The background evolution of the Universe

Milestone I, AST5220
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Code available at: https://github.com/jowborowska/AST5220

1 Introduction

The goal of the first project milestone is to model the evolution of the uniform background in the universe. The components of interest are baryonic matter, dark matter, radiation and dark energy. We will follow the evolution of the corresponding relative density parameters, as well as the behaviour of Hubble parameter and conformal time, from the very early stage of the universe (with corresponding scale factor $a_{min}(t) = 10^{-7}$) until today ($a_0 = a(t_0) = 1$).

2 Methods

2.1 Equations governing the background evolution

Hubble parameter and density components

The evolution of Hubble parameter is given by the first Friedmann equation, which can be written in the following form

$$H(a) = H_0\sqrt{(\Omega_{b,0} + \Omega_{\mathrm{CDM},0})\left(\frac{a_0}{a}\right)^3 + (\Omega_{r,0} + \Omega_{\nu,0})\left(\frac{a_0}{a}\right)^4 + \Omega_{k,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}}, \quad (1)$$

where H_0 is the value of the Hubble parameter today and $a = a(t)$ is the scale factor, measuring the size of the universe relative to the present one, $a(t_0) = a_0 = 1$. $\Omega_{i,0}$'s in the equation above are the relative density parameters today, corresponding to the baryonic matter ($i = b$), dark matter ($i = \mathrm{CDM}$), radiation ($i = r$), neutrinos ($i = \nu$), spatial curvature ($i = k$) and dark energy ($i = \Lambda$).

In general, the relative density parameter of a given component is defined as its density in units of critical density,

$$\Omega_i = \frac{\rho_i}{\rho_c}, \quad (2)$$

where $\rho_c = 3H^2/(8\pi G)$ is the critical density for a given value of Hubble parameter, H.

In the project, we will consider a flat universe with no neutrinos, setting $\Omega_{k,0} = \Omega_{\nu,0} = 0$. Following Callin (2006), we choose a model with $\Omega_{b,0} = 0.046$, $\Omega_{\text{CDM},0} = 0.224$ and $\Omega_{\Lambda,0} = 0.72995$ as input parameters and calculate $\Omega_{r,0}$ from the CMB temperature (see Winther, 2020) as

$$\Omega_{r,0} = \frac{\pi^2}{15}\frac{(k_b T_{\text{CMB}})^4}{\hbar^3 c^5} \cdot {\rho_{c,0}}^{-1} = \frac{\pi^2}{15}\frac{(k_b T_{\text{CMB}})^4}{\hbar^3 c^5} \cdot \frac{8\pi G}{3H_0^2} \approx 5.043 \cdot 10^{-5}, \tag{3}$$

where $T_{\text{CMB}} = 2.725$ K and $H_0 = h \cdot 100\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ with $h = 0.7$.

The evolution of energy density of each component follows from the equation of adiabatic expansion, which gives

$$\rho_i = \rho_{i,0}\left(\frac{a_0}{a}\right)^{3(1+w_i)}, \tag{4}$$

where w_i is a constant corresponding to component i. Dividing both sides of the equation by $\rho_c \rho_{c,0}$ yields

$$\begin{aligned}
\frac{\rho_i}{\rho_c}\frac{1}{\rho_{c,0}} &= \frac{\rho_{i,0}}{\rho_{c,0}}\frac{1}{\rho_c}\left(\frac{a_0}{a}\right)^{3(1+w_i)}, \\
\Omega_i \frac{8\pi G}{3H_0^2} &= \Omega_{i,0}\frac{8\pi G}{3H^2}\left(\frac{a_0}{a}\right)^{3(1+w_i)}, \\
\frac{\Omega_i}{H_0^2} &= \frac{\Omega_{i,0}}{H^2}\left(\frac{a_0}{a}\right)^{3(1+w_i)}, \\
\Rightarrow \Omega_i(a) &= \Omega_{i,0}\left(\frac{H_0}{H(a)}\right)^2\left(\frac{a_0}{a}\right)^{3(1+w_i)},
\end{aligned} \tag{5}$$

where we have used definitions of critical density and Ω (eq. 2) and $H(a)$ can be calculated from the Friedmann equation (eq. 1).
Setting $a_0 = 1$ and $w_b = w_{\text{CDM}} = 0$, $w_r = 1/3$ and $w_\Lambda = -1$, we find

