Near MDS and near quantum MDS codes via orthogonal arrays

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Abstract — Near MDS (NMDS) codes are closely related to interesting objects in finite geometry and have nice applications in combinatorics and cryptography. But there are many unsolved problems about construction of NMDS codes. In this paper, by using symmetrical orthogonal arrays (OAs), we construct a lot of NMDS, m-MDS and almost extremal NMDS codes. We establish a relation between asymmetrical OAs and quantum error correcting codes (QECCs) over mixed alphabets. Since quantum maximum distance separable (QMDS) codes over mixed alphabets with the dimension equal to one have not been found in all the literature so far, the definition of a near quantum maximum distance separable (NQMDS) code over mixed alphabets is proposed. By using asymmetrical OAs, we obtain many such codes.

Key words — orthogonal array; NMDS; NQMDS code over mixed alphabets.

1 Introduction

In digital communication, due to various interferences, errors occur during the transmission of information, which requires that the information is encoded so that it has the ability to self-correct. MDS codes are a kind of error correcting code with good performance. However, since the parameters of an MDS code are limited by the size of the field, it is desirable to study codes nearly meeting the Singleton bound with more flexible parameters [?]. For a linear code $C = [n, k, d]_s$ define $S(C) = n - k + 1 - d$. If $S(C) = S(C^{\perp}) = m$, we call C is m-MDS. Particularly, if $S(C) = 1$, C is almost MDS (AMDS), and $S(C) = S(C^{\perp}) = 1$, C is near MDS (NMDS) [?]. An AMDS code and a linear orthogonal array are equivalent [?]. Thus AMDS and NMDS codes are valuable and interesting as they have special geometric properties [?]. The first NMDS code was the $[11, 6, 5]$ ternary Golay code discovered in 1949 by Golay [?], which has applications in group theory and combinatorics. Some recent progress on theory and applications of NMDS codes were made in [?,?,?,?,?,?,?,?,?,?]. In [?], Ding and Tang constructed infinite families of NMDS codes which hold t-designs, $t = 2, 3, 4$. Ding also constructed t-designs from some geometry codes containing AMDS ones [?]. In [?], several families of NMDS codes which are both distance-optimal and dimension-optimal locally recoverable codes were studied. In [?], the authors used NMDS codes to construct secret sharing schemes which have good security properties. The error detection capability of AMDS and NMDS codes was studied in [?] and conditions for the codes to be good for error detection were established. In [?], the authors constructed MDS symbol-pair codes from AMDS codes. In [?], based on cyclic subgroups of $F_{q^2}^*$, the authors constructed MDS, NMDS and AMDS codes. There are still a lot of NMDS codes remaining unknown.

In particular, NMDS codes with parameters $[2q+k, k+1, 2q-1]$ over $GF(q)$ are said to be almost extremal. Almost extremal NMDS codes with $k > q$ are all known. But the existence and construction of $k \leq q$ are still open [?].

In this paper, we explicitly construct almost extremal NMDS codes through OAs such as $[6,3,3]$ NMDS code over $GF(2)$, $[8,3,5]$ NMDS code over $GF(3)$.

As in the classical transmission of data, it is inevitable that errors occur in quantum information processing [?]. Since QECCs [?,?,?] could fight against various quantum noises, it has been attracting a great deal of attentions [?,?]. Numerous classes of QMDS codes over a single alphabet have been constructed mainly from Galois field, Euclidean construction, Hermitian construction and OAs [?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?,?]. However, it is expected that we frequently face a more complicated situation that quantum resources, in which quantum information is encoded, have different dimensions. In particular, we often use hybrid systems [?,?,?] with different dimensions to store, transmit, and process the quantum information. Thus it is quite necessary to generalize QECCs over a single alphabet to mixed alphabets [?]. Construction of such codes has become one of the most important tasks in quantum coding theory [?,?]. The QECCs $((n, K, d))_{s_1, s_2, \ldots, s_n}$ over mixed alphabets have been studied in [?,?], and the quantum Singleton bound is also generalized. In [?], Yan et al. obtained some QECCs over mixed alphabets based on OAs. However, other than the above papers [?,?,?,?], there are little work on QECCs over mixed alphabets particularly on QMDS codes because of their harder construction. Besides, in all the literature, QMDS codes over mixed alphabets are all for $K > 1$ [?,?] while such codes for $K = 1$ have not been found so far.

