The recombination history of the Universe

Milestone II, AST5220
Spring 2020

Jowita Borowska

Code available at: https://github.com/jowborowska/AST5220

1 Introduction

The Universe was opaque at its early stages. All the atoms were fully ionized, so that photons were Thomson scattered by free electrons. At the time of recombination (when most of free electrons were captured by free protons, forming neutral hydrogen atoms), photons decoupled from the ordinary matter and could travel freely through the transparent Universe. These photons, scattered last time around the time of recombination, can be observed today as CMB.

In this project, we will look more closely on quantities associated with electron density, making it possible to gather some information about the time of recombination and compute the CMB power spectrum later on. We will follow the evolution of fractional electron density, optical depth and visibility function from the early times (with corresponding scale factor $a_{min}(t) = 10^{-7} \Rightarrow x_{min} = \ln(10^{-7})$) until today ($a(t_0) = a_0 = 1 \Rightarrow x_0 = 0$). The mathematical description of all these quantities will be presented in the Methods section, together with some details about the numerical implementation. Thereafter, we will present results produced by the programs (Results section).

2 Methods

2.1 Equations of interest

The equations associated with the epoch of recombination will be presented in this subsection. We will closely follow the discussion in Callin (2016) and Winther (2020). However, all the expressions have undergone dimensional analysis, transforming them from natural units (where $c = \hbar = k_b = 1$) to SI units.

Fractional electron density, X_e

First of all, we want to compute the fractional electron density,

$$Xe \equiv \frac{n_e}{n_H} \approx \frac{n_e}{n_b}, \tag{1}$$

where n_e is the electron density - number of free electrons per cubic meter - and n_H is the proton density (equal to the baryon density, $n_H = n_b$, if we assume that all baryons are protons and ignore helium). The baryon density is a function of scale factor, a, and can be found as

$$n_b(a) \approx \frac{\rho_b(a)}{m_H} = \frac{\Omega_{b,0}\rho_c}{m_H a^3}, \tag{2}$$

where $\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G}$ is the critical density of the universe today, $\Omega_{b,0} = 0.046$ is the relative density parameter of baryons today and m_H is the mass of hydrogen atom. In order to approximate X_e before recombination - system in strong thermodynamic equilibrium ($X_e < 0.99$), we will use Saha equation. During and after recombination ($X_e > 0.99$), the Peebles equation will be used.

Saha equation

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{m_e T_b k_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/(T_b k_b)}, \tag{3}$$

where m_e is the mass of electron, $\epsilon_0 = 13.6$ eV is the ionization energy of hydrogen and T_b is the baryon temperature, that can be approximated by

$$T_b(a) = \frac{T_{\text{CMB}}}{a}, \tag{4}$$

with $T_{\text{CMB}} = 2.725$ K. The Saha equation is then a simple quadratic equation on the form

$$\begin{aligned} &0 = AX_e^2 + BX_e + C, \\ \text{with :}& \\ &A = -1 \\ &B = -C \\ &C = \frac{1}{n_b}\left(\frac{m_e T_b k_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/(T_b k_b)} \end{aligned}$$

and the solution reads

$$X_e = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm B\sqrt{1 - \frac{4AC}{B^2}}}{2A} = \frac{C \mp C\sqrt{1 + \frac{4}{C}}}{-2} = \frac{-C \pm C\sqrt{1 + \frac{4}{C}}}{2}. \tag{5}$$

Since $X_e > 0$ at all times, we use the upper "+" sign. Moreover, at early times C is huge, so that we can approximate the expression by

$$X_e = \frac{-C + C\sqrt{1 + \frac{4}{C}}}{2} \approx \frac{-C + C(1 + \frac{2}{C})}{2} = \frac{-C + C + 2}{2} = 1. \tag{6}$$

Peebles equation

Peebles equation is a first order differential equation on the form

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1-X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right], \tag{7}$$

that can be initialized with the last X_e value calculated in the Saha regime.
We will follow Ma and Bertschinger (1995), describing terms of the above equation. First of all, $C_r(T_b)$ is the ratio of the net decay rate to the sum of the decay and ionization rates from the 2nd level,

$$C_r(T_b) = \frac{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha}{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}. \tag{8}$$

Here, $\Lambda_{2s\rightarrow 1s} = 8.227\text{s}^{-1}$ is the rate of 2-photon decay from the 2s level and Λ_α is the rate of Lyman alpha production allowed by cosmological redshifting of photons out of the line,

$$\Lambda_\alpha = H(x)\frac{(3\epsilon_0)^3}{(8\pi)^2(\hbar c)^3 n_{1s}} \tag{9}$$

$$n_{1s} = (1-X_e)n_H, \tag{10}$$

where $H(x)$ is the Hubble parameter, evolving as presented in the previous milestone, and n_{1s} is the number density of hydrogen atoms in the 1s state. Moreover,

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b k_b} \tag{11}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b k_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/T_b k_b} \tag{12}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2\hbar^2}{m_e^2 c}\sqrt{\frac{\epsilon_0}{T_b k_b}}\phi_2(T_b) \tag{13}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b k_b), \tag{14}$$

where $\beta(T_b)$ is the collisional ionization rate from the ground state and $\alpha^{(2)}(T_b)$ is the recombination rate to excited states, with the fine structure constant $\alpha \approx \frac{1}{137}$.