$$\Omega_b(a) = \Omega_{b,0}\left(\frac{H_0}{H(a)}\right)^2 a^{-3}, \tag{6}$$

$$\Omega_{\text{CDM}}(a) = \Omega_{\text{CDM},0}\left(\frac{H_0}{H(a)}\right)^2 a^{-3}, \tag{7}$$

$$\Omega_r(a) = \Omega_{r,0}\left(\frac{H_0}{H(a)}\right)^2 a^{-4}, \tag{8}$$

$$\Omega_\Lambda(a) = \Omega_{\Lambda,0}\left(\frac{H_0}{H(a)}\right)^2. \tag{9}$$

Besides the scale factor, we will consider other time variables, including the redshift,

$$z = \frac{a_0}{a(t)} - 1, \tag{10}$$

as well as the logarithm of the scale factor,

$$x \equiv \ln(a) \Rightarrow a \equiv e^x.$$

We also introduce a scaled Hubble parameter, $\mathcal{H}(a) = aH(a)$, which (using eq. 1) can be written as

$$\mathcal{H}(a) = aH(a) = H_0\sqrt{(\Omega_{b,0} + \Omega_{\mathrm{CDM},0})a^{-1} + \Omega_{r,0}a^{-2} + \Omega_{\Lambda,0}a^2}, \tag{11}$$

or equivalently

$$\mathcal{H}(x) = e^x H(x) = H_0\sqrt{(\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} + \Omega_{r,0}e^{-2x} + \Omega_{\Lambda,0}e^{2x}}, \tag{12}$$

with associated derivatives:

$$\frac{d\mathcal{H}}{dx} = \frac{H_0}{2}\left((\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} + \Omega_{r,0}e^{-2x} + \Omega_{\Lambda,0}e^{2x}\right)^{-\frac{1}{2}}\left(-(\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} - 2\Omega_{r,0}e^{-2x} + 2\Omega_{\Lambda,0}e^{2x}\right), \tag{13}$$

$$\frac{d^2\mathcal{H}}{dx^2} = -\frac{H_0}{4}\left((\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} + \Omega_{r,0}e^{-2x} + \Omega_{\Lambda,0}e^{2x}\right)^{-\frac{3}{2}}\left(-(\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} - 2\Omega_{r,0}e^{-2x} + 2\Omega_{\Lambda,0}e^{2x}\right)^2$$
$$+\frac{H_0}{2}\left((\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} + \Omega_{r,0}e^{-2x} + \Omega_{\Lambda,0}e^{2x}\right)^{-\frac{1}{2}}\left((\Omega_{b,0} + \Omega_{\mathrm{CDM},0})e^{-x} + 4\Omega_{r,0}e^{-2x} + 4\Omega_{\Lambda,0}e^{2x}\right). \tag{14}$$

Conformal time

Another parameter that can be used as a time variable is so-called conformal time, η. We have

$$\frac{d\eta}{dt} = \frac{c}{a},$$

which can be rewritten as

$$\begin{aligned}\frac{d\eta}{dt} = \frac{d\eta}{da}\frac{da}{dt} &= \frac{d\eta}{da}a\frac{\dot{a}}{a} = \frac{d\eta}{da}aH \\ \Rightarrow \frac{d\eta}{da}aH &= \frac{c}{a} \\ \Rightarrow \frac{d\eta}{da} &= \frac{c}{a^2H} = \frac{c}{a\mathcal{H}},\end{aligned}$$

being an ordinary differential equation for η, that can be solved numerically. For our purposes, we rewrite this equation again, now in terms of variable x,

$$\frac{d\eta}{dx} = \frac{d\eta}{da}\frac{da}{dx} = \frac{d\eta}{da}\frac{d(e^x)}{dx} = \frac{d\eta}{da}e^x = \frac{d\eta}{da}a$$

$$\Rightarrow \frac{d\eta}{dx} = \frac{c}{a\mathcal{H}}a = \frac{c}{\mathcal{H}}. \tag{15}$$