An orthogonal array OA$(N, n, s_1^{n_1} s_2^{n_2} \ldots s_v^{n_v}, k)$ of strength k is an $N \times n$ matrix, having n_i columns with s_i levels, $i = 1, 2, \ldots, v$, v is an integer, $n = \sum_{i=1}^{v} n_i$, and $s_i \neq s_j$ for $i \neq j$, with the property that, in any $N \times k$ submatrix, all possible combinations of k symbols appear equally often as a row. The orthogonal array is called a mixed orthogonal array if $v \geq 2$. Otherwise, the array is called symmetrical. OAs play a prominent role in the design of experiments which were introduced by Rao [?,?]. As is often the case, they can be useful for quantum information theory. In recent years, many new classes of OAs, especially high strength OAs have been obtained [?,?,?,?,?,?]. The relationship among OAs, classical error correcting codes (CECCs), quantum uniform states and QECCs was further revealed [?,?,?,?,?,?,?,?]. An OA$(N, n, s_1^{n_1}, \ldots, s_v^{n_v}, k)$ having $n = n_1 + \cdots + n_v$ columns is called an irredundant orthogonal array (IrOA), if every subset of $n-k$ columns contains a different sequence of $n-k$ symbols in every row [?]. IrOAs play an important role in the construction of quantum uniform states and QECCs. A lot of symmetrical or asymmetrical IrOAs, quantum uniform states and QECCs over a single alphabet or mixed alphabets including QMDS codes have also been constructed [?,?,?,?]. It is these new developments in OAs that suggest the possibility of constructing NMDS, almost extremal

NMDS and NQMDS codes.

In this paper, we present sufficient and necessary conditions for a symmetrical OA to be an NMDS or m-MDS code. Then we construct a lot of NMDS codes including almost extremal NMDS codes and m-MDS codes. Further, we establish a relation between asymmetrical OAs and QECCs over mixed alphabets. In addition, a near quantum MDS (NQMDS) code is defined. From an OA$(s^k, 2k+1, s^{2k}2^1, k)$ for even s, we can construct an NQMDS code $((2k+1, 1, k+1))_{s^{2k}2^1}$ such as $((3,1,2))_{8^2 2^1}$, $((3,1,2))_{12^2 2^1}$, $((5,1,3))_{16^4 2^1}$, $((5,1,3))_{20^4 2^1}$.

The rest of this paper is organized as follows. In Section ??, we introduce some basic notations and useful results on OAs, CECCs, QECCs and NQMDS codes. Main results are given in Section ??. In Section ??, we present sufficient and necessary conditions for a symmetrical OA to be an NMDS or m-MDS code. And then we construct a lot of NMDS codes including almost extremal NMDS codes and m-MDS codes. In Section ??, we construct NQMDS codes over mixed alphabets through asymmetrical OAs. The paper is concluded in Section ??.

2 Preliminaries

First, the notations used in this paper are listed as follows.

Let Z_s^n denote the n-dimensional space over a ring $Z_s = \{0, 1, \ldots, s-1\}$. When s is a prime power, let F_s be a Galois field containing s elements with binary operations ($+$ and $\cdot$). If $A = (a_{ij})_{n\times m}$ and $B = (b_{uv})_{s\times t}$ with elements from a Galois field, the Kronecker sum $A \oplus B$ is defined as $A \oplus B = (a_{ij} + B)_{ns\times mt}$ where $a_{ij} + B$ represents the $s \times t$ matrix with entries $a_{ij} + b_{uv} (1 \le u \le s, 1 \le v \le t)$ and the Kronecker product $A \otimes B$ is defined as $A \otimes B = (a_{ij} \cdot B)_{ns\times mt}$ where $a_{ij} \cdot B$ represents the $s \times t$ matrix with entries $a_{ij} \cdot b_{uv}$ $(1 \le u \le s,\ 1 \le v \le t)$. Let $(\mathbb{C}^s)^{\otimes n} = \underbrace{\mathbb{C}^s \otimes \mathbb{C}^s \otimes \cdots \otimes \mathbb{C}^s}_{n}$.

Some basic knowledge about OA, CECC and QECC is given.

