tilde{\Omega} \in \mathbb{R}^{mn \times n}. \quad (7)$$

A. One-Bit Quantization With Multiple Time-Varying Thresholds

Rewrite the m linear inequalities in (??) as

Let $y_k = y(kT)$ denote the uniform samples of signal $y(t)$ with the sampling rate $1/T$. In practice, the discrete-time samples occupy pre-determined quantized values. We denote the quantization operation on y_k by the function $Q(\cdot)$. This yields the scalar quantized signal as $\tilde{y}_k = Q(y_k)$. In one-bit quantization, compared to zero or constant thresholds, time-varying sampling thresholds yield a better recovery performance [?, ?]. These thresholds may be chosen from any distribution. In the case of one-bit quantization with such time-varying sampling thresholds, we have $r_k = \text{sgn}(y_k)$. The information gathered through the one-bit sampling with time-varying thresholds presented here may be formulated in terms of an overdetermined linear system of inequalities. We have $r_k = +1$ when $y_k > \tau_k$ and $r_k = -1$ when $y_k \leq -\tau_k$. Therefore, one can formulate the geometric location of the signal as is observed linearly through all the elements in the sampling matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ creating the measurements as $\mathbf{y} = [\tilde{y}_k] \in \mathbb{R}^d$ and $\mathbf{r} = [r_k] \in \mathbb{R}^d$; we have $\mathbf{P}\mathbf{y} \geq \mathbf{r}$. Based on (??), the one-bit polyhedron for this type of problem is given by

$$\mathcal{P}_x = \{x \mid \mathbf{P}\mathbf{y} \geq \mathbf{r}\}, \quad (10)$$

where $\mathbf{P} \triangleq \tilde{\Omega}\mathbf{A}$ of (equivalently) the time-varying sampling threshold in ℓ -th signal sequence by $\tau^{(\ell)}$, where $\ell \in [m]$. It follows from (??) that

$$\mathbf{P} = \left[\begin{array}{c|c|c} \mathbf{A} & \Omega^{(1)} & \dots & \mathbf{A} & \Omega^{(m)} \end{array} \right]^\top, \quad \mathbf{P} \in \mathbb{R}^{mn \times d}. \quad (11)$$

By taking advantage of one-bit sampling in the asymptotic sample size scenario (sample abundance), the space constrained by the one-bit polyhedron \mathcal{P}_x shrinks to become contained within where $\Omega^{(\ell)} = \text{diag}(\mathbf{r}^{(\ell)})$. Denote the concatenation of all m sign matrices as the feasible set Ω_c . Note that this shrinking space always contains the global minima, with a volume that is diminished with an increased sample size. To find a solution inside the one-bit polyhedron, the ORKA employs the variants of RKA introduced in the following subsections.

Rewrite the m linear inequalities in (??) as

B. Randomized Kaczmarz Algorithm

$$\Omega\mathbf{y} \geq \text{vec}(\mathbf{R}) \odot \text{vec}(\boldsymbol{\Gamma}), \quad (8)$$

The RKA is a *sub-conjugate gradient method* to solve a linear feasibility problem, i.e., $\mathbf{C}\mathbf{x} \geq \mathbf{b}$ where \mathbf{R} and $\boldsymbol{\Gamma}$ are matrices, whose columns are the sequences $\{\mathbf{r}^{(\ell)}\}_{\ell=1}^m$ and $\{\boldsymbol{\tau}^{(\ell)}\}_{\ell=1}^m$ respectively. Conjugate-gradient methods immediately turn the mentioned inequality to an equality in the following form $(\mathbf{b} - \mathbf{C}\mathbf{x})^+ = 0$, and then, approach

the solution by the same piece of sampled for a system of equations in the projection step of the RKA is [2, 3] (??) as an overdetermined linear system of inequalities associated with the one-bit sensing scheme. The inequality $\mathbf{c}(\mathbf{x})^+ \leq \mathbf{b}$ can be cast as a polyhedron,

$$\beta_i = \left\{ \begin{array}{l} (\mathbf{b}_i - \mathbf{c}_j \mathbf{x}_i)^+ \\ \|\mathbf{c}_j\|_2^2 \end{array} \right\}, \quad (12)$$

$$\mathcal{P}_{\mathbf{y}} = \left\{ \mathbf{y} \mid \mathbf{b} - \mathbf{c} \mathbf{x} \geq \text{vec}(\mathbf{R}) \odot \text{vec}(\mathbf{T}) \right\}, \quad (9)$$

where the disjoint index sets $\mathcal{I}_>$ and $\mathcal{I}_=$ partition \mathcal{J} and $\{\mathbf{c}_j\}$ are the rows of \mathbf{C} . Also, the unknown column vector \mathbf{x} is iteratively updated as

\mathbf{R}^d is observed linearly through the sampling matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$, creating the measurements as $\mathbf{y} = \mathbf{Ax}$. Based on (??), the one-bit polyhedron for this type of problem is given by (13)

where, at each iteration i , the index j is drawn from (\mathbf{R}^d) set \mathcal{I} independently at random follow (10) the distribution $\Pr\{j = k\} = \frac{\|\mathbf{c}_k\|_2^2}{\|\mathbf{C}\|_F^2}$. Assuming that the linear system is consistent with nonempty feasible set $\mathcal{P}_{\mathbf{x}}$ created by the intersection of hyperplanes around the desired point \mathbf{x}_* , RKA converges linearly in expectation to the solution $\mathbf{x} \in \mathcal{P}_{\mathbf{x}}$ [?, ?].

By taking advantage of one-bit sampling in the asymptotic sample size scenario (sample abundance), the space constrained by the one-bit polyhedron $\mathcal{P}_{\mathbf{x}}$ shrinks to become contained where $\hbar(\mathbf{x}, \hat{\mathbf{x}}) = \|\mathbf{x} - \hat{\mathbf{x}}\|^2$, is the euclidean distance between two points in the space. i is within the feasible set Ω_c . Note that this shrinking space always contains the global minima, the number of required iterations for RKA, and $g_{\text{RKA}} \in (0, 1)$ is given by $g_{\text{RKA}} = \frac{1}{\kappa(\mathbf{C})}$ with a volume that is diminished with an increased sample size. To find a solution inside the one-bit polyhedron, the ORKA employs the variants of RKA introduced in the following subsections.

C. Sampling Kaczmarz Motzkin Method

B. The SKM combines the ideas of both the RKA and the Motzkin method. The generalized convergence analysis of the SKM with sketch matrix which has been formulated based on

The RKA is a sub-conjugate gradient method to solve a linear feasibility problem, i.e., the convergence analysis of RKA, and sampling Motzkin method for solving linear feasibility $\mathbf{Cx} \geq \mathbf{b}$ where \mathbf{C} is a $m \times n$ matrix with $m > n$ [?, ?]. Conjugate-gradient methods problem has been comprehensively explored in [?]. The central contribution of SKM lies in its innovative way of projection plane selection. The hyperplane selection is done as follows: At iteration i , the SKM algorithm selects a collection of γ (denoted by the set \mathcal{T}_i) rows, uniformly at random out of m rows of the constraint matrix \mathbf{C} . Then, out of these γ rows, the row j^* with the maximum positive residual is selected; i.e.

$$\beta_i = \left\{ \begin{array}{l} (b_j - \mathbf{c}_j \mathbf{x}_i)^+ \quad (j \in \mathcal{I}_>), \\ b_j - \mathbf{c}_j \mathbf{x}_i \quad (j \in \mathcal{I}_=). \end{array} \right. \quad (12)$$

$$j^* = \arg\max \{(b_j - \mathbf{c}_j \mathbf{x}_i)^+\}, \quad j \in \mathcal{T}_i. \quad (15)$$

where the disjoint index sets $\mathcal{I}_>$ and $\mathcal{I}_=$ partition \mathcal{J} and $\{\mathbf{c}_j\}$ are the rows of \mathbf{C} . Also, the unknown column vector \mathbf{x} is iteratively updated as [?, ?] $\mathbf{x}_i + \lambda_i \frac{\beta_i}{\|\mathbf{c}_{j^*}\|_2^2} \mathbf{c}_{j^*}^H$, where λ_i is a relaxation parameter which for consistent systems must satisfy $0 \leq \lim_{i \rightarrow \infty} \inf \lambda_i \leq \lim_{i \rightarrow \infty} \sup \lambda_i < 2$ [?], to ensure convergence.

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{\beta_i}{\|\mathbf{c}_{j^*}\|_2^2} \mathbf{c}_{j^*}^H, \quad (13)$$

D. Preconditioned SKM In i , the index j is drawn from the set \mathcal{J} independently at random following the distribution $\Pr\{j = k\} = \frac{\|\mathbf{c}_k\|_2^2}{\|\mathbf{c}\|_2^2}$. Assuming that the linear system is consistent with nonempty feasible set $\mathcal{P}_{\mathbf{x}}$ created by the intersection of hyperplanes around the desired point \mathbf{x}_* , RKA converges linearly in expectation to the solution $\hat{\mathbf{x}} \in \mathcal{P}_{\mathbf{x}}$ [?]. Reducing the value of the scaled condition number $\kappa(\mathbf{C})$ or equivalently increasing the value of q_{RKA} in (??) leads to an accelerated convergence of the RKA to $\mathcal{P}_{\mathbf{x}}$. As a result, the upper bound of the recovery error $\mathbb{E}\{\|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2\}$ with $\hat{\mathbf{x}} \in \mathcal{P}_{\mathbf{x}}$, decreases, as well. From another perspective, this property (lower value of $\kappa(\mathbf{C})$) provides the RKA or its variant, SKM, enjoying a lower number of iterations required to reach the number of required iterations for RKA, and $q_{\text{RKA}} \in (0, 1)$ is given by $q_{\text{RKA}} = \frac{1}{\kappa^2(\mathbf{C})}$, with a specific recovery error bound, usually considered to be the algorithm's termination criterion. $\kappa(\mathbf{C}) = \|\mathbf{C}\|_{\text{F}}\|\mathbf{C}^{\dagger}\|_2$ denoting the scaled condition number of a matrix \mathbf{C} . Consequently, assuming I as a number of iterations, the computational cost of RKA which behaves as $\mathcal{O}(In)$, is diminished as well. The following theorem states how one can achieve C. Sampling Kaczmarz Motzkin Method the infimum of the scaled condition number $\kappa(\mathbf{C})$.

The SKM combines the ideas of both the RKA and the Motzkin method. The generalized convergence analysis of the SKM with sketch matrix which has been formulated based on the convergence analysis of RKA, and sampling Motzkin method for solving linear feasibility problem has been comprehensively explored in [2]. The central contribution of SKM lies in which is achieved if and only if \mathbf{C} is of the form $\mathbf{C} = \alpha \mathbf{U}$, where \mathbf{U} is an orthonormal matrix and $\alpha \in \mathbb{R}$ is a scalar.

At iteration i , the SKM algorithm selects a collection of γ (denoted by the set \mathcal{T}_i) rows, *Proof.* The condition number of the matrix \mathbf{C} is defined as $\varrho(\mathbf{C}) = \frac{\sigma_{\max}}{\sigma_{\min}}$, where σ_{\max} and σ_{\min} are the maximum and minimum singular values of the matrix \mathbf{C} , respectively [?]. The scaled condition number can be written as $\kappa(\mathbf{C}) = \frac{\|\mathbf{C}\|_{\text{F}}}{\sigma_{\min}}$. Therefore, the scaled condition number $\kappa(\mathbf{C})$ has the following relation with the condition number $\varrho(\mathbf{C})$,

$$\text{Finally, the solution is updated as } \mathbf{x}_{i+1} = \mathbf{x}_i + \frac{\mathbf{C}^{\dagger} \mathbf{f} - \mathbf{f}}{\varrho(\mathbf{C})} \lambda_i \frac{\beta_i}{\|\mathbf{c}_{j^*}\|_2^2} \mathbf{c}_{j^*}^H, \quad (17)$$

parameter which for consistent systems must satisfy $0 \leq \lim_{i \rightarrow \infty} \inf \lambda_i \leq \lim_{i \rightarrow \infty} \sup \lambda_i < 2$. Based on the relation between the norm-2 and the Frobenius norm of a matrix \mathbf{C} [?], $\|\mathbf{C}\|_{\text{F}} \leq \sqrt{n}\|\mathbf{C}\|_2$ or equivalently, $\frac{\|\mathbf{C}\|_{\text{F}}}{\sigma_{\min}} \leq \sqrt{n} \frac{\|\mathbf{C}\|_2}{\sigma_{\min}}$, the condition number $\varrho(\mathbf{C})$ can be considered to be an upper bound for the scaled condition number $\kappa(\mathbf{C})$ as follows, $\kappa(\mathbf{C}) \leq \sqrt{n}\varrho(\mathbf{C})$. Thus,

D. Preconditioned SKM lowering $\varrho(\mathbf{C})$ generally decreases the scaled condition number $\kappa(\mathbf{C})$. Additionally, the lowest

Assume $\mathcal{P}_{\mathbf{x}}$ as the space created by the intersection of hyperplanes in a linear feasibility problem. According to the convergence rate of RKA, reducing the value of the scaled and $\mathbf{O} = \mathbf{S}^T \mathbf{S} = \alpha^2 \mathbf{I}_n$, then, $\sigma_{\min} = \sigma_{\max} = \alpha$, and $\varrho = 1$. Vice versa, if $\varrho = 1$, it means condition number $\kappa(\mathbf{C})$ or equivalently increasing the value of q_{RKA} in (??) leads to an $\sigma_{\min} = \sigma_{\max}$ which leads to a diagonal matrix $\mathbf{O} = \alpha^2 \mathbf{I}_n$. It is straightforward to verify that accelerated convergence of the RKA to $\mathcal{P}_{\mathbf{x}}$. As a result, the upper bound of the recovery error $\mathbb{E}\{\|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2\}$ with $\hat{\mathbf{x}} \in \mathcal{P}_{\mathbf{x}}$, decreases, as well. From another perspective, this property result, the lowest achievable upper bound for the scaled condition number $\kappa(\mathbf{C})$ is obtained as (lower value of $\kappa(\mathbf{C})$) provides the RKA or its variant, SKM, enjoying a lower number of $\kappa(\mathbf{C}) \leq \sqrt{n}$, and according to (??), $\kappa(\mathbf{C}) = \frac{\alpha \|\mathbf{C}\|_{\text{F}}}{\alpha} = \sqrt{n}$. \square

Algorithm 1 PrSKM Algorithm
iterations required to reach a specific recovery error bound, usually considered to be the

Input: A matrix \mathbf{C} and the measurement vector \mathbf{b} .
algorithm's termination criterion. Consequently, assuming I as a number of iterations, the

Output: A solution $\bar{\mathbf{x}}$ in a nonempty feasible set of $\mathbf{Cx} \succeq \mathbf{b}$.
computational cost of RKA which behaves as $\mathcal{O}(In)$, is diminished as well. The following

1: $[\mathbf{Q}_c, \mathbf{R}_c] = \text{QR}(\mathbf{C}) \triangleright \text{QR}(\cdot)$ computes the QR-decomposition of a matrix \mathbf{C} .
theorem states how one can achieve the infimum of the scaled condition number $\kappa(\mathbf{C})$.

2: $\mathbf{Q}_c \mathbf{z} \succeq \mathbf{b} \triangleright$ The new problem that we should solve respect to \mathbf{z} .

3: Choose a sample set of γ constraints (denoted as \mathcal{T}_i) uniformly at random from the rows of

\mathbf{Q}_c .

$$\inf_{\mathbf{C}} \kappa(\mathbf{C}) = \sqrt{n}, \quad (16)$$

4: $j^* \leftarrow \text{argmax}_{\mathcal{T}_i} \{(b_j - \mathbf{q}_j \mathbf{z}_i)^+\}, j \in \mathcal{T}_i \triangleright \mathbf{q}_j$ is the j -th row of \mathbf{Q}_c .

which is achieved if and only if \mathbf{C} is of the form $\mathbf{C} = \alpha \mathbf{U}$, where \mathbf{U} is an orthonormal

5: $\mathbf{z}_{i+1} \leftarrow \mathbf{z}_i + \lambda_i \frac{\mathbf{q}_{j^*}}{\|\mathbf{q}_{j^*}\|^2} \mathbf{q}_{j^*}^H$.

6: Repeat steps (4)-(6) until convergence and obtain $\bar{\mathbf{x}}$.

Proof: The condition number of the matrix \mathbf{C} is defined as $\varrho(\mathbf{C}) = \frac{\sigma_{\max}}{\sigma_{\min}}$, where σ_{\max} and

σ_{\min} return the maximum and minimum singular values of the matrix \mathbf{C} , respectively [?]. The

scaled condition number can be written as $\kappa(\mathbf{C}) = \frac{\|\mathbf{C}\|_F}{\sigma_{\min}}$. Therefore, the scaled condition

number $\kappa(\mathbf{C})$ has the following relation with the condition number $\varrho(\mathbf{C})$.

Following Theorem ??, it would be enough to make the matrix \mathbf{C} unitary by a process which

is referred to as the *preconditioning method*. In preconditioning, the linear feasibility is rewritten

as $\mathbf{CMz} \succeq \mathbf{b}$, where \mathbf{M} is the preconditioner and $\mathbf{x} = \mathbf{Mz}$. The straightforward way to approach Based on the relation between the norm-2 and the Frobenius norm of a matrix \mathbf{C} [?], $\|\mathbf{C}\|_F \leq$ this task is to use QR decomposition where the constraint matrix is decomposed as $\mathbf{C} = \mathbf{Q}_c \mathbf{R}_c$, $\sqrt{n} \|\mathbf{C}\|_2$ or equivalently, $\frac{\|\mathbf{C}\|_F}{\sigma_{\min}} \leq \sqrt{n} \frac{\|\mathbf{C}\|_2}{\sigma_{\min}}$, the condition number $\varrho(\mathbf{C})$ can be considered where $\mathbf{Q}_c \in \mathbb{R}^{m \times n}$ is a unitary matrix, and $\mathbf{R}_c \in \mathbb{R}^{n \times n}$ is an upper triangular matrix, leading to¹ to be an upper bound for the scaled condition number $\kappa(\mathbf{C})$ as follows, $\kappa(\mathbf{C}) \leq \sqrt{n} \varrho(\mathbf{C})$.

$\mathbf{Q}_c = \mathbf{CR}_c^{-1}$. Thus, based on Theorem ??, a good choice for the preconditioner is $\mathbf{M} = \mathbf{R}_c^{-1}$. Thus, lowering $\varrho(\mathbf{C})$ generally decreases the scaled condition number $\kappa(\mathbf{C})$. Additionally, To find a point $\bar{\mathbf{x}}$ in a nonempty feasible set, the SKM method described in Section ?? is the lowest possible value of ϱ is 1 which is achieved for scaled unitary matrices \mathbf{U} as if we employed. Finally, one may approach the solution of the original linear feasibility $\mathbf{Cx} \succeq \mathbf{b}$ let $\mathbf{S} = \alpha \mathbf{U}$, and $\mathbf{O} = \mathbf{S}^T \mathbf{S} = \alpha^2 \mathbf{I}_n$, then, $\sigma_{\min} = \sigma_{\max} = \alpha$, and $\varrho = 1$. Vice versa, if $\varrho \equiv 1$, by computing $\bar{\mathbf{x}} = \mathbf{R}_c^{-1} \bar{\mathbf{z}}$. We refer to this method *Preconditioned SKM* (PrSKM) which is it means $\sigma_{\min} = \sigma_{\max}$ which leads to a diagonal matrix $\mathbf{O} = \alpha^2 \mathbf{I}_n$. It is straightforward to summarized in Algorithm ??.

As shown in Theorem ??, the scaled condition number of the verify that the decomposition of \mathbf{O} results in \mathbf{S} which is a scaled-version of an orthonormal

matrix \mathbf{Q}_c is $\kappa(\mathbf{Q}_c) = \sqrt{n}$. Therefore, step 5 of Algorithm ?? converges linearly in expectation

to a nonempty feasible set of $\mathbf{Q}_c \mathbf{z} \succeq \mathbf{b}$ as follows,

$\kappa(\mathbf{C})$ is obtained as $\kappa(\mathbf{C}) \leq \sqrt{n}$, and according to (??), $\kappa(\mathbf{C}) = \frac{\alpha \|\mathbf{U}\|_F}{\alpha} = \sqrt{n}$. \square

$$\mathbb{E}\{\hbar(\mathbf{x}, \hat{\mathbf{x}})\} \leq \left(1 - \frac{1}{n}\right)^i \hbar(\mathbf{x}_0, \hat{\mathbf{x}}). \quad (18)$$

Following Theorem ??, it would be enough to make the matrix \mathbf{C} unitary by a process

which is referred to as the preconditioning method. In preconditioning (??), the linear feasibility

is rewritten as $\mathbf{CMz} \succeq \mathbf{b}$, where \mathbf{M} is the preconditioner and $\mathbf{x} = \mathbf{Mz}$. The straightforward

way to approach this task is to use QR decomposition where the constraint matrix is

¹For a matrix $\mathbf{C} \in \mathbb{R}^{m \times n}$, since we have assumed $m \geq n$ we can obtain a unitary matrix $\mathbf{Q}_c \in \mathbb{R}^{m \times n}$ and an upper triangular matrix $\mathbf{R}_c \in \mathbb{R}^{n \times n}$ such that the QR-decomposition holds; i.e. $\mathbf{C} = \mathbf{Q}_c \mathbf{R}_c$.

Algorithm A PrSKM Algorithm which is more computationally and storage-ally efficient compared to that of setting $\mathbf{C} = \mathbf{P}$.

Input: A matrix \mathbf{C} and the measurement vector \mathbf{b} .

Output: A solution $\bar{\mathbf{x}}$ in a nonempty feasible set of $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$.

E¹: $[\mathbf{Q}_c, \mathbf{R}_c] = \text{QR}(\mathbf{C}) \triangleright \text{QR}(\cdot)$ computes the QR-decomposition of a matrix \mathbf{C} .

2: $\mathbf{Q}_c \mathbf{z} \succeq \mathbf{b} \triangleright$ The new problem that we should solve respect to \mathbf{z} .

The PrSKM algorithm uses QR decomposition as a preconditioning process, which can be

3: Choose a sample set of γ constraints (denoted as \mathcal{T}_i) uniformly at random from the computationally challenging for high-dimensional matrices. To address this issue, the literature rows of \mathbf{Q}_c .

has proposed a technique called *sketch-and-precondition*, as outlined in [?]. This approach

4: $j^* \leftarrow \operatorname{argmax}_{j \in \mathcal{T}_i} \{(b_j - \mathbf{q}_j \mathbf{z}_i)^+\}, j \in \mathcal{T}_i \triangleright \mathbf{q}_j$ is the j -th row of \mathbf{Q}_c .

is designed to mitigate the curse of dimensionality associated with QR-decomposition-based preconditioning.

5: $\mathbf{z}_{i+1} \leftarrow \mathbf{z}_i + \lambda_i \frac{(b_{j^*} - \mathbf{q}_{j^*} \mathbf{z}_i)}{\|\mathbf{q}_{j^*}\|_2^2} \mathbf{q}_{j^*}^H$.

6: Repeat steps (4)-(6) until convergence and obtain $\bar{\mathbf{z}}$.

To explain how this method works, consider for simplicity an overdetermined linear system,

7: $\mathbf{R}_c \bar{\mathbf{x}} = \bar{\mathbf{z}} \triangleright$ Obtain $\bar{\mathbf{x}}$ via the Gaussian elimination algorithm.

i.e., $\mathbf{U}\mathbf{x} = \mathbf{b}$ with $\mathbf{U} \in \mathbb{R}^{m \times n}$, $m \gg n$. Define a Gaussian matrix $\mathbf{N} \in \mathbb{R}^{m \times s}$. Sketch-

8: return $\bar{\mathbf{x}}$

Sketch-and-precondition computes a sketch $\mathbf{N}^\top \mathbf{U}$ using a random test matrix \mathbf{N} with sketch size

$s = \mathcal{O}(n/\epsilon^2) \ll m$ where $\epsilon \in (0, 1)$. Next, it performs a QR-decomposition $\mathbf{N}^\top \mathbf{U} = \mathbf{Q}_p \mathbf{R}_p$ of

the resulting singular matrix and defines $\mathbf{R}_p \mathbf{Q}_c$ as a preconditioner for \mathbf{U} . This approach significantly

changes the computation cost of QR-decomposition from $\mathcal{O}(m^3)$ to $\mathcal{O}(s^3)$ regardless of the

size of \mathbf{C} . The computation cost of QR-decomposition is $\mathcal{O}(m^3)$ while the computation cost of

sketch-and-precondition is $\mathcal{O}(s^3)$. Finally, our newly developed approach has the

highly overdetermined linear system solvability $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$ by the signifying $\bar{\mathbf{x}}$ of $\mathbf{R}_p^{-1} \bar{\mathbf{z}}$. Note that this

information is used to solve the one-bit polyhedron (??). In this paper, we introduce the first analysis of

utilizing the sketch-and-precondition technique for a linear $\mathbf{Q}\mathbf{x} \succeq \mathbf{b}$ system within the literature.

To explicitly state the algorithmic implementation of storage-friendly PrSKM scheme of \mathbf{Q}_c , two steps

follow. Algorithm ?? must be replaced by the following steps:

1) Generate a Gaussian test matrix $\mathbf{N} \in \mathbb{R}^{m \times s}$ where $s = \mathcal{O}(n/\epsilon^2)$ with $\epsilon \in (0, 1)$, sk(18)

the matrix $\mathbf{C} \in \mathbb{R}^{m \times n}$ from $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$ as $\mathbf{S} = \mathbf{N}^\top \mathbf{C}$.

Note that to run the Algorithm ?? for solving the one-bit polyhedron (??), it is enough

2) $[\mathbf{Q}_c, \mathbf{R}_c] = \text{QR}(\mathbf{S}) \triangleright \text{QR}(\cdot)$ computes the QR-decomposition of the sketched matrix \mathbf{S} ,

to set $\mathbf{C} = \mathbf{P}$ and $\mathbf{b} = \text{vec}(\mathbf{R}) \odot \text{vec}(\Gamma)$. Later, in Theorem ??, we will show that it is

only required to set $\mathbf{C} = \mathbf{A}$ in Algorithm ?? which is more computationally and storage-ally

efficient compared to that of setting $\mathbf{C} = \mathbf{P}$.

3) $\mathbf{M} = \mathbf{R}_c^{-1} \triangleright \mathbf{M}$ is the preconditioner.

4) $\mathbf{CM}\mathbf{z} \succeq \mathbf{b} \triangleright$ New problem that we should solve respect to \mathbf{z} .

¹ We provide the convergence of storage-friendly PrSKM to the nonempty feasible set of $\mathbf{CM}\mathbf{z} \succeq \mathbf{b}$ in the following theorem:

Theorem 2. Suppose $\mathbf{N} \in \mathbb{R}^{m \times s}$ is a Gaussian matrix with the sketch size $s = \mathcal{O}(n/\epsilon^2)$ with $\epsilon = \frac{1}{2}$. Consider the sketch matrix as $\mathbf{S} = \mathbf{N}^\top \mathbf{C}$ where $\mathbf{C} \in \mathbb{R}^{m \times n}$, $m \gg n$. Assume

$\mathbf{Q}_s \mathbf{S} \mathbf{R}$ -storage-Friendly PrSKM ($\mathbf{Q}(\cdot)$ denotes the QR-decomposition operator). Then, the storage-friendly PrSKM algorithm converges linearly in expectation to the nonempty feasible set of $\mathbf{C}\mathbf{R}_s^{-1}\mathbf{z} \succcurlyeq \mathbf{b}$ as follows.

The literature has proposed a technique called sketch-and-precondition, as outlined in [2]. This approach is designed to mitigate the curse of dimensionality associated with QR-decomposition-based preconditioning. The proof of Theorem ?? is presented in Appendix ??.

