Comments on "A Linear Time Algorithm for the Optimal Discrete IRS Beamforming"

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Abstract—Comments on [?] are provided. Updated necessary and sufficient conditions for its Lemma 1 are given. Consequently, an updated Algorithm 1 is provided with full specification. Simulation results with improved performance over the implementation of Algorithm 1 are provided.

Index Terms—Intelligent reflective surface (IRS), reconfigurable intelligent surface (RIS), discrete beamforming for IRS/RIS.

I. Introduction

Reference [?] presented an algorithm to solve the problem of finding the values $\theta_1, \theta_2, \ldots, \theta_N$ to maximize $|h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}|$ where $\theta_n \in \Phi_K$ and $\Phi_K = \{\omega, 2\omega, \ldots, K\omega\}$ with $\omega = \frac{2\pi}{K}$ and $j = \sqrt{-1}$. The set Φ_K can equivalently be described as $\{0, \omega, 2\omega, \ldots, (K-1)\omega\}$. In [?], the values $h_n \in \mathbb{C}$, $n = 1, 2, \ldots, N$ are the channel coefficients and θ_n are the phase values added to the corresponding h_n by an intelligent reflective surface (IRS), also known as reconfigurable intelligent surface (RIS).

II. Two Statements from [?]

Towards achieving its goal, [?] introduced the following lemma.

Lemma 1: For an optimal solution $(\theta_1^*, \ldots, \theta_n^*)$ to problem (8), each θ_n^* must satisfy

$$\theta_n^* = \arg\min_{\theta_n \in \Phi_K} |(\theta_n + \alpha_n - \angle\mu) \mod 2\pi| \tag{11}$$

where $\angle\mu$ stands for the phase of μ in (10)[1].

In [?], problem (8) is defined as

$$\underset{\boldsymbol{\theta}}{\text{maximize}} \; f(\boldsymbol{\theta}) \tag{8a}$$

$$\text{subject to } \theta_n \in \Phi_K \quad \text{for} \quad n = 1, 2, \ldots, N \tag{8b}$$

where

$$f(\boldsymbol{\theta}) = \frac{1}{\beta_0^2} \left| \beta_0 e^{j\alpha_0} + \sum_{n=1}^{N} \beta_n e^{j(\alpha_n + \theta_n)} \right|^2, \tag{7b}$$

$h_n = \beta_n e^{j\alpha_n}$ for $n = 0, 1, \ldots, N$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$. Also, g is defined as

$$g \equiv h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n^*} \tag{9}$$

and μ as

$$\mu = \frac{g}{|g|}. \tag{10}$$

Lemma 1 does not hold. This can be seen by numerical examples. We give one such example in Table ??. In this table, we look at the simple case of $K = 2$, $N = 2$. According to Lemma 1 in [?], the condition in (11) should satisfy (8) for this simple case. We draw values of h_n according to the first paragraph of Sec. IV in [?]. We list these values in rows 2–4 of Table ??. We define

$$g_0(\theta_1, \theta_2) = h_0 + \sum_{n=1}^{2} h_n e^{j\theta_n} \tag{19}$$

and list the values of $g_0(\theta_1, \theta_2)$ for all possible $\theta_1, \theta_2 \in \{0, \pi\}$. There are four such values and they are listed in rows 5–8 of Table ??. The set of values for θ_1 and θ_2 that maximize $|g_0|$, or equivalently, that achieve g in (9), are $\theta_1 = \theta_2 = \pi$ as in row 8 of Table ??. Note that this operation results in $\angle\mu = 2.3719$ radians as shown in column 5 of row 8 of Table ??.

At this point, we would like to emphasize that [?] uses a particular convention for the phases of complex numbers. They are defined to be in $[0, 2\pi)$, see the text that follows (2) in [?]. We use the same convention in generating Table ??, see its column 5, as well as in generating Table ??. With this convention, we list $\theta_n + \alpha_n - \angle\mu$ and $(\theta_n + \alpha_n - \angle\mu) \mod 2\pi$ for possibilities of $\theta_n = 0$ and $\theta_n = \pi$ and $n = 1, 2$ in rows 1–8 of Table ??.[2] It can be seen from rows 1–4 of Table ?? that the method results in $\theta_1 = \pi$ as the potential θ_1^*, which we know from the discussion in the previous paragraph to be correct. When we carry out the calculation $(\theta_2 + \alpha_2 - \angle\mu) \mod 2\pi$ in rows 5–8 of Table ??, we find that the method suggests $\theta_2 = 0$ should be θ_2^*. However, we know from the exhaustive search in rows 5–8 of Table ?? that $\theta_2^* = \pi$. Thus, Lemma 1 is not correct.

It is possible to come up with a correct lemma similar to Lemma 1. We specify this lemma below.