Definition 2.1. *[?] Let $R_1, \ldots, R_N$ be the rows of an $N \times t$ matrix A, with entries at the ith column from $Z_{s_i} = \{0, 1, \ldots, s_i - 1\}$, where $s_i \ge 2$ and $i = 1, 2, \ldots, t$. The Hamming distance $Hd(R_u, R_v)$ between $R_u = (a_{u1}, \ldots, a_{ut})$ and $R_v = (a_{v1}, \ldots, a_{vt})$ is defined as follows:*

$$Hd(R_u, R_v) = |\{r : 1 \le r \le t, a_{ur} \ne a_{vr}\}|.$$

In this paper, $md(L)$ denotes the minimum Hamming distance between two distinct rows of an OA L.

Definition 2.2. *[?] Let A be the orthogonal array $\mathrm{OA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$ and $\{A_1, A_2, \ldots, A_u\}$ be a set of orthogonal arrays $\mathrm{OA}(\frac{N}{u}, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k_1)$. If $\bigcup_{i=1}^{u} A_i = A$ and $A_i \bigcap A_j = \varnothing$ for $i \ne j$, then $\{A_1, A_2, \ldots, A_u\}$ is said to be an orthogonal partition of strength k_1 of A. In particular, when $k_1 = 0$, $\{A_1, A_2, \ldots, A_u\}$ is still an orthogonal partition of A of strength 0.*

Definition 2.3. *[?] An $((n, K, d))_s$ QECC has the quantum Singleton bound:*

$$K \le s^{n-2d+2}. \tag{1}$$

An $((n,K,d))_{s_1,s_2,\ldots,s_n}$ QECC *satisfies the quantum Singleton bound:*

$$K \le \min\{\prod_{j\epsilon C} s_j \mid C \subset \{1,2,\ldots,n\}, |C| = n-2(d-1)\} \tag{2}$$

for $n \ge 2(d-1)+1$, *and*

$$K \le 1 \tag{3}$$

for $n = 2(d-1)$.

A QECC that achieves the equality in Eq. (??), Eq. (??) or Eq. (??) is called a quantum MDS (QMDS) code.

Definition 2.4. *An* $((n,K,d))_{s_1,s_2,\ldots,s_n}$ *is called a* near quantum MDS (NQMDS) code if

$$K = \min\{\prod_{j\epsilon C} s_j \mid C \subset \{1,2,\ldots,n\}, |C| = n-2(d-1)\} - 1$$

for $n \ge 2(d-1)+1$.

2.1 Important properties of OAs

Lemma 2.1. *[?] The minimal distance of an* $\mathrm{OA}(s^k,n,s,k)$ *is* $n-k+1$ *for* $s \ge 2$ *and* $k \ge 1$.

Lemma 2.2. *For a prime power* s, *let* $(a_1,a_2,\ldots,a_m) = ((s)\oplus 0_{s^{m-1}}, 0_s \oplus (s) \oplus 0_{s^{m-2}}, \ldots, 0_{s^{m-1}} \oplus (s))$. $b_n = c_{i_1}a_{i_1} + \cdots + c_{i_{u-1}}a_{i_{u-1}} + a_{i_u}$ $(1 \le n \le \frac{s^m-1}{s-1} - m, c_{i_v} \ \epsilon\ F_s, 1 \le u \le m, 1 \le v \le u-1)$. *Then*

$$A = (a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_{\frac{s^m-1}{s-1}-m})$$

is a saturated orthogonal array $\mathrm{OA}(s^m, \frac{s^m-1}{s-1}, s, 2)$.

Proof. It follows from linear independence of any two columns of A. $\square$

Remark 2.1. *The method of Lemma* ?? *is called independent columns method, abbreviated* IC *method.*

Lemma 2.3. *[?] Assume that* A *is an* $\mathrm{OA}(N_1,n,s_1,t)$ *with* $md(A) = h_1$, *and that* B *is an* $\mathrm{OA}(N_2,n,s_2,$ $t)$ *with* $md(B) = h_2$. *Let* $h = \min\{h_1,h_2\}$. *Then there exists an* $\mathrm{OA}(N_1N_2,n,s_1s_2,t)$ *with* $md = h$.