It is worth noticing that $k_b T_b = k_b T_{\text{CMB}}/a$ becomes very small as we approach present times ($a(t_0) = a_0 = 1$ today). The exponential term in $\beta^{(2)}(T_b)$ can then easily overflow. In order to prevent that, we sharply set $\beta^{(2)}(T_b)$ equal to zero for large exponent values (i.e. when $3\epsilon_0/4k_b T_b > 200$).

Optical depth, $\tau(x)$

Optical depth, $\tau(x)$, is a measure of attenuation of the intensity of radiation emitted by a distant source,

$$I_{\text{observed}}(x) = I_{\text{emitted}}\, e^{-\tau(x)}, \tag{15}$$

where x measures the 'distance' that light travels through medium between the source and the observer. If $\tau(x) \ll 1$ the medium is called optically thin and if $\tau(x) \gg 1$ the medium is optically thick. Transition between these two regimes, $\tau(x_*) \approx 1$, marks the surface of last scattering (we will compute the time x_*, as well as the corresponding redshift, z_*).

In cosmology, the main cause of radiation intensity attenuation by the medium is Thomson scattering of photons by free electrons. Therefore, the expression for optical depth on differential form reads

$$\tau' = \frac{d\tau}{dx} = -\frac{n_e \sigma_T c}{H(x)}, \tag{16}$$

where $\sigma_T = \frac{8\pi}{3}\frac{\alpha^2\hbar^2}{m_e^2 c^2}$ is Thomson cross-section. This is again a first order differential equation. We will initialize it with a random optically thick value, $\tau(x = \ln(10^{-7})) = 100$, and subtract the result we get for $\tau(x = 0)$ at the end, just to ensure that the optical depth today is zero, $\tau(x = 0) = 0$.

Visibility function, $\tilde{g}(x)$

The visibility function is defined as

$$\tilde{g}(x) = -\tau' e^{-\tau}. \tag{17}$$

This is a probability distribution,

$$\int_{-\infty}^{0} \tilde{g}(x)dx = 1, \tag{18}$$

describing the probability density for a given photon to scatter at time x. The visibility function is sharply peaked around the surface of last scattering, where $\tau(x_*) \approx 1$, as we will see later.

2.2 Numerical implementation

The second project milestone is based on modifying the RecombinationHistory class in the corresponding files from the given set of C++ templates. To create the class object, we have to provide it with h, $\Omega_{b,0}$ and T_{CMB} parameters, as well as the object of BackgroundCosmology class (to fetch H(x) computed there) and helium fraction, Y_p, that is set to zero throughout the whole project.

This class has several functions. First of all, there is a group of methods computing fractional electron density, X_e, by implementing Saha and Peebles equations. In the solve_number_density_electrons() we set up a grid of 4000 x-values in range $x_{min} = \ln(a_{min} = 10^{-7})$ to $x_{max} = \ln(a_0 = 1) = 0$, as well as the empty arrays to store $n_e(x)$ and $X_e(x)$. Thereafter, we loop over all x-values, firstly computing a set of corresponding X_e and n_e through electron_fraction_from_saha_equation() function, where the equations 1, 3 - 6 are implemented. Next, we check if the given X_e-value is still in the Saha regime ($X_e > 0.99$). If it satisfies this condition, we assign the present results to the corresponding arrays, if it does not - we switch to the Peebles regime. The differential equation is set up by rhs_peebles_ode, implementing eqations 7-14, and solved by an ODE solver. Finally, we use natural logarithms of X_e and n_e to create splines through the resulting data-points ($\ln(X_e(x))$ and $\ln(n_e(x))$ vary more slowly with x, so that they are easier to interpolate).

Next, we have method solve_for_optical_depth_tau(), that sets up a grid of 1000 x-values in range $x_{min} = \ln(a_{min} = 10^{-7})$ to $x_{max} = \ln(a_0 = 1) = 0$, as well as empty arrays for storing $\tau(x)$, $\tau'(x)$ and $\tilde{g}(x)$ values. Again, we set up an ordinary differential equation solver to find τ and compute its derivative, τ', according to Eq. 16. Then, the visibility function, $\tilde{g}$, is calculated (Eq. 17). Finally, splines for τ, τ' and $\tilde{g}$ are created through the resulting data points.