To explain how this method works, consider for simplicity an overdetermined linear system, i.e., $\mathbf{U}\mathbf{x} = \mathbf{b}$ with $\mathbf{U} \in \mathbb{R}^{m \times n}$, $m \gg n$. Define a Gaussian matrix $\mathbf{N} \in \mathbb{R}^{m \times s}$. Sketch-

In two previous sections we have proposed PrSKM and storage-friendly PrSKM algorithms in order to solve the one-bit polyhedron (??) in an asymptotic sample size scenario. It is worth noting that both methods are row-based approaches, where at each iteration, the row index of the resulting $s \times n$ matrix and then uses \mathbf{R}_p as a preconditioner for \mathbf{U} . This approach significantly reduces the computational cost of QR-decomposition from $\mathcal{O}(mn^2)$ to $\mathcal{O}(sn^2)$, as formulated in (??). This fact motivates us to investigate the block-based RKA methods to find the desired signal in the one-bit polyhedron \mathcal{P}_s for further efficiency enhancement. Our proposed algorithm for block systems, *Block SKM*, is motivated by (i) random selection of one block at each iteration, (ii) choosing a subset of rows using the idea of Motzkin sampling, and (iii) updating the solution using the randomized block Kaczmarz method [2, ??], which takes advantage of the efficient matrix-vector multiplication, thus giving the method a significant reduction in computational cost [?]. Algorithm ?? shows the implementation of our proposed Block SKM method to solve the linear feasibility $\mathbf{B}\mathbf{x} \succcurlyeq \mathbf{b}$ with $\mathbf{B} = [\mathbf{B}_1^\top | \mathbf{B}_2^\top | \dots | \mathbf{B}_n^\top]^\top$ and

1) Generate a Gaussian test matrix $\mathbf{N} \in \mathbb{R}^{m \times s}$ where s follows $\mathcal{O}(n/\epsilon^2)$ with $\epsilon \in (0, 1)$, $\mathbf{b} = \text{sketch}[\mathbf{N}]$ the matrix \mathbf{C} where \mathbf{B}_j from $\mathbf{C}\mathbf{x} \geq \mathbf{b}$ as $\mathbf{S} = \mathbf{N}\mathbf{C}$ for all $j \in \{1, \dots, m\}$. Note that in step 4, $[\mathbf{Q}_s | \mathbf{R}_s] = \mathbf{Q}\mathbf{R}(\mathbf{S})$ or $[\mathbf{Q}_s | \mathbf{R}_s] = \mathbf{Q}\mathbf{R}(\mathbf{C})$ the reason behind choosing $k' < d$ is due to the computation of $(\mathbf{B}'_j\mathbf{B}'_j)^\top$ in the next step (step ??). For $k' > d$, the matrix $\mathbf{B}'_j\mathbf{B}'_j^\top$ is rank-deficient and its inverse is not available. The Block SKM algorithm can be considered to be a special case of the more general sketch-and-project method with a sparse block sketch matrix as defined in [?]. We provide the convergence of Block SKM algorithm with the sparse Gaussian sketch in the case of $k' = 1$ is presented in the following lemma:

Lemma 1 *The Block SKM algorithm with the sparse Gaussian sketch in the case of $k' = 1$ converges linearly in expectation to the nonempty feasible set of $\mathbf{B}\mathbf{x} \succcurlyeq \mathbf{b}$, $\mathbf{B} \in \mathbb{R}^{mn \times d}$ with $\epsilon = \frac{1}{2}$. Consider the sketch matrix as $\mathbf{S} = \mathbf{N}\mathbf{C}$ where $\mathbf{C} \in \mathbb{R}^{m \times n}$, $m \gg n$. Assume follows, $[\mathbf{Q}_s | \mathbf{R}_s] = \mathbf{Q}\mathbf{R}(\mathbf{S})$ where $\mathbf{Q}\mathbf{R}(\cdot)$ denotes the QR-decomposition operator. Then, the storage-friendly PrSKM algorithm converges linearly in expectation to the nonempty feasible set*

Algorithm 2 Block SKM Algorithm

Input: A block matrix $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1^\top \\ \vdots \\ \mathbf{B}_m^\top \end{bmatrix}^\top$, $\mathbb{E}\{\hat{h}(\mathbf{x}_i, \hat{\mathbf{x}})\} \leq \left(1 - \frac{\|\mathbf{B}_i\|_F^2}{3n}\right)^i \hat{h}(\mathbf{x}_0, \hat{\mathbf{x}})$, and the measurement vector $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1^\top \\ \vdots \\ \mathbf{b}_m^\top \end{bmatrix}^\top$.

Output: A solution \mathbf{x} in a nonempty feasible set of $\mathbf{Bx} \succeq \mathbf{b}$.

- 1: Choose a block \mathbf{B}_j uniformly at random with the probability $\Pr\{j = k\} = \frac{\|\mathbf{B}_k\|_F^2}{\|\mathbf{B}\|_F^2}$.
- 2: **Block SKM**

In two previous sections, we have proposed PrSKM and storage-friendly PrSKM algorithms in order to solve the one-bit polyhedron (??) in an asymptotic sample size scenario. It is worth noting that both methods are row-based approaches, where at each iteration, the row index of the matrix \mathbf{P} is chosen independently at random. However, the matrix \mathbf{P} in (??) has a block structure as formulated in (??). This fact motivates us to investigate the block-based RKA methods to find the desired signal in the one-bit polyhedron $\mathcal{P}_{\mathbf{x}}$ for further efficiency enhancement. Our proposed algorithm for block systems, Block SKM, is motivated by (i) random selection of one block at each iteration, (ii) choosing a subset of rows using the idea of Motzkin sampling, and (iii) updating the solution using the randomized block Kaczmarz method [?, ?], which takes advantage of the efficient matrix-vector multiplication, thus giving the method a significant reduction in computational cost [?]. Algorithm ?? shows the implementation of our proposed Block SKM method to solve the linear feasibility $\mathbf{Bx} \succeq \mathbf{b}$ with $\mathbf{B} = \begin{bmatrix} \mathbf{B} & \mathbf{B}^\top & \mathbf{B} \end{bmatrix}^\top$ and $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1^\top \\ \vdots \\ \mathbf{b}_m^\top \end{bmatrix}^\top$

where $\mathbf{B}_j \in \mathbb{R}^{n \times d}$ and $\mathbf{b}_j \in \mathbb{R}^n$ for all $j \in \{1, \dots, m\}$. Note that in step 4 of Algorithm ??, where c is a positive constant value, \mathbf{B}' is the $n \times d$ submatrix of \mathbf{B} (one of the candidates of the reason behind choosing $k' \leq d$ is due to the computation of $(\mathbf{B}'\mathbf{B}'^\top)^{-1}$ in the next step \mathbf{B}_j in Algorithm ??), and K is the number of update process for the linear inequalities. (step 5). For $k' > d$, the matrix $\mathbf{B}'\mathbf{B}'^\top$ is rank-deficient and its inverse is not available. The Block SKM algorithm can be considered to be a special case of the more general sketch-and-project method with a sparse block sketch matrix as defined in [?]. The convergence of the RKA (i.e., SKM, Block SKM and etc.) throughout the paper, we represent each iterate of the update process for RKA, PrSKM, and Block SKM, aimed at solving the feasibility problem

$\mathbf{Bx} \succeq \mathbf{b}$, as $\text{KA}_r(\mathbf{x})$, $\text{KA}_p(\mathbf{x})$, and $\text{KA}_b(\mathbf{x})$, respectively. To examine the performance of the Lemma 1, The Block SKM algorithm with the sparse Gaussian sketch in the case of $k' = 1$, Block SKM, we will compare it with the PrSKM, SKM and RKA. converges linearly in expectation to the nonempty feasible set of $\mathbf{Bx} \succeq \mathbf{b}$, $\mathbf{B} \in \mathbb{R}^{mn \times d}$, as follows.

G. Comparing RKA, SKM, PrSKM and Block SKM

In this section, we numerically compare the RKA, SKM, PrSKM, and Block SKM in linear system of inequalities. Accordingly, we utilize ORKA to make a linear equation $\mathbf{Bx} = \mathbf{y}$ a linear

Algorithm 2 Block SKM Algorithm

Input: A block matrix $\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_m \end{bmatrix}^\top$, and the measurement vector $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1^\top & \cdots & \mathbf{b}_m^\top \end{bmatrix}^\top$.

Output: A solution $\bar{\mathbf{x}}$ in a nonempty feasible set of $\mathbf{B}\mathbf{x} \succeq \mathbf{b}$.

1: Choose a block \mathbf{B}_j uniformly at random with the probability $\Pr\{j = k\} = \frac{\|\mathbf{B}_k\|_F^2}{\|\mathbf{B}\|_F^2}$.

2: $\mathbf{e} \leftarrow \mathbf{B}_j \mathbf{x} - \mathbf{b}_j$.

3: $\mathbf{e}' \leftarrow \text{sort}(\mathbf{e}) \triangleright \text{sort}(\cdot)$ is the operator that sorts the elements of the vector \mathbf{e} from e_{\max} (the maximum element of \mathbf{e}) to e_{\min} (the minimum element of \mathbf{e}). This step is inspired by the idea of the Motzkin sampling, presented in Section ??, to have an accelerated

Figure 1. Comparing the recovery performance of the two proposed Kaczmarz algorithms, namely the PrSKM and the Block SKM, with that of SKM and RKA for a linear inequality system.

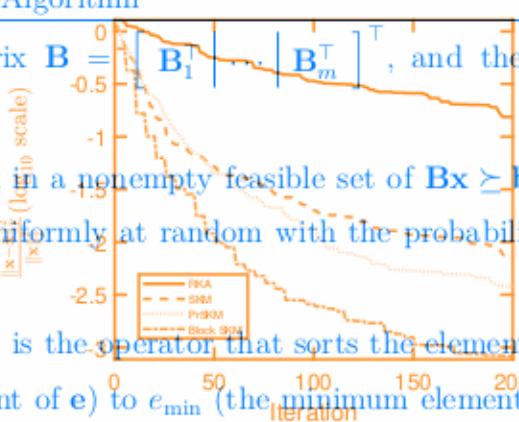
4: $[\mathbf{B}'_j, \mathbf{b}'_j] = \text{select}_{k'}(\mathbf{B}_j, \mathbf{b}_j, \mathbf{e}') \triangleright \text{select}_{k'}(\cdot)$ is the operator which selects the first $k' < d$ inequality system, where the number of bits problem \mathbf{B}'_j sampling the \mathbf{B}_j sequentially and $\mathbf{b}'_j \in \mathbb{R}^{10}$. $\mathbf{B}'_j \leftarrow \mathbf{B}'_j \mathbf{x} (\mathbf{B}'_j \mathbf{B}'_j^\top)^{-1}$ and $\mathbf{y} \in \mathbb{R}^{10}$ is each Moore-Penrose generalized inverse of $\mathbf{B}'_j(0, \mathbf{I}_{10})$. Also, the desired signal \mathbf{x} is generated $\mathbf{x} \sim \mathcal{N}(0, \mathbf{I}_{10})$. All time-varying sampling threshold sequences are generated according to until convergence $\{\mathbf{e}_t\}_{t=1}^m$. The performance of the RKA, SKM, PrSKM, and Block SKM is illustrated in Fig. ???. Similar to the linear feasibility of equalities, it can be seen that the Block SKM has a better accuracy in recovering the desired signal \mathbf{x} in the one-bit polyhedron (??) compared to the other three approaches. The NMSE results in Fig. ?? where c is a positive constant value, \mathbf{B} is the $n \times d$ submatrix of \mathbf{B} (one of the candidates are averaged over 1000 experiments of \mathbf{B}_j in Algorithm ??), and K is the number of update process for the linear inequalities.

IV. PROBABILISTIC EFFECT OF SAMPLE ABUNDANCE IN ONE-BIT SENSING
The proof of Lemma ?? is provided in Appendix ???. In the rest of the paper, we will derive all theoretical guarantees for the RKA. Similar guarantees can also be derived for the variants of the RKA, i.e. SKM, Block SKM and etc. Throughout the paper, we represent one-bit CS and one-bit low-rank matrix recovery. In Section ??, we will provide the convergence of ORKA based on the theoretical results obtained in Section ???. To examine the performance of the Block SKM, we will compare it with the PrSKM, SKM and RKA.

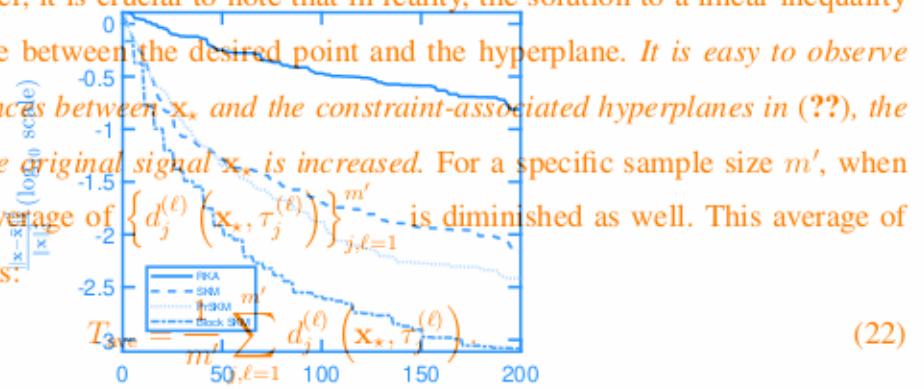
A. Finite Volume Property

G. Comparing RKA, SKM, PrSKM and the j -th hyperplane presented in (???)

as In this section, we numerically compare the RKA, SKM, PrSKM, and Block SKM in linear system of inequalities. Accordingly, we utilize ORKA to make a linear equation $\mathbf{B}\mathbf{x} = \mathbf{y}$ a linear inequality classify that the analysis of adopting the stopping criterion for the distances between $\mathbf{B}\mathbf{x}$ desired point and the solution \mathbf{R} . This approach of Besides that solution to lie in $\Delta_p(0, c)$ is



on the hyperplane. However, it is crucial to note that in reality, the solution to a linear inequality system can exist anywhere between the desired point and the hyperplane. It is easy to observe that by reducing the distances between \mathbf{x}_* and the constraint-associated hyperplanes in (??), the possibility of capturing the original signal \mathbf{x}_* is increased. For a specific sample size m' , when $\text{vol}(\mathcal{P}_{\mathbf{x}})$ is reduced, the average of $\left\{ d_j^{(\ell)} \left(\mathbf{x}_*, \tau_j^{(\ell)} \right) \right\}_{j,\ell=1}^{m'}$ is diminished as well. This average of distances can be written as:



The one-bit phase retrieval problem, as investigated in [?], derived a *general Hoeffding's bound* [?, Theorem 2.6.2] to quantify the likelihood of achieving a finite volume and determine the necessary number of samples for one-bit signal reconstruction. In Theorem ??, we utilize this

result to address the problem of one-bit sensing. Specifically, we consider the distance between the original signal \mathbf{x}_* and the j -th hyperplane within the polyhedron defined in (??), as described sequences are generated according to $\{\tau^{(\ell)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{10})\}_{\ell=1}^m$. The performance of the RKA, in (??).

SKM, PrSKM, and Block SKM is illustrated in Fig. ???. Similar to the linear feasibility Theorem 3 (Finite volume property (FVP)). Assume the distances $\left\{ d_j^{(\ell)} \left(\mathbf{x}_*, \tau_j^{(\ell)} \right) \right\}_{j,\ell=1}^{m'}$ defined in (??) between the desired point \mathbf{x}_* and the hyperplanes of the one-bit polyhedron (??) are independent random variables with $\mathbb{E}(T_{\text{ave}}) = \mu$ and $\left\| d_j^{(\ell)} \left(\mathbf{x}_*, \tau_j^{(\ell)} \right) \right\|_{\psi_2}^2 \leq K$. Then, based on the general Hoeffding's inequality, the probability of the finite volume created by hyperplanes lying within a ball $B_{\rho}(\mathbf{x}_*)$ centered at the original signal, with a radius of ρ , is bounded by:

In Section ??, we will introduce the concept of FVP and subsequently obtain the required number of one-bit samples m' to accurately capture the solution in the case of sample abundance, one-bit CS, and one-bit low-rank matrix recovery. In Section ??, we will provide the convergence of ORKA based on the theoretical results obtained in Section ??.

Theorem ?? provides the probability of the finite volume, created by hyperplanes, being contained within the ball around the original signal. The positive number C serves to consider the A. Finite Volume Property

distances beyond the radius ρ of the ball. These distances correspond to ineffective hyperplanes

that are incapable of forming a finite volume around the original signal. Mathematically, based

on the general Hoeffding's inequality presented in (??), it follows $\delta = C\rho - \mu$ for a positive constant δ . We utilize this constant, δ , in all the theoretical guarantees that will be obtained in

this paper. In the remainder of the paper, our objective is to derive theoretical guarantees that enable us to achieve the uniform perfect reconstruction, as defined in the following definition. This approach considers the solution to lie precisely on the hyperplane. However, it is crucial to note that in reality, the solution to a

Definition 1 (Uniform system perfect reconstruction) **b** Uniform perfect reconstruction is defined as follows. In this polyhedron \mathcal{P}_x , the set of all possible recovery solutions for $\bar{x} \in \mathcal{B}_\rho(\mathbf{x}_*)$ with associated hyperplanes in (??), the possibility of capturing the original signal \mathbf{x}_* is increased. For a specific sample size m' , when $\text{vol}(\mathcal{P}_x)$ is reduced, the average of $\left\{d_j^{(\ell)}(\mathbf{x}_*, \tau_j^{(\ell)})\right\}_{j,\ell=1}^{m'}$ is satisfied $\bar{x} \in \mathcal{B}_\rho(\mathbf{x}_*)$ for all \mathbf{x}_* in the space. This average of distances can be written as:

Note that the sign preservation $T_{\text{ave}} = \frac{1}{m'} \sum_{j=1}^{m'} d_j^{(1)}(\mathbf{x}_*, \tau_j^{(1)})$ mentioned in our work, as stated in Definition 1, is equivalent to the solution provided by a linear feasibility solver, denoted as \bar{x} , satisfying all the given inequalities, meaning $\bar{x} \in \mathcal{P}_x$. To fulfill Definition 1, it appears sufficient to acquire the number of samples that ensures the creation of a finite volume of intersections of hyperplanes with the maximum radius ρ . This central idea underlies all of our theorems. In the following theorem, we utilize this result to address the problem of one-bit sensing. Specifically, we consider the distance between the original signal \mathbf{x}_* and the j -th hyperplane within the polyhedron defined in (??), as described in (??).

Theorem 4. Assume a $n \times d$ sampling matrix \mathbf{A} such that each distance defined in (??), is Theorem 3 (Finite volume property (FVP)). Assume the distances $\left\{d_j^{(\ell)}(\mathbf{x}_*, \tau_j^{(\ell)})\right\}_{j,\ell=1}^m$ defined in (??) between the desired point \mathbf{x}_* and the hyperplanes of the one-bit polyhedron (??) are independent random variables with $\mathbb{E}\{T_{\text{ave}}\} = \mu$ and $\left\|d_j^{(\ell)}(\mathbf{x}_*, \tau_j^{(\ell)})\right\|^2$ varying sampling threshold sequences. Denote δ and C_1 as positive constants. If the number of one-bit samples obeys

hyperplanes lying within a ball $\mathcal{B}_\rho(\mathbf{x}_*)$ centered at the original signal, with a radius of (25) is bounded by:

then with a probability of at least $1 - \eta$, we achieve the uniform perfect reconstruction with $\bar{x} \in \mathcal{B}_\rho(\mathbf{x}_*)$.

where C and c_1 are positive numbers.

The proof of Theorem ?? is presented in Appendix ???. Following the discussion provided in Theorem ?? provides the probability of the finite volume created by hyperplanes, being contained within the ball around the original signal. The positive number C serves to Corollary ?? presents the result of Theorem ?? in the case of the DCT sensing matrix \mathbf{A} and consider the distances beyond the radius ρ of the ball. These distances correspond to ineffective hyperplanes that are incapable of forming a finite volume around the original

Corollary 1. Consider a $n \times d$ DCT matrix \mathbf{A} and the time-varying thresholds which are generated according to $\{\tau_j^{(\ell)} \sim \mathcal{U}(-\tilde{b}, \tilde{b})\}_{j,\ell=1}^m$ such that $\tilde{b} > 0$. Denote f_j as the theoretical coefficient of \mathbf{x}_* . If we have guarantees that will be obtained in this paper. In the remainder of the paper, our objective is to derive theoretical guarantees that enable us to achieve the uniform perfect reconstruction as defined in the following definition:

Definition 1 (Probability of perfect reconstruction) *Under the perfect reconstruction of two signals from one-bit and dithered CDT, is a given point value possible recovery solutions for $\bar{\mathbf{x}} \in \mathcal{P}_{\mathbf{x}}$ with*

$$\operatorname{sgn}(\mathbf{a}_j \mathbf{x}_* - \sum_{j=1}^{n(\ell)} b_j) = \tilde{b} \operatorname{sgn}(\mathbf{a}_j \bar{\mathbf{x}} - \tilde{b}) \tau_j^{\ell} + \tilde{b} \in [n], \quad \ell \in [m], \quad (24)$$

$$(27)$$

satisfy $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$, for all \mathbf{x}_* in the space.

$$+ I(f_j \geq b)(2\tilde{b}f_j) + I(f_j \leq -\tilde{b})(-2\tilde{b}f_j).$$

Note that the sign preservation mentioned in our work as stated in Definition 1 is equivalent to the solution provided by a linear feasibility solver, denoted as $\bar{\mathbf{x}}$, satisfying the result of Theorem ?? for the Gaussian sampling matrix with Gaussian dithering scheme:

all the given inequalities, meaning $\bar{\mathbf{x}} \in \mathcal{P}_{\mathbf{x}}$. To fulfill Definition 1, it appears sufficient to **Corollary 2.** Consider a $n \times d$ sampling matrix $\mathbf{A} = [\mathbf{a}_j]$ with a_{ij} independently drawn from the acquire the number of samples that ensures the creation of a finite volume of intersections of standard Gaussian distribution, $\mathcal{N}(0, 1)$, and the time-varying thresholds which are generated of hyperplanes with the maximum radius ρ . This central idea underlies all of our theorems. according to $\{\tau_j^{\ell} \sim \mathcal{N}(0, \sigma^2)\}_{j=1}^m$. If we have.

In the following theorem, we establish the minimum number of one-bit samples m' required

in the sample abundance scenario to achieve an accurate recovery:

$$m' \geq C_1 \delta^{-2} (\sigma^2 + 1) \left(\frac{3d}{\rho} + \log \left(\frac{1}{\eta} \right) \right), \quad (28)$$

Theorem 4. Assume a $n \times d$ sampling matrix \mathbf{A} , such that each distance defined in (??) is a then with a probability of at least $1 - \eta$, we achieve the uniform perfect reconstruction with sub-Gaussian random variable, and the one-bit sampling matrix $\mathbf{P} = \tilde{\Omega} \mathbf{A} \in \mathbb{R}^{m' \times d}$, where $\mathbf{x} \in \mathcal{B}_\rho(\mathbf{x}_*)$, and $\delta = C\rho - \mu$ for a positive value C with $\rho \leq \sqrt{\frac{1}{\pi}(\sigma^2 + 1)}$.

$m' = mn$ denotes the total number of one-bit samples, and m represents the number of

The proof of Corollary ?? is presented in Appendix ?? It is important to note that in the time-varying sampling threshold sequences. Denote δ and C_1 as positive constants. If the absence of dithering, creating a finite volume around the desired solution becomes impossible, rendering the FVP ineffective. Furthermore, in such scenarios, certain information, such as the signal amplitude, may be lost. However, in specific circumstances where the signal is s -sparse

then the probability of at least one sampling achieve the uniform perfect reconstruction with Theorem ??.

The existing literature provides numerous theoretical derivations for the case where the input signal possesses an s -sparse structure, primarily focusing on the zero threshold scenario (e.g., in [?] regarding the benefit of uniform dithering in the case of bounded measurements, the [?, Theorem 2]). However, in order to determine the minimum number of samples required for Corollary ?? presents the result of Theorem ?? in the case of the DCT sensing matrix \mathbf{A} achieving perfect reconstruction of sparse signals, we need to introduce the distance within the and uniform dither:

s -dimensional space. Let us consider an s -support column-submatrix of the sampling matrix,

Corollary 1A Consider a $n \times d$ DCT matrix \mathbf{A} represented by a varying $\mathbf{a}_j^{(s)}$ and thresholds $\tau_j^{(s)}$, we have an generated by a fixed \mathbf{U} denoted as $\mathbf{x} \in \mathcal{U}(\tilde{\mathbf{b}}, \tilde{\mathbf{d}})$. The distances between the s -sparse original signal and the hyperplane of \mathbf{x}_* given by

$$d_j^{(s)} \left(m_* \geq \frac{1}{2} \delta^{-2} \left(\tilde{b} \left| \tau_j^{(s)} \sqrt{a_j^{(s)}} \right|^2 \mathbf{x}_* \left(\frac{3d}{\rho} + \log \left(\frac{1}{\eta} \right) \right) \right) \right) \in [d]. \quad (29)$$

Then import probability that at least s samples signal lie we have a total perfect possible choices with the distance, and other words, for the positive value support problem, we can select $\binom{d}{s}$ different submatrices \mathbf{A}_s from the sampling matrix. By utilizing the general Hoeffding's inequality along with the union bound for these $\binom{d}{s}$ possibilities, the following theorem provides the necessary number of one-bit samples to achieve uniform perfect reconstruction of a s -sparse signal:

Theorem 5 Assume a $n \times d$ sampling matrix \mathbf{A} such that each distance defined in (27), is a Gaussian random variable. Assume that the signal is s -sparse and \mathbf{x}_* is the ball centered at the original signal $\mathbf{x}_* \in \mathbb{R}^d$. Consider a fixed $0 < \eta < 1$ and δ_s to be a positive constant value. Then the minimum number of one-bit samples m' required to accurately capture a s -sparse solution $\bar{\mathbf{x}}$, satisfies

from the standard Gaussian distribution, $\mathcal{N}(0, 1)$, and the time-varying thresholds which are generated according to $\tau_j^{(t)} \sim \mathcal{N}(0, \sigma^2)$ for $j, t = 1, \dots, m'$

$$m' \geq C_1 \delta^{-2} (\sigma^2 + 1) \left(\frac{3d}{\rho} + \log \left(\frac{1}{\eta} \right) \right), \quad (28)$$

The proof of Theorem ?? is provided in Appendix ?? In the following corollary, we provide then with a probability of at least $1 - \eta$, we achieve the uniform perfect reconstruction with the result of Theorem ?? in the case of Gaussian sensing matrix with a Gaussian dithering scheme:

The proof of Corollary ?? is presented in Appendix ?? It is important to note that in the **Corollary 3**. Assume a $n \times d$ sampling matrix $\mathbf{A} = [a_{ij}]$ with a_{ij} independently drawn from the standard Gaussian distribution, $\mathcal{N}(0, 1)$, and the time-varying thresholds which are generated according to $\tau_j^{(t)} \sim \mathcal{N}(0, \sigma^2)$ for $j, t = 1, \dots, m'$. However, in specific circumstances where the signal is s -sparse and confined within a unit ball some sampling theorems have been derived in the literature [?, Theorem 2],

with $s \ll d$. Denote $\delta = C\rho - \sqrt{\frac{2}{\pi}(\sigma^2 + 1)}$ for a positive value C and $0 < \rho < 1$. For a positive constant C_1 , if $m' \geq C_1 \delta^{-2} (\sigma^2 + 1) \left(\log \left(\frac{1}{\eta} \right) + s \left(\log \left(\frac{ed}{s} \right) + \frac{3}{2} \right) \right)$, then with probability of input signal possesses an s -sparse structure, primarily focusing on the zero threshold scenario at least $1 - \eta$, and for all $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{H}_{d,s}$ we achieve the uniform perfect reconstruction with (e.g., [?, Theorem 2]). However, in order to determine the minimum number of samples $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$.

required for achieving perfect reconstruction of sparse signals, we need to introduce the distance within the d -dimensional space. Let us consider an off-diagonal column of the matrix of the sampling matrix denoted as \mathbf{A}_j . The rows of this submatrix are represented by $\mathbf{a}_j^{(s)}$. Additionally, we have \mathbf{x}_* from a set of possibilities, such that $\mathbf{x}_* \in \mathcal{X}_k$. The distance between the sparse original signal and the hyperplane \mathcal{H}_2 , previous studies, such as [?, Lemma 3.1], have demonstrated that $d_j^{(K)}(\mathbf{x}_*, \mathbf{a}_j^{(s)}) = \left| r_j^{(s)} \mathbf{a}_j^{(s)} \mathbf{x}_*^{(s)} - r_j^{(s)} b_j^{(s)} \right|$, $s \in [d]$.

Hoeffding's inequality with the union bound applied to all $\binom{d}{s}$ possible submatrices for the following theorem. In other words, in the case of a s -sparse problem, we can select $\binom{d}{s}$ different

submatrices \mathbf{A}_s from the sampling matrix. By utilizing the general Hoeffding's inequality **Theorem 6.** Assume a $n \times n_1^2$ sampling matrix \mathbf{A} such that each distance defined in (??) is along with the union bound for these $\binom{d}{s}$ possibilities, the following theorem provides the a sub-Gaussian random variable. Assume that the desired signal is r -rank matrix and ρ is the necessary number of one-bit samples to achieve uniform perfect reconstruction of a s -sparse radius of a ball centered at the original signal $\mathbf{X}_* \in \mathbb{R}^{n_1 \times n_1}$. Consider a fixed $0 < \eta < 1$ and signal:

δ_r to be a positive constant value. Then the minimum number of one-bit samples m' required to

Theorem 5 captures one-rank solution implies matrix \mathbf{A} such that each distance defined in (??) is

a sub-Gaussian random variable $m' \geq \delta_r \left(\frac{18\rho r}{\rho} + \log \left(\frac{1}{\eta} \right) \right)$, signal is s -sparse and ρ is the radius of a ball centered at the original signal $\mathbf{x}_* \in \mathbb{R}^d$. Consider a fixed $0 < \eta < 1$ and δ_s with a probability higher than $1 - \eta$.

to be a positive constant value. Then the minimum number of one-bit samples m' required

The proof of Theorem ?? is provided in Appendix ??.

In the following corollary, we provide the result of Theorem ?? in the case of Gaussian sensing matrix with a Gaussian dithering scheme:

$$m' \geq \delta_s \left(\log \left(\frac{1}{\eta} \right) + s \left(\log \left(\frac{1}{s} \right) + \frac{3}{\rho} \right) \right), \quad (30)$$

with a probability higher than $1 - \eta$.