Lemma 2.4. *[?](Expansive replacement method) Suppose* A *is an* OA *of strength* k *with column* 1 *having* d_1 *levels and that* B *is an* OA *of strength* k *with* d_1 *rows. After making a one-to-one mapping between the levels of column* 1 *in* A *and the rows of* B, *if each level of column* 1 *in* A *is replaced by the corresponding row from* B, *we can obtain an* OA *of strength* k.

2.2 Important properties of CECCs and QECCs

Lemma 2.5. *[?] If* C *is an* $(n,N,d)_s$ CECC *over* F_s *with dual distance* $d^{\perp}$, *then the codewords of* C *form the rows of an* $\mathrm{OA}(N,n,s,d^{\perp}-1)$ *with entries from* F_s. *Conversely, the rows of a linear* $\mathrm{OA}(N,n,s,k)$ *over* F_s *form an* $(n,N,d)_s$ CECC *over* F_s *with dual distance* $d^{\perp} \ge k+1$. *If the orthogonal array has strength* k *but not* $k+1, d^{\perp}$ *is precisely* $k+1$.

Lemma 2.6. *[?] Assume that there exists an* $\mathrm{OA}(N, n, s, k)$ *with md=h and an orthogonal partition* $\{A_1, \ldots, A_K\}$ *of strength* k_0. *Let* $d = \min\{k_0, h-1\}$. *Then, there exists an* $((n, K, d+1))_s$ QECC.

Lemma 2.7. *[?] Let* Q *be a subspace of* $\mathbb{C}^{s_1} \otimes \mathbb{C}^{s_2} \otimes \cdots \otimes \mathbb{C}^{s_n}$. *If* Q *is an* $((n, K, k+1))_{s_1, s_2, \ldots, s_n}$ QECC, *then for any* k *parties, the reductions of all states in* Q *to the* k *parties are identical. The converse is true. Further if* Q *is pure, then any state in* Q *is a* k*-uniform state. The converse is also true. In particular, when* $s_1 = s_2 = \cdots = s_n$, Q *is an* $((n, K, k+1))_{s_1}$ QECC.

The Lemma **??** can be regarded as the definition of a QECC $((n, K, k+1))_{s_1, s_2, \ldots, s_n}$, where n is the number of qudits, K is the dimension of the encoding state, $k+1$ is the minimum distance, and $s_1, s_2, \ldots, s_n$ are the alphabet size.

Lemma 2.8. *[?] If* $L = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{Nn} \end{pmatrix}$ *is an* $\mathrm{IrOA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$, *then the superposition of* N *product states,* $|\phi_{s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}}\rangle = |a_{11}a_{12}\ldots a_{1n}\rangle + |a_{21}a_{22}\ldots a_{2n}\rangle + \cdots + |a_{N1}a_{N2}\ldots a_{Nn}\rangle$ *is a* k*-uniform state.*

3 Main Results

In this section, we construct NMDS codes including almost extremal NMDS codes, m-MDS codes and NQMDS codes over mixed alphabets through OAs. Here we first give the relationship between OAs and QECCs. There exists a perfect match between the parameters of an $\mathrm{OA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$, A, with an orthogonal partition $\{A_1, A_2, \ldots, A_K\}$ of strength k_1 and the parameters of an $((n, K, d))_{s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}}$ QECC, which is listed in Table **??**.

Table 1: Correspondence between parameters of OAs and QECCs.

	$\mathrm{OA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$	QECC $((n, K, d))_{s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}}$
n	Number of factors	Length of code
K	Number of partitioned blocks	Dimension of code
d	$\min\{k_1 + 1, md(A)\}$	Minimum distance of code
$s_1, s_2, \ldots, s_v$	Number of levels	Alphabet size

3.1 Construction of MDS, NMDS and m-MDS CECCs through orthogonal arrays

Theorem 3.1. *For a prime power* s, *suppose* A *is an* $\mathrm{OA}(s^k, n, s, t)$ $(n \geq k)$ *constructed by* IC *method. The rows of* A *form a* $C = [n, k, d]_s$ CECC. *Then*

(1). *C is* MDS *if and only if the strength of A is k;*

(2). *C is* NMDS *if and only if C is* AMDS *and the strength of A is $k-1$;*

(3). *C is m-*MDS *if and only if $S(C) = m$ and the strength of A is $k-m$ $(k > m)$.*

Proof. Suppose C is an $\mathrm{OA}(s^k, n, s, t)$ constructed from linear combination of k independent columns $((s) \oplus 0_{s^{k-1}}, 0_s \oplus (s) \oplus 0_{s^{k-2}}, \ldots, 0_{s^{k-1}} \oplus (s))$. Because the row rank of C is equal to its column rank, C is a linear code $[n, k, d]_s$.