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This work is partially supported by NSF grant 2030029.

[1] To prevent confusion, we will use the same equation numbers (7)–(13) in [?]. Our own equation numbers, not available in [?], will begin at (19) and will be incremented from that number on. Similarly, we will introduce Lemma 2 and Algorithm 2. Note that a lemma or an algorithm with number 2 does not exist in [?].

[2] Note that absolute value signs in (11) are not needed since the argument of the operation in (11) is in $[0, 2\pi)$.

	$\mathrm{Re}[\cdot]$	$\mathrm{Im}[\cdot]$	$\lvert\cdot\rvert$	$\angle \cdot \in [0, 2\pi)$ (rad.)
h_0	-2.8267×10^{-7}	2.7376×10^{-7}	3.9350×10^{-7}	2.3722
h_1	1.0958×10^{-10}	-1.0501×10^{-11}	1.1008×10^{-10}	6.1876
h_2	-1.2238×10^{-11}	-2.6605×10^{-11}	2.6634×10^{-10}	4.6664
$g_0(\theta_1 = 0, \theta_2 = 0)$	-2.8257×10^{-7}	2.7348×10^{-7}	3.9324×10^{-7}	2.3725
$g_0(\theta_1 = 0, \theta_2 = \pi)$	-2.8255×10^{-7}	2.7401×10^{-7}	3.9359×10^{-7}	2.3715
$g_0(\theta_1 = \pi, \theta_2 = 0)$	-2.8279×10^{-7}	2.7350×10^{-7}	3.9341×10^{-7}	2.3729
$g_0(\theta_1 = \pi, \theta_2 = \pi)$	-2.8277×10^{-7}	2.7403×10^{-7}	$\mathbf{3.9377 \times 10^{-7}}$	**2.3719**

Table 1: Sample calculation for attempting to find optimum θ_1^*, θ_2^* to maximize $|g_0|$ where $g_0(\theta_1, \theta_2) = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}$ with $\theta_n \in \Phi_K = \{0, \frac{2\pi}{K}, \ldots, (K-1)\frac{2\pi}{K}\}$, $n = 1, 2, \ldots, N$ for $K = 2$ and $N = 2$. Channel coefficients h_n, $n = 0, 1, 2$ are calculated using the technique described in [?]. Rows 5–8 present all values of g_0 with all combinations of $\theta_1, \theta_2 \in \Phi_2$, showing that $|g| = \max |g_0(\theta_1, \theta_2)|$ is achieved with $\theta_1^* = \theta_2^* = \pi$.

$(\theta_1 = 0) + \alpha_1 - \angle\mu$	3.8158
mod $((\theta_1 = 0) + \alpha_1 - \angle\mu, 2\pi)$	3.8158
$(\theta_1 = \pi) + \alpha_1 - \angle\mu$	6.9574
mod $((\theta_1 = \pi) + \alpha_1 - \angle\mu, 2\pi)$	**0.67417**
$(\theta_2 = 0) + \alpha_2 - \angle\mu$	2.2945
mod $((\theta_2 = 0) + \alpha_2 - \angle\mu, 2\pi)$	**2.2945**
$(\theta_2 = \pi) + \alpha_2 - \angle\mu$	5.4361
mod $((\theta_2 = \pi) + \alpha_2 - \angle\mu, 2\pi)$	5.4361
$\cos((\theta_1 = 0) + \alpha_1 - \angle\mu)$	-0.7812
$\cos((\theta_1 = \pi) + \alpha_1 - \angle\mu)$	**0.7812**
$\cos((\theta_2 = 0) + \alpha_2 - \angle\mu)$	-0.6672
$\cos((\theta_2 = \pi) + \alpha_2 - \angle\mu)$	**0.6672**

Table 2: Continuation of the sample calculation for attempting to find optimum θ_1^*, θ_2^* to maximize $|g_0|$. Rows 1–8 present the calculation of mod $(\theta_n + \alpha_n - \angle\mu, 2\pi)$ for $n = 1, 2$, as specified in [?] to attempt to find the optimum values of θ_n. This calculation results in values $\theta_1^* = 0$ and $\theta_2^* = \pi$, which are not optimal. Rows 9–12 present the calculation of $\max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu)$ to find θ_1^*, θ_2^* as discussed in this comment. This technique finds the optimum values of θ_n, $n = 1, 2, \ldots, N$.

Lemma 2: For an optimal solution $(\theta_1^*, \theta_2^*, \ldots, \theta_n^*)$, it is necessary and sufficient that each θ_n^* satisfy

$$\theta_n^* = \arg \max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu) \tag{20}$$

where $\angle\mu$ stands for the phase of μ in (10).