Corollary 4. Assume a $n \times n_1^2$ sampling matrix $\mathbf{A} = [a_{ij}]$ with a_{ij} independently drawn from the standard Gaussian distribution, $\mathcal{N}(0, 1)$, and the time-varying thresholds which are generated according to $\{\tau_j \sim \mathcal{N}(0, \sigma^2)\}_{j=1}^r$. Define the set $\mathcal{K}_{n_1, r}$ as

$$\mathcal{K}_{n_1, r} = \{ \mathbf{X} \in \mathbb{R}^{n_1 \times n_1} \mid \text{rank}(\mathbf{X}) \leq r, \|\mathbf{X}\|_F \leq 1 \}. \quad (33)$$

Denote $\delta = C\rho - \sqrt{\frac{2}{\pi}(\sigma^2 + 1)}$ for a positive value C and $0 < \rho < 1$. For a positive constant C_1 , if $m \geq C_1 \delta^{-2} (\sigma^2 + 1) \left(\frac{18\rho r}{\rho} + \log \left(\frac{1}{\eta} \right) \right) m'$, then with a probability of at least $1 - \eta$, and for generated according to $\{\tau_j \sim \mathcal{N}(0, \sigma^2)\}_{j=1}^r$. Define the set $\mathcal{H}_{d, s}$ as all $\mathbf{X}_*, \mathbf{X} \in \mathcal{K}_{n_1, r}$, we achieve the uniform perfect reconstruction with $\mathbf{X} \in \mathcal{B}_\rho(\text{vec}(\mathbf{X}_*))$.

$$\mathcal{H}_{d, s} = \{ \mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_1 \leq \sqrt{s}, \|\mathbf{x}\|_2 \leq 1 \}. \quad (31)$$

The proof of Corollary ?? follows directly from the proof of Theorem ?? and the proof of Corollary ?? Due to the aforementioned structure of the signal, positive sparsity or low rank $\geq C_1 \delta^{-2} (\sigma^2 + 1) \left(\log \left(\frac{1}{\eta} \right) + \text{probability of } \frac{3}{\rho} \right)$, creating with probability and the original signal using one-bit hypothesis planes. This structure and perfect reconstruction in the required number of samples for achieving accurate reconstruction. Note that Theorems ??, ??, and ?? present uniform reconstruction results, indicating that with high probability, all vectors

The proof of Corollary ?? follows directly from the proof of Theorem ?? and the proof of can be reconstructed. This differs from a nonuniform result where each vector is individually Corollary ?? To investigate the necessary number of one-bit samples for r -rank matrices, reconstructed with high probability. one can identify a value $k \in [K]$ from a set of K possibilities, such that \mathbf{X}_k lies within a ball

To have a robust recovery performance in one-bit signal sensing, it is necessary to design centered at the original signal with a radius of ρ . Assume $n_1 = n_2$, previous studies, such the time-varying thresholds such that the dynamic range (DR) of such thresholds covers the DR

of the high-rank but jointed measurements in the following proposition inequality $K \leq (1 + \frac{6}{\rho})^{(2n+1)r}$. By combining getting general RKA_r from Gaussian inequality with the dithering bound applied to all K possibilities, we can establish the following theorem:

Lemma 2. Assume a $n \times d$ sub-Gaussian sensing matrix \mathbf{A} with the rows $\{\mathbf{a}_j\}_{j=1}^n$ and the time-varying thresholds which are generated sampling matrix \mathbf{A} such that (each) $\{\mathbf{a}_j\}_{j=1}^n$ and $\{\tilde{b}_j\}_{j=1}^n$ defined in $(\tilde{\Omega}, \tilde{b})$ is $\mathcal{N}(0, 1)$. Then high-resolution random variables $\tilde{\Omega}$ represent the desired Asymptotic-Krank matrix \mathbf{A} and \tilde{b} . If the radius of a shallower centered $\mathcal{O}\left(\sqrt{\frac{8}{3}}KR\right)$ in the original Gaussian dither and consider $\mathcal{O}(\log(2)KR)$ in the Uniform dither a positive effect the same value. Then the minimum number of one-bit samples m' required to precisely capture a r -rank solution, satisfies

The proof of Lemma ?? is presented in Appendix ???. In the case of the Gaussian sensing matrix $\mathbf{A} = [a_{ij}]$ with a_{ij} independently drawn from the standard Gaussian distribution, $\mathcal{N}(0, 1)$, the result of Lemma ?? is simplified to $\sigma = \mathcal{O}(R)$ in the Gaussian dither and $\tilde{b} = \mathcal{O}\left(R \log(2) \sqrt{\frac{8}{3}}\right)$ with a probability higher than $1 - \eta$.

The proof of Theorem ?? is provided in Appendix ???. In the following corollary, we provide the result of Theorem ?? in the case of Gaussian sensing matrix with a Gaussian dithering scheme:

Corollary 4. Assume a $n \times n_1^2$ sampling matrix $\mathbf{A} = [a_{ij}]$ with a_{ij} independently drawn from the standard Gaussian distribution, $\mathcal{N}(0, 1)$, and the time-varying thresholds which are generated according to $\{\tilde{b}_j\}_{j=1}^{m'} \sim \mathcal{N}(0, \sigma^2)$. Define the set $\mathcal{K}_{n_1, r}$ as

As the scaled condition number is the central parameter governing the recovery error of the RKA or its variants, in the following theorem we will evaluate the scaled condition number of the matrix \mathbf{P} defined in (??) to unveil the connection between the convergence bounds of the RKA (or its variants) and ORKA.

Theorem 7. Consider the one-bit polyhedron (??) associated with the linear system of equations for all $\mathbf{X}_*, \mathbf{X} \in \mathcal{K}_{n_1, r}$, we achieve the uniform perfect reconstruction with $\mathbf{X} \in \mathcal{B}_\rho(\text{vec}(\mathbf{X}_*))$. $\mathbf{A}\mathbf{x} = \mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{n \times d}$ and m denotes the number of time-varying sampling threshold sequences. Then, the scaled condition number of the matrix \mathbf{P} is equal to that of \mathbf{A} ,

The proof of Corollary ?? follows directly from the proof of Theorem ?? and the proof of Corollary ???. As per the aforementioned theorems, the specific structure of the signal, such as sparsity or low-rank, has a direct impact on the probability of creating a finite volume around the original signal using one-bit hyperplanes. This structural characteristic

The proof of Theorem ?? is provided in Appendix ???. Considering $\mathbf{A} = \mathbf{I}$ corresponds to manifests itself in the required number of samples for achieving accurate reconstruction. the one-bit sampled signal sensing problem with $\mathbf{P} = \bar{\Omega}$ as formulated in (??). In the following Note that Theorems ??, ??, and ?? present uniform reconstruction results, indicating that corollary, we present the scaled condition number of the matrix $\bar{\Omega}$ in the light of Theorem ??; with high probability, all vectors can be reconstructed. This differs from a nonuniform result

where each vector is individually reconstructed with high probability.

Corollary 5. *For a recovery performance threshold $\tilde{\Omega}$ sampled using a reconstruction topology design function \tilde{h} (varying), thresholds such that the boundary of the range $\left[\Omega^{(1)}(\text{DR}) \mid \Omega^{(2)}\right]^\top \tilde{\Omega}^{\text{max}}$ thresholds covers the $\Omega(\text{DR})$ of the high-resolution measurements condition following proposition, in the case our analysis to the sub-Gaussian sensing matrix with both Gaussian and Uniform dithering:*

Based on Theorem ?? and the convergence bound of the RKA, it is concluded that ORKA converges to the feasible set of PC Gaussian (Sensing) if it is within the threshold depends on the sampling-varying \tilde{h} . This follows the convergence bound of ORKA to independent $\mathcal{N}(0, \sigma^2)$ number of time-varying \tilde{h}_j sampling threshold sequences which means it represents takes into account the effect of increasing the number of time-varying threshold sequences. Our objective is to recover $\hat{\mathbf{x}} = \max_j \|\mathbf{a}_j \hat{\mathbf{x}}\|_{\ell_2^2}$. If $\|\mathbf{x}\|_2 \leq R$, then setting $\sigma = \mathcal{O}(\sqrt{s} R H)$ in the Gaussian dither and $\tilde{b} = \mathcal{O}(\log(2) R H)$ in the Uniform dither guarantee the cover property.

From a mathematical perspective, we seek a signal $\bar{\mathbf{x}}$ that lies within a ball space centered

The proof of Lemma ?? is presented in Appendix ???. In the case of the Gaussian sensing at the desired signal, characterized by a small radius ρ . In other words, we aim for \mathbf{x} to belong matrix $\mathbf{A} = [a_{ij}]$ with a_{ij} independently drawn from the standard Gaussian distribution, to the ball space $\mathcal{B}_\rho(\mathbf{x}_*)$. To achieve that, we are required to increase the number of samples in $\mathcal{N}(0, 1)$, the result of Lemma ?? is simplified to $\sigma = \mathcal{O}(R)$ in the Gaussian dither and the polyhedron (??), as typically provided via one-bit sensing, to make a non-linear constraint $\tilde{b} = \mathcal{O}(R \log(2) \sqrt{\frac{s}{\rho}})$ in the Uniform dither. For the example provided in Corollary ?? regarding the DCT sampling matrix and uniform dithering, we should have $\tilde{b} = \mathcal{O}(\sqrt{2})$ to RKA appears to be insufficient since we must have enough number of samples to fulfill costly cover the DR of high-resolution measurements (DCT coefficients). The reason behind this constraints (or equivalently, we must have enough number of samples so that the finite-volume can be seen in the proof of Corollary ??, space created by the intersection of hyperplanes of (??) be inside the desired signal's ball space

$\mathcal{B}_\rho(\mathbf{x}_*)$. Define $\text{vol}(\mathcal{P}_x)$ as the volume space created by the intersection of hyperplanes in (??).

B. Recovery Error Upper Bound for ORKA

In the following proposition, we present the convergence rate of ORKA:

As the scaled condition number is the central parameter governing the recovery error

Proposition 1. *(Convergence rate of ORKA) Consider the one-bit polyhedron defined on the number of sampling matrix \mathbf{P} defined in system (??) equivalent to the connection between $\mathbb{R}^{n \times d}$. Then, a convergence bound of the original signal $\mathcal{B}_\rho(\mathbf{x}_*)$ and ORKA \mathcal{P}_x , a convergence rate for ORKA may be formulated as:*

Theorem 7. Consider the one-bit polyhedron (??) associated with the linear system of equations $\mathbf{Ax} = \mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{m \times d}$ and m denotes the number of time-varying sampling threshold sequences. Then, the scaled condition number of the matrix \mathbf{P} is equal to that of \mathbf{A} ,

Proof: As demonstrated in [?], the solution obtained from RKA lies in the space formed by the hyperplanes of the linear inequality problem with the following convergence rate:

$$\kappa(\mathbf{P}) = \kappa(\mathbf{A}). \quad (34)$$

$$\mathbb{E}\{\tilde{h}(\mathbf{x}_i, \hat{\mathbf{x}})\} \leq \left(1 - \frac{1}{\kappa(\mathbf{A})}\right)^i \tilde{h}(\mathbf{x}_0, \hat{\mathbf{x}}). \quad (35)$$

The proof of Theorem ?? is provided in Appendix ???. Considering $\mathbf{A} = \mathbf{I}$ corresponds to the one-bit sampled signal sensing problem with $\mathbf{P} = \tilde{\Omega}$ as formulated in (??). In the

following is a point inside the space created by one polyhedron \mathcal{P}_x of the conic region Ω in the ORKA. This means that the solution will lie within the space created by the hyperplanes, not necessarily

within the ball around the desired solution. However, in order to guarantee a perfect reconstruction, Corollary 5. For $\mathbf{A} = \mathbf{I}$, corresponding to the one-bit sampled signal reconstruction problem, and ensure that the solution of the linear feasibility problem lies within the ball around the desired solution with radius ρ , it is essential to have a sufficient number of samples. Until we reach the required number of samples, the upper bound of convergence to a solution lying within the ball around the desired point, is given by $\text{vol}(\mathcal{P}_x) \leq \text{vol}(\mathcal{B}_\rho)$. However, conclude obtain

a sufficient number of samples to have of the volume (\mathbf{P}_x) created by the intersection of hyperplanes inside the sampling matrix (\mathbf{A}). Therefore, we can then have only ORKA at the independent of the midpoint of time, varying sampling thresholds discrepancy between the two scenarios take into account the effect of this depending on the difference between threshold, as follows: Our objective is to recover the signal $\hat{\mathbf{x}}$ within the polyhedron defined in (??), aiming for its proximity to the original signal \mathbf{x}_* . From a mathematical perspective, we seek a signal $\hat{\mathbf{x}}$

that lies within a ball space centered at the desired signal, characterized by a small radius where from (??) and the fact that the error between $\hat{\mathbf{x}}$ and the original signal remains deterministic ρ . In other words, we aim for $\hat{\mathbf{x}}$ to belong to the ball space $\mathcal{B}_\rho(\mathbf{x}_*)$. To achieve that, we are with respect to each iteration, we can write

required to increase the number of samples in the polyhedron (??), as typically provided via one-bit sensing to make the non-linear constraint redundant in the original problem (38)

interest. In such a case, the offered convergence rate for the RKA appears to be insufficient. The convergence to \mathbf{x}_* is ensured only when the second term, $\|\mathbf{x}_* - \hat{\mathbf{x}}\|_2^2$, is bounded. As since we must have enough number of samples to fulfill costly constraints (or equivalently, demonstrated in Theorem ??, with a minimum probability of $1 - e^{-\frac{1}{\kappa^2} m^2}$, we establish that we must have enough number of samples so that the finite-volume space created by the $\text{vol}(\mathcal{P}_x) \subseteq \text{vol}(\mathcal{B}_\rho(\mathbf{x}_*))$ and consequently have $\|\mathbf{x}_* - \hat{\mathbf{x}}\|_2^2 \leq \rho^2$, which proves the proposition. intersection of hyperplanes of (??) be inside the desired signal's ball space $\mathcal{B}_\rho(\mathbf{x}_*)$. Define

$\text{vol}(\mathcal{P}_x)$ as the volume space created by the intersection of hyperplanes in (??). In the

In the following corollary, we present the required number of iterations i such that ORKA following proposition, we present the convergence rate of ORKA: obtains an upper recovery bound $\mathbb{E}\{\|\mathbf{x}_i - \mathbf{x}_*\|_2^2\} \leq \epsilon_0$ at the i -th iteration:

Proposition 1 (Convergence rate of ORKA). Consider the one-bit polyhedron \mathcal{P}_x obtained Corollary 6. Based on the assumptions in Proposition ??, ORKA meets an upper recovery bound in (??) associated with the linear system of equations $\mathbf{Ax} = \mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{n \times d}$. Consider a ball centered at the original signal $\mathcal{B}_\rho(\mathbf{x}_*)$ and $\hat{\mathbf{x}} \in \mathcal{P}_x$, a convergence rate for ORKA may be formulated as:

$$\mathbb{E}\{\|\mathbf{x}_i - \mathbf{x}_*\|_2^2\} \leq \left(\frac{1}{\kappa^2(\mathbf{A})}\right)^i h(\mathbf{x}_0, \hat{\mathbf{x}}) + \rho^2, \quad (35)$$

Based on (??) and its proof, it is evident that the constant δ_s contains ρ^{-2} , which leads to the conclusion that $\rho = \mathcal{O}(m^{-\frac{2}{3}})$ and $\rho^2 = \mathcal{O}(m'^{-\frac{2}{3}})$. As a result, the upper bound of ORKA

error probability with a rate of $\mathcal{O}\left(\frac{1}{m}\right)^{\frac{2}{\kappa^2}}$, the solution is obtained from RKA if it satisfies the Span condition. Note that the hypothesis of the bounding inequality (??) is violated with the following consequence: the constant δ_r also contains ρ^{-2} , the relation includes ρ as well. Hence, the upper bound of ORKA for low-rank matrix sensing decays with a rate of $\mathcal{O}\left(\frac{1}{m}\right)^{\frac{2}{\kappa^2}}$.

Note that if any other randomized algorithm is utilized for one-bit sensing instead of RKA only ensures that the solution will lie within the space created by the hyperplanes, not to achieve uniform reconstruction, the convergence rate will maintain the same structure as Proposition ???. However, there will be a difference in the first term, which will be substituted by the algorithm's convergence rate to a point inside the feasible space of hyperplanes. The convergence rate (??) where $\hat{\mathbf{x}}$ is a point inside the space created by a polyhedron \mathcal{P}_x . The convergence rate (??) only ensures that the solution will lie within the space created by the hyperplanes, not necessarily within the ball around the desired solution. However, in order to guarantee a perfect reconstruction and ensure that the solution of the linear feasibility problem lies within the ball around the desired solution with radius ρ , it is essential to have a sufficient number of samples. Until we reach the **ORKA With N Samples**, the upper bound of convergence to a solution lying within the space \mathcal{P}_x differs (and is smaller than) from that of the convergence to a solution within the ball around the desired point, since $\text{vol}(\mathcal{P}_x) \subsetneq \text{vol}(\mathcal{B}_\rho(\mathbf{x}_*))$. However, once we obtain a sufficient number of samples to have the volume created by the intersections of hyperplanes inside the ball, i.e., $\text{vol}(\mathcal{P}_x) \subseteq \text{vol}(\mathcal{B}_\rho(\mathbf{x}_*))$, we then have $\hat{\mathbf{x}}$ lying within the ball around the desired point \mathbf{x}_* , i.e., $\hat{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$. To address this discrepancy between the two scenarios, we introduce a second term that is dependent on the difference between $\hat{\mathbf{x}}$ and \mathbf{x}_* , as follows:

MLE-based recovery is computationally more complex for high-dimensional signals. Herein, we formulate the noisy version of one-bit sampling with time-varying thresholds. Denote $\mathbf{z} = [z_j] \in \mathbb{R}^n$ as a noise vector which has been added to the linear system of equations $\mathbf{y} = \mathbf{Ax}$. Then,

the corresponding noisy one-bit samples are generated as

where from (??) and the fact that the error between $\hat{\mathbf{x}}$ and the original signal remains deterministic with respect to each iteration, we can write

$$r_j^{(\ell)} = \begin{cases} 1 & \text{if } \mathbf{a}_j^\top (\mathbf{x}_* - \hat{\mathbf{x}}) \geq \tau_j^{(\ell)}, \\ 0 & \text{otherwise,} \end{cases} \quad j \in [n], \ell \in [m], \quad (39)$$

$$\mathbb{E}\{\|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2\} \leq \mathbf{a}_j^\top \left(1 + \frac{z_j - \mathbf{a}_j^\top \hat{\mathbf{x}}}{\kappa^2(\mathbf{A})}\right) \mathbf{h}(\mathbf{x}_0, \hat{\mathbf{x}}) + \|\mathbf{x}_* - \hat{\mathbf{x}}\|_2^2, \quad (38)$$

where \mathbf{a}_j denotes the j -th row of a sampling matrix \mathbf{A} . Consequently, the one-bit polyhedron

The convergence to \mathbf{x}_* is ensured only when the second term, $\|\mathbf{x}_* - \hat{\mathbf{x}}\|_2^2$, is bounded. As associated with (??) is rewritten as

demonstrated in Theorem ??, with a minimum probability of $1 - e^{-c_1(C\rho - \mu)^2/m'}$, we establish

that $\text{vol}(\mathcal{P}_x) \subseteq \text{vol}(\mathcal{B}_\rho(\mathbf{x}_*))$ and consequently have $\|\mathbf{x}_* - \hat{\mathbf{x}}\|_2^2 \leq \rho^2$, which proves the proposition.

where \mathbf{P} is defined in (??) and $\mathbf{v} = \tilde{\Omega}\mathbf{z}$ is the noise of our system with $\tilde{\Omega}$ defined in (??).

For instance, assuming a zero-mean Gaussian noise vector $\mathbf{z} \sim \mathcal{N}(0, \Sigma_\sigma)$ with the covariance matrix Σ_σ , the distribution of \mathbf{v} will be $\mathcal{N}\left(0, \tilde{\Omega}\Sigma_\sigma\tilde{\Omega}^H\right)$ at the i -th iteration:

The robustness of the RKA against noise has been demonstrated in [?] and [?]. Furthermore, the authors of [?] specifically explored the performance of the RKA in the presence of Gaussian

and of Poirier. Based on highlighting its unique robustness, Proposition ?? deals with ORKA with Poisson noisy measurements. In the discussion in Section ?? we will explore the one-bit sensing consistency of the defined noisy system of linear equations. It also manifests itself in the recovery error of the RKA. Next, in Section ?? we will propose a novel algorithm to have a robust recovery performance in the presence of impulsive noise.

The proof of Corollary ?? is provided in Appendix ??.

Based on (??) and its proof, it is evident that the constant δ_s contains ρ^{-2} , which leads to the conclusion that $\rho = \mathcal{O}(m'^{-\frac{1}{3}})$ and $\rho^2 = \mathcal{O}(m'^{-\frac{2}{3}})$. As a result, the upper bound of ORKA error decays with a rate of $\mathcal{O}(m'^{-\frac{2}{3}})$ with respect to the number of samples.

Given a linear system of equations $\mathbf{U}\mathbf{x} = \mathbf{b}$ that is highly over-determined and subject to a noise vector $\mathbf{n} = [n_j]$ resulting in a corrupted system of equations $\mathbf{U}\mathbf{x} \approx \mathbf{b} + \mathbf{n}$. The convergence rate of the noisy RKA was comprehensively discussed in [?, Theorem 2.1] for the case of bound of ORKA for low-rank matrix sensing decays with a rate of $\mathcal{O}(m'^{-\frac{2}{3}})$.

$\mathbf{U}\mathbf{x} \approx \mathbf{b} + \mathbf{n}$. The primary contrast between the convergence rates of RKA and noisy RKA, as

Note that if any other randomized algorithm is utilized for one-bit sensing instead of RKA demonstrated in [?, Theorem 2.1], lies in the second term of convergence rate $\kappa^2 \max_j \frac{n_j}{\|\mathbf{u}_j\|^2}$. to achieve uniform reconstruction, the convergence rate will maintain the same structure as This term indicates the degree to which the error in the corrupted system $\mathbf{U}\mathbf{x} \approx \mathbf{b} + \mathbf{n}$ deviates Proposition ??.

However, there will be a difference in the first term, which will be substituted from the main solution.

by the algorithm's convergence rate to a point inside the feasible space of hyperplanes.

Drawing inspiration from the convergence rate of the noisy RKA, we can similarly derive the convergence rate of noisy RKA in the case of noisy linear systems of inequalities $\mathbf{C}\mathbf{x} + \mathbf{n} \succeq \mathbf{b}$ using the following proposition:

In addition to the theoretical assurances offered by our proposed algorithms, it is crucial to assess their effectiveness in the presence of noise. Previous studies, such as [?, ?], have examined one-bit noisy models with a linear measurement framework incorporating additional Gaussian noise. In these models, the input signal was recovered using a MLE approach, employing the Gaussian likelihood function. However, when dealing with non-Gaussian contamination, the MLE objective becomes nonconcave, leading to non-unique solutions for signal recovery. Moreover, MLE-based recovery is computationally more complex for where $\gamma_j = \frac{\langle \mathbf{c}_j, \hat{\mathbf{x}} \rangle^2}{\|\mathbf{c}_j\|^2}$.

high-dimensional signals. Herein, we formulate the noisy version of one-bit sampling with

The proof of Proposition ?? is presented in Appendix ??.

Based on the noisy one-bit polyhedron (??), the parameters $\mathbf{C} = \mathbf{P}^\top \mathbf{A} \mathbf{x}$ and $\mathbf{b} = \text{vec}(\mathbf{B})$, $\text{vec}(\Gamma)$ can be replaced in the convergence rate presented in (??) to get the similar result in the one-bit noisy scenario. Note that as can be observed in (??), a small perturbation in the linear feasibility problem $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$ may slightly deviate the solution of the noisy RKA from the main solution.

where \mathbf{a}_j denotes the j -th row of a sampling matrix \mathbf{A} . Consequently, the one-bit polyhedron associated with (??) is rewritten as

B. Upper Quantile-Based ORKA

In Section ??, the robustness of the noisy RKA in the presence of a small perturbation has been discussed. However, as can be observed in the convergence rate of the noisy RKA ??, a large perturbation in the linear feasibility $Cx \succeq b$ can lead to a significant error in the input signal recovery. In such a scenario, an accurate input signal recovery is quite a challenging task. A well-known example of large perturbation is impulsive noise. Impulsive noise poses substantial challenges in the realm of signal processing and imaging applications. Within the domain of audio signal processing, it manifests as disruptive events such as clicks, pops, or random bursts, of Gaussian and Poisson noise, highlighting its robustness even when dealing with Poisson leading to a degradation in sound quality [2, 2]. In the field of magnetic resonance imaging (MRI), the presence of impulsive noise gives rise to unwanted anomalies and distortions in the acquired images [2]. In this section, our goal is to propose a novel RKA-based algorithm which is robust to impulsive noise. The noisy linear inequality feasibility problem is defined as $Cx + n \succeq b$, where n is the noise of our system. If the noise does not have a significant impact

An Robustness of ORKA against Noise

Given a linear system of equations $Ux = b$ that is highly over-determined and subject to handle it differently. In this section, specifically, we introduce an algorithm that identifies the “orthants” that are immune to corruption (where noise cannot change their direction), and only incorporate them in the updates of RKA.

The probability of orthants that are corrupted with noise is formulated as the following upper noisy RKA, as demonstrated in [?, Theorem 2.1], lies in the second term of convergence rate quantile $\kappa \max_j \frac{n_j^2}{\|u_j\|_2^2}$. This term indicates the degree to which the error in the corrupted system $Ux \approx b + n$ deviates from the main solution.

Drawing inspiration from the convergence rate of the noisy RKA, we can similarly derive the convergence rate of noisy RKA in the case of noisy linear system of inequalities $Cx + n \succeq b$ using the following proposition. The threshold is calculated based on the empirical q -quantile of the noise:

Proposition 2. Let $C \in \mathbb{R}^{Q \times p}$ have full column rank and assume \hat{x} is the solution of the noisy linear feasibility problem $Cx + n \succeq b$. Let \bar{x}_i be the i -th iterate of the noisy RKA. If a residual exceeds this threshold, it suggests that the noise may not have been strong enough run with $Cx \succeq b$, and let n_j denote the j -th element of n , respectively. Then we have to change its direction. By applying the thresholding process, we ensure that a sufficient distance is maintained between $\{n_j(\bar{x}_i) \text{ and } b_j\} \leq \left(\frac{1}{\kappa^2(C)}\right)^i$ to prevent any noise from impacting the inequality system. This helps to preserve the integrity of the solution throughout the algorithm. Assume p_j is the j -th row of P randomly chosen at each iteration i , the proposed algorithm for the noisy one-bit

sampled systems. Proposition 9 is also presented ORKA appendices. Based on (i) update then RKA projection if $(\mathbf{p})_{\mathbf{x}_i}$ the parameters $\mathbf{G}_i \in \mathbf{B}$ and (ii) when $\mathbf{v} = \text{vec}(\mathbf{R}) \otimes \text{vec}(\mathbf{T})$ can be replaced in the convergence rate presented in (??) to get the similar result in the one-bit noisy scenario.

V. JUDICIOUS SAMPLING WITH ADAPTIVE THRESHOLDING FOR ORKA Note that as can be observed in (??), a small perturbation in the linear feasibility problem the spirit of using the iterative RKA suitable time-varying RKA sampling thresholds can be selected in order to enhance the recovery performance. In the sample abundance ORKA, we face a highly over-determined linear feasibility problem creating a finite-volume space. To capture the desired signal \mathbf{x} , more efficiently, the right-hand side of the inequalities in (??) i.e. $\text{vec}(\mathbf{R}) \odot \text{vec}(\mathbf{T})$ must be determined in a way that each associated hyperplane passes through the desired feasible region within \mathcal{F}_x . Therefore, an algorithm is proposed to ensure that this occurs in practice. Accordingly, we propose an iterative algorithm generating adaptive sampling thresholds to accurately obtain the desired solution. To have a smaller area of the finite-volume space around the desired signal \mathbf{x} , one can somehow choose thresholds to reduce the distances between the desired point and the associated hyperplanes in (??). To do so, we update the time-varying thresholds such as clicks, pops, or random bursts, leading to a degradation in sound quality [?, ?].

In the field of magnetic resonance imaging (MRI), the presence of impulsive noise gives rise to unwanted anomalies and distortions in the acquired images [?]. In this section, our goal is to propose $\tau_k^{(t)}$ denoted RKA-based algorithm which is robust to impulsive noise. The design of the algorithm is based on the probability of an orthogonal vector $\mathbf{r}_k^{(t)}$ to the residual $\epsilon_k^{(t)}$ defined as $\text{Pr}[\mathbf{r}_k^{(t)} \perp \epsilon_k^{(t)}]$. By neglecting the thresholding step, if the thresholding algorithm has a significant impact on each distance quality in (??) results in the system resulting in a large error. Therefore, we will handle the noise by the potential of higher threshold detection of the inequalities given in ORKA to handle it differently. Performance and especially non-uniform sampling thresholds which is proposed "exhaustive" constraints in the condition of the heavy in sampling threshold direction, be definitely incorporated by choosing the adaptive RKA thresholds with closer hyperplanes to the desired point.

The probability of orthants that are corrupted with noise is formulated as the following upper quantile

VII. ONE-BIT LOW-RANK MATRIX SENSING

Low-rank matrix sensing is an excellent example for problems that assume the form in (??) and that can be tackled using our methodology. In Section ??, we first briefly introduce the where b_j and n_j are j -th elements of \mathbf{b} and \mathbf{n} , respectively. By using this formulation, we nuclear norm minimization form of the low-rank matrix sensing problem. Subsequently, we may be able to determine a threshold for our residuals $\{\mathbf{c}_j \mathbf{x} - b_j\}$ that can be used to apply ORKA to this problem without considering the associated costly constraints. As mentioned previously, there exists a trade-off between the number of samples and the computational

Algorithm 3 Adaptive Thresholding for ORKA identifies the corrupted ones. The threshold is calculated based on the empirical q -quantile

Input: One-bit data, time-varying sampling thresholds, and the sampling matrix. of the noise:

Output: A solution $\bar{\mathbf{x}}$ in the one-bit polyhedron (??). $\mathcal{Q}(\mathbf{x}) \triangleq q\text{-quantile}\{|\mathbf{c}_j\mathbf{x} - b_j|, j \in [m]\}$. (43)

$$1: \mathbf{b}^{(\ell)} \leftarrow \boldsymbol{\tau}_k^{(\ell)}$$

If 2: $\mathcal{P}_k \leftarrow \{\mathbf{x}_k \mid r_j^{(\ell)} \mathbf{a}_j \mathbf{x}_k \geq r_j^{(\ell)} b_j, j \in [n], \ell \in [m]\}$. It suggests that the noise may not have been strong enough to change its direction. By applying the thresholding process, we ensure that a

3: Obtain \mathbf{x}_k in \mathcal{P}_k by RKA, PrSKM or Block SKM.