(1). If C is MDS, from [?], $C^\perp$ is also MDS. So $d^\perp = k+1$. It follows from Lemma ?? that $t = k$. Conversely, if $t = k$, from Lemma ?? we have $d = n - k + 1$. Thus C is MDS.

(2). If $C = [n, k, d]_s$ is NMDS, then both C and $C^\perp = [n, n-k, d^\perp]_s$ are AMDS. Thus $d^\perp = n - (n-k) = k$. It follows from Lemma ?? that $t = d^\perp - 1 = k - 1$. So C is AMDS and $t = k-1$. Conversely, if C is AMDS and $t = k-1$, we have $C^\perp$ is an $[n, n-k, k]_s$ CECC. i.e. $C^\perp$ is AMDS. Thus C is NMDS.

(3). If $C = [n, k, d]_s$ is m-MDS, that is $S(C) = S(C^\perp) = m$ where $C^\perp = [n, n-k, d^\perp]_s$. Since $S(C^\perp) = n-(n-k)+1-d^\perp$, we have $d^\perp = k+1-m$. It follows from Lemma ?? that $t = d^\perp - 1 = k-m$. So $S(C) = m$ and $t = k-m$. Conversely, if $S(C) = m$ and $t = k-m$, then $C^\perp$ is an $[n, n-k, k-m+1]_s$ CECC. Obviously, $S(C^\perp) = n - (n-k) + 1 - (k-m+1) = m$. That is, $S(C) = S(C^\perp) = m$. Thus C is m-MDS. □

Example 3.1. *Let $s = 2$ and $k = 3$ in Theorem ??. Let $(a_1, a_2, a_3) = ((2) \oplus 0_4, 0_2 \oplus (2) \oplus 0_2, 0_4 \oplus (2))$.*

(i). Suppose $t = k = 3$. According to Lemma ??, $A = (a_1, a_2, a_3)$ and $B = (a_1, a_2, a_3, a_1+a_2+a_3)$ are $\mathrm{OA}(8, 3, 2, 3)$ *and* $\mathrm{OA}(8, 4, 2, 3)$*, respectively. Then, we have $[3, 3, 1]_2$ and $[4, 3, 2]_2$* MDS *codes through Theorem ?? (1).*

(ii). Suppose $t = k - 1 = 2$. From Lemma ??, $A_1 = (a_1, a_2, a_3, a_2 + a_3)$, $A_2 = (a_1, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3)$, $A_3 = (a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3)$ and $A_4 = (a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3)$ are $\mathrm{OA}(8, 4, 2, 2)$, $\mathrm{OA}(8, 5, 2, 2)$, $\mathrm{OA}(8, 6, 2, 2)$ *and* $\mathrm{OA}(8, 7, 2, 2)$*, respectively. Then, we have $[4, 3, 1]_2$, $[5, 3, 2]_2$, $[6, 3, 3]_2$ and $[7, 3, 4]_2$* NMDS *codes according to Theorem ?? (2).*

Example 3.2. *Let $s = 2$ and $k = 4$ in Theorem ??. Let $(a_1, a_2, a_3, a_4) = ((2) \oplus 0_8, 0_2 \oplus (2) \oplus 0_4, 0_4 \oplus (2) \oplus 0_2, 0_8 \oplus (2))$.*

(i). Suppose $t = k = 4$. According to Lemma ??, $A = (a_1, a_2, a_3, a_4)$ and $B = (a_1, a_2, a_3, a_4, a_1 + a_2 + a_3 + a_4)$ are $\mathrm{OA}(16, 4, 2, 4)$ *and* $\mathrm{OA}(16, 5, 2, 4)$*, respectively. Then, we have $[4, 4, 1]_2$ and $[5, 4, 2]_2$* MDS *codes through Theorem ?? (1).*