2.2 Numerical implementation

The first project milestone is based on modifying the file BackgroundCosmology.cpp from the given set of C++ templates. The only change made in Main.cpp is implementing the values of cosmological parameters (h, T_{CMB}, $\Omega_{b,0}$, $\Omega_{\mathrm{CDM},0}$, $\Omega_{\Lambda,0}$), which are subsequently used in BackgroundCosmology class, for instance to calculate H_0 and $\Omega_{r,0}$, as given by Eq. 3. This class has several functions. Function solve() sets up a grid of 500 x-points in range $x_{min} = \ln(a_{min} = 10^{-7})$ to $x_{max} = \ln(a_0 = 1) = 0$ and computes conformal time, $\eta(x)$, by solving Eq. 15. with use of an ordinary differential equation solver. Moreover, it computes a spline through the resulting data points. Function info() prints out the information about the model and function output() writes the data of interest to a file. Data is calculated and returned by a series of "get" functions, where we implement the equations governing the evolution of relative density parameters (Eq. 6-9), Hubble paramter, $H(x)$ (analogously to Eq. 1), scaled Hubble parameter, $\mathcal{H}(x)$ (Eq. 12) and its derivatives (Eq. 13, 14). Running the whole script is timed to take approximately 0.07 s.

The generated text file with data (cosmology.txt) is thereafter read by a program written in Python (milestone1.py), also used to create all the plots included in this report. Codes to both programs (BackgroundCosmology.cpp, milestone1.py), as well as the cosmology.txt file with outupt data are available in the following github repository:
https://github.com/jowborowska/AST5220

3 Results

The results of computations performed by the programs are presented in this section in form of plots. Figure 1 shows the behaviour of Hubble parameter as a function of x (left panel) and z (right panel). As expected, the value of Hubble parameter decreases with the logarithm of scale factor (towards today) and increases with redshift. The evolution of conformal time is shown on figure 2. As this parameter represents the particle horizon (η is the distance that light may have travelled since the Big Bang), it clearly increases with time (towards today), in agreement with the plot shown on figure 2. From figure 3, showing the evolution of relative density parameters for various components, we can f.ex. read off the time-span of radiation dominated era, followed by the epoch of matter domination, until the present day, dominated by the dark energy component.

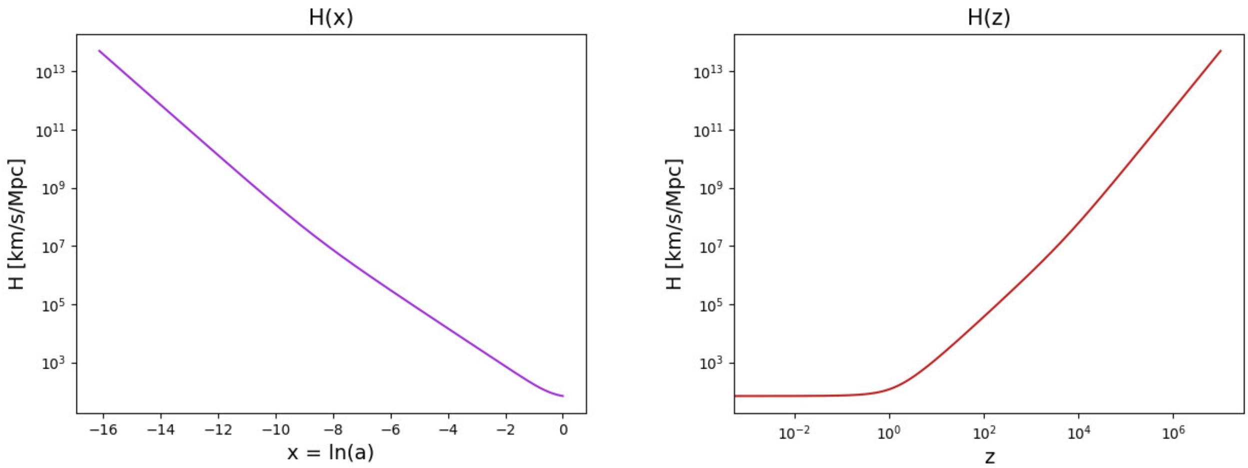

Figure 1: The evolution of Hubble parameter, H, as a function of logarithm of the scale factor, $x = \ln(a)$ (left panel) and redshift, $z = a^{-1} - 1$ (right panel). The Hubble parameter is given on a logarithmic y-axis in units km/s/Mpc.