4: $\epsilon_k^{(\ell)} \leftarrow \Gamma \odot (\mathbf{A}\mathbf{x}_k - \mathbf{b}^{(\ell)})$. sufficient distance is maintained between $\mathbf{c}_j\mathbf{x}$ and \mathbf{b}_j to prevent any noise from impacting

5: $\boldsymbol{\tau}_{k+1}^{(\ell)} \leftarrow \mathbf{A}\mathbf{x}_k - \frac{1}{2}(\mathbf{r}^{(\ell)} \odot \epsilon_k^{(\ell)})$. This helps to preserve the integrity of the solution throughout the algorithm. Assume \mathbf{p}_j is the j -th row of \mathbf{P} randomly chosen at each iteration i , the proposed

6: Increase k by one. algorithm for the noisy one-bit sampled systems (??) upper quantile-based ORKA is written

7: Repeat Steps (1)-(6) until $\sum_{\ell=1}^m \|\boldsymbol{\tau}_{k+1}^{(\ell)} - \boldsymbol{\tau}_k^{(\ell)}\|_2^2 \leq \delta$. as follows: (i) update the RKA projection if $|\mathbf{p}_j\mathbf{x}_i - r_j^{(\ell)}\boldsymbol{\tau}_j^{(\ell)}| \geq \mathcal{Q}(\mathbf{x}_i)$, and (ii) otherwise,

8: return $\bar{\mathbf{x}}$. set $\mathbf{x}_i = \mathbf{x}_{i+1}$.

complexity of the reconstruction algorithm in the problem of signal parameter recovery. In Section ??, we specifically address the scenario where, due to practical limitations or other factors, an adequate number of measurements or one-bit samples may not be available to satisfy the FVP described in Section ?? To address this challenge, we will introduce a novel algorithm called *SVP-ORKA*.

To capture the desired signal \mathbf{x}_* more efficiently, the right-hand side of the inequalities in (??), i.e. $\text{vec}(\mathbf{R}) \odot \text{vec}(\boldsymbol{\Gamma})$, must be determined in a way that each associated hyperplane passes through the desired feasible region within $\mathcal{F}_{\mathbf{X}}$. Therefore, an algorithm is proposed to ensure that this occurs in practice. Accordingly, we propose an iterative algorithm generating adaptive sampling thresholds to accurately obtain the desired solution. To have a smaller

A. *Problem Formulation* area of the finite-volume space around the desired signal \mathbf{x}_* , one can somehow choose where $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ is the matrix of unknowns, $\mathbf{y} \in \mathbb{R}^n$ is the measurement vector, and \mathcal{A} is thresholds to reduce the distances between the desired point and the associated hyperplanes a linear transformation such that $\mathcal{A}: \mathbb{R}^{n_1 \times n_2} \mapsto \mathbb{R}^n$. In general, Ω_c can be chosen such as the in (??). To do so, we update the time-varying thresholds as

set of semi-definite matrices, symmetric matrices, upper or lower triangle matrices, Hessenberg

matrices and a specific constraint on the matrix elements $\|\mathbf{X}\|_{\infty} \leq \alpha$ or on its eigenvalues, (ie)

$\lambda_i \leq \epsilon$ where $\{\lambda_i\}$ are eigenvalues of \mathbf{X} [?, ?, ?]. The problem (??) can be rewritten as an

optimization problem. Denote the k -th updates of \mathbf{x} and $\boldsymbol{\tau}^{(\ell)}$ in our proposed adaptive threshold

design algorithm. The proposed sampling algorithm is summarized in Algorithm ?? By em-

ploying this adaptive thresholding algorithm, it is notable that each distance defined in (??)

This problem is known to be NP-hard, whose solution is difficult to approximate [?, ?]. Recall

tends to decrease, resulting in a reduction of T_{ave} with a high probability. Therefore, a smaller

that the rank of \mathbf{X} is equal to the number of nonzero singular values. In the case when similar

Algorithm 3 Adaptive Thresholding for ORKA

Input: One-bit data, time-varying sampling thresholds, and the sampling matrix.

Output: A solution $\bar{\mathbf{x}}$ in the one-bit polyhedron (??).

1: $\mathbf{b}^{(\ell)} \leftarrow \tau_k^{(\ell)}$
replace the rank function with the sum of the singular values of \mathbf{X} ; i.e., its nuclear norm. The
2: $\mathcal{P}_k \leftarrow \left\{ \mathbf{x}_k \mid r_k^{(\ell)} \mathbf{a}_j \mathbf{x}_k \geq r_k^{(\ell)} b^{(\ell)}, j \in [n], \ell \in [m] \right\}$
nuclear norm minimization alternative of the problem is given by [?, ?, ?]:

3: Obtain \mathbf{x}_k in \mathcal{P}_k by RKA, PrSKM or Block SKM.

$$4: \mathbf{e}_k^{(\ell)} \leftarrow \mathbf{r}^{(\ell)} \odot (\mathbf{A}\mathbf{x}_k - \mathbf{b}^{(\ell)}) \underset{\mathbf{X} \in \Omega_c}{\text{minimize}} \quad \|\mathbf{X}\|_* \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}. \quad (47)$$

In this problem, the feasible set $\mathcal{F}_{\mathbf{X}}$ is obtained as $\mathcal{F}_{\mathbf{X}} = \{\mathcal{P}_{\mathbf{X}}^* \cap \Omega_c\}$, where $\mathcal{P}_{\mathbf{X}}^*$ is defined as follows.
Increase k by one.

$$7: \text{Repeat Steps (1)-(6) until } \|\sum_{\ell=1}^m \|\mathbf{X} \tau_k^{(\ell)} \mathbf{X}^\top \tau_k^{(\ell)}\|_2 \leq \delta_r \in \mathbb{R}^+. \quad (48)$$

8: return $\bar{\mathbf{x}}$

Next, we will apply ORKA to (??) to make its costly constraints redundant by using abundant number of one-bit samples m' .

recovery performance. Additionally, non-informative sampling thresholds, which appear as

extra inequality constraints in the random time-varying sampling thresholds scenario, may

be efficiently removed by choosing the adaptive thresholds with closer hyperplanes to the

desired point. $\mathbf{A}^{n_1 \times n_2}$ is the j -th sensing matrix. The one-bit polyhedron for the low-rank matrix sensing is given by

VII. One-Bit Low-Rank Matrix Sensing

$$\mathcal{P}^{(M)} = \left\{ \mathbf{X} \mid r_j^{(\ell)} \text{Tr}(\mathbf{A}_j^\top \mathbf{X}) \geq r_j^{(\ell)} \tau_j^{(\ell)}, j \in [n], \ell \in [m] \right\}. \quad (50)$$

Low-rank matrix sensing is an excellent example for problems that assume the form

Writing the update process of ORKA in matrix form helps the following representation briefly

introduce the nuclear norm minimization (??) form of the low-rank matrix sensing problem.

Subsequently, we apply ORKA to this problem without considering the associated costly

constraints. As mentioned previously, there exists a trade-off between the number of samples and the computational complexity of the reconstruction algorithm in the problem of signal recovery. In Section ?? we specifically address the scenario where, due to practical limitations or other factors, an adequate number of measurements or one-bit samples may not be available to satisfy the FVP described in Section ?? To address this challenge, we will introduce a novel algorithm called SVP-ORKA.

A numerical investigation of (??) reveals that by increasing the number of time-varying sampling threshold sequences m , the space formed by the intersection of half-spaces (inequality

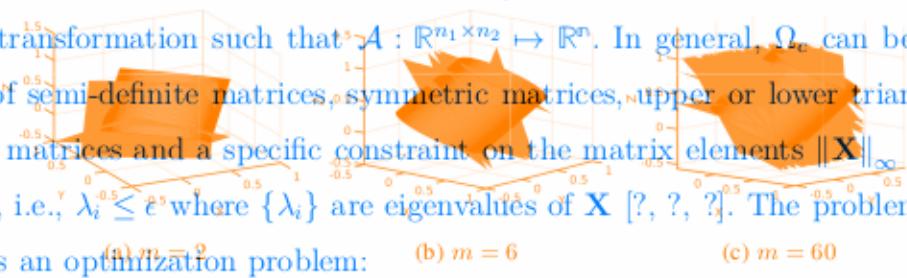
A. Problem Formulation

constraints) can fully shrink to the desired signal \mathbf{X}_* inside the feasible region of (??) which is

The problem of the low-rank matrix sensing is formulated as: shown by the cylindrical space [?].—see Fig. ?? for an illustrative example of this phenomenon.

As can be seen in this figure, the blue subject displaying (\mathbf{X}_*) the linear rank (SVD) form a finite-volume

where $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ is the matrix of unknowns, $\mathbf{y} \in \mathbb{R}^n$ is the measurement vector, and \mathcal{A} is a linear transformation such that $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \mapsto \mathbb{R}^n$. In general, Ω_c can be chosen such as the set of semi-definite matrices, symmetric matrices, upper or lower triangle matrices, Hessenberg matrices and a specific constraint on the matrix elements $\|\mathbf{X}\|_\infty \leq \alpha$ or on its eigenvalues, i.e., $\lambda_i \leq \epsilon$ where $\{\lambda_i\}$ are eigenvalues of \mathbf{X} [?, ?, ?]. The problem (??) can be rewritten as an optimization problem:



$$\underset{\mathbf{X} \in \Omega_c}{\text{minimize}} \quad \text{rank}(\mathbf{X}) \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}. \quad (46)$$

This problem is known to be NP-hard, whose solution is difficult to approximate [?, ?]. Recall that the rank of \mathbf{X} is equal to the number of nonzero singular values. In the case when the singular values are all equal to one, the sum of the singular values is equal to the rank. When the singular values are less than or equal to one, the sum of the singular values

is a convex function that is strictly less than the rank. Therefore, it has been popular for shown with black cylindrical region and its red contours, when the number constraints (samples) grows large. The arrows point this problem to replace the rank function with the sum of the singular values of \mathbf{X} ; i.e., its nuclear norm. The nuclear norm minimization alternative of the problem is given by [?, ?]:

$$\underset{\mathbf{X} \in \Omega_c}{\text{minimize}} \quad \|\mathbf{X}\|_* \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{y}. \quad (47)$$

In this problem, the feasible set $\mathcal{F}_{\mathbf{X}}$ is obtained as $\mathcal{F}_{\mathbf{X}} = \{\mathcal{P}_{\mathbf{X}}^* \cap \Omega_c\}$, where $\mathcal{P}_{\mathbf{X}}^*$ is defined space around the original signal displayed by the yellow circle inside the cylinder (the elliptical as follows

region) by growing the number of threshold sequences or one-bit samples. In (a)/(d), constraints

$$\mathcal{P}_{\mathbf{X}}^* = \{\mathbf{X} \mid \|\mathbf{X}\|_* \leq \tau\}, \quad \tau \in \mathbb{R}^+. \quad (48)$$

are not enough to create a finite-volume space, whereas in (b)/(e) such constraints can create

Next, we will apply ORKA to the (??) space which, however, cannot be represented by a cylinder

abundant number of one-bit samples m .

Finally, in (f), the one-bit volume space shrinks to be fully inside the cylinder. The

theoretical discussion regarding the required number of one-bit samples m to accurately recover

the low-rank matrix from the polyhedron (??) has been presented in Section ??.

$$\mathcal{A}(\mathbf{X}) = \frac{1}{\sqrt{n}} [\text{Tr}(\mathbf{A}_1^\top \mathbf{X}) \cdots \text{Tr}(\mathbf{A}_n^\top \mathbf{X})], \quad (49)$$

where $\mathbf{A}_j \in \mathbb{R}^{n_1 \times n_2}$ is the j -th sensing matrix. The one-bit polyhedron for the low-rank

B. ORKA With Low-Rank Matrix Factorization

matrix sensing is given by

Although Kaczmarz algorithms use a simple and low-complexity update process, the computational cost can still be prohibitively large for a matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ when n_1 and n_2 are

large values, typically requiring $\mathcal{O}(n_1 n_2)$ operations. To address this issue, we employ the well-

written update process of ORKA in matrix form yields the following representation:

known *low-rank matrix factorization* technique. Instead of using the full matrix \mathbf{X} , we use its low-rank factorization $\mathbf{X} = \mathbf{LW}^\top$ where $\mathbf{L} \in \mathbb{R}^{n_1 \times r}$ and $\mathbf{W} \in \mathbb{R}^{n_2 \times r}$. The factors \mathbf{L} and \mathbf{W} are low-rank factors. (The

$$\|\mathbf{A}_j\|_F^2$$

By applying this approach is it possible to obtain a representation (?) of the utilizing the fact that a lower storage requirement allows updates in parallel that of RKA. Therefore, it is possible to parallelize it. If this makes sense depends on the availability of RKA and the memory bandwidth, especially when using iterative Krylov methods like ORKA. The roots of RKA, from Skarink and Block [2014], show that the gradient descent problem has been extensively discussed in the literature [?].

In this section we explain the integration of what they have done with RKA and the alternating thresholding (AltMin) algorithm discussed by [1]. The intersection of half-spaces (inequalities) one-bit polyhedron fully shrinkage to the signal \mathbf{x} that is factorizable approach of (written) is shown by the cylindrical space [?]-see Fig. ?? for an illustrative example of this phenomenon. As can be seen in this figure, the blue lines displaying the linear feasibility

$$\mathcal{P}_0^{(M)} = \left\{ \mathbf{L}, \mathbf{W} \mid r_j^{(\ell)} \text{Tr} (\mathbf{A}_j^\top \mathbf{L} \mathbf{W}) \geq r_j^{(\ell)} \tau_j^{(\ell)}, j \in [n], \ell \in [m] \right\}. \quad (52)$$

form a finite-volume space around the original signal displayed by the yellow circle inside the cylinder (the elliptical region). To find the solution from $\mathcal{P}_0^{(M)}$, we use the idea of AltMin algorithm. We split $\mathcal{P}_0^{(M)}$ into two linear feasibility sub-problems with respect to \mathbf{L} and \mathbf{W} , respectively. Specifically, with respect to \mathbf{L} when \mathbf{W} is fixed we have:

in (a)/(d), constraints are not enough to create a finite-volume space, whereas in (b)/(e) such constraints can create the desired finite-volume polyhedron space which, however, is not fully inside the cylinder. Lastly, in (e)/(f), the created finite-volume space shrinks to be fully inside the cylinder. The theoretical discussion regarding the required number of one-bit samples m' to accurately recover the low-rank matrix from the polyhedron

$$\mathcal{P}_{\mathbf{W}}^{(M)} = \left\{ \mathbf{W} \mid r_i^{(\ell)} \text{Tr} (\mathbf{A}_i^\top \mathbf{L} \mathbf{W}^\top) \geq r_i^{(\ell)} \tau_i^{(\ell)}, \quad j \in [n], \quad \ell \in [m] \right\}. \quad (54)$$

In either ORKA or low-Rank Matrix Factorization with respect to the other variable is achieved via ARKA as indicated in Algorithm 22. Although ARKA uses a simple and low-complexity update process, the computational cost can still be prohibitively large for a matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ when n_1 and n_2 are large values, typically requiring $\mathcal{O}(n_1 n_2)$ operations. To address this issue, we employ the Whillie algorithm in low-rank number of completion not typically in issue for one-bit sampling as it is very practical to avoid using excess samples, particularly in signal processing applications where only a small number of measurements may be permitted. Adhering to the analysis presented in the introduction, the low-rank matrix factorization results in lower storage requirements and does not require multiplications by one-bit signals. This section focuses on enabling low-rank matrix factorization using ORKA-like algorithms for low-rank matrix setting with a scaled number of one-bit samples. Specifically, when using one-bit optimization between ORKA and The proposed algorithm of complexity and that is characterized by the grid diagram to address the challenges associated with a limited number of samples. In this section, we explore the integration of low-

Algorithm 4 ORKA (with Low-Rank Matrix Factorization)

Input: One-bit samples, time-varying thresholds and the sensing matrices presented in (??), total number of required iterations T , and $\text{rank}(\mathbf{X}_*) = r$.

Output: A solution $\bar{\mathbf{X}} \in \mathbb{R}^{n_1 \times n_2}$ in the polyhedron (??).

Note: $\mathbf{L}_t \in \mathbb{R}^{n_1 \times r}$ and $\mathbf{W}_t \in \mathbb{R}^{n_2 \times r}$ denote the obtained matrices at t -th iteration, $\mathbf{H}_{i+1} = \text{KA}_r(\mathbf{H}_i; \tilde{\mathbf{H}})$ denotes the update process of ORKA using the RKA when $\tilde{\mathbf{H}}$ is fixed.

1: $\mathbf{W}_{t+1} \leftarrow \text{KA}_r(\mathbf{W}_t; \mathbf{L}_t) \triangleright$ Update process for polyhedron (??).

2: $\mathbf{L}_{t+1} \leftarrow \text{KA}_r(\mathbf{L}_t; \mathbf{W}_{t+1}) \triangleright$ Update process for polyhedron (??).

3: Repeat steps (1) and (2) until convergence.

4: $\mathbf{L}_* \leftarrow \mathbf{L}_T$ and $\mathbf{W}_* \leftarrow \mathbf{W}_T$.

5: **return** $\bar{\mathbf{X}} = \mathbf{L}_* \mathbf{W}_*^\top$.

(d) $m = 2$

(e) $m = 6$

(f) $m = 60$

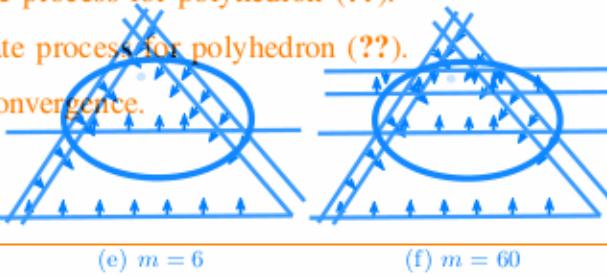


Figure 2. Shrinkage of the one-bit polyhedron (??) in blue, ultimately placed within the unit ball of the nuclear norm. Define $\mathbf{X}_* \leq \mathbf{X}$ as the predefined rank of the unknown matrix \mathbf{X} . In order to obtain the solution within a reduced number of samples in the polyhedron $\mathcal{P}^{(M)}$ defined in (??), we impose a rank constraint, $\text{rank}(\mathbf{X}) \leq r$, to shrink the entire space, as shown by the following polyhedron: not forming a finite-value polyhedron; (b) and (e) medium sample-size regime, constraints forming a finite-volume polyhedron, parts of which are outside the cylinder; (c) and (f) large sample-size regime, constraints forming a finite-volume polyhedron inside the nuclear norm cylinder, making its constraint redundant. The original signal representing the signal to be recovered is shown by yellow, where $j \in [n]$, $\ell \in [m]$. To tackle this problem, we apply the SVP method to ORKA. The SVP was introduced as a solution to the general affine rank minimization problem (ARMP). In [?], rank matrix factorization with ORKA and the alternating minimization (AltMin) algorithm it was demonstrated that the SVP can effectively recover the minimum rank solution even in the presence of noise and when the affine constraints satisfy RIP. Moreover, some theoretical guarantees for this approach were also established. This method utilizes the operator P_r to modify the gradient descent process at each iteration. The operator P_r calculates the r largest singular values of a matrix and subsequently rewrites its SVD based on these r singular values and their corresponding singular vectors. Similar to Section ??, we use the idea of AltMin algorithm to update process of ORKA using RKA defined in (32) through the integral form of SVP into specifically with ORKA, we can have the following update process:

$$\mathcal{P}_L^{(M)} = \left\{ \mathbf{L} \mid r_j^{(\ell)} \text{Tr} \begin{cases} \mathbf{Z}_{i+1}^\top \mathbf{A}_j^\top \mathbf{L} \geq r_j^{(\ell)} \tau_j^{(\ell)}, & j \in [n], \ell \in [m] \\ \mathbf{X}_{i+1} = P_r(\mathbf{Z}_{i+1}) \end{cases} \right\}, \quad (53)$$

and with respect to \mathbf{W} when \mathbf{L} is fixed we have:

The first step of the update process (??), $\mathbf{Z}_{i+1} = \text{KA}_r(\mathbf{X}_i)$, can be also replaced with $\mathbf{Z}_{i+1} = \text{KA}_p(\mathbf{X}_i)$ or $\mathbf{Z}_{i+1}^{(M)} = \text{KA}_b(\mathbf{X}_i)$ to be consistent with the updates of PrSKM and Block SKM,

$$\mathcal{P}_W^{(M)} = \left\{ \mathbf{W} \mid r_j^{(\ell)} \text{Tr} (\mathbf{A}_j^\top \mathbf{L} \mathbf{W}^\top) \geq r_j^{(\ell)} \tau_j^{(\ell)}, j \in [n], \ell \in [m] \right\}. \quad (54)$$

Algorithm 4 ORKA (with Low-Rank Matrix Factorization)

Input: One-bit samples, time-varying thresholds and the sensing matrices presented in lemma:

(??), total number of required iterations T , and $\text{rank}(\mathbf{X}_*) = r$.

Lemma 3. *The update process of SVP-ORKA presented in (??) converges linearly in expectation to a ball centered at the original signal \mathbf{x} (vec(\mathbf{X}_*)) with the number of samples satisfying*

Note: $\mathbf{L}_t \in \mathbb{R}^{n \times d}$ and $\mathbf{W}_t \in \mathbb{R}^{d \times n}$ denote the obtained matrices at t -th iteration;

Theorem ?? and probability exceeding $1 - \alpha$ as follows: if ORKA using the RKA when $\tilde{\mathbf{H}}$ is fixed.

$$\mathbb{E}\{\|\mathbf{X}_{i+1} - \mathbf{X}_*\|_F^2\} \leq \left(1 - \frac{1}{\kappa^2(\mathbf{V})}\right)^i \|\mathbf{X}_0 - \hat{\mathbf{X}}\|_F^2 + \rho^2, \quad (57)$$

1: $\mathbf{W}_{t+1} \leftarrow \text{KA}_r(\mathbf{W}_t; \mathbf{L}_t)$ ▷ Update process for polyhedron (??).

where $\hat{\mathbf{X}} \in \mathcal{P}_1^{(M)}$ (and \mathbf{V} is the matrix with vectorized sensing matrices $\{\text{vec}(\mathbf{A}_j)\}_{j=1}^n$ as its rows)

2: Repeat steps (1) and (2) until convergence.

3: $\mathbf{L}_* \leftarrow \mathbf{L}_T$ and $\mathbf{W}_* \leftarrow \mathbf{W}_T$. The proof of Lemma (??) is studied in Appendix ??.

4: return $\bar{\mathbf{X}} \leftarrow \mathbf{L}_* \mathbf{W}_*^\top$.

VIII. ONE-BIT COMPRESSED SENSING:

FROM OPTIMIZATION TO LINEAR FEASIBILITY

If either \mathbf{L} or \mathbf{W} are fixed, finding the solution with respect to the other variable is achieved

CS is an interesting and rapidly growing area of research that has attracted considerable via ORKA as indicated in Algorithm ??.

attention in electrical engineering, applied mathematics, statistics, and computer science [?, ?]. In

CS, the objective is to recover a sparse high-dimensional signal from incomplete measurements,

which may be formulated as follows [?]:

While acquiring a large number of samples is not typically an issue for one-bit sampling, it is still practical to avoid using too many samples, particularly in signal processing applications

where access to sufficient measurements may be limited. Adhering to the law of parsimony, where $\mathbf{A} \in \mathbb{R}^{n \times d}$, and $n \ll d$.

“Entia non sunt multiplicanda praeter necessitatem,” i.e., entities should not be multiplied

The concise summary of CS theory mentioned earlier implies that, under ideal circumstances, beyond necessity, this section focuses on developing ORKA to facilitate low-rank matrix the measurement vector \mathbf{y} is assumed to be represented with infinite precision. However, real-sensing with a reduced number of one-bit samples. As with any technique, there is a trade-off between the number of samples and the complexity of that technique, and this section purposes such as storing, transmitting, or processing acquired observations.

aims to address the challenges associated with a limited number of samples.

Following our motivation in one-bit low-rank matrix recovery, herein we are inspired to

Define r as the predefined rank of the unknown matrix \mathbf{X} . In order to obtain the solution investigate the one-bit CS problem in (i) sample abundance, and (ii) sample restricted scenarios. within a reduced number of samples in the polyhedron $\mathcal{P}^{(M)}$ defined in (??), we impose In Section ??, our goal is to examine the reconstruction performance of one-bit CS with a rank constraint, $\text{rank}(\mathbf{X}) \leq r$, to shrink the entire space, as shown by the following multiple time-varying threshold sequences under a sample-abundance regime without any form of polyhedron:

regularization. Beyond the sample abundance scenario, in Section ?? we also develop the ORKA

$\mathcal{P}_1^{(M)} = \left\{ \mathbf{X} \mid r_i^{(\ell)} \text{Tr}(\mathbf{A}_i^\top \mathbf{X}) \geq r_i^{(\ell)} \tau_i^{(\ell)}, \text{rank}(\mathbf{X}) \leq r \right\}, \quad (55)$

for conditions where enough measurements to create a highly-overdetermined feasibility problem

are not available. It is crucial to highlight that the choice of a particular one-bit reconstruction

scheme for compressed sensing measure profile in heavily fluctuating SVP method still ORKA available SVP update more sources, a completion to the complexity and cost considerations possible (ARMP). In [3], it was demonstrated that the SVP can effectively recover the minimum rank solution even in the presence of noise and when the affine constraints satisfy RIP. Moreover, some *A. One-Bit CS via Sample Abundance* theoretical guarantees for this approach were also established. This method utilizes the operator P_r to modify the gradient descent process at each iteration. The operator P_r calculates the r largest singular values of a matrix and subsequently rewrites its SVD based constrained one-bit polyhedron: on these r singular values and their corresponding singular vectors. Similar to Section ??, denote $\text{KA}_r(\cdot)$ as the update process of ORKA $^{(\ell)}$ using the RKA $^{(\ell)}$ defined in (??). Through (59) the integration of SVP into each iteration of ORKA, we can achieve the following update process: In other words, instead of solving an optimization problem with costly constraints, the problem may be tackled by the following update process:

$$\begin{cases} \mathbf{Z}_{i+1} = \text{KA}_r(\mathbf{X}_i), \\ \mathbf{x}_{i+1} = P_r(\mathbf{Z}_{i+1}). \end{cases} \quad (60)$$

where $\text{KA}(\cdot)$ denotes the RKA update process (??), $\mathbf{Z}_{i+1} = \text{KA}_r(\mathbf{X}_i)$, can be also replaced with $\mathbf{Z}_{i+1} = \text{KA}_p(\mathbf{X}_i)$ or $\mathbf{Z}_{i+1} = \text{KA}_b(\mathbf{X}_i)$ to be consistent with the updates of PrSKM and Block SKM, respectively. The theoretical discussion about the minimum number of one-bit samples m' to precisely recover the sparse vector from the polyhedron (??) has been presented in Section ??.

Lemma 3. The update process of SVP-ORKA presented in (??) converges linearly in

B. Regularized ORKA for One-Bit CS The original signal $\mathcal{B}_\rho(\text{vec}(\mathbf{X}_*))$ with the number of samples satisfying Theorem ?? and probability exceeding $1 - n$ as follows:

In this section, we continue our exploration of algorithms for one-bit CS by introducing the ℓ_1 regularized ORKA. Building on the motivation behind our earlier proposal of the SVP-ORKA, ORKA is developed to efficiently recover a sparse solution from the one-bit CS polyhedron where $\widehat{\mathbf{X}} \in \mathcal{P}_1^{(M)}$ and \mathbf{V} is the matrix with vectorized sensing matrices $\{\text{vec}(\mathbf{A}_j)\}_{j=1}^n$ as its rows. The number of measurements is severely limited. The regularized one-bit CS polyhedron is obtained as

The proof of Lemma (??) is studied in Appendix ??.

$$\mathcal{P}_1^{(C)} = \left\{ \mathbf{x} \mid r_j^{(\ell)} \mathbf{a}_j \mathbf{x} \geq r_j^{(\ell)} \tau_j^{(\ell)}, \|\mathbf{x}\|_1 \leq \kappa \right\}, \quad (61)$$

where $j \in [n]$ and $\ell \in [m]$. Define the soft thresholding (ST) operator as $S_\kappa(\mathbf{x}) = \text{sgn}(\mathbf{x})(|\mathbf{x}| - \mathbf{t}_1)^+$,

where \mathbf{t}_1 is the predefined threshold. The ST-ORKA utilizes the ST operator S_κ to project each iteration of ORKA to the rapidly growing the following update process:

$$\begin{cases} \mathbf{z}_{i+1} = \text{KA}_r(\mathbf{x}_i), \\ \mathbf{x}_{i+1} = S_\kappa(\mathbf{z}_{i+1}), \end{cases} \quad (62)$$

[Where KACS) denotes the RKA update over the first step of the empirical process from (32) in (33) is replaced with \mathbf{z}_i , which may be formulated as follows: consistent with the updates of PrSKM and Block SKM, respectively. The convergence rate of ST-ORKA is studied in the following lemma:

where $\mathbf{A} \in \mathbb{R}^{n \times d}$, and $n \ll d$.

Lemma 4. Consider a ball centered at the original signal $\mathcal{B}_\rho(\mathbf{x}_*)$ and $\hat{\mathbf{x}} \in \mathcal{P}_1^{(C)}$. Under ideal circumstances, the measurement vector \mathbf{y} is assumed to be represented with infinite precision.