Proof: We can rewrite (9) as

$$|g| = \beta_0 e^{j(\alpha_0 - \angle\mu)} + \sum_{n=1}^{N} \beta_n e^{j(\alpha_n + \theta_n - \angle\mu)} \tag{21}$$

$$\begin{aligned} &= \beta_0 \cos(\alpha_0 - \angle\mu) + j\beta_0 \sin(\alpha_0 - \angle\mu) \\ &+ \sum_{n=1}^{N} \beta_n \cos(\theta_n + \alpha_n - \angle\mu) \\ &+ j \sum_{n=1}^{N} \beta_n \sin(\theta_n + \alpha_n - \angle\mu). \end{aligned} \tag{22}$$

Because $|g|$ is real-valued, the second and fourth terms in (22) sum to zero, and

$$|g| = \beta_0 \cos(\alpha_0 - \angle\mu) + \sum_{n=1}^{N} \beta_n \cos(\theta_n + \alpha_n - \angle\mu) \tag{23}$$

from which (??) follows as a necessary and sufficient condition for Lemma 2 to hold. ■

Rows 9–12 of Table ?? illustrate that this method finds θ_1^* and θ_2^*. More extensive calculations can be carried out to show that an exhaustive search as in rows 5–8 of Table ?? confirms that Lemma 2 holds for a wide set of K and N values as well as a wide set of channel coefficients $h_0, h_1, \ldots, h_N$.

Reference [?] attempts to decide a range of μ for which $\theta_n^* = k\omega$ must hold, making use of Lemma 1. Towards that end, it first defines a sequence of complex numbers with respect to each $n = 1, 2, \ldots, N$ as

$$s_{nk} = e^{j(\alpha_n + (k-0.5)\omega)}, \text{ for } k = 1, 2, \ldots, K. \tag{12}$$

Then, [?] defines, for any two points a and b on the unit circle C, $\mathrm{arc}(a : b)$ to be the unit circular arc with a as the initial end and b as the terminal end in the counterclockwise direction; in particular, it defines $\mathrm{arc}(a : b)$ as an open arc with the two endpoints a and b excluded. With this definition, [?] states the following proposition follows from Lemma 1.

Proposition 1: A sufficient condition for $\theta_n^* = k\omega$ is

$$\mu \in \mathrm{arc}(s_{nk} : s_{n,k+1}). \tag{13}$$

Reference [?] states that "letting $\theta_n = k\omega$ is guaranteed to minimize the gap $|(\theta_n + \alpha_n - \angle\mu) \bmod 2\pi|$ whenever μ lies in its associated arc, and thus $k\omega$ must be optimal according to Lemma 1."

Now, let $K = 2$ and thus $\omega = \frac{2\pi}{K} = \pi$, and the two possibilities for θ are $\theta^1 = \pi$ and $\theta^2 = 2\pi$, or equivalently $\theta^2 = 0$. According to (12), we have

$$s_{n1} = e^{j(\alpha_n + \frac{\pi}{2})}, \quad s_{n2} = e^{j(\alpha_n + \frac{3\pi}{2})}. \tag{24}$$

According to Proposition 1, if $\mu \in \mathrm{arc}(s_{n1} : s_{n2})$ then $\theta_n^* = \omega = \pi$ should hold. Assume μ is in $\mathrm{arc}(s_{n1} : s_{n2})$. Then it can be observed that $\alpha_n - \angle\mu \in (\frac{\pi}{2}, \frac{3\pi}{2})$, paying attention to the change of order due to the subtraction of $\angle\mu$. In particular, let μ be such that $\alpha_n - \angle\mu \in (\frac{\pi}{2}, \pi)$. When this is the case, note that $(\theta^1 + \alpha_n - \angle\mu) \in (\frac{3\pi}{2}, 2\pi)$ while $(\theta^2 + \alpha_n - \angle\mu) \in (\frac{\pi}{2}, \pi)$. Thus, $|(\theta^2 + \alpha_n - \angle\mu) \bmod 2\pi| < |(\theta^1 + \alpha_n - \angle\mu) \bmod 2\pi|$, and according to Lemma 1, $\theta_n^* = \theta^2 = 0$, in contradiction with Proposition 1. On the other

$$\angle\mu \in \left(\alpha_n + \left(k - \frac{1}{2}\right)\omega, \alpha_n + \left(k + \frac{1}{2}\right)\omega\right) \tag{25}$$

and, Proposition 1 is compatible with Lemma 2. To see this, assume μ satisfies (12). Then,

$$\angle\mu \in \left(\alpha_n + \left(k - \frac{1}{2}\right)\omega, \alpha_n + \left(k + \frac{1}{2}\right)\omega\right). \quad (25)$$

Since $\omega = \frac{2\pi}{K}$,

$$\alpha_n - \angle\mu \in \left((-2k - 1)\frac{\pi}{K}, (-2k + 1)\frac{\pi}{K}\right) \quad (26)$$

considering the reversal of order due to the substraction of $\angle\mu$. Now, let $\theta_n = k\omega = 2k\frac{\pi}{K}$. Then

$$\theta_n + \alpha_n - \angle\mu \in \left(-\frac{\pi}{K}, \frac{\pi}{K}\right) \quad (27)$$

and thus $\cos(\theta_n + \alpha_n - \angle\mu)$ is the largest among all other possibilities for θ_n because the slice $(-\frac{\pi}{K}, \frac{\pi}{K})$ corresponds to the largest values of the cosine function among all slices corresponding to different values of $\theta_k \in \Phi_K$ for $k = 1, 2, \ldots, K$.