(ii). Suppose $t = k - 1 = 3$. From Lemma ??, [illegible] *are* $\mathrm{OA}(16, 5, 2, 3)$, $\mathrm{OA}(16, 6, 2, 3)$, $\mathrm{OA}(16, 7, 2, 3)$ *and* $\mathrm{OA}(16, 8, 2, 3)$*, respectively. Then, we have $[5, 4, 1]_2$, $[6, 4, 2]_2$, $[7, 4, 3]_2$ and $[8, 4, 4]_2$* NMDS *codes according to Theorem ?? (2).*

In particular, $[8, 4, 4]_2$ is NMDS *self-dual.*

(iii). Suppose $m = 2$ and $t = k - 2 = 2$. From Lemma ??, $A_1 = (a_1, a_2, a_3, a_4, a_1 + a_2, a_1 + a_3)$, $A_2 = (a_1, a_2, a_3, a_4, a_1 + a_2, a_1 + a_3, a_2 + a_3)$, $A_3 = (a_1, a_2, a_3, a_4, a_1 + a_2, a_1 + a_3, a_2 + a_4, a_3 + a_4)$ *and* $A_4 = (a_1, a_2, a_3, a_4, a_1 + a_2, a_1 + a_3, a_2 + a_4, a_3 + a_4, a_1 + a_2 + a_3 + a_4)$ *are* $\mathrm{OA}(16, 6, 2, 2)$, $\mathrm{OA}(16, 7, 2, 2)$, $\mathrm{OA}(16, 8, 2, 2)$ *and* $\mathrm{OA}(16, 9, 2, 2)$, *respectively. Then, from Theorem ?? (3), we can get* $[6, 4, 1]_2$, $[7, 4, 2]_2$, $[8, 4, 3]_2$ *and* $[9, 4, 4]_2$ 2-MDS *codes.*

Example 3.3. *Let $s = 3$ and $k = 3$ in Theorem ??. Let* $(a_1, a_2, a_3) = ((3) \oplus 0_9, 0_3 \oplus (3) \oplus 0_3, 0_9 \oplus (3))$.

(i). Suppose $t = k = 3$. According to Lemma ??, $A = (a_1, a_2, a_3)$ *and* $B = (a_1, a_2, a_3, a_1 + a_2 + a_3)$ *are* $\mathrm{OA}(27, 3, 3, 3)$ *and* $\mathrm{OA}(27, 4, 3, 3)$, *respectively. Then, from Theorem ?? (1), there are two* MDS *codes* $[3, 3, 1]_3$ *and* $[4, 3, 2]_3$.

(ii). Suppose $t = k - 1 = 2$. From Lemma ??, $A_1 = (a_1, a_2, a_3, a_1 + a_2, a_1 + a_2 + a_3, 2a_1 + a_3, 2a_2 + a_3, 2a_1 + a_2 + a_3, a_1 + 2a_2 + a_3)$, $A_2 = (a_1, a_2, a_3, a_1 + a_2, a_1 + a_2 + a_3, 2a_1 + a_3, 2a_2 + a_3, 2a_1 + a_2 + a_3)$, $A_3 = (a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, 2a_1 + a_3, 2a_1 + a_2)$, $A_4 = (a_1, a_2, a_3, a_1 + a_2, 2a_1 + a_3, 2a_2 + a_3)$, $A_5 = (a_1, a_2, a_3, a_1 + a_2)$ *and* $A_6 = (a_1, a_2, a_3)$ *are* $\mathrm{OA}(27, 9, 3, 2)$, $\mathrm{OA}(27, 8, 3, 2)$, $\mathrm{OA}(27, 7, 3, 2)$, $\mathrm{OA}(27, 6, 3, 2)$, $\mathrm{OA}(27, 5, 3, 2)$ *and* $\mathrm{OA}(27, 4, 3, 2)$, *respectively. Then, according to Theorem ?? (2), there exist six* NMDS *codes* $[9, 3, 6]_3$, $[8, 3, 5]_3$, $[7, 3, 4]_3$, $[6, 3, 3]_3$, $[5, 3, 2]_3$ *and* $[4, 3, 1]_3$.

Here, both $[6, 3, 3]_2$ and $[8, 3, 5]_3$ are almost extremal NMDS according to the definition of almost extremal NMDS codes.