However, real-world sensing models typically involve digitalization and finite precision data representations for purposes such as storing, transmitting, or processing acquired with a probability exceeding $1 - \eta$, and the number of samples satisfying Theorem ?? observations.

The proof of Lemma ?? is studied in Appendix ?? Note that to have a guaranteed convergence, one can integrate ORKA with different operators that satisfy Lipschitz continuity. This is important in various applications and provides a chance to go beyond just ST-ORKA and SVP-ORKA. By doing so, we can achieve a convergence rate that is described in the following lemma:

we also develop the ORKA for conditions where enough measurements to create a highly-overdetermined feasibility problem are not available. It is crucial to highlight that the choice of a particular one-bit reconstruction scheme for compressed sensing measurements is heavily influenced by factors such as available computational resources, implementation complexity, and cost considerations.

Lemma 5. Assume $f(\cdot)$ is an operator such that for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$ we have $\|f(\mathbf{x}_1) - f(\mathbf{x}_2)\|_2^2 \leq L \|\mathbf{x}_1 - \mathbf{x}_2\|_2^2$. Then the integration of RKA with the operator f in each iteration of solving the feasibility problem $\mathbf{Cx} \succeq \mathbf{b}$ with $\mathbf{C} \in \mathbb{R}^{n \times d}$ has the following convergence rate:

is determined by the sensor- $\hat{\mathbf{x}}$ associated with the sensor- $\hat{\mathbf{x}}$:

$$\mathbb{E}\{\|\mathbf{x}_{i+1} - \mathbf{x}_*\|_2^2\} \leq \left(1 - \frac{1}{\kappa^2(\mathbf{C})}\right)^i \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2. \quad (64)$$

A notable example for Lemma ?? is the integration of ORKA with the hard thresholding (HT) operator $\mathcal{T}_s(\cdot)$ to reconstruct the s -sparse signal in the one-bit CS problem. This operator selects the best s -sparse approximation of the solution at each iteration. The algorithm is named HT-ORKA with the update process presented as

highly-constrained one-bit polyhedron:

$$\mathcal{P}_1^{(C)} = \left\{ \mathbf{x} \mid r_j^{(t)} \begin{cases} \mathbf{z}_{i+1} = \mathbf{K}\mathbf{A}_r(\mathbf{x}_i), \\ \mathbf{a}_j \mathbf{x} \geq r_j^{(t)} \tau_j^{(t)}, \quad j \in [n], \ell \in [d] \\ \mathbf{x}_{i+1} = \mathcal{T}_s(\mathbf{z}_{i+1}). \end{cases} \right\}. \quad (65)$$

In other words, instead of solving an optimization problem with costly constraints, the problem may be tackled by the following update process:

Lemma 6. Assume $\mathcal{B}_\rho(\mathbf{x}_*)$ be a ball centered at the original signal and $\hat{\mathbf{x}} \in \mathcal{P}_1^{(C)}$, the convergence of HT-ORKA presented in (??) is given by

where $\mathbf{K}\mathbf{A}_r(\cdot)$ denotes the RKA update process presented as $\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{(r_j^{(t)} \tau_j^{(t)} - r_j^{(t)} \mathbf{a}_j \mathbf{x}_i)^+}{\|\mathbf{a}_j\|_2^2} \mathbf{a}_j^H$.

The update process in (??) can be also replaced with $\mathbf{x}_{i+1} = \mathbf{K}\mathbf{A}_p(\mathbf{x}_i)$ or $\mathbf{x}_{i+1} = \mathbf{K}\mathbf{A}_b(\mathbf{x}_i)$

to the probability higher than updates of PrSKM and Block SKM, respectively. The theoretical

discussion about the minimum number of one-bit samples m' to precisely recover the sparse

The proof of Lemma ?? is provided in Appendix ???. In their work, the authors of [?] vector from the polyhedron (??) has been presented in Section ???. introduced the random hyperplane tessellations theorem for the ditherless scenario in one-bit

CS, specifically targeting the reconstruction of signal direction. This theorem was developed in B. Regularized ORKA for One-Bit CS: ST-ORKA

[?] for one-bit CS with Gaussian dithering, where based on the augmentation trick [?, ?], random

In this section, we continue our exploration of algorithms for one-bit CS by introducing hyperplane tessellations are limited to Gaussian random variables for the sampling matrix and the ℓ_1 regularized ORKA. Building on the motivation behind our earlier proposal of the normal distribution for the dithering. The random hyperplane tessellations theorem has previously SVP-ORKA, ORKA is developed to efficiently recover a sparse solution from the one-bit been extended for certain non-Gaussian sampling matrices in [?, ?]. In their work, the authors CS polyhedron using a reduced number of one-bit samples. This feature proves particularly derived theoretical guarantees for Uniform dithering and focused on the Hamming distance. The useful in scenarios where the number of measurements is severely limited. The regularized main distinction between the FVP and random hyperplane tessellations lies in the modeling of the one-bit CS polyhedron is obtained as

created finite volume around the original signal. While random hyperplane tessellations utilize

Hamming distance to capture the direction of measurement $\|x\|_1 \leq \kappa$

between the one-bit hyperplanes and the original signal, considering the importance of amplitude in dithered one-bit sensing. By using the Euclidean distances between the original signal and the $\text{sgn}(x)(|x| - t_1)^+$, where t_1 is the predefined threshold. The ST-ORKA utilizes the ST surrounding hyperplanes as the basis of our work, we were able to obtain theoretical guarantees operator S_κ to project each iteration of ORKA to the set $\{\|x\|_1 \leq \kappa\}$ with the following beyond Uniform dithering. This is a crucial distinction, as it highlights the significance of the update process:

difference between measurements and thresholds in one-bit sensing. Intuitively, if measurements and thresholds become closer and closer the recovery performance improves.

where $\text{KA}_r(\cdot)$ denotes the RKA updates. The first step of the update process (??) can be

IX. NUMERICAL INVESTIGATIONS

also replaced with $\mathbf{z}_{i+1} = \text{KA}_p(\mathbf{x}_i)$ or $\mathbf{z}_{i+1} = \text{KA}_b(\mathbf{x}_i)$ to be consistent with the updates of

In this section, we conduct numerical evaluations to assess the performance of our proposed PrSKM and Block SKM, respectively. The convergence rate of ST-ORKA is studied in the algorithms in two distinct scenarios: (i) sample abundance and (ii) sample restriction. All pre-following lemma:

presented results are averaged over 1000 experiments.

Lemma 4. Consider a ball centered at the original signal $\mathcal{B}_p(\mathbf{x}_*)$ and $\hat{\mathbf{x}} \in \mathcal{P}_1^{(C)}$, the

convergence of ST-ORKA presented in (??) is given by

In this particular scenario, we examine two examples: one-bit low-rank matrix sensing and one-bit CS. For both cases, we employ the Block SKM algorithm to recover the desired signal and with a probability exceeding $1 - \eta$, and the number of samples satisfying Theorem ???. then evaluate its performance against the adaptive threshold strategy introduced in [?, Section VI].

The only proof of Lemma ??, in this study, Appendix ?? does not prove that one-bit data guarantees an

exponentially low complexity ORKA implementation. In all the scenarios, we have taken into

This is important since various applications and provides a new perspective beyond just ST-ORKA

and SVP-ORKA. By doing so, we can achieve a convergence rate that is described in the following lemma:

Lemma 5. Assume $f(\cdot)$ is an operator such that for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$ we have $\|f(\mathbf{x}_1) - f(\mathbf{x}_2)\|_2^2 \leq L \|\mathbf{x}_1 - \mathbf{x}_2\|_2^2$. Then the integration of RKA with the operator f in each iteration of solving the feasibility problem $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$ with $\mathbf{C} \in \mathbb{R}^{n \times d}$ has the following convergence rate:

$$\mathbb{E}\{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2\} \leq L \left(1 - \frac{1}{10^{3/2}(\mathbf{C})}\right)^i \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2. \quad (64)$$

A notable example for Lemma ?? is the integration of ORKA with the hard thresholding (HT) operator $\mathcal{T}_s(\cdot)$ to reconstruct the s -sparse signal in the one-bit CS problem. This

operator selects the best s -sparse approximation of the solution at each iteration. The Figure 3. Comparison between the recovery performance of Block SKM using random thresholds and adaptive thresholds in the one-bit low-rank matrix sensing problem. The update process presented as

standard deviation of 0.1.
$$\begin{cases} \mathbf{z}_{i+1} = \mathbf{K}\mathbf{A}_r(\mathbf{x}_i), \\ \mathbf{x}_{i+1} = \mathcal{T}_s(\mathbf{z}_{i+1}). \end{cases} \quad (65)$$

One-bit low-rank matrix sensing. We considered a set of sampling matrices $\{\mathbf{A}_j\}_{j=1}^{1800}$, where each entry is independently drawn from a standard normal distribution. We have generated the

desired matrix $\mathbf{X}_* \in \mathbb{R}^{30 \times 30}$ such that $\text{rank}(\mathbf{X}_*) = 2$. The number of time-varying sampling threshold sequences were set to $m \in \{1, 10, 20, 30\}$. Accordingly, we have generated sequences of time-varying sampling thresholds as $\{\boldsymbol{\tau}^{(\ell)} \sim \mathcal{N}\left(0, \frac{\beta_y^2}{9} \mathbf{I}_m\right)\}_{\ell=1}^m$, where β_y denotes the dynamic range of the high-resolution measurements \mathbf{y} . Figure (A2)(a) illustrates a comparison between the

recovery probability of Block SKM and the random threshold and adaptive Thresholds?? It is

evident that the utilization of adaptive thresholds enhances the recovery performance compared to random thresholds.

The proof of Lemma ?? is provided in Appendix ???. In their work, the authors of [?] introduced the random hyperplane tessellations theorem for the ditherless scenario in one-bit **One-bit CS**. We have generated a sensing matrix $\mathbf{A} \in \mathbb{R}^{500 \times 100}$ in which each element follows CS, specifically targeting the reconstruction of signal direction. This theorem was developed in [?] for one-bit CS with Gaussian dithering, where based on the augmentation trick [?, ?], sparsity $s = 10$. The settings for time-varying sampling thresholds were considered to be the random hyperplane tessellations are limited to Gaussian random variables for the sampling same as the one-bit low-rank matrix sensing case. Fig. ??(b) displays the recovery performance matrix and normal distribution for the dithering. The random hyperplane tessellations of Block SKM using random thresholds in comparison with adaptive thresholds. Consistent with theorem has previously been extended for certain non-Gaussian sampling matrices in [?, ?]. previous observations, the utilization of adaptive thresholds improves the recovery performance. In their work, the authors derived theoretical guarantees for Uniform dithering and focused on the Hamming distance. The main distinction between the FVP and random hyperplane tessellations lies in the modeling of the created finite volume around the original signal. While random hyperplane tessellations utilize Hamming distance to capture the direction

B. Sample Frenquency we incorporate Euclidean distance between the one-bit hyperplanes and the original signal considering the importance of amplitude in dithered one-bit sensing. By similar to sample abundance, herein we investigate the performance of our proposed methods using the Euclidean distances between the original signal and the surrounding hyperplanes for one-bit low-rank matrix sensing and one-bit CS when we have a limited number of samples. Note that in all experiments, the high-resolution measurements were contaminated by the additive Gaussian noise with the standard deviation 0.1 except the case related to low-rank matrix sensing between measurements and thresholds in one-bit sensing. Intuitively, if measurements and thresholds become closer and closer, the recovery performance improves.

One-bit low-rank matrix sensing. We generated a collection of sampling matrices $\{\mathbf{A}_j\}_{j=1}^n$, where each entry is independently sampled from a standard normal distribution. The desired

matrix $\mathbf{X}_* \in \mathbb{R}^{30 \times 30}$ was generated with $\text{rank}(\mathbf{X}_*) = 2$. Define the oversampling factor as $\lambda = \frac{n_1 r}{n_1 r} = \frac{30}{60}$.

In this section, we conduct numerical evaluations to assess the performance of our proposed algorithms in two distinct scenarios: (i) sample abundance and (ii) sample restriction. All sampling threshold sequences was fixed at $m = 1$. The generation of time-varying sampling thresholds followed the same procedure as in the previous cases. Fig. ??(a) compares the recovery

performance of SVP-ORKA with hard singular value thresholding (HSVT) algorithm [?] in the A. Sample abundance

noiseless scenario. As can be observed, SVP-ORKA outperforms HSVT over different values of

In this particular scenario, we examine two examples: one-bit low-rank matrix sensing and one-bit CS. For both cases, we employ the Block-SKM algorithm to recover the desired matrix signal and then evaluate its performance against the adaptive threshold strategy introduced in [?, Section VI]. The only distinction is that in this study we did not update the one-bit data to account for an exceedingly low-complexity hardware implementation. In all our simulations, we have set $\beta = \{5, 10, 15, 20\}$. As can be seen in Fig. ??(b), the recovery performance of Algorithm ?? enhances as the value of the oversampling factor β grows large. known as prequantization error, with a standard deviation of 0.1

One-bit CS. Each element of the sensing matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ was independently drawn from One-bit low-rank matrix sensing. We considered a set of sampling matrices $\{\mathbf{A}_j\}_{j=1}^{1800}$ where each entry is independently drawn from a standard normal distribution. We have generated the desired matrix $\mathbf{X}_* \in \mathbb{R}^{30 \times 30}$ such that $\text{rank}(\mathbf{X}_*) = 2$. The number of time-varying sampling threshold sequences were set to $m \in \{1, 10, 20, 30\}$. Accordingly, we have generated sequences of time-varying sampling thresholds as $\{\tau_j^{(t)} \sim \mathcal{N}(0, \frac{\beta_j^2}{m} \mathbf{I})\}_{j=1}^m$ where β_j denotes the dynamic range of the high-resolution measurements \mathbf{y} . Figure ??(a) illustrates the recovery performance of HT-ORKA, ST-ORKA, and BIHT algorithm with random time-varying sampling thresholds [?]. It is evident that HT-ORKA outperforms both ST-ORKA and BIHT and adaptive thresholds. It is evident that the utilization of adaptive thresholds enhances in recovering the s -sparse signal in the one-bit CS problem. In the next example, we examine the recovery performance compared to random thresholds.

One-bit CS. We have generated a sensing matrix $\mathbf{A} \in \mathbb{R}^{500 \times 100}$ in which each element follows a standard normal distribution. The desired signal $\mathbf{x}_* \in \mathbb{R}^{100}$ was assumed to have the level

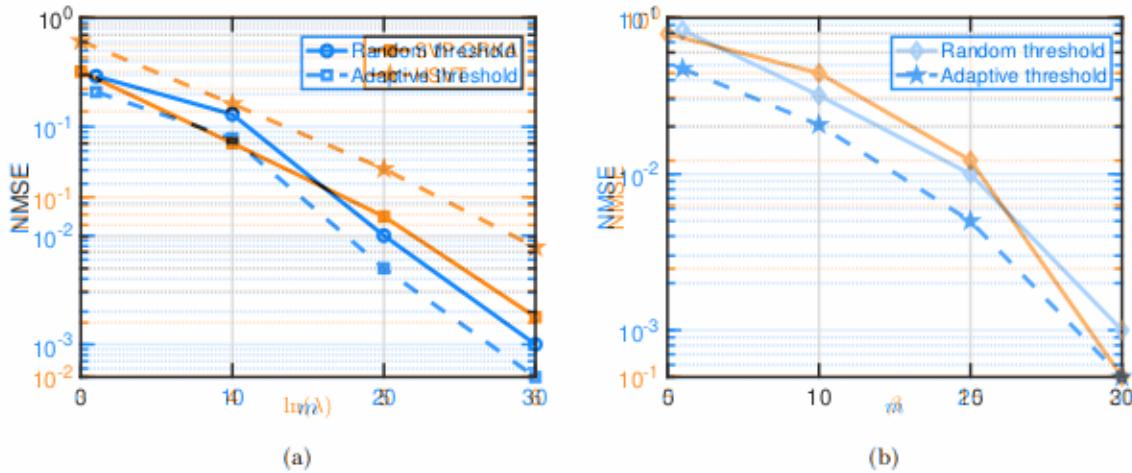


Figure 43. (a) Comparison between the recovery performance of SVP-ORKA and HSVT algorithm thresholds and adaptive thresholding in the sample abundance scenario. (b) Recovery performance of Algorithm ?? to ORKA with low-rank matrix sensing and (a) noiseless or (b) noisy CS over different values of sampling factor β .

of sparsity $s = 10$. The settings for time-varying sampling thresholds were considered to

be the same as the one-bit low-rank matrix sensing case. Fig. ??(b) displays the recovery performance of Block SKM using random thresholds in comparison with adaptive thresholds. Consistent with previous observations, the utilization of adaptive thresholds improves the recovery performance.

B. Sample Scarcity

Similar to sample abundance, herein we investigate the performance of our proposed methods for one-bit low-rank matrix sensing and one-bit CS when we have a limited number of samples. Note that in all experiments, the high-resolution measurements were

contaminated by the additive Gaussian noise with the standard deviation 0.1 except the (b) NBIHT in ditherless scenario.

case related to low-rank matrix sensing by SVP-ORKA which was considered to be noiseless.

One-bit low-rank matrix sensing. We generated a collection of sampling matrices $\{\mathbf{A}_n\}_{n=1}^N$ where each entry is independently sampled from a standard normal distribution. The desired matrix $\mathbf{X}_* \in \mathbb{R}^{30 \times 30}$ was generated with $\text{rank}(\mathbf{X}_*) = 2$. Define the oversampling factor as $\lambda = N/\beta$. In Fig. ??(b), we compare the recovery performance of HT-ORKA, ST-ORKA and the NBIHT algorithm [7]. In our experiments, we have set $\log(\lambda) \in \{3, 4, 5, 6\}$. The numbers of time-varying sampling threshold sequences was fixed at 10. The generation of super-recovery performance compared to ST-ORKA and the NBIHT method. The generation of time-varying sampling thresholds followed the same procedure as in the previous cases. Fig. ??(a) compares the recovery performance of SVP-ORKA with hard singular value thresholding (HSVT) algorithm [?] in the noiseless scenario. As can be observed, SVP-

X. DISCUSSION

In this paper, we have established the theoretical guarantees for uniform perfect reconstruction in dithered one-bit sensing. Our approach involves transforming the one-bit signal reconstruction problem into a linear feasibility problem. We then introduced the FVP theorem to analyze the possibility of creating a finite volume formed by the hyperplanes around the original signal. The FVP theorem allows us to determine the minimum probability of achieving perfect reconstruction and the required number of samples to attain uniform perfect reconstruction. What sets this theorem apart from others, such as the random hyperplane tessellations theorem, is that it approaches one-bit sensing from the perspective of a linear feasibility problem. It investigates the number of samples needed to capture the original signal within the space of hyperplanes.

Figure 4. (a) Comparison between the recovery performance of SVP-ORKA and HSVT algorithm over different values of oversampling factor λ . (b) Recovery performance of Algorithm ?? (ORKA with low-rank matrix factorization) over different values of sampling factor β .

Uniform dithering and restricted sampling matrices considered in previous efforts. Leveraging this theorem, we were able to derive theoretical guarantees for deterministic matrices like DCT as well. The only restriction in the FVP theorem is that the distances between the original signals and the surrounding hyperplanes defined in (??), should be considered as sub-Gaussian. It remains an open problem for other distributions, which may be explored in future research.

This work represents a pioneering effort in the literature, as it explores the performance of randomized algorithms in one-bit sensing for the first time. We introduced two novel variations of the Kaczmarz algorithm, PrSKM and Block SKM, which served as the foundation for our proposed algorithm, ORKA. In our investigation of ORKA, we analyzed its upper recovery bound and demonstrated that it decays concerning the number of measurements. Specifically,

for both compressed sensing and low-rank matrix recovery, the decay rate is $\mathcal{O}(m'^{-\frac{2}{3}})$. These findings contribute valuable insights into the potential of randomized algorithms in one-bit sensing applications. To the best of our knowledge, we are the first to derive the convergence

ORKA RKA performs HSVT in different systems. This finding highlights the robustness of the algorithm setting gain the presence of this performance of Algorithm ?? (ORKA with low-rank matrix factorization) improved update process for designing the thresholding process (X). Unlike the sigma-delta thresholding, it signs discrete and therefore the sparsity is not applicable. Note that in this updating defined bit data sampling fact the used for feedback. In the sampling scheme, through some (local) iterations, we showed in Fig. ?? (d) the thresholding process can be an algorithm ?? enhancement. The performance of the oversampling through numerical experiments, we experimentally

Demonstrated that the proposed algorithm exhibits super-AoR reconstruction performance by dithering GS in standard the NBIHT distribution with the Flesske noise signal. BIHT is adapted with dithering for the sparsity level of 5. Furthermore, the overampling factor is set to 100 (selected SVPs). ORKA output performance is compared with the HSYTA algorithm over sampling factors of 10, 50, 100, 200. The number of time-varying sampling thresholds is set to 100, and the iteration of quantizing sampling addressed is followed by the sparsest projections. For the previous we further developed the proposed randomized algorithm performances of HT-ORKA, ST-ORKA, and BIHT algorithms with a low-tolerance factor of 10. This threshold is sufficient for efficient HT-ORKA. It is noted that the proposed algorithms push the limits of the dithering of the sparse signal in the practical setting. The In the next example, we extend the low-rank matrix factorization proposal algorithm, HT-ORKA, and ST-ORKA to sparse signals using cyclic algorithms of ATOMS, which can generate shift-invariant signals. A going topic was generated in the literature [3] manner as in the previous example. The number of high-resolution samples was considered to be $n \in \{1000, 1500, 2000, 2500\}$. The desired signal $\mathbf{x}_* \in \mathbb{R}^{256}$ was assumed to have a sparsity level of $s = 25$. In Fig. ??(b), we compare the recovery performance of HT-ORKA, ST-ORKA and the NBIHT algorithm [?]. Once again, similar to the previous example, HT-ORKA exhibits superior performance than the other two methods. The proposed algorithm also has the NP-hardability for all $\mathbf{x} \in \mathbb{R}^n$ [?].

$$(1 - \epsilon) \|\mathbf{C}\mathbf{x}\|_2^2 \leq \|\mathbf{N}^\top \mathbf{C}\mathbf{x}\|_2^2 \leq (1 + \epsilon) \|\mathbf{C}\mathbf{x}\|_2^2. \quad (67)$$

As studied in [?], we can use this result to show that the preconditioned system \mathbf{CR}_s^{-1} preserves the length of vectors in the range of \mathbf{C} . To see how substitute \mathbf{x} in (22) with $\mathbf{R}_s^{-1}\mathbf{z}$. By the subspace embedding property problem into a linear feasibility problem. We then introduced the FVP theorem to analyze the possibility of creating a finite-volume formed by the hyperplanes around the original signal. The FVP theorem allows us to determine the minimum probability of achieving perfect reconstruction and the required number of samples to attain uniform perfect reconstruction. What sets this theorem apart from others, such as the random hyperplane tessellations theorem, is that it approaches one-bit sensing from which bounds the condition number $\varrho(\mathbf{CR}_s^{-1}) \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}$. Setting $\epsilon = \frac{1}{2}$, we have $\varrho(\mathbf{CR}_s^{-1}) \leq \sqrt{3}$ the perspective of a linear feasibility problem. It investigates the number of samples needed and based on (??), $\kappa(\mathbf{CR}_s^{-1}) \leq \sqrt{3n}$. Therefore, according to (??) and the result that we have to capture the original signal within the space of hyperplanes. An intriguing aspect of obtained here, one can conclude the convergence bound (??).

The FVP theorem is its capability to provide guarantees not only for Uniform dithering and restricted sampling matrices considered in previous efforts. Leveraging this theorem, we were able to derive theoretical guarantees for deterministic matrices like DCT as well.

The only restriction in the FVP theorem is that the distances between the original signals and the corresponding hyperspheres defined by $\|\mathbf{x} - \mathbf{x}_i\|_2^2 \leq \lambda_i^2$ should be sparse Gaussian [22]. It remains an open problem for other distributions, which may be explored in future research.

The Block SKM algorithm can be considered to be a special case of the more general *sketch-and-project* method with a sparse block sketch matrix as defined in [?]. This work represents a pioneering effort in the literature, as it explores the performance of randomized algorithms in one-bit sensing for the first time. We introduced two novel variations of the Kaczmarz algorithm, PrSKM and Block SKM, which served as the foundation for our proposed algorithm, CORKA. In block sensing, at each iteration, the sketch matrix \mathbf{B} is updated similarly to step 3 of Algorithm ??, and the second step of sketching and projection follows the Mitz Specimally, where the compressed sensing in each iteration is matrix recovery, the decay rate is $\mathcal{O}\left(m'^{-\frac{2}{3}}\right)$. These findings contribute valuable insights into the potential of randomized algorithms in one-bit sensing applications. To the best of our knowledge, we are the first to derive the convergence rate of RKA for a noisy linear inequality system of Algorithm ?? with $k' = 1$. In the Block SKM algorithm, the sketch matrix is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{n \times p} & \text{[improved update process for designing]} & \mathbf{G}^{T,n} \end{bmatrix}^T, \quad \mathbf{S} \in \mathbb{R}^{mn \times n}, \quad (72)$$

Unlike the sigma-delta thresholding design discussed in references [?, ?], our approach does not require updating the one-bit data, eliminating the need for feedback in the sampling scheme. Through numerical demonstrations, we showed that this adaptive thresholding process enhances signal reconstruction performance from one-bit data. Through numerical Gaussian sketch matrix as follows:

experiments, we experimentally demonstrated that the proposed algorithm exhibits superior

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{n \times p} & \mathbf{G}^{T,n} \end{bmatrix}^T, \quad \mathbf{S} \in \mathbb{R}^{mn \times n}, \quad (73)$$

and BIHT adapted with dithering for the dithered scenario. Furthermore, our results show that the proposed SVP-ORKA outperforms the HSVT algorithm in terms of recovery accuracy. In this framework, we are able to provide some theoretical guarantees by taking advantage of the favorable properties of Gaussian random variables. Assume that \mathcal{P}_x denotes a nonempty solution set of $\mathbf{Bx} \geq \mathbf{b}$ and $\mathbf{x} \in \mathcal{P}_x$. Considering the sparse Gaussian sketch (??) which satisfies the set

we also address scenarios with sample restrictions. For these scenarios, we further develop the proposed randomized algorithms into storage-friendly approaches, such as random sketching and low-rank matrix factorization. This extension allows for efficient handling of limited samples, broadening the applicability of the algorithms to a wider range of practical settings. The convergence rate of ORKA with low-rank matrix factorization remains an open problem. This is because it employs the idea of cyclic algorithms or AltMin, whose convergence is still an ongoing topic of research in the literature [?].

of inequalities $\mathbf{S}^\top \mathbf{Bx} \succeq \mathbf{S}^\top \mathbf{b}$, we have, Appendix A

$$\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2 = \text{Proof of Theorem ??}$$

Since $\mathbf{N} \in \mathbb{R}^{m \times s}$ is a Gaussian matrix with sketch size $s = \Theta(n^2/\epsilon^2)$ and $\epsilon = \frac{1}{2}$, the ϵ -subspace embedding property holds with high probability for all $\mathbf{x} \in \mathbb{R}^n$ [?],

$$(1 - \epsilon) \|\mathbf{Cx}\|_2^2 \leq \|\mathbf{N}^\top \mathbf{Cx}\|_2^2 \leq (1 + \epsilon) \|\mathbf{Cx}\|_2^2. \quad (67)$$

As studied in [?], we can use this result to show that the preconditioned system \mathbf{CR}_s^{-1} preserves the length of vectors in the range $\|\mathbf{SCB}\|_2^2$ (see how, substitute \mathbf{x} in (??) with $\mathbf{R}_s^{-1}\mathbf{z}$). By the subspace embedding property,

$$\frac{2((\mathbf{S}^\top \mathbf{b})_{j_i^*} - (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i)^+ (\mathbf{S}^\top \mathbf{B})_{j_i^*}}{1 + \epsilon} \|\mathbf{N}^\top \mathbf{CR}_s^{-1} \mathbf{z}\|_2^2 \leq \|\mathbf{CR}_s^{-1} \mathbf{z}\|_2^2 \leq \frac{1}{1 - \epsilon} \|\mathbf{N}^\top \mathbf{CR}_s^{-1} \mathbf{z}\|_2^2. \quad (68)$$

Now using $\|\mathbf{N}^\top \mathbf{CR}_s^{-1} \mathbf{z}\|_2^2 \leq \|\mathbf{Q}_i \mathbf{z}\|_2^2$ the relation (??) becomes, therefore, the left-hand side of (??) is less than or equal to,

$$\frac{1}{1 + \epsilon} \|\mathbf{z}\|_2^2 \leq \|\mathbf{CR}_s^{-1} \mathbf{z}\|_2^2 \leq \frac{1}{1 - \epsilon} \|\mathbf{z}\|_2^2, \quad (69)$$

which bounds the condition number $\varrho(\mathbf{CR}_s^{-1}) \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}$. Setting $\epsilon = \frac{1}{2}$, we have $\varrho(\mathbf{CR}_s^{-1}) \leq \sqrt{3}$ and based on (??), $\kappa(\mathbf{CR}_s^{-1}) \leq \sqrt{3n}$. Therefore, according to (??) and the result that we have obtained here, one can conclude the convergence bound (??).

$$\frac{2((\mathbf{S}^\top \mathbf{b})_{j_i^*} - (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i)^+ ((\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i - (\mathbf{S}^\top \mathbf{b})_{j_i^*})}{\|\mathbf{A}(\mathbf{S}^\top \mathbf{B})_{j_i^*}\|_2^2} \quad (75)$$

Convergence analysis of Block SKM with the sparse Gaussian sketch (??)

The Block SKM algorithm can be considered to be a special case of the more general sketch-and-project method with a sparse block sketch matrix as defined in [?].