We have also series of "get" functions, used to return quantities like $X_e(x)$, $n_e(x)$, $\tau(x)$, $\tau'(x)$, $\tau''(x)$, $\tilde{g}(x)$, $\tilde{g}'(x)$, $\tilde{g}''(x)$ and $n_b(x)$ (from Eq.2). Function `info()` prints out the information about the model and function `output()` writes the data of interest to a file. Running the whole script is timed to take approximately 0.12 s.

The generated text file with data (recombination.txt) is thereafter read by a program written in Python (milestone2.py), also used to create all the plots included in this report. In addition, we find some quantities here, like the time of last scattering, $x_* = x(\tau \approx 1)$, and half-time of recombination, $x_{rec} = x(Xe \approx 0.5)$, with corresponding redshifts.

Codes to both programs (RecombinationHistory.cpp, milestone2.py) are available in the following github repository: https://github.com/jowborowska/AST5220

3 Results

Fractional electron density

The fractional electron density as a function of logarithm of the scale factor, $X_e(x)$, is shown on Figure 1. The Saha regime holds until x = -7.37, then we switch to the Peebles regime. As we approach present times, X_e settles on some small value ('freeze out'). The time when recombination is half way ($X_e = 0.5$) is $x_{rec} = -7.16$, with corresponding redshift $z_{rec} = 1/\exp(x_{rec}) - 1 = 1291$. The prediction from the Saha equation used entire time (without switching to Peebles equation for $X_e < 0.99$) is $x_{rec,S} = -7.23$ and $z_{rec,S} = 1380$. Saha-treatment predicts then recombination at earlier times (Xe is cut off more sharply).

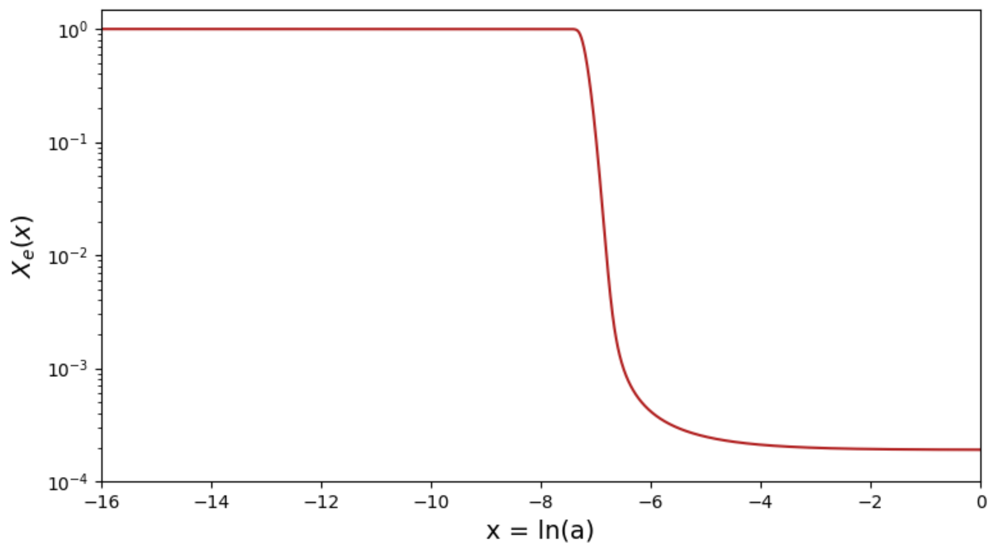

Figure 1: The fractional electron density as a function of logarithm of the scale factor, $X_e(x)$. Logarithmic y-axis has been used.

Optical depth and visibility function

Figure 2 shows the optical depth as a function of logarithm of the scale factor, $\tau(x)$, as well as its first- and second derivative. The optical depth decreases towards today, which is reasonable - there is less medium absorbing radiation between the source emitting at time x and today as we approach $x = 0$. We have found that the last scattering, for which $\tau(x_*) = 1$, happens around $x_* = -6.99$ (with corresponding redshift $z_* = 1/\exp(x_*) - 1 = 1081$ and $X_e(x_*) = 0.096$). This time is also marked by the peak in visbility function, as shown on Figure 3. We have plotted $\tilde{g}(x)$ together with its first- and second derivative. All of them are appropriately scaled by dividing with the corresponding absolute maximal value, so that they can fit into the same plot. It is worth noticing that these normalization factors are $\max|\tilde{g}| \approx 4.9$, $\max|\tilde{g}'| \approx 50.8$ and $\max|\tilde{g}''| \approx 998.8$, so that in reality derivatives peak much higher than the visibility function itself.