III. New Algorithm

We now specify Algorithm 2 to replace Algorithm 1 in [?]. In doing so, not only do we incorporate Lemma 2 instead of Lemma 1 but also we eliminate the many uncertainties present in Algorithm 1 of [?].

Algorithm 2 Update for Algorithm 1

1: **Initialization:** Compute $s_{nk} = e^{j(\alpha_n+(k-0.5)\omega)}$ for $n = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, K$.
2: Eliminate duplicates among s_{nk} and sort to get $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_L < 2\pi$.
3: Let, for $l = 1, 2, \ldots, L$, $\mathcal{N}(\lambda_l) = \{n | s_{nk} = \lambda_l\}$.
4: Set $\angle\mu = 0$. For $n = 1, 2, \ldots, N$, calculate $\theta_n = \arg\max_{\theta_n \in \Phi_K} \cos(\theta_n + \alpha_n - \angle\mu)$.
5: Set $g_1 = h_0 + \sum_{n=1}^{N} h_n e^{j\theta_n}$, `absgmax` $= 0$.
6: **for** $l = 2, 3, \ldots, L$ **do**
7: For each $n \in \mathcal{N}(\lambda_l)$, let $(\theta_n + \omega \leftarrow \theta_n) \bmod \Phi_K$.
8: Let $g_l = g_{l-1} + \sum_{n \in \mathcal{N}(\lambda_l)} h_n\left(e^{j\theta_n} - e^{j(\theta_n - \omega) \bmod \Phi_K}\right)$
9: **if** $|g_l| >$ `absgmax` **then**
10: Let `absgmax` $= |g_l|$
11: Store θ_n for $n = 1, 2, \ldots, N$
12: **end if**
13: **end for**
14: Read out θ_n^* as the stored θ_n, $n = 1, 2, \ldots, N$.

IV. Results and Remarks

Because its description is based on Lemma 1, which does not provide an equivalency condition for finding $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$, the performance of Algorithm 1 will in general not achieve the optimum result for SNR Boost [?].

We have implemented Algorithm 1 to the best of our interpretation. We have also implemented Algorithm 2. We present the CDF results for SNR Boost [?] in Fig. 1 for $K = 2$ and $N = 16$, 64, and 256, using the average of 1,000 realizations of the channel. Clearly, Algorithm 1 is not optimal as Algorithm 2 performs better than Algorithm 1 although the gains decrease with N. Plots for $K = 4$ show smaller gains as compared to $K = 2$, but still Algorithm 2 always performs better than Algorithm 1 for the same K and N.

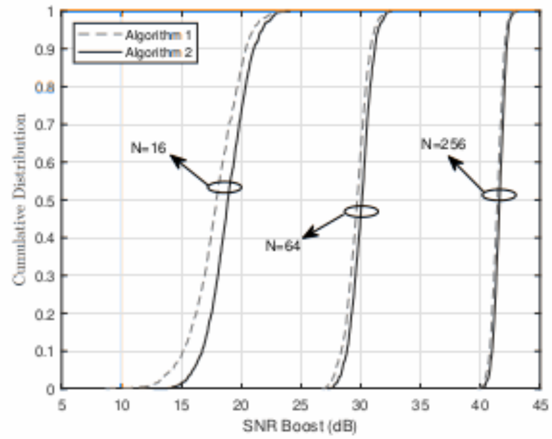

Fig. 1: CDF plots for SNR Boost with Algorithm 1 and Algorithm 2, $K = 2$.

We note that it is possible to convert the maximization of $\cos(x)$ into the minimization of a simple expression. For example, minimization of $f_1(x) = \pi - |(x \bmod 2\pi) - \pi|$ is the same as maximization of $\cos(x)$ within the context of Lemma 2. However, this is different than minimization of $|x \bmod 2\pi|$ proposed in Lemma 1 of [?]. The reason can be seen by plotting these functions against x. While $f_1(x)$ and $\cos(x)$, in addition to being periodic with period 2π, have even symmetry around odd multiples of π, $|x \bmod 2\pi|$ (or equivalently, $(x \bmod 2\pi)$) does not have this symmetry.