Based on the definition of j_i^* , one can rewrite (??) as

$$\underset{\mathbf{x}}{\arg\min} \|\mathbf{x} - \mathbf{x}_i\|_2^2 \text{ subject to } \mathbf{S}^\top \mathbf{Bx} \succeq \mathbf{S}^\top \mathbf{b}, \quad (70)$$

where $\mathbf{S} \in \mathbb{R}^{mn \times n}$ is the sketch matrix choosing a block uniformly at random from the main matrix \mathbf{B} similar to step 1 of Algorithm ??.

Following the Motzkin's proof, the row index j_i^* is chosen in i -th iteration as

$$\mathbb{E}_{\mathbf{S}} \left\{ \|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2 \right\} \leq \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 - \mathbb{E}_{\mathbf{S}} \left\{ \frac{\|(\mathbf{S}^\top \mathbf{b} - \mathbf{S}^\top \mathbf{Bx}_i)^+\|_\infty^2}{\|\mathbf{S}^\top \mathbf{B}\|_2^2} \right\}. \quad (71)$$

with $(\cdot)_j$ denoting the j -th row of the matrix/vector argument (this step is similar to steps 2-4 of Algorithm ?? with $k' = 1$). In the Block SKM algorithm, the sketch matrix is given by

$$\mathbb{E}_{\mathbf{S}} \left\{ \left\| \begin{pmatrix} \mathbf{S}^\top \mathbf{B} \\ \mathbf{0}_{n \times p} \end{pmatrix} \mathbf{I}_n^2 \right\|_2^2 \right\} = \sum_{l=1}^d \mathbb{E}_{\mathbf{S}} \left\{ \left(\sum_{i=1}^{mn} \mathbf{S}_{j_i^* l} \mathbf{B}_{lk} \right)^2 \right\}, \quad (72)$$

or equivalently, in terms of Grin (??). Note that the literature does not offer any theoretical guarantees for the convergence of the Block²SKM with the identity matrix [?]. To derive our theoretical guarantees for the Block SKM algorithm, we change the sketch matrix to the sparse Gaussian sketch matrix as follows:

$$\mathbf{S} = \left[\begin{array}{c|c} \mathbf{0}_{n \times p} & \mathbf{G} \\ \hline & \mathbf{0}_{n \times (mn+1-n-p)} \end{array} \right] \sum_{k=1}^n \sum_{l=1}^n \mathbb{E}_G \left\{ \left(\mathbf{G}_{j_l k}^\top \mathbf{B}_{lk} \right)^2 \middle| \mathbf{S} \in \mathbb{R}^{mn \times n} \right\}, \quad (73)$$

where $\mathbb{E}_G \left\{ \left(\mathbf{G}_{j_l k}^\top \mathbf{B}_{lk} \right)^2 \right\} = 1$, which helps to simplify (??) as the entries are i.i.d. following the distribution $\mathcal{N}(0, 1)$. In this framework, we are able to provide some theoretical guarantees by taking advantage of the favorable properties of Gaussian random variables. Assume that \mathcal{P}_x denotes a nonempty solution set of $\mathbf{B}\mathbf{x} \succeq \mathbf{b}$ and $\hat{\mathbf{x}} \in \mathcal{P}_x$. Considering the sparse Gaussian sketch where $\hat{\mathbf{B}}$ is the $n \times d$ submatrix of \mathbf{B} (one of the candidates of \mathbf{B}_i in Algorithm ??). Due to the fact that the second term in the right-hand side of (??) is an expectation over the convex function $f(x, y) = x^2/y$, we can apply Jensen's inequality as follows:

$$\mathbb{E}_S \left\{ \frac{\left\| (\mathbf{S}^\top \mathbf{b} - \mathbf{S}^\top \mathbf{B}\mathbf{x}_i) \right\|_\infty^2}{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2} \right\} \geq \frac{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2 \left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2}{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2}. \quad (81)$$

Consider the following lemma regarding the estimate of the maximum of independent normal random variables:

$$= \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 + \frac{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2}{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2} + \quad (74)$$

Lemma 7. [?, Section 2.5.2] Let $X(\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i$ be independent $\mathcal{N}(0, 1)$ random variables. Then we have

$$\mathbb{E} \left\{ \max_{i \leq n} X_i^2 \right\} \geq c \sqrt{\log n}, \quad (82)$$

We can observe that $(\mathbf{S}^\top \mathbf{B})_{j_i^*} (\mathbf{x}_i - \hat{\mathbf{x}}) \leq (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i - (\mathbf{S}^\top \mathbf{b})_{j_i^*}$, therefore, the left-hand side of (??) is less than or equal to,

$$\text{By taking advantage of Lemma ??, we have } \left((\mathbf{S}^\top \mathbf{b})_{j_i^*} - (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i \right)^+ \leq \left\| \mathbf{x}_i - \hat{\mathbf{x}} \right\|_2^2 + \frac{\left\| (\mathbf{S}^\top \mathbf{b} - \mathbf{S}^\top \mathbf{B}\mathbf{x}_i)^+ \right\|_\infty^2}{\left\| (\mathbf{S}^\top \mathbf{B})_{j_i^*} \right\|_2^2} \left\| \max_{t \in [n]} ((\mathbf{s}_t, \mathbf{b} - \mathbf{B}\mathbf{x}_i))^+ \right\|^2 + \quad (75)$$

$$2 \left((\mathbf{S}^\top \mathbf{b})_{j_i^*} - (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i \right)^+ \left(\mathbb{E}_S \left\{ \max_{t \in [n]} \langle \mathbf{s}_t, (\mathbf{b} - \mathbf{B}\mathbf{x}_i) \rangle^+ \right\} \right)^2 \quad (83)$$

$$= \mathbb{E}_G \left\{ \max_{t \in [n]} \left\langle \mathbf{g}_t, \left(\hat{\mathbf{b}} - \hat{\mathbf{B}}\mathbf{x}_i \right)^+ \right\rangle \right\}$$

$$= \left\| \left((\mathbf{S}^\top \mathbf{b})_{j_i^*} - (\mathbf{S}^\top \mathbf{B})_{j_i^*} \mathbf{x}_i \right)^+ \right\|_2^2 \sqrt{\log n},$$

Based on the definition of $\hat{\mathbf{b}}$, \mathbf{b}_i and \mathbf{g}_i are the i -th columns of \mathbf{S} and \mathbf{G} , respectively, $[n] = \{1, 2, \dots, n\}$, and c is a positive value. By plugging the inequality (??) into (??) and using the Hoffman bound [?, Theorem 4.2], we have

$$\|(\mathbf{S}^\top \mathbf{B})_{j_i^*} - \hat{\mathbf{B}} \mathbf{x}_i\|_\infty^2. \quad (76)$$

$$\mathbb{E}\{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2\} \leq \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 - \frac{c\|(\mathbf{S}^\top \mathbf{B})_{j_i^*}\|_2^2 \log n}{\|\hat{\mathbf{B}}\|_F^2}.$$

By taking the expectation over the error, we have

$$\begin{aligned} \mathbb{E}_{\mathbf{S}}\{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2\} &\leq \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 - \mathbb{E}_{\mathbf{S}}\left\{\frac{c\sigma_{\min}^2(\mathbf{S}^\top \mathbf{B})_{j_i^*} + \|\hat{\mathbf{B}}\|_F^2}{\|\hat{\mathbf{B}}\|_F^2}\right\}^2 \\ &\leq \left(1 - \frac{c\sigma_{\min}^2(\hat{\mathbf{B}}) \log n}{\|\hat{\mathbf{B}}\|_F^2}\right) \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2, \end{aligned} \quad (77)$$

In addition, we have that

$$\mathbb{E}_{\mathbf{S}}\left\{\left\|(\mathbf{S}^\top \mathbf{B})_{j_i^*}\right\|_2^2\right\} = \sum_{k=1}^n \mathbb{E}_{\mathbf{S}}\left\{\left(\sum_{l=1}^n \mathbf{S}_{j_i^* k} \mathbf{B}_{lk}\right)^2\right\}, \quad (78)$$

$$\mathbb{E}\{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2\} \leq \left(1 - \frac{c\sigma_{\min}^2(\hat{\mathbf{B}}) \log n}{\|\hat{\mathbf{B}}\|_F^2}\right) \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2. \quad (85)$$

$$\sum_{k=1}^d \mathbb{E}_{\mathbf{G}}\left\{\left(\sum_{l=1}^n \mathbf{G}_{j_i^* l}^\top \mathbf{B}_{lk}\right)^2\right\} = \text{APPENDIX C}$$

To prove Theorem ??, define the set $\mathcal{T} = \left\{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_2 \leq 1\right\}^2$. Assume that $\mathbb{E}\{T_{\text{ave}}\} = \mu$ and $\left\|d_j^{(\ell)}\right\|_{\psi_2}^2 \leq K$. Then our goal is to obtain the required number of one-bit samples m' to achieve the perfect reconstruction criterion with $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$ considering all $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$. The condition $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$ or equivalently $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ implies a creation of finite-volume in \mathbb{R}^d with a maximum radius of ρ . To see the connection between this condition, $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$

$$\sum_{k=1}^d \sum_{l=1}^n \mathbf{B}_{lk}^2 = \|\mathbf{B}\|_F^2, \quad (80)$$

and T_{ave} , there exists a set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d^2$ which contains

where $(\hat{\mathbf{B}})$ is the minimum distances of \mathbf{B} such that one of the candidates of \mathbf{B} or in the complement of the

set \mathcal{G} denoted by $\bar{\mathcal{G}}$, we have $d_j^{(\ell)} \geq \rho$ for all $j \in \bar{\mathcal{G}}$ which leads to an expectation over the

function $f(x, y) = x^2/y$. We can apply Jensen's inequality as follows:

$$\mathbb{E}_{\mathbf{S}}\left\{\left\|(\mathbf{S}^\top \mathbf{B})_{j_i^*} - \hat{\mathbf{B}} \mathbf{x}_i\right\|_\infty^2\right\} \geq \left(\mathbb{E}_{\mathbf{S}}\left\{\left\|(\mathbf{S}^\top \mathbf{B})_{j_i^*}\right\|_2^2\right\}\right)^2. \quad (81)$$

Lemma 8. In \mathbb{R}^d , with probability $\mathbb{P}\left\{\left\|(\mathbf{S}^\top \mathbf{B})_{j_i^*}\right\|_2^2\right\}$ there exists a set $\left\{\left\|(\mathbf{S}^\top \mathbf{B})_{j_i^*}\right\|_2^2\right\}$ with the minimum cardinality $\text{card}(\mathcal{G}) \sim d$ which contains $\text{card}(\mathcal{G})$ minimum distances $d_j^{(\ell)}$ such that $\frac{1}{\text{card}(\mathcal{G})} \sum_{j \in \mathcal{G}} d_j^{(\ell)} \sim \rho$. Consider the following lemma regarding the estimate of the maximum of independent normal random variables:

Proof: For simplicity denote $T_{\text{ave}}^G = \frac{1}{\text{card}(\mathcal{G})} \sum_{j \in \mathcal{G}} d_j^{(\ell)}$. If $\exists \rho' > \rho$ such that $T_{\text{ave}}^G \approx \rho'$, Lemma 7. [?, Section 2.5.2] Let X_1, \dots, X_n be independent $\mathcal{N}(0, 1)$ random variables. Then then $\exists \mathcal{G}' \subseteq \mathcal{G}$ such that $T_{\text{ave}}^{\mathcal{G}'} \sim \rho$ with $\text{card}(\mathcal{G}') < \text{card}(\mathcal{G}) \sim d$. This property, $\text{card}(\mathcal{G}') <$ we have

$$\mathbb{E}\left\{\max_{i \leq n} X_i\right\} \geq c\sqrt{\log n}. \quad (82)$$

²In \mathbb{R}^d , we need at least $d + 1$ number of hyperplanes to create a finite-volume space.

where \mathcal{G} is a contradiction of a finite-volume with the maximum radius of ρ in \mathbb{R}^d .

Therefore, we can only assume that $T_{\text{ave}}^{\mathcal{G}} < \rho$. Define a set $\tilde{\mathcal{G}} = \mathcal{G} \cup \bar{\mathcal{G}}$ such that $T_{\text{ave}}^{\tilde{\mathcal{G}}} \sim \rho$, where

By taking advantage of Lemma ??, we have

$\bar{\mathcal{G}}$ is a set with J minimum distances of the set $\tilde{\mathcal{G}}$. If $J < \text{card}(\bar{\mathcal{G}})$, then such set $\hat{\mathcal{G}}$ exists which

partitions all distances $d_j^{(\ell)}$ into $\hat{\mathcal{G}}$ with $T_{\text{ave}}^{\hat{\mathcal{G}}} \sim \rho$ and $\bar{\mathcal{G}} \setminus \hat{\mathcal{G}}$ with $T_{\text{ave}}^{\bar{\mathcal{G}} \setminus \hat{\mathcal{G}}} \geq \rho$. The existence of the

set $\hat{\mathcal{G}}$ with $\text{card}(\hat{\mathcal{G}}) \sim d + J$ such that $T_{\text{ave}}^{\hat{\mathcal{G}}} \sim \rho$ implies the creation of a finite-volume space

with the maximum radius of ρ in \mathbb{R}^d . If $J > \text{card}(\bar{\mathcal{G}})$, then the set $\hat{\mathcal{G}} \cup \bar{\mathcal{G}}$ does not exist which informs

$T_{\text{ave}} < \rho$. Since $m' \geq \text{card}(\mathcal{G}) \sim d$, the result $T_{\text{ave}} < \rho$ implies the creation of a finite-volume

space in \mathbb{R}^d with the maximum radius less than ρ with high probability. In fact, one can apply

the general Hoeffding's inequality [?, Theorem 2.6.2] to the random variable T_{ave} as follows:

$$\Pr(T_{\text{ave}} \leq \mu - \delta) \leq e^{-\frac{c_1 \delta^2 m'}{K}} \quad (86)$$

where $\hat{\mathbf{b}} \in \mathbb{R}^n$ is a block of \mathbf{b} , \mathbf{s}_t and \mathbf{g}_t are the t -th columns of \mathbf{S} and \mathbf{G} , respectively,

where $\{1, 2, \dots, c_1 n\}$ are positive consecutive values. By plugging this probability ?? into ?? and

using the Hoffman-Banach theorem [?], we have $\geq \text{card}(\mathcal{G}) \sim d$, we can conclude that

$T_{\text{ave}} \geq \rho$ with high probability. This, in turn, indicates $\|\hat{\mathbf{b}} - \hat{\mathbf{B}}\|_F^2 \leq \rho^2$ for set \mathcal{G} with the minimum

cardinality $\text{card}(\mathcal{G}) \sim d$ such that $T_{\text{ave}}^{\mathcal{G}} \sim \rho$. ■

Based on Lemma ??, existing a set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d$ such that

$\frac{1}{\text{card}(\mathcal{G})} \sum_{d_j^{(\ell)} \in \mathcal{G}} d_j^{(\ell)} \sim \rho$ with high probability informs that $\|\hat{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$. To meet $\|\hat{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$,

$\exists C > 0$ such that $T_{\text{ave}} \leq C\rho$, where $C\rho = \frac{c\sigma_{\min}^2(\hat{\mathbf{B}}) \log n}{\|\hat{\mathbf{B}}\|_F^2}$ with a positive constant c and δ . To derive the

probability of the event $\|\hat{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ for a specific $\mathbf{x}_* \in \mathcal{T}$, we apply the general Hoeffding's

which can be recast as the following convergence rate, after K updates:

$$\mathbb{E}\{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2\} \leq \left(1 - \frac{c\sigma_{\min}^2(\hat{\mathbf{B}}) \log n}{\|\hat{\mathbf{B}}\|_F^2}\right)^K \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2. \quad (87)$$

Consider the following lemma:

Appendix C

Lemma 9. Define $T_{\text{ave}}(\mathbf{x}_*)$ as in (??). Then for any $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$ we have

To prove Theorem ??, define the set $\mathcal{T} = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\|_2 \leq 1\}$. Assume that $\mathbb{E}\{T_{\text{ave}}\} = \mu$

and $\|\mathbf{d}_j^{(\ell)}\|_{\psi_2}^2 \leq K$. Then our goal is to obtain the required number of one-bit samples

where $\mathbf{e}_A = \frac{1}{n} \sum_{j=1}^n \|\mathbf{a}_j\|_2$ perfect reconstruction criterion with $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$ considering all

$\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$. The condition $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$ or equivalently $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ implies a creation of finite-

Proof: For any $j \in [n]$, $\ell \in [m]$ and any $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$ we have

volume in \mathbb{R}^d with a maximum radius of ρ . To see the connection between this condition, $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$, and T_{ave} , there exists a set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d^2$

$$\leq |\tau_j^{(\ell)}| + \|\mathbf{a}_j\|_2 \|\mathbf{x}_* - \bar{\mathbf{x}}\|_2,$$

²In \mathbb{R}^d , we need at least $d + 1$ number of hyperplanes to create a finite-volume space.

which contains step 3(G) derived based distance Cauchy such that inequality $\sum_{d_j^{(\ell)} \in \mathcal{G}} d_j^{(\ell)}$. By averaging the left and the right-hand sides of (22) over ball $\bar{\mathcal{G}}$, we have $d_j^{(\ell)} \geq \rho$ for all $d_j^{(\ell)} \in \bar{\mathcal{G}}$ which leads to $\frac{1}{\text{card}(\mathcal{G})} \sum_{d_j^{(\ell)} \in \bar{\mathcal{G}}} d_j^{(\ell)} \geq \rho$. Therefore, we can conclude that $T_{\text{ave}} \geq \rho$. To see why the set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d$ exists such that $\frac{1}{\text{card}(\mathcal{G})} \sum_{d_j^{(\ell)} \in \mathcal{G}} d_j^{(\ell)} \approx \rho$, we present (90) the following lemma:

$$= \frac{1}{m'} \sum_{\ell=1}^m \|\boldsymbol{\tau}^{(\ell)}\|_1 + c_A \|\mathbf{x}_* - \bar{\mathbf{x}}\|_2,$$

Lemma 8. In \mathbb{R}^d , with high probability there exists a set \mathcal{G} with the minimum cardinality where $T_{\text{ave}}(\mathbf{x}_* - \bar{\mathbf{x}})$ contains $\frac{1}{\text{card}(\mathcal{G})} \sum_{j=1}^{m'} d_j^{(\ell)}$ minimum distances $d_j^{(\ell)}$ such that $T_{\text{ave}}(\mathbf{x}_* - \bar{\mathbf{x}})$ and $T_{\text{ave}}(\mathbf{x}_*)$ are related because $\|\mathbf{x}_*\|_2 \leq \rho$.

$$\left| a_j (\mathbf{x}_* - \bar{\mathbf{x}}) - \tau_j^{(\ell)} \right| = \left| a_j \mathbf{x}_* - \tau_j^{(\ell)} - a_j \bar{\mathbf{x}} \right|$$

Proof: For simplicity denote $T_{\text{ave}}^{\mathcal{G}} = \frac{1}{\text{card}(\mathcal{G})} \sum_{d_j^{(\ell)} \in \mathcal{G}} d_j^{(\ell)}$. If $\exists \rho' > \rho$ such that $T_{\text{ave}}^{\mathcal{G}} \sim \rho'$ then $\exists \mathcal{G}' \subseteq \mathcal{G}$ such that $T_{\text{ave}}^{\mathcal{G}'} \sim \rho$ with $\text{card}(\mathcal{G}') < \text{card}(\mathcal{G}) \sim d$. This property, $\text{card}(\mathcal{G}') < \text{card}(\mathcal{G}) \sim d$, contradicts the creation of a finite-volume with the maximum radius of ρ in \mathbb{R}^d . Therefore, we can only assume that $T_{\text{ave}}^{\mathcal{G}} < \rho$. Define a set $\tilde{\mathcal{G}} = \mathcal{G} \cup \bar{\mathcal{G}}$ such that $T_{\text{ave}}^{\tilde{\mathcal{G}}} \sim \rho$, where $\tilde{\mathcal{G}}$ is a set with J minimum distances of the set $\bar{\mathcal{G}}$. If $J < \text{card}(\bar{\mathcal{G}})$, then such set $\tilde{\mathcal{G}}$ exists which partitions all distances $d_j^{(\ell)}$ into $\tilde{\mathcal{G}}_{j,\ell}$ with $T_{\text{ave}}^{\tilde{\mathcal{G}}_{j,\ell}} \sim \rho$ and $\tilde{\mathcal{G}} \setminus \tilde{\mathcal{G}}_{j,\ell}$ with $T_{\text{ave}}^{\tilde{\mathcal{G}} \setminus \tilde{\mathcal{G}}_{j,\ell}} \geq \rho$. The existence of the set $\tilde{\mathcal{G}}$ with $\text{card}(\tilde{\mathcal{G}}) \leq T_{\text{ave}}(\mathbf{x}_*) + c_A \|\mathbf{x}_*\|_2$ implies the creation of a finite-volume space with the maximum radius of ρ in \mathbb{R}^d . If $J > \text{card}(\bar{\mathcal{G}})$, the set $\tilde{\mathcal{G}}$ does not exist which informs $T_{\text{ave}} < \rho$. Since $m' \geq \text{card}(\mathcal{G}) \sim d$ the result $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ implies the creation of a finite-volume space in \mathbb{R}^d with the maximum radius less than ρ with high probability. In fact, one can apply the general Hoeffding's inequality [?, Theorem 2.6.2] to the random variable T_{ave} as follows:

$$\Pr(T_{\text{ave}} \geq \rho) \leq e^{-c_1 \delta^2 m'}, \quad (93)$$

Now to include all possible $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$ that satisfy the uniform perfect reconstruction criterion (86) with $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$, we consider a ρ -net $\{\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_K\}$ for the set \mathcal{T} which means that, for any $\mathbf{x}_* \in \mathcal{T}$, one can find $k \in [1 : K]$ such that $\|\mathbf{x}_* - \bar{\mathbf{x}}_k\|_2 \leq \rho$. We can take $K \leq \left(1 + \frac{2}{\rho}\right)^d$ [?] and we have $T_{\text{ave}} \leq \rho$. Based on this result and the fact $m' \geq \text{card}(\mathcal{G}) \sim d$, we can conclude employ the union bound to obtain the general case of (??) as follows:

that $T_{\text{ave}} \geq \rho$ with high probability. This, in turn, indicates the existence of a set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d$ such that $T_{\text{ave}}^{\mathcal{G}} \sim \rho$. ■

Based on Lemma ??, existing a set \mathcal{G} with the minimum cardinality $\text{card}(\mathcal{G}) \sim d$ such that $\frac{1}{\text{card}(\mathcal{G})} \sum_{d_j^{(\ell)} \in \mathcal{G}} d_j^{(\ell)} \sim \rho$ with high probability informs that $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$. To meet (94) $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$, $\exists C > 0$ such that $T_{\text{ave}} \leq C\rho$, where $C\rho = \mu + \delta$ with a positive constant δ . To derive the probability of the event $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ for a specific $\mathbf{x}_* \in \mathcal{T}$, we apply the

$$\leq e^{\frac{3d}{\rho} - \frac{c_1(C\rho - \mu)^2}{K} m'}.$$

Consider Hoeffding's inequality (??). The achievable rate $T_{\text{ave}} \leq C\rho$ as follows sufficient to ensure that the upper bound of (??) is lower than $\frac{c_1 \delta^2}{K} m'$, which results in

$$\Pr(T_{\text{ave}} \geq C\rho) \leq e^{-\frac{c_1 \delta^2}{K} m'}. \quad (87)$$

Consider the following lemma: $m' \geq \frac{K}{c_1} \delta^{-2} \left(\frac{3d}{\rho} + \log \left(\frac{1}{\eta} \right) \right)$,

Therefore, the constant C_1 in Theorem ?? is $C_1 = \frac{K}{c_1}$. Then for any $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$ we have

$$T_{\text{ave}}(\mathbf{x}_*) \leq \frac{1}{m'} \sum_{\ell=1}^m \|\mathbf{a}_j^{(\ell)}\|_1 \Delta_{\mathbf{A}} (\|\mathbf{x}_* - \bar{\mathbf{x}}\|_2 + 1), \quad (88)$$

where $c_{\mathbf{A}} = \frac{1}{n} \sum_{j=1}^n \|\mathbf{a}_j\|_2$.

PROOF OF COROLLARY ??

To prove Corollary ??, we alternatively define the distances in (??) as

Proof: For any $j \in [n], \ell \in [m]$ and any $\mathbf{x}_*, \bar{\mathbf{x}} \in \mathcal{T}$ we have

$$d_{\mathbf{a}_j}(\mathbf{x}_*, \tau_j^{(\ell)}) = |\mathbf{a}_j \mathbf{x}_* - \tau_j^{(\ell)}| \leq |\mathbf{a}_j \mathbf{x}_*| + |\mathbf{a}_j \bar{\mathbf{x}}|, \quad (89)$$

Note that the definition presented in (??) is exactly the same with the one in (??) owing to the fact that each $\tau_j^{(\ell)}$ is either $+1$ or -1 . The dct coefficients $\{f_j = \mathbf{a}_j \mathbf{x}_*\}$ are defined as

left and the right-hand sides of (??) over all $j \in [n], \ell \in [m]$ we have

$$f_j = \sqrt{\frac{2}{d}} \sum_{i=1}^d \Lambda_i \cos \left(\frac{\pi i}{2d} (2j+1) \right) (\mathbf{x}_*)_i, \quad j \in [n], \quad (97)$$

where $\Lambda_i = \frac{1}{m'} \sum_{j,\ell=1}^{m'} |\mathbf{a}_j (\mathbf{x}_* - \bar{\mathbf{x}}) - \tau_j^{(\ell)}| \leq \frac{1}{m'} \sum_{\ell=1}^m \|\mathbf{a}_j^{(\ell)}\|_1 + \|\mathbf{x}_* - \bar{\mathbf{x}}\|_2 \left(\frac{1}{n} \sum_{j=1}^n \|\mathbf{a}_j\|_2 \right)$

$$\Lambda_i = \begin{cases} \frac{1}{\sqrt{d}} & \text{for } i = 0, \\ \frac{1}{m'} \sum_{\ell=1}^m \|\mathbf{a}_j^{(\ell)}\|_1 + c_{\mathbf{A}} \|\mathbf{x}_* - \bar{\mathbf{x}}\|_2, & \text{otherwise.} \end{cases} \quad (98)$$

Notethatwhenthe \mathbf{x}_* distances $\frac{1}{m'} \sum_{j,\ell=1}^{m'} |\mathbf{a}_j (\mathbf{x}_* - \bar{\mathbf{x}}) - \tau_j^{(\ell)}|$ are bounded. NotethatVariables \mathbf{x}_* and $\bar{\mathbf{x}}$ and $T_{\text{ave}}(\mathbf{x}_*)$ are related because $\Pr(T_{\text{ave}} \geq C\rho) \leq e^{-\frac{-2(C\rho-\mu)^2}{K} m'}$ [?, Theorem 2]. Therefore, to prove the Corollary ??, we only need to obtain the parameters μ and b . It is easy to verify that

$$b = \max \left\{ \left| \sup_{j \in [n]} f_j + b \right|, \inf_{j \in [n]} |f_j - b| \right\}. \quad (99)$$

To obtain the first term in (??), we need to compute $\sup_{j \in [n]} f_j$. By the assumption $\|\mathbf{x}_*\|_2 \leq 1$, we have $\|\mathbf{x}_*\|_1 \leq \sqrt{d} \|\mathbf{x}_*\|_2 \leq \sqrt{d}$. Therefore, $\sup_{j \in [n]} f_j$ can be computed as

$$\frac{1}{m'} \sum_{j,\ell=1}^{m'} |\mathbf{a}_j (\mathbf{x}_* - \bar{\mathbf{x}}) - \tau_j^{(\ell)}| \geq \frac{1}{m'} \sum_{j,\ell=1}^{m'} |\mathbf{a}_j \mathbf{x}_* - \tau_j^{(\ell)}| - c_{\mathbf{A}} \|\bar{\mathbf{x}}\|_2, \quad (92)$$

which informs $T_{\text{ave}}(\mathbf{x}_*) \leq T_{\text{ave}}(\mathbf{x}_* - \bar{\mathbf{x}}) + c_{\mathbf{A}} \|\bar{\mathbf{x}}\|_2$. Combining this result with (??) completes the proof.

$$\leq \sqrt{\frac{2}{d}} \sum_{i=1}^d \Lambda_i (\mathbf{x}_*)_i \leq \sqrt{\frac{2}{d}} \sum_{i=1}^d (\mathbf{x}_*)_i. \quad (100)$$

Based on Lemma ??, to ensure that the event $T_{\text{ave}} \leq C\rho = \mu + \delta$ implies $\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2 \leq \rho$ with a failure probability at most $e^{-\frac{-c_1 \delta^2}{K} m'}$, we should have

$$\delta \ll \frac{1}{m'} \sum_{\ell=1}^m \|\mathbf{a}_j^{(\ell)}\|_1 - \mu + c_{\mathbf{A}} (\rho + 1). \quad (93)$$

Therefore include all possible $\bar{\mathbf{x}}_k$. Similarly that satisfy the uniform perfect reconstruction criterion have $\bar{\mathbf{x}} \in \mathcal{B}_\rho(\mathbf{x}_*)$. For simplicity, denote $(\bar{\mathbf{x}}_j, (\mathbf{x}_*, \bar{\mathbf{x}}_K))$ for the set \mathcal{T} which means child, compute $\mathbf{x}_* \in \mathcal{T}$, one can find $k \in [1 : K]$ such that $\|\mathbf{x}_* - \bar{\mathbf{x}}_k\|_{2\ell} \leq \rho$. We can take $K \leq \left(1 + \frac{3}{\rho}\right)^d$ [?], and employ the union bound to obtain the general case of f_j as follows:

$$\begin{aligned} & \Pr(T_{\text{ave}} \geq C\rho \text{ for some } k \in [1 : K]) \\ & \leq K e^{-\frac{c_1(C\rho-\mu)^2}{K}m'} \mathbb{E} \left\{ \frac{1}{n} \sum_{j=1}^n |f_j - \tau_j^{(\ell)}| \right\}. \end{aligned} \quad (101)$$

Based on (??), we should consider three scenarios: (i) $-\tilde{b} \leq f_j \leq \tilde{b}$, (ii) $f_j \geq \tilde{b}$, and (iii) $f_j \leq -\tilde{b}$. For simplicity, denote $\tau_j^{(\ell)} = \tau$. Under scenario (i), $\mathbb{E}\{|f_j - \tau|\}$ can be computed as:

$$\leq e^{d \log\left(1 + \frac{3}{\rho}\right) - \frac{c_1(C\rho-\mu)^2}{K}m'}.$$

$$\mathbb{E}\{|f_j - \tau|\} = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} e^{\frac{3d}{\rho} + \frac{|f_j - \tau|^2}{C\rho - \tau}} d\tau.$$

Considering the bound on δ in (??), to achieve a minimum probability of $1 - \eta$, it is sufficient to ensure that the upper bound of (??) is lower than η which results in (102).