3.2 Construction of near quantum MDS codes over mixed alphabets through orthogonal arrays

Theorem 3.2. *Assume that there exists an* $\mathrm{OA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$ *with $md = h$ and an orthogonal partition* $\{A_1, \dots, A_K\}$ *of strength k_0. Let $d = \min\{k_0, h - 1\}$. Then, there exists an* $((n, K, d + 1))_{s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}}$ QECC.

Proof. By Definition ??, the $\mathrm{OA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, k)$ and A_i $(i = 1, \dots, K)$ are an $\mathrm{IrOA}(N, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, d)$ and an $\mathrm{IrOA}(\frac{N}{K}, n, s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}, d)$, respectively. From the link between IrOAs and uniform states in [?] and $\{A_1, \dots, A_K\}$, we can obtain K d-uniform states $\{|\phi_1\rangle, \cdots, |\phi_K\rangle\}$, which can be used as an orthogonal basis. By Lemma ??, the complex subspace spanned by the orthogonal basis is an $((n, K, d + 1))_{s_1^{n_1} s_2^{n_2} \cdots s_v^{n_v}}$ QECC. □

In fact, if there exists an $\mathrm{OA}(N, 2k, s, k)$ with $md = k + 1$, from Lemma ??, there exists a $((2k, 1, k + 1))_s$ QECC which is a quantum MDS code according to Definition ??.

Sometimes, it is difficult to construct QECCs over mixed alphabets which achieve quantum Singleton bound, we will obtain near quantum MDS codes according to Definition ??. We have the following results.

Theorem 3.3. *If there exists an* $\mathrm{OA}(s^k, 2k + 1, s, k)$ *for even s, then there exists an* NQMDS *code* $((2k + 1, 1, k + 1))_{s^{2k} 2^1}$.

Proof. From Lemma ??, the minimal distance of $\mathrm{OA}(s^k, 2k+1, s, k)$ is $k+2$. We can obtain an $\mathrm{OA}(s^k, 2k+1, s^{2k}2^1, k)$ with $md = k+1$ after an s-level column of $\mathrm{OA}(s^k, 2k+1, s, k)$ is replaced by a two-level column through expansive replacement method in Lemma ??. By Theorem ??, we have a QECC $((2k+1, 1, k+1))_{s^{2k}2^1}$ which is also an NQMDS code according to Definition ??. $\square$

Corollary 3.4. *Suppose there exist two arrays* $\mathrm{OA}(s_1^k, 2k+1, s_1, k)$ *and* $\mathrm{OA}(s_2^k, 2k+1, s_2, k)$. *Let* $s = s_1 s_2$ *be even. Then, there exists an* NQMDS *code* $((2k+1, 1, k+1))_{s^{2k}2^1}$.

Proof. From Lemma ??, we have an $\mathrm{OA}(s^k, 2k+1, s, k)$. From Theorem ??, we have the corollary is true. $\square$

Corollary 3.5. *Suppose there exist* $m \geq 3$ *arrays* $\mathrm{OA}(s_1^k, 2k+1, s_1, k)$, $\mathrm{OA}(s_2^k, 2k+1, s_2, k)$, $\ldots$, $\mathrm{OA}(s_m^k, 2k+1, s_m, k)$. *Let* $s = s_1 s_2 \cdots s_m$ *be even. Then, there exists an* NQMDS *code* $((2k+1, 1, k+1))_{s^{2k}2^1}$.

Proof. Repeatedly using Corollary ??, we have the corollary is true. $\square$

Theorem 3.6. *An* NQMDS *code* $((5, 1, 3))_{s^4 2^1}$ *exists for even* $s \geq 4$ *and* $s \neq 6$.

Proof. From [?], we have the following conclusions: An $\mathrm{OA}(s^2, k, s, 2)$ exists if and only if $k-2$ pairwise orthogonal Latin squares of order s exist; There exist $s-1$ pairwise orthogonal Latin squares for prime power s; There exist more than 2 pairwise orthogonal Latin squares of order $s \geq 12$ which is not a prime power. When $s = 10$, we have an $\mathrm{OA}(100, 5, 10^4 2^1, 2)$ with $md = 3$ in [?]. So $\mathrm{OA}(s^2, 5, s^4 2^1, 2)$ with $md = 3$ for $s \geq 4$ and $s \neq 6$ can be obtained after an s-level column of $\mathrm{OA}(s^2, 5, s, 2)$ is replaced by a two-level column through expansive replacement method in Lemma ??. The proof is complete. $\square$

Example 3.4. *Let* $s = 4$ *and* $k = 2$. *We have an* $\mathrm{OA}(16, 5, 4, 2)$. *Then, we can obtain an* NQMDS *code* $((5, 1, 3))_{4^4 2^1}$ *through Theorem* ??.