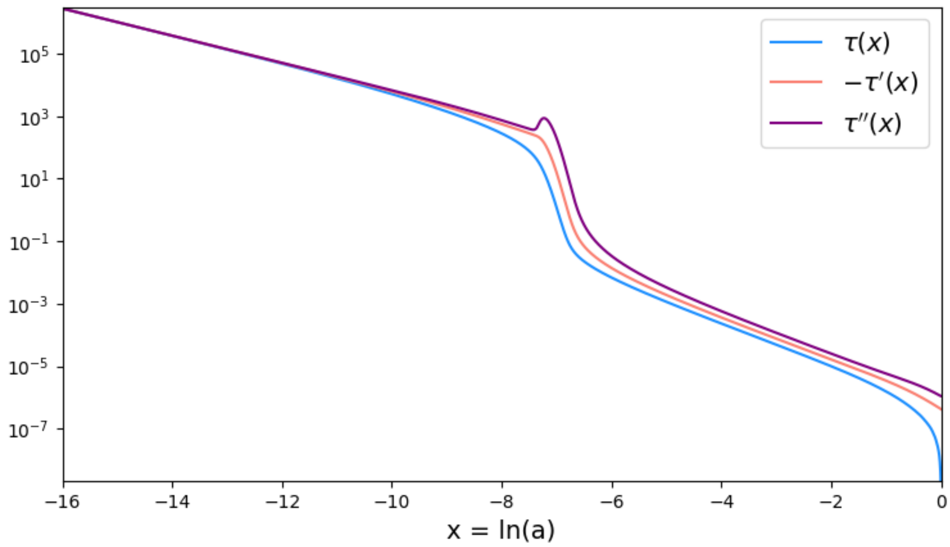

Figure 2: The optical depth as a function of logarithm of the scale factor, $\tau(x)$, plotted together with its first- and second derivative. Logarithmic y-axis has been used.

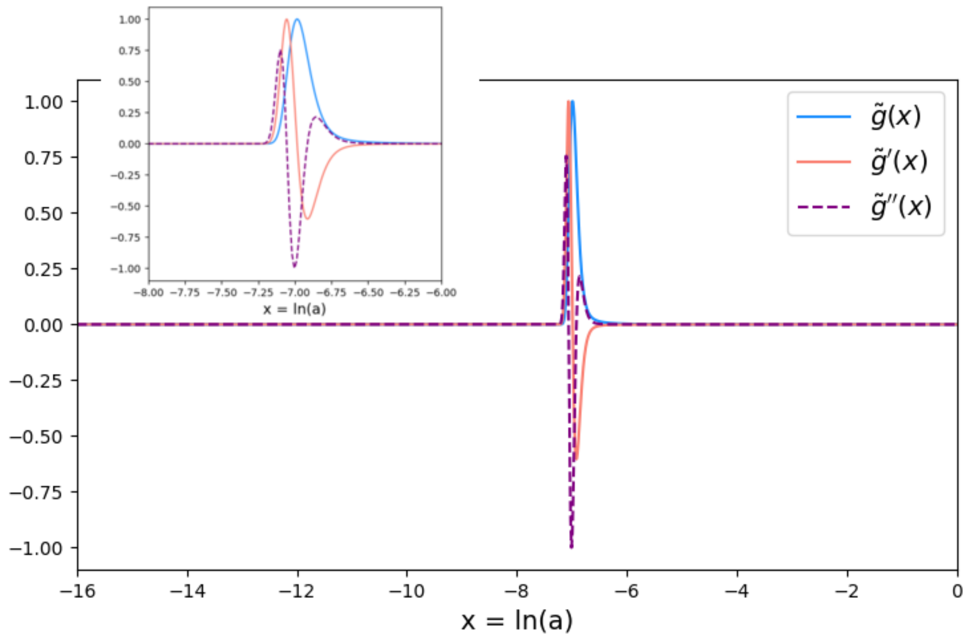

Figure 3: The visibility function, $\tilde{g}(x)$, plotted together with its first- and second derivative. All three curves are normalized with the corresponding absolute maximal value, so that the plot shows $\tilde{g}(x)/max(|\tilde{g}(x)|)$ and so on. Zoom on the central part is included in the upper left corner.

BIBLIOGRAPHY

Callin, P. (2006) *How to calculate the CMB spectrum.* Retrieved from: https://arxiv.org/pdf/astro-ph/0606683.pdf

Ma, C.-P. and Bertschinger, E. (1995) *Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges.* Retrieved from: https://arxiv.org/pdf/astro-ph/9506072.pdf

Winther, H. (2020) *Overview: Milestone II.* Retrieved from: http://folk.uio.no/hansw/AST5220/notes/milestone2.html