Under scenario (ii), we have $m' \geq \frac{K}{c_1} \delta^{-2} \left(\frac{3d}{\rho} + \log\left(\frac{1}{\eta}\right) \right)$, (95)

Therefore, the constant C_T in Theorem ?? is $|C_T| = \frac{K}{c_1} \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} f_j - \tau d\tau = f_j$, (103)

and in scenario (iii)

Appendix D

$$\mathbb{E}\{|f_j - \tau|\} = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} |f_j - \tau| d\tau = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} \tau - f_j d\tau = -f_j. \quad (104)$$

To prove Corollary ??, we alternatively define the distances in (??) as

Combining (??), (??) and (??) with (??) leads to (??) which completes the proof.

$$d_j^{(\ell)}(\mathbf{x}_*, \tau_j^{(\ell)}) = |\mathbf{a}_j \mathbf{x}_* - \tau_j^{(\ell)}|, \quad j \in [n], \ell \in [m]. \quad (96)$$

Note that the definition presented in APPENDIX E the same with the one in (??) owing to the fact that each $r_j^{(\ell)}$ is either PROOF OF THEOREM ?? coefficients $\{f_j = \mathbf{a}_j \mathbf{x}_*\}$ are defined as

To prove Corollary ??, assume $\|\mathbf{x}_*\|_2^2 \leq \sqrt{\frac{2}{d} \sum_i^d \Lambda_i \cos\left(\frac{\pi j}{2d}(2i+1)\right)}$. Then, it can be easily derived that $\mathbf{a}_j \mathbf{x}_* \sim \mathcal{N}(0, M')$. Due to the independency of $\mathbf{a}_j \mathbf{x}_*$ and $\tau_j^{(\ell)}$, $(\mathbf{a}_j \mathbf{x}_* - \tau_j^{(\ell)})$ is distributed as where Λ_i is $\left(\mathbf{a}_j \mathbf{x}_* - \tau_j^{(\ell)}\right) \sim \mathcal{N}(0, \alpha^2)$, where $\alpha^2 = M' + \sigma^2$. Consequently, each distance in (??) follows the folded normal distribution characterized by a location parameter zero and a scale parameter $\sqrt{\frac{1}{\alpha^2}}$ for $i = 0$, 1 otherwise. (98)

To complete the proof of Corollary ??, our remaining task is to determine the parameters μ and K . Note that when the distances defined in (??) are bounded random variables, $0 \leq d_j^{(\ell)}(\mathbf{x}_*, \tau_j^{(\ell)}) \leq b$ we will have $\Pr(T_{\text{ave}} \geq C\rho) \leq e^{-\frac{c_1(C\rho-\mu)^2}{K^2}m'}$ with the folded normal distribution as previously described. Corollary ?? we only need to obtain the parameters μ and b . It is easy to verify that the value of μ satisfies

$$\mu = \frac{b}{\alpha} \sqrt{\frac{2}{\pi}} = \max \left\{ \sqrt{\sup_{j \in [n]} (f_j^2 + \tilde{b}^2)}, \inf_{j \in [n]} f_j \sqrt{\frac{2}{\pi} (\tilde{b}^2 + 1)} \right\}. \quad (105)$$

To find the first term in (??) we use the Gaussian property of the folded normal random variable. To achieve this, we begin by expressing the sub-Gaussian norm of the Gaussian random variable

$$f_j = \sqrt{\frac{2}{d}} \sum_i \Lambda_i \cos\left(\frac{\pi j}{2d}(2i+1)\right) (\mathbf{x}_*)_i$$

$$\|X\|_{\psi_2} = \inf \left\{ t > 0 : \mathbb{E}\left\{ e^{X^2/t^2} \right\} \leq 2 \right\}, \quad (106)$$

which leads to

$$\leq \sqrt{\frac{2}{d}} \sum_i \Lambda_i (\mathbf{x}_*)_i \leq \sqrt{\frac{2}{d}} \sum_i (\mathbf{x}_*)_i \quad (100)$$

$$\|X\|_{\psi_2} = \inf \left\{ t > 0 : \frac{1}{\sqrt{\frac{2}{d}} \sum_i (\mathbf{x}_*)_i^2} \int_{-\infty}^{\infty} e^{x^2/t^2} e^{-x^2/(2\alpha^2)} dx \leq 2 \right\}$$

$$\leq \sqrt{\frac{2}{d}} \sum_i (\mathbf{x}_*)_i^2 = \sqrt{\frac{2}{d}} \|\mathbf{x}_*\|_1 \leq \sqrt{2}. \quad (107)$$

Therefore, $\sup_{j \in [n]} f_j = \sqrt{2}$. Similarly, we can obtain $\inf_{j \in [n]} f_j = -\sqrt{2}$. Based on (??), we [B, Page=28] this presented simplicity holding (??) leads to $\|X\|_{\psi_2} \leq c_2$. To obtain a positive, we should

To compute the value of c_2 that satisfies $\|X\|_{\psi_2} = c_2 \alpha$, we should find $\inf c_2 > 0$ such that

$$\mu = \mathbb{E}\left\{ \frac{1}{\sqrt{2\pi\alpha^2}} \sum_{j=1}^{m'} d_j^{(\ell)} e^{x^2/2\alpha^2} \mathbb{E}\left\{ \frac{1}{m'} \sum_{j=1}^{m'} |f_j - \tau_j^{(\ell)}| \right\} \right\} \quad (108)$$

or equivalently

$$\frac{1}{\sqrt{2\pi\alpha^2}} \int_{-\infty}^{\infty} e^{-x^2/(2\alpha^2)} \left[\frac{1}{m'} \sum_{j=1}^{m'} |f_j - \tau_j^{(\ell)}| \right] dx \leq 2. \quad (109)$$

Based on (??), we should consider three scenarios: (i) $-\tilde{b} \leq f_j \leq \tilde{b}$, (ii) $f_j \geq \tilde{b}$, and (iii) $f_j \leq -\tilde{b}$. For simplicity, denote $\tau_{j,\infty}^{(\ell)} = \tau_1$. Under scenario (i), $\mathbb{E}\{|f_j - \tau|\}$ can be computed as:

$$\frac{1}{\beta} \int_{-\infty}^{\tilde{b}} \frac{1}{\sqrt{2\pi\alpha^2}} e^{-u^2/(2\alpha^2)} du \leq 2, \quad (110)$$

which results in $c_2 \geq \sqrt{\frac{8}{3}}$. Therefore, we have $\|X\|_{\psi_2} = \sqrt{\frac{8}{3}}\alpha$. The probability density function (PDF) of the folded normal random variable $|X|$ with a location parameter zero and a scale parameter α^2 can be presented as $f_X(x) = \frac{1}{2\tilde{b}} \sqrt{\frac{2}{\pi\alpha^2}} \tilde{b}^{-x^2/(2\alpha^2)}$ for $x \geq 0$. Then, based on the definition of the sub-Gaussian norm, we have

$$\mathbb{E}\{|f_j - \tau|\} = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} |f_j - \tau| d\tau = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} f_j - \tau d\tau = f_j, \quad (102)$$

which is the same as the result we have obtained in (??). As a result, the sub-Gaussian norm of the folded normal random variable $|X|$ is $\|X\|_{\psi_2} = \sqrt{\frac{8}{3}}\alpha$. By setting $C_1 = \frac{8}{3c_1}$ and combining this result with (??), the proof is now complete.

$$\mathbb{E}\{|f_j - \tau|\} = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} |f_j - \tau| d\tau = \frac{1}{2\tilde{b}} \int_{-\tilde{b}}^{\tilde{b}} \tau - f_j d\tau = -f_j. \quad (104)$$

Combining (??), (??) and (??) with (??) leads to (??) which completes the proof.

Appendix E

PROOF OF COROLLARY ??

For simplicity of later analysis, assume $\|x\|_2^2 \leq d_j^{(\ell)s}$. Then we can derive that $a_j x_* \sim \mathcal{N}(0, M')$. Due to the independency of $a_j x_*$ and $\tau_j^{(\ell)}$, $(a_j x_* - \tau_j^{(\ell)})$ is distributed as $(a_j x_* - \tau_j^{(\ell)}) \sim \mathcal{N}(0, \alpha^2)$, where $\alpha^2 = \frac{T_{\text{ave}}^s}{M' + \sigma^2} \sum_{j,\ell=1}^{d_j^{(\ell)s}}$. Consequently, each distance in (??) follows the folded normal distribution characterized by a location parameter zero and a scale α .

Assume that $\mathbb{E}\{T_{\text{ave}}^s\} = \mu_s$ and $\|\mathbf{d}^{(\ell)s}\|_2^2 \leq K_s$. Fix an index set S of size s and identify the parameter α^2 . To complete the proof of Corollary ??, our remaining task is to determine the space $\Sigma_S = \{\mathbf{x} \in \mathbb{R}^n : \text{supp}(\mathbf{x}) \subseteq S, \|\mathbf{x}\|_2 \leq 1\}$ with \mathbb{R}^s . Then our goal is to obtain the required parameters μ and K based on the proof of Theorem ?? for each distance stated in (??). number of hyperplanes (one-bit data) to achieve the uniform perfect reconstruction criterion with It can be easily confirmed that, for each distance mentioned in (??) with the folded normal $\mathbf{x}' \in \mathcal{B}_\rho(\mathbf{x}_*)$ considering all $\mathbf{x}_*, \mathbf{x}' \in \Sigma_S$. Based on this criterion and the conclusion made in Lemma ??, to see the connection between $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$ and T_{ave}^s , there exists a set \mathcal{G}_s with the minimum cardinality $\text{card}(\mathcal{G}_s) \sqrt{\frac{2}{\pi}} \sqrt{\frac{\alpha^2 + M'}{m}}$ which contains $\text{card}(\mathcal{G}_s) \sqrt{\frac{2}{\pi}} \sqrt{\frac{\alpha^2 + M'}{m}}$ minimum distances $d_j^{(\ell)s}$ such that

To find $\sum_{j \in \mathcal{G}_s} d_j^{(\ell)s}$ we need to obtain the sub-Gaussian norm of the folded normal random variable X . Existing such a set \mathcal{G}_s with high probability implies $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$. For the complement of the set \mathcal{G}_s denoted by $\bar{\mathcal{G}}_s$ we have $d_j^{(\ell)s} \geq \rho$ for all $j \in \bar{\mathcal{G}}_s$ which leads to $\sum_{j \in \bar{\mathcal{G}}_s} d_j^{(\ell)s}$ characterized by $X \sim \rho_s \mathcal{N}(0, \alpha^2)$. Therefore, we can conclude that $T_{\text{ave}}^s \geq \rho_s$. To meet the condition $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$, $\exists C > 0$ such that $T_{\text{ave}}^s \leq C \rho_s$ where $C \rho_s = \mu_s + \delta$ with a positive constant δ . To derive the probability of the event $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$ for a specific $\mathbf{x}_* \in \Sigma_S$, we apply the general Hoeffding's inequality [?, Theorem 2.6.2] to the event $T_{\text{ave}}^s \leq C \rho_s$ as follows:

$$\|X\|_{\psi_2} = \inf \left\{ t > 0 : \Pr(T_{\text{ave}}^s \geq C \rho_s) \leq e^{-t^2/2\alpha^2} \right\} \quad (113)$$

Based on Lemma ??, to ensure that the event $T_{\text{ave}}^s \leq C \rho_s$ implies $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$ with a failure probability at most $e^{-t^2/2\alpha^2}$, we should have

In [?, Page 28], it is presented that solving (??) leads to $\|X\|_{\psi_2} \leq c_2 \alpha$ for a positive value c_2 . To obtain the value of c_2 that satisfies $\|X\|_{\psi_2} = c_2 \alpha$, we should find $\inf c_2 > 0$ such that

Now to include all possible $\mathbf{x}' \in \Sigma_S$ that satisfy the uniform perfect reconstruction criterion, we consider a ρ_s -net $\{\mathbf{x}'_1, \dots, \mathbf{x}'_K\}$ for the set Σ_S which means that, for any $\mathbf{x}_* \in \Sigma_S$, one can find $k \in [1 : K]$ such that $\|\mathbf{x}_* - \mathbf{x}'_k\|_2 \leq \rho_s$. We can take $K \leq \left(1 + \frac{3}{\rho_s}\right)^s$ and employ the union

Set $\beta^2 = 1 - 2/c_2^2$ and $u = \beta x$, we can rewrite (??) as

$$\frac{1}{\beta} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha^2}} e^{-u^2/(2\alpha^2)} du \leq 2, \quad (110)$$

which results in $c_2 \geq \sqrt{\frac{8}{3}}$. Therefore, we have $\|X\|_{\psi_2} = \sqrt{\frac{8}{3}} \alpha$. The probability density function (PDF) of the folded normal random variable $|X|$ with a location parameter zero

and to derive the general case of (39) we follow if $X(x) = \sqrt{\frac{2}{\pi\alpha^2}}e^{-x^2/(2\alpha^2)}$ for $x \geq 0$. Then, based on the definition of the sub-Gaussian norm we have $\Pr(T_{\text{ave}} \geq C\rho_s \text{ for some } k \in [1, K])$

$$\|X\|_{\psi_2} = \inf \left\{ \begin{array}{l} \text{if } \sqrt{\frac{-c_1(C\rho_s - \mu_s)}{K_s}} \leq 0 \\ \text{then } \sqrt{\frac{2}{\pi\alpha^2}} \int_0^\infty e^{x^2/t^2} e^{-x^2/(2\alpha^2)} dx \leq 2 \end{array} \right\}, \quad (111)$$

which is the same as the result we have obtained in (??). As a result, the sub-Gaussian norm of the folded normal random variable $|X|$ is $\|X\|_{\psi_2} = \sqrt{\frac{8}{3}\alpha}$. By setting $C_1 = \frac{8}{3c_1}$ and combining this result with (??), the proof is now complete.

$$\leq e^{\frac{3s}{\rho_s} - \frac{c_1(C\rho_s - \mu_s)^2}{K_s} m'}$$

The probability mentioned earlier corresponds to a single possibility of a fixed index set S . However, to account for all possible s -sparse solutions, we unfix the index set S and employ the following simplicity and express the general case of (D). Define the coverage of such distances as

$$\Pr(T_{\text{ave}}^s \geq C\rho_s \text{ for some } S \subseteq [1 : m'] \text{ with } \text{card}(S) = s) = \binom{d}{s} e^{\frac{3s}{\rho_s} - \frac{c_1(C\rho_s - \mu_s)^2}{K_s} m'} \sum_{j,\ell=1}^{m'} d_j^{(\ell)s}. \quad (112)$$

Assume that $\mathbb{E}\{T_{\text{ave}}^s\} = \mu_s$ and $\|\mathbf{d}^{(\ell)s}\|^2 \leq K_s$. Fix an index set S of size s and identify the space $\Sigma_S = \{\mathbf{x} \in \mathbb{R}^n : \text{supp}(\mathbf{x}) \subseteq S, \|\mathbf{x}\|_2 \leq 1\}$ with \mathbb{R}^s . Then our goal is to obtain the required number of hyperplanes (one-bit data) to achieve the uniform perfect reconstruction criterion with $\mathbf{x}' \in \mathcal{R}_s(\mathbf{x}_*)$, considering all $\mathbf{x} \in \Sigma_S$. Based on this criterion and the conclusion made in Lemma ??, it is sufficient to ensure that the upper bound of (??) is lower than η which results in

$$\frac{1}{m'} \geq \frac{K_s}{\delta^2} \left(\log\left(\frac{1}{\delta}\right) + \frac{1}{s} \left(\log\left(\frac{ed}{\rho_s}\right) + \frac{c_1(C\rho_s - \mu_s)^2}{K_s m'} \right) \right). \quad (117)$$

Therefore, the constant δ in Theorem ?? leads to $\frac{1}{\text{card}(\mathcal{G}_s)} \sum_{j \in \mathcal{G}_s} d_j^{(\ell)s} \geq \rho_s$. Therefore, we can conclude that $T_{\text{ave}}^s \geq \rho_s$. To meet the condition $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$, $\exists C > 0$ such that $T_{\text{ave}}^s \leq C\rho_s$, where $C\rho_s = \mu_s + \delta$ with a positive constant δ . To derive the probability of the event $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$ for a specific $\mathbf{x}_* \in \Sigma_S$, we apply the general Hoeffding's inequality [?, Theorem ??]. Define the event $T_{\text{ave}}^s \leq C\rho_s$ as follows:

$$\Pr(T_{\text{ave}}^s \leq C\rho_s) \geq \Pr\left(\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s, \|\mathbf{x}'\|_F \leq 1\right). \quad (118)$$

As shown in Lemma ??, based on the uniform law of large numbers in Section ??, we define the distance $\|\mathbf{x}' - \mathbf{x}_*\|_2 \leq \rho_s$ with a failure probability at most $e^{-\frac{c_1\delta^2}{K_s} m'}$, we should have

$$d_j^{(\ell)} = \left| \text{Tr}\left(\mathbf{A}_j^\top \mathbf{x}_*\right) - \tau_j^{(\ell)} \right|, \quad j \in [n], \quad \ell \in [m], \quad (119)$$

$$\delta \ll \frac{1}{m'} \sum_{\ell=1}^m \|\boldsymbol{\tau}^{(\ell)}\|_1 - \mu_s + c_{\mathbf{A}_s} (\rho_s + 1). \quad (114)$$

Where the knowledge of possible distances is formulated as satisfy the uniform perfect reconstruction criterion, we consider a ρ_s -net $\{\mathbf{x}'_1, \dots, \mathbf{x}'_K\}$ for the set Σ_S which means that, for any $\mathbf{x}_* \in \Sigma_S$, one can find $k \in [1 : K]$ such that $\sum_{j,t=1}^{T_{\text{ave}}^r} d_j^{(\ell)r} \|\mathbf{x}_* - \mathbf{x}'_k\|_2 \leq \rho_s$. We can take $K \leq \left(1 + \frac{3}{\rho_s}\right)$ and employ the union bound to derive the general case of (??) as follows:

Assume that $\mathbb{E}\{T_{\text{ave}}^r\} = \mu_r$ and $\left\|d_j^{(\ell)r}\right\| \leq K_r$. Our goal is to obtain the required number of hyperplanes to achieve the uniform perfect reconstruction criterion with $\mathbf{X}' \in \mathcal{B}_\rho(\text{vec}(\mathbf{X}_*))$ considering all $\mathbf{X}_*, \mathbf{X}' \in \mathcal{K}_{n_1,r}$. Based on this criterion and the conclusion made in Lemma ??, we establish the connection between $\mathbf{X}' - \mathbf{X}_*$ and T_{ave}^r by identifying a set \mathcal{G}_r with a minimum cardinality $\text{card}(\mathcal{G}_r) \sim n_1 r^\rho$, containing $\text{card}(\mathcal{G}_r)$ minimum distances $d_j^{(\ell)r}$ such that $\frac{1}{\text{card}(\mathcal{G}_r)} \sum_{d_j^{(\ell)r} \in \mathcal{G}_r} d_j^{(\ell)r} \sim \rho_r$. Existing such a set \mathcal{G}_r with high probability implies $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$. For the complement of the set \mathcal{G}_r denoted by $\bar{\mathcal{G}}_r$, we have $d_j^{(\ell)r} \geq \rho_r$ for all $d_j^{(\ell)r} \in \bar{\mathcal{G}}_r$ which leads to $\frac{1}{\text{card}(\bar{\mathcal{G}}_r)} \sum_{d_j^{(\ell)r} \in \bar{\mathcal{G}}_r} d_j^{(\ell)r} \geq \rho_r$. Consequently, we can conclude that $T_{\text{ave}}^r \geq C\rho_r$. To meet The probability mentioned earlier corresponds to a single possibility of a fixed index set S . $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$, $\exists C > 0$ such that $T_{\text{ave}}^r \leq C\rho_r$, where $C\rho_r = \mu_r + \delta$ with a positive constant. However, to account for all possible s -sparse solutions, we unfix the index set S and employ δ . Note that in the probabilistic sense, to derive the probability of the event $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$ for the union bound and express the general case of (??) as follows:

a specific $\mathbf{X}_* \in \mathcal{K}_{n_1,r}$ we can utilize the probability of $T_{\text{ave}}^r \leq C\rho_r$. Therefore, by the general Hoeffding's inequality [?, Theorem 2.6.2], we have

$$= \Pr(T_{\text{ave}}^r \geq C\rho_r \geq C\|\mathbf{X}' - \mathbf{X}_*\|_F) \leq e^{-\frac{c_1(C\rho_r - \mu_r)^2}{K_r} m'}. \quad (116)$$

Based on Lemma ??, to ensure that the event $T_{\text{ave}}^r \leq C\rho_r = \mu_r + \delta$ implies $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$ with a failure probability at most $e^{-(\log(\frac{ed}{\delta}) + \frac{3}{\rho_s} s - \frac{c_1(C\rho_r - \mu_r)^2}{K_r} m')}$, we should have

where we have used the inequality $\frac{1}{m'} \sum_{s=1}^m \binom{d}{s} \tau^s \leq \left(\frac{ed}{1-s}\right)^s \mu_r + c\sqrt{(\mu_r + 1)}$, Considering the bound on δ in (??), we achieve a minimum probability of $1 - \eta$, it is sufficient to ensure that the upper bound of where \mathbf{V} is defined in Lemma ??, Now to include all possible $\mathbf{X}_*, \mathbf{X}' \in \mathcal{K}_{n_1,r}$ that satisfy the

uniform perfect reconstruction criterion, we consider a ρ_r -net $\{\mathbf{x}'_1, \dots, \mathbf{x}'_K\}$ for the set $\mathcal{K}_{n_1,r}$ which means that, for any $\mathbf{x}_* \in \mathcal{K}_{n_1,r}$, one can find $k \in [1 : K]$ such that $\|\mathbf{x}_* - \mathbf{x}'_k\|_F \leq \rho_r$. According to [2, Lemma 3.1] we can take K is $\delta_s \left(1 + \frac{K_6}{C\rho_r} \delta\right)^{(2n_1+1)r}$. Then by employing the union bound, the general case of (??) can be expressed as

$$\begin{aligned} & \Pr(T_{\text{ave}}^r \geq C\rho_r \text{ for some } k \in [1 : K]) \\ & \quad \text{Proof of Theorem ??} \\ & \leq K e^{-\frac{c_1(C\rho_r - \mu_r)^2}{K_r} m'} \end{aligned} \quad (123)$$

Define the set $\mathcal{K}_{n_1,r}$ as

$$\mathcal{K}_{n_1,r} = \left\{ \mathbf{X} \in \mathbb{R}^{n_1 \times n_1} \mid \text{rank}(\mathbf{X}) \leq r, \|\mathbf{X}\|_F \leq 1 \right\}. \quad (118)$$

$$\leq e^{\frac{18n_1 r}{\rho_r} - \frac{c_1(C\rho_r - \mu_r)^2}{K_r} m'}.$$

Assuming the bound based in (32), for a given solution probability of the distance sufficient to ensure that the upper bound of (22) is lower than η , which results in

$$d_j^{(\ell)r} = \left| \text{Tr} (\mathbf{A}_j^\top \mathbf{X}_*) - \tau_j^{(\ell)} \right|, \quad j \in [n], \quad \ell \in [m], \quad (119)$$

$$\text{where } \frac{m'}{m} \geq \frac{K_r}{\delta^{-2}} \left(\frac{18n_1r}{\mu_r} + \log \left(\frac{1}{\eta} \right) \right). \quad (124)$$

Therefore, the constant δ_r in Theorem ?? is $\delta_1 = \frac{mK_r}{m'} \delta^{-2}$.

$$T_{\text{ave}}^r = \frac{1}{m'} \sum_{j,\ell=1}^{m'} d_j^{(\ell)r}. \quad (120)$$

APPENDIX H

Assume that $\mathbb{E}\{T_{\text{ave}}^r\} = \mu_r$ and $\|d_j^{(\ell)r}\|^2 \leq K_r$. Our goal is to obtain the required number of

hyperplanes to achieve the uniform perfect reconstruction criterion with $\mathbf{X}' \in \mathcal{B}_p(\text{vec}(\mathbf{X}_*))$

Consider the following lemma:

considering all $\mathbf{X}_*, \mathbf{X}' \in \mathcal{K}_{n_1,r}$. Based on this criterion and the conclusion made in Lemma ??,

Lemma 16 [2, Section 2] Between $\|\mathbf{X}_1\| \mathbf{X}_2 - \mathbf{X}_*\|_F$ and T_{ave}^r by identifying a random set \mathcal{G}_r with the minimum cardinality $\text{card}(\mathcal{G}_r)$ for $m_1 \geq 1$ containing $\text{card}(\mathcal{G}_r)$ minimum distances $d_j^{(\ell)r}$ such that $\frac{1}{\text{card}(\mathcal{G}_r)} \sum_{d_j^{(\ell)r} \in \mathcal{G}_r} d_j^{(\ell)r} \gtrsim \rho_r$. Existing such a set \mathcal{G}_r with high probability implies $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$. For the complement of the set \mathcal{G}_r denoted by $\bar{\mathcal{G}}_r$, we have $d_j^{(\ell)r} \geq \rho_r$ for all $d_j^{(\ell)r} \in \bar{\mathcal{G}}_r$ which leads to $\frac{1}{\text{card}(\mathcal{G}_r)} \sum_{d_j^{(\ell)r} \in \bar{\mathcal{G}}_r} d_j^{(\ell)r} \geq \rho_r$. Consequently, we can conclude that

$T_{\text{ave}}^r \geq \rho_r$. To meet $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$, $\exists C > 0$ such that $T_{\text{ave}}^r \leq C\rho_r$, where $C\rho_r = \mu_r + \delta$. Denote the uniform samples of \mathbf{y} by $\{y_j\}_{j=1}^m$. The DR of the high-resolution measurements is with a positive constant δ . Note that in the probabilistic sense, to derive the probability defined as $\text{DR}_y = \|y\|_\infty$. Based on Lemma ??, the expected value of the random variable DR_y of the event $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$ for a specific $\mathbf{X}_* \in \mathcal{K}_{n_1,r}$ we can utilize the probability of $T_{\text{ave}}^r \leq C\rho_r$. Therefore, by the general Hoeffding's inequality [?, Theorem 2.6.2], we have

$$\Pr(T_{\text{ave}}^r \geq C\rho_r \geq C\|\mathbf{X}' - \mathbf{X}_*\|_F) \leq e^{-\frac{-c_1(C\rho_r - \mu_r)^2}{C^2} m'}. \quad (121)$$

$$(122)$$

Based on Lemma ??, to ensure that the event $T_{\text{ave}}^r \leq C\rho_r = \mu_r + \delta$ implies $\|\mathbf{X}' - \mathbf{X}_*\|_F \leq \rho_r$ with a failure probability at most $e^{-\frac{-c_1\delta^2}{C^2} m'}$, we should have

where $K = \max_j \|\mathbf{a}_j \mathbf{x}\|_{\psi_2} \delta$. Similar to this, we can obtain the bound on the DR of Gaussian dither as $\mathbb{E}\{\max_{j \leq n} |\tau_j^g|\} \leq C\sqrt{\frac{8}{3}\sigma\sqrt{\log n}}$. For the DR of the Uniform dither we have where \mathbf{V} is defined in Lemma ???. Now to include all possible $\mathbf{X}_*, \mathbf{X}' \in \mathcal{K}_{n_1,r}$ that satisfy the $\mathbb{E}\{\max_{j \leq n} |\tau_j^u|\} \leq C\sqrt{\frac{\tilde{b}}{K^2}\log n}$ [?, Section 2.5.2]. Therefore, by setting $\sigma = \mathcal{O}(\sqrt{\frac{3}{8}KR})$ in uniform perfect reconstruction criterion, we consider a ρ_r -net $\{\mathbf{X}_1, \dots, \mathbf{X}_K\}$ for the set $\mathcal{K}_{n_1,r}$ the Gaussian dither and $\tilde{b} = \mathcal{O}(\log(2)KR)$ in the Uniform dither, we can guarantee that the DR which means that, for any $\mathbf{X}_* \in \mathcal{K}_{n_1,r}$, one can find $k \in [1 : K]$ such that $\|\mathbf{X}_* - \mathbf{X}_k\|_F \leq \rho_r$. of high-resolution measurements \mathbf{y} can be covered with a high probability.

According to [?, Lemma 3.1], we can take $\Lambda \leq \left(1 + \frac{6}{\rho_r}\right)^{(2n_1+1)r}$. Then by employing the union bound, the general case of PROOF OF THEOREM ?? can be expressed as

Assume that $\{\sigma_{iP}\}$ and $\{\sigma_{iA}\}$ denote the singular values of the matrices P and A , respectively.