Table ?? list plenty of NQMDS codes constructed by Theorem ??, ??, Corollary ??, ??.

4 Conclusion

In this paper, by using OAs, we construct NMDS codes, m-MDS codes and NQMDS codes over two distinct alphabets. In the future, we will study construction of QMDS and NQMDS codes over more distinct alphabets from asymmetrical OAs.

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Table 2: Near quantum MDS codes

Parameters		NQMDS code
k	s	$((2k+1,1,k+1))_{s^{2k}2^1}$
1	4	$((3,1,2))_{4^2 2^1}$
1	6	$((3,1,2))_{6^2 2^1}$
1	8	$((3,1,2))_{8^2 2^1}$
1	10	$((3,1,2))_{10^2 2^1}$
1	12	$((3,1,2))_{12^2 2^1}$
1	14	$((3,1,2))_{14^2 2^1}$
1	16	$((3,1,2))_{16^2 2^1}$
1	$s=2t$ $(t \geq 9)$	$((3,1,2))_{s^2 2^1}$
2	4	$((5,1,3))_{4^4 2^1}$
2	8	$((5,1,3))_{8^4 2^1}$
2	10	$((5,1,3))_{10^4 2^1}$
2	12	$((5,1,3))_{12^4 2^1}$
2	14	$((5,1,3))_{14^4 2^1}$
2	16	$((5,1,3))_{16^4 2^1}$
2	18	$((5,1,3))_{18^4 2^1}$
2	20	$((5,1,3))_{20^4 2^1}$
2	$s=2t$ $(t \geq 11)$	$((5,1,3))_{s^4 2^1}$
3	8	$((7,1,4))_{8^6 2^1}$
3	16	$((7,1,4))_{16^6 2^1}$
3	32	$((7,1,4))_{32^6 2^1}$
3	56	$((7,1,4))_{56^6 2^1}$
3	64	$((7,1,4))_{64^6 2^1}$
3	72	$((7,1,4))_{72^6 2^1}$
3	88	$((7,1,4))_{88^6 2^1}$
3	104	$((7,1,4))_{104^6 2^1}$
3	112	$((7,1,4))_{112^6 2^1}$
3	$s=8\times 2^u \times p_1^{v_1} \times p_2^{v_2} \times \cdots \times p_m^{v_m}$ (p_i is a prime and $p_i^{v_i} \geq 7$)	$((7,1,4))_{s^6 2^1}$
4	8	$((9,1,5))_{8^8 2^1}$
4	16	$((9,1,5))_{16^8 2^1}$
4	32	$((9,1,5))_{32^8 2^1}$
4	64	$((9,1,5))_{64^8 2^1}$
4	72	$((9,1,5))_{72^8 2^1}$
4	88	$((9,1,5))_{88^8 2^1}$
4	104	$((9,1,5))_{104^8 2^1}$
4	128	$((9,1,5))_{128^8 2^1}$
4	$s=8\times 2^u \times p_1^{v_1} \times p_2^{v_2} \times \cdots \times p_m^{v_m}$ (p_i is a prime and $p_i^{v_i} \geq 9$)	$((9,1,5))_{s^8 2^1}$
5	16	$((11,1,6))_{16^{10} 2^1}$
5	32	$((11,1,6))_{32^{10} 2^1}$
5	64	$((11,1,6))_{64^{10} 2^1}$
5	128	$((11,1,6))_{128^{10} 2^1}$
5	176	$((11,1,6))_{176^{10} 2^1}$
5	208	$((11,1,6))_{208^{10} 2^1}$
5	$s=16\times 2^u \times p_1^{v_1} \times p_2^{v_2} \times \cdots \times p_m^{v_m}$ (p_i is a prime and $p_i^{v_i} \geq 11$)	$((11,1,6))_{s^{10} 2^1}$
$\cdots$	$\cdots$	$\cdots$

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