To obtain the singular values of P , the matrix $W = P^T P$ is computed as

$$W = P^T P \quad (123)$$

$$= \left[A^T \Omega^{(1)} \left| \dots \left| \frac{6}{\rho_r} A^T \Omega^{(m)} \right| \dots \right| \right] e^{-\frac{c_1(C\rho_r - \mu_r)^2}{K_r} m'} \quad (127)$$

$$= A^T \Omega^{(1)} \left(\frac{6}{\rho_r} \right)^{m'} A + \dots + A^T \Omega^{(m)} \Omega^{(m)} A$$

Considering the bound on δ in (??), to achieve a minimum probability of $1 - \eta$, it is sufficient to ensure that the upper bound of (??) is lower than η which results in

which means that the singular values of P are $\{\sigma_{iP}\} = \sqrt{m} \{\sigma_{iA}\}$. Also, the Frobenius norm of P is obtained as

$$m' \geq \frac{K_r}{c_1} \delta^{-2} \left(\frac{18n_1 r}{\rho_r} + \log \left(\frac{1}{\eta} \right) \right). \quad (124)$$

Therefore, the constant δ_r in Theorem ?? is $\delta_r = \frac{K_r}{c_1} \delta^{-2}$.

$$= \text{Tr}(mA^T A) = m \|A\|_F^2.$$

Plugging in (??) and (??) in the definition of the scaled condition number provided in Section ??, one can conclude that $\kappa(P) = \kappa(A)$.

Consider the following lemma:

APPENDIX J

Lemma 10. [?, Section 2.5.2] Let X_1, X_2, \dots, X_n be a sequence of sub-Gaussian random variables with $K = \max_i \|X_i\|_{\psi_2}$. Then for $n \geq 2$ we have

Based on the convergence rate of ORKA in the sample abundance scenario (??), the upper recovery bound of ORKA $\mathbb{E} \left\{ \max_{i+1 \leq n} \frac{\|X_i\|_2}{\|\mathbf{x}_*\|_2} \right\} \leq C K \sqrt{\log n}$ can be met when $q^i \omega_0 + \rho^2 \leq \epsilon_0$, where where C is a constant.

$$i \geq \frac{\log(\epsilon_0 - \rho^2)}{\log(1 - \frac{\rho^2}{\kappa^2(A)})}. \quad (129)$$

Denote the uniform samples of y by $\{y_j\}_{j=1}^n$. The DR of the high-resolution measurements is defined as $\text{DR}_y = \|\mathbf{y}\|_\infty$. Based on Lemma ??, the expected value of the random variable DR_y is bounded by

$$\mathbb{E} \left\{ \text{DR}_y = \|\mathbf{y}\|_\infty \right\} = \mathbb{E} \left\{ \max_j \|\mathbf{a}_j \mathbf{x}\| \right\} \approx \kappa^2(A) \log \left(\frac{1}{\epsilon_0 - \rho^2} \right). \quad (130)$$

For a well conditioned sensing matrix A , i.e. $\varrho(A) = \mathcal{O}(1)$, we have $\kappa^2(A) = \mathcal{O}(d)$. Therefore, the required number of iterations for ORKA to meet the aforementioned upper recovery bound is $i = \mathcal{O} \left(d \log \left(\frac{1}{\epsilon_0 - \rho^2} \right) \right)$.

$$\leq CK \|\mathbf{x}\|_2 \sqrt{\log n}$$

$$\leq CKR \sqrt{\log n},$$

where $K = \max_j \|\mathbf{a}_j \mathbf{x}\|_{\psi_2}$. Similar to this, we can obtain the bound on the DR of Gaussian dither as $\mathbb{E} \left\{ \max_{j \leq n} |\tau_j^g| \right\} \leq C \sqrt{\frac{8}{3}} \sigma \sqrt{\log n}$. For the DR of the Uniform dither we have

$\mathbb{E} \{\max_{j \leq n} |\tau_j^u|\} \leq C \frac{\tilde{b}}{\log(2)} \sqrt{\log n}$ [?, Section 2.5.2]. Therefore, by setting $\sigma = \mathcal{O}(\sqrt{\frac{3}{8}} KR)$ in the Gaussian dither and $\tilde{b} = \mathcal{O}(\text{Pro}(2) KR)$ [Proposition ??] in dither, we can guarantee that the DR of high-resolution measurements can be covered with a high probability.

We begin the proof by presenting the following lemma.

Lemma 11. Let $\mathcal{H}_j = \{\mathbf{x} : \mathbf{c}_j \mathbf{x} \geq b_j - n_j\}$. Then the solution spaces of the noisy linear inequalities. Let $\bar{\mathcal{H}}_j = \{\mathbf{x} : \mathbf{c}_j \mathbf{x} \geq b_j\}$ be the solution of the system run by the noisy RKA. Then $\bar{\mathcal{H}}_j = \{\mathbf{x} + \alpha_j \mathbf{c}_j^H \mid \mathbf{x} \in \mathcal{H}_j\}$, where $\alpha_j = \frac{n_j}{\|\mathbf{c}_j\|_2^2}$ denote the singular values of the matrices \mathbf{P} and \mathbf{A} , respectively.

Proof. Assume $\mathbf{x} \in \mathcal{H}_j$, then we can write

$$\mathbf{W} = \mathbf{P}^\top \mathbf{P} \quad \mathbf{c}_j (\mathbf{x} + \alpha_j \mathbf{c}_j^H) \geq b_j - n_j + \alpha_j \|\mathbf{c}_j\|_2^2. \quad (131)$$

$$\text{By plugging } \alpha_j = \frac{n_j}{\|\mathbf{c}_j\|_2^2} \left[i \mathbf{A}^\top \Omega^{(1)} \text{ we can obtain } \right] \left[\mathbf{A}^\top \Omega^{(1)} \mid \dots \mid \mathbf{A}^\top \Omega^{(m)} \right]^\top \quad (127)$$

$$= \mathbf{A}^\top \Omega^{(1)} \Omega^{(1)} \mathbf{A} + \mathbf{c}_j (\mathbf{x} + \mathbf{A}^\top \Omega^{(m)} \mathbf{A}) \geq b_j, \quad (132)$$

which implies $\mathbf{x} + \alpha_j \mathbf{c}_j^H \in \mathcal{H}_j$.

■

which means that the singular values of \mathbf{P} are $\{\sigma_i \mathbf{P}\} = \sqrt{m} \{\sigma_i \mathbf{A}\}$. Also, the Frobenius norm

Assume $\bar{\mathbf{x}}_i$ denotes the i -th iterate of the noisy RKA run with $\mathbf{C} \mathbf{x} \geq \mathbf{b}$. Denote $\hat{\mathbf{x}} \in \cap_{j=1}^m \mathcal{H}_j$.

of \mathbf{P} is obtained as

Based on Lemma ??, we can write $\|\mathbf{P}\|_F^2 = \text{Tr}(\mathbf{P}^\top \mathbf{P})$

$$(128)$$

$$\bar{\mathbf{x}}_i = \text{Tr}(m \mathbf{A}^\top \mathbf{A}) + \alpha_i \mathbf{c}_i^H \|\mathbf{A}\|_F^2. \quad (133)$$

Plugging in ?? and ?? in the definition of the scaled condition number provided in Section ??, one can conclude that $\kappa(\mathbf{P}) = \kappa(\mathbf{A})$.

As a result, we modify α_j in ?? to $(\alpha_j)^+$ to take into account our above discussion.

Appendix J

Taking the norm-2 of ?? leads to

Proof of Corollary ??

$$\|\bar{\mathbf{x}}_i - \hat{\mathbf{x}}\|_2^2 = \|\mathbf{x}_i - \hat{\mathbf{x}} + (\alpha_j)^+ \mathbf{c}_j^H\|_2^2$$

Based on the convergence rate of ORKA in the sample abundance scenario ??, the upper recovery bound of ORKA $\mathbb{E} \{\|\mathbf{x}_{i+1} - \hat{\mathbf{x}}\|_2^2 + \|(\alpha_j)^+ \mathbf{c}_j^H\|_2^2\} \leq \epsilon_0$ can be met when $q^i \omega_0 + \rho^2$ (134)

where $q = 1 - \kappa^{-2}(\mathbf{A})$. This leads to $= \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 + \frac{((n_j)^+)^2}{\log(\epsilon_0 - \rho^2) \|\mathbf{c}_j\|_2^2}$.

$$i \geq \frac{\log(\epsilon_0 - \rho^2)}{\log(1 - \kappa^{-2}(\mathbf{A}))}. \quad (129)$$

Denote the scaled condition number of the matrix \mathbf{C} by $\kappa(\mathbf{C}) = \sqrt{R}$. Then by considering the convergence rate of the RKA in ?? and drawing inspiration from [?, Lemma 2.2] in the context

of linear system of inequalities, we have $\frac{\log(\epsilon_0 - \rho^2)}{\log(1 - \kappa^{-2}(\mathbf{A}))} \approx \kappa^2(\mathbf{A}) \log\left(\frac{1}{\epsilon_0 - \rho^2}\right)$.

For a well conditioned sensing matrix $\mathbf{A} \in \mathbb{R}^{d \times n}$, i.e., $\|\mathbf{c}_j\|_2^2 \geq \frac{1}{R}$, we have $\kappa^2(\mathbf{A}) = \mathcal{O}(d)$.

Therefore, the required number of iterations for ORKA to meet the aforementioned upper recovery bound is $i = \mathcal{O}\left(d \log\left(\frac{1}{\epsilon_0 - \rho^2}\right)\right) + \frac{1}{R} \|\bar{\mathbf{x}}_{i-1} - \hat{\mathbf{x}}\|_2^2 + \max_j \gamma_j$,

where γ_j is defined in Proposition ??, and \mathbb{E} expectation is conditioned upon the choice of the random selections in the first $i - 1$ iteration. By applying this recursive relation iteratively and taking full expectation, we have. We begin the proof by presenting the following lemma:

Lemma 11. Let $\mathcal{H}_j = \{\mathbf{x} : \mathbf{c}_j \mathbf{x} \geq b_j\} \leq \left(1 - \frac{1}{R}\right)^i \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2$ be the solution spaces of the noisy linear inequalities. Let $\bar{\mathcal{H}}_j = \{\mathbf{x} : \mathbf{c}_j \mathbf{x} \geq b_j\}$ be the solution spaces of the system run by the noisy RKA. Then $\bar{\mathcal{H}}_j = \{\mathbf{x} + \alpha_j \mathbf{c}_j^H : \mathbf{x} \in \mathcal{H}_j\}$, where $\alpha_j = \sum_{k=0}^{j-1} \left(1 - \frac{1}{R}\right)^{i-k} \max_j \gamma_j$. (136)

Proof: Assume $\mathbf{x} \in \mathcal{H}_j$, then we can write

$$\begin{aligned} &\leq \left(1 - \frac{1}{R}\right)^i \|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2 + R \max_j \gamma_j, \\ \mathbf{c}_j (\mathbf{x} + \alpha_j \mathbf{c}_j^H) &\geq b_j - n_j + \alpha_j \|\mathbf{c}_j\|_2^2. \end{aligned} \quad (131)$$

which completes the proof.

By plugging $\alpha_j = \frac{n_j}{\|\mathbf{c}_j\|_2^2}$ in (??), we can obtain

$$\mathbf{c}_j (\mathbf{x} + \alpha_j \mathbf{c}_j^H) \geq b_j, \quad (132)$$

CONVERGENCE ANALYSIS OF ST-ORKA, SVP-ORKA, AND HT-ORKA

which implies $\mathbf{x} + \alpha_j \mathbf{c}_j^H \in \bar{\mathcal{H}}_j$. ■

In this section, we prove the convergence, and derive the convergence rates of ST-ORKA, SVP-ORKA, and HT-ORKA. At the i -th iterate of the noisy RKA run with $\mathbf{C}\mathbf{x} \succeq \mathbf{b}$. Denote $\hat{\mathbf{x}} \in \cap_{j=1}^m \mathcal{H}_j$. Based on Lemma ??, we can write

A. Convergence Analysis for ST-ORKA

$$\hat{\mathbf{x}}_i - \hat{\mathbf{x}} = \mathbf{x}_i - \hat{\mathbf{x}} + \alpha_j \mathbf{c}_j^H. \quad (133)$$

As discussed in [?], if Ω_c is a closed and convex set in any Hilbert space \mathcal{H} , then the projection where $\mathbf{x}_i \in \mathcal{H}_j$. For $\mathbf{x}_i \in \mathcal{H}_j$, we have $\mathbf{c}_j \mathbf{x}_i \geq b_j - n_j$. Then, assuming $n_j \leq 0$, we still have operator $P : \mathcal{H} \rightarrow \Omega_c$ is non-expansive, i.e., for any vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{H}$, $\|P(\mathbf{x}_1) - P(\mathbf{x}_2)\|_2 \leq \mathbf{c}_j \mathbf{x}_i \geq b_j$ which informs $\mathcal{H}_j \subseteq \mathcal{H}_i$ and there is no inequality flip in the i -th iteration of $\|\mathbf{x}_1 - \mathbf{x}_2\|_2$. Due to the fact that the ST operator projects the signal to $\|\mathbf{x}\|_1 \leq \epsilon$ which is a closed the noisy RKA. As a result, we modify α_j in (??) to $(\alpha_j)^+$ to take into account our above and convex set, we can write that for any vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{H}$, $\|S_k(\mathbf{x}_1) - S_k(\mathbf{x}_2)\|_2 \leq \|\mathbf{x}_1 - \mathbf{x}_2\|_2$. discussion. Taking the norm-2 of (??) leads to

Define $\mathbf{e}_i = \mathbf{x}_i - \hat{\mathbf{x}}$, $t_j^{(\ell)} = r_j^{(\ell)} \tau_j^{(\ell)}$ and $\mathbf{p}_j^{(\ell)} = r_j^{(\ell)} \mathbf{a}_j$, we derive the convergence rate of ST-ORKA as

$$\begin{aligned} &\leq \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 + \|(\alpha_j)^+ \mathbf{c}_j^H\|_2^2 \\ \|\mathbf{e}_{i+1}\|_2^2 &= \|S_k(\mathbf{z}_{i+1}) - S_k(\hat{\mathbf{x}})\|_2^2 \\ &= \|\mathbf{x}_i - \hat{\mathbf{x}}\|_2^2 + \frac{((n_j)^+)^2}{\|\mathbf{c}_j\|_2^2}. \end{aligned} \quad (134)$$

Denote the scaled condition number of the matrix \mathbf{C} by $\kappa(\mathbf{C}) = \sqrt{R}$. Then by considering the convergence rate of the RKA in (??) and drawing inspiration from [?, Lemma 2] in the context of linear system of inequalities, we have

$$\begin{aligned} &= \mathbb{E} \left\{ \|\bar{\mathbf{x}}_i - \left(\frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right) \right\}^2 \\ &= \|\mathbf{e}_i\|_2^2 + \frac{\left(\left(\frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right)^2 - \left(\frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right) \mathbf{e}_i \right)^2}{\left\| \frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right\|_2^2} \\ &\leq \left\| \frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right\|_2^2 \|\bar{\mathbf{x}}_i - \left(\frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right) \|^2 + \frac{\left(\frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right)^2 \|\mathbf{e}_i\|_2^2}{\left\| \frac{(t_j^{(\ell)})^+ \mathbf{c}_j^H}{\|\mathbf{c}_j\|_2^2} \right\|_2^2} \\ &\leq \left(1 - \frac{1}{R} \right) \|\bar{\mathbf{x}}_{i-1} - \hat{\mathbf{x}}\|_2^2 + \max_j \gamma_j, \end{aligned} \quad (135)$$

Since $\mathbf{p}_j^{(\ell)}$ is defined in Proposition ??, $\mathbf{p}_j^{(\ell)}$ and the expectation of $\mathbf{p}_j^{(\ell)}$ is $\mathbf{p}_j^{(\ell)}$. Therefore, upon the choice of \mathbf{P} , the random selections in the first $i - 1$ iterations. Then applying this recursive relation iteratively and taking full expectation, we have

$$\|\mathbf{e}_{i+1}\|_2^2 \leq \mathbb{E}\{\|\bar{\mathbf{x}}_i\|_2^2 + \hat{\mathbf{x}}\|_2^2\} \leq \frac{\left(\frac{t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i}{1 - \frac{1}{R}}\right)^+^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2} \|\mathbf{x}_0\|_2^2 + \frac{2\|\hat{\mathbf{x}}\|_2^2 - \mathbf{p}_j^{(\ell)} \mathbf{x}_i}{\|\mathbf{p}_j^{(\ell)}\|_2^2} \mathbf{p}_j^{(\ell)} \mathbf{e}_i \quad (136)$$

$$\leq \|\mathbf{e}_i\|_2^2 + \frac{\left(\frac{(t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+}{1 - \frac{1}{R}}\right)^k \max_j \gamma_j}{\|\mathbf{p}_j^{(\ell)}\|_2^2 \left(1 - \frac{1}{R}\right)^2} (t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+ (t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i) \quad (138)$$

which completes the proof.

$$= \|\mathbf{e}_i\|_2^2 - \frac{\left((t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+\right)^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2} \quad \text{Appendix L}$$

Convergence analysis of ST-ORKA, SVP-ORKA, and HT-ORKA

In this section, we prove the convergence, and derive the convergence rates of ST-ORKA, both sides of ?? results in SVP-ORKA, and HT-ORKA.

$$\text{A. Convergence Analysis for ST-ORKA} \quad \mathbb{E}\{\|\mathbf{e}_{i+1}\|_2^2\} \leq \|\mathbf{e}_i\|_2^2 \sum_{\ell=1}^{m'} \frac{\|\mathbf{p}_j^{(\ell)}\|_2^2}{\|\mathbf{P}\|_F^2} \frac{\left((t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+\right)^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2}$$

As discussed in [?], if Ω_c is a closed and convex set in any Hilbert space \mathcal{H} , then the projection operator $P : \mathcal{H} \rightarrow \Omega_c$ is non-expansive, i.e., for any vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{H}$, $\|P(\mathbf{x}_1) - P(\mathbf{x}_2)\|_2 \leq \|\mathbf{x}_1 - \mathbf{x}_2\|_2$. Due to the fact that the ST operator projects the signal based on the Hoffman closed and Theorem ??, we have $\|\mathbf{e}_i\|_2^2 \leq \|\mathbf{S}_\kappa(\mathbf{x}_i)\|_2^2$ for any vectors $\mathbf{x}_1, \mathbf{x}_2$ where $\|\mathbf{S}_\kappa(\mathbf{x}_1)\|_2^2 \leq \|\mathbf{S}_\kappa(\mathbf{x}_2)\|_2^2$. Combining this result with ?? leads to

$$\text{Define } \mathbf{e}_i = \mathbf{x}_i - \hat{\mathbf{x}}, \quad t_j^{(\ell)} = r_j^{(\ell)} \tau_j^{(\ell)} \text{ and } \mathbf{p}_j^{(\ell)} = \sigma_{\min}^{(\ell)}(\mathbf{P}) \mathbf{e}_i \text{ we derive the convergence rate of ST-ORKA as} \quad (140)$$

$$\|\mathbf{e}_{i+1}\|_2^2 = \|\mathbf{S}_\kappa(\mathbf{z}_{i+1}) - \bar{\mathbf{S}}_\kappa(\hat{\mathbf{x}})\|_2^2 \frac{1}{\kappa^2(\mathbf{P})} \|\mathbf{e}_i\|_2^2.$$

According to Theorem ??, we know $\kappa(\mathbf{A}) = \kappa(\mathbf{P})$. Thus, after i iterations we have

$$= \left\| \mathbf{e}_i + \frac{\left(\mathbb{E}^{\ell}(\|\mathbf{e}_i\|_2^2 \mathbf{x}_i)\right)^+}{\|\mathbf{p}_j^{(\ell)}\|_2^2} \right\|_2^2 \left(1 - \frac{1}{\kappa^2(\mathbf{A})}\right)^i \|\mathbf{e}_0\|_2^2. \quad (141)$$

It is evident that the rest of the proof follows the same logic as the proof of Proposition ??.

Note that the convergence rate ?? can be derived for any operator follows $P : \mathcal{H} \rightarrow \Omega_c$. This underscores the versatility of the algorithm and its ability to accommodate various types of operators.

SinCorporated **Convergence Analysis for SVP-ORKA** ($\mathbf{x}_i - \hat{\mathbf{x}} \leq \mathbf{p}_j^{(\ell)} \mathbf{x}_i - t_j^{(\ell)}$). Therefore, one can rewrite (??) as To prove the convergence of SVP-ORKA, consider an operator function \mathcal{G}_f applied to a matrix \mathbf{X} with rank r' as follows:

$$\|\mathbf{e}_{i+1}\|_2^2 \leq \|\mathbf{e}_i\|_2^2 + \frac{\left(\left(\mathcal{G}_f(\mathbf{X}) - \mathbf{P}_j^{(\ell)} \sum_{k=1}^{r'} f(\sigma_k) \mathbf{u}_k \mathbf{v}_k^\top (\mathbf{t}_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+ \mathbf{p}_j^{(\ell)} \right)^+ \right)^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2} \mathbf{e}_i \quad (142)$$

where $\{\sigma_k, \mathbf{u}_k, \mathbf{v}_k\}_{k=1}^{r'}$ are singular values of \mathbf{X} and its corresponding singular vectors, and f is a L -Lipschitz continuous function. As comprehensively discussed in [?] and [?, Theorem 4.2], the following relation holds for two matrices \mathbf{X}_1 and \mathbf{X}_2 belonging to the Hilbert space \mathcal{H} :

$$\left\| \left(\left(\mathcal{G}_f(\mathbf{X}) - \mathbf{P}_j^{(\ell)} \sum_{k=1}^{r'} f(\sigma_k) \mathbf{u}_k \mathbf{v}_k^\top (\mathbf{t}_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+ \mathbf{p}_j^{(\ell)} \right)^+ \right)^2 \right\|_{\text{F}} \leq L \|\mathbf{X}_1 - \mathbf{X}_2\|_{\text{F}}. \quad (143)$$

In SVP-ORKA, f is an operator which only chooses the r -largest singular values. It is straightforward to verify that such f satisfies (??) with $L = 1$. Therefore, one can conclude

Define $\mathbf{t} = \begin{bmatrix} \mathbf{t}_1^\top & \cdots & \mathbf{t}_{r'}^\top \end{bmatrix}^\top$, where $\mathbf{t}_\ell = \begin{bmatrix} t_\ell^{(\ell)} \end{bmatrix}^n$ for $\ell \in [m]$. Taking the expectation from both sides of (??) results in

Since $\|\mathbf{X}_1 - \mathbf{X}_2\|_{\text{F}} = \|\text{vec}(\mathbf{X}_1) - \text{vec}(\mathbf{X}_2)\|_2$, the convergence proof of SVP-ORKA is identical to that of ST-ORKA from this point forward. It is worth noting that using the SVT operator instead of SVP can lead to the same convergence proof for finding the solution in $\mathcal{P}_1^{(M)}$. This is because in SVT operator, the function $f(\cdot)$ is $f(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\tau})^+$, where $\boldsymbol{\tau}$ is a predefined threshold. As discussed for the SVP operator in Appendix ??, this function also satisfies the Lipschitz continuity.

Based on the Hoffman bound [?, Theorem 4.2], we have $\|\mathbf{e}_i\|_2^2 \leq L_1 \|\mathbf{(t} - \mathbf{P}\mathbf{x}_i)^+\|_2^2$, where

Convergence Analysis for HT-ORKA Combining this result with (??) leads to

Define $\mathbf{e}_i = \mathbf{x}_i - \hat{\mathbf{x}}$, $t_j^{(\ell)} = \mathbb{E} \left\{ \frac{t_j^{(\ell)}}{\|\mathbf{e}_{i+1}\|_2} \right\}$ and $\mathbf{p}_j^{(\ell)} = \frac{1}{\|\mathbf{P}\|_{\text{F}}^2} \frac{r_j^{(\ell)} \sigma_{\min}^2(\mathbf{P})}{\|\mathbf{P}\|_{\text{F}}^2} \mathcal{T}_s(\mathbf{e}_i)$. Since $\mathcal{T}_s(\cdot)$ is an operator that selects the best s -sparse approximation of the solution at each iteration, we can determine the convergence rate of HT-ORKA as follows:

$$= \left(1 - \frac{1}{\kappa^2(\mathbf{P})} \right) \|\mathbf{e}_i\|_2^2.$$

According to Theorem ??, we know $\kappa(\mathbf{A}) = \kappa(\mathbf{P})$. Thus, after i iterations we have

$$\mathbb{E} \left\{ \|\mathbf{e}_{i+1}\|_2^2 \right\} \leq \|\mathcal{T}_s(\mathbf{z}_{i+1}) - \hat{\mathbf{x}}\|_2^2 + \|\mathbf{z}_{i+1} - \hat{\mathbf{x}}\|_2^2 \quad (145)$$

$$\leq \left(1 - \frac{1}{\kappa^2(\mathbf{A})} \right) \|\mathbf{e}_0\|_2^2. \quad (146)$$

It is evident that the rest of the proof follows the same logic as the proof of Proposition ??.

As presented earlier, the term $\|\mathbf{z}_{i+1} - \hat{\mathbf{x}}\|_2^2$ can be bounded as

Note that the convergence rate (??) can be derived for any operator follows $P : \mathcal{H} \rightarrow \Omega_c$.

This underscores the versatility of the algorithm and its ability to accommodate various types of operators.

$$\|\mathbf{z}_{i+1} - \hat{\mathbf{x}}\|_2^2 \leq \|\mathbf{e}_i\|_2^2 - \frac{\left(\left(\mathcal{G}_f(\mathbf{X}) - \mathbf{P}_j^{(\ell)} \sum_{k=1}^{r'} f(\sigma_k) \mathbf{u}_k \mathbf{v}_k^\top (\mathbf{t}_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+ \mathbf{p}_j^{(\ell)} \right)^+ \right)^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2}, \quad (146)$$

End Convergence Analysis for SVP-ORKA

To prove the convergence of SVP-ORKA, consider an operator function \mathcal{G}_f applied to a matrix \mathbf{X} with rank r' as follows:
According to the Hoffman bound [?, Theorem 4.2] and (??), we have

$$\mathbb{E} \left\{ \|\mathbf{e}_{i+1}\|_2^2 \right\} \leq 2 \sum_{k=1}^{r'} \frac{f(\sigma_k) \|\mathbf{u}_k \mathbf{v}_k^H\|^2}{\kappa^2(\mathbf{A})} \|\mathbf{e}_0\|_2^2. \quad (142)$$

It is straightforward to verify that a similar roadmap to the proof of Proposition ?? may be followed. As comprehensively discussed in [?] and [?, Theorem 4.2], the following relation holds for two matrices \mathbf{X}_1 and \mathbf{X}_2 belonging to the Hilbert space \mathcal{H} :

$$\|\mathcal{G}_f(\mathbf{X}_1) - \mathcal{G}_f(\mathbf{X}_2)\|_F \leq L \|\mathbf{X}_1 - \mathbf{X}_2\|_F. \quad (143)$$

In SVP-ORKA, f is an operator which only chooses the r -largest singular values. It is straightforward to verify that such f satisfies (??) with $L = 1$. Therefore, one can conclude

$$\|P_r(\mathbf{X}_1) - P_r(\mathbf{X}_2)\|_F \leq \|\mathbf{X}_1 - \mathbf{X}_2\|_F, \quad \forall \mathbf{X}_1, \mathbf{X}_2 \in \mathcal{H}. \quad (144)$$

Since $\|\mathbf{X}_1 - \mathbf{X}_2\|_F = \|\text{vec}(\mathbf{X}_1) - \text{vec}(\mathbf{X}_2)\|_2$, the convergence proof of SVP-ORKA is identical to that of ST-ORKA from this point forward. It is worth noting that using the SVT operator instead of SVP can lead to the same convergence proof for finding the solution in $\mathcal{P}_1^{(M)}$. This is because in SVT operator, the function $f(\cdot)$ is $f(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\tau})^+$, where $\boldsymbol{\tau}$ is a predefined threshold. As discussed for the ST operator in Appendix ??, this function also satisfies the Lipschitz continuity.

C. Convergence Analysis for HT-ORKA

Define $\mathbf{e}_i = \mathbf{x}_i - \widehat{\mathbf{x}}$, $t_j^{(\ell)} = r_j^{(\ell)} \tau_j^{(\ell)}$, and $\mathbf{p}_j^{(\ell)} = r_j^{(\ell)} \mathbf{a}_j$. Since $\mathcal{T}_s(\cdot)$ is an operator that selects the best s -sparse approximation of the solution at each iteration, we can determine the convergence rate of HT-ORKA as follows:

$$\begin{aligned} \|\mathbf{e}_{i+1}\|_2^2 &= \|\mathcal{T}_s(\mathbf{z}_{i+1}) - \widehat{\mathbf{x}}\|_2^2 \\ &\leq \|\mathcal{T}_s(\mathbf{z}_{i+1}) - \widehat{\mathbf{x}}\|_2^2 + \|\mathbf{z}_{i+1} - \widehat{\mathbf{x}}\|_2^2 \\ &\leq 2 \|\mathbf{z}_{i+1} - \widehat{\mathbf{x}}\|_2^2. \end{aligned} \quad (145)$$

As presented earlier, the term $\|\mathbf{z}_{i+1} - \widehat{\mathbf{x}}\|_2^2$ can be bounded as

$$\|\mathbf{z}_{i+1} - \widehat{\mathbf{x}}\|_2^2 \leq \|\mathbf{e}_i\|_2^2 - \frac{\left((t_j^{(\ell)} - \mathbf{p}_j^{(\ell)} \mathbf{x}_i)^+ \right)^2}{\|\mathbf{p}_j^{(\ell)}\|_2^2}, \quad (146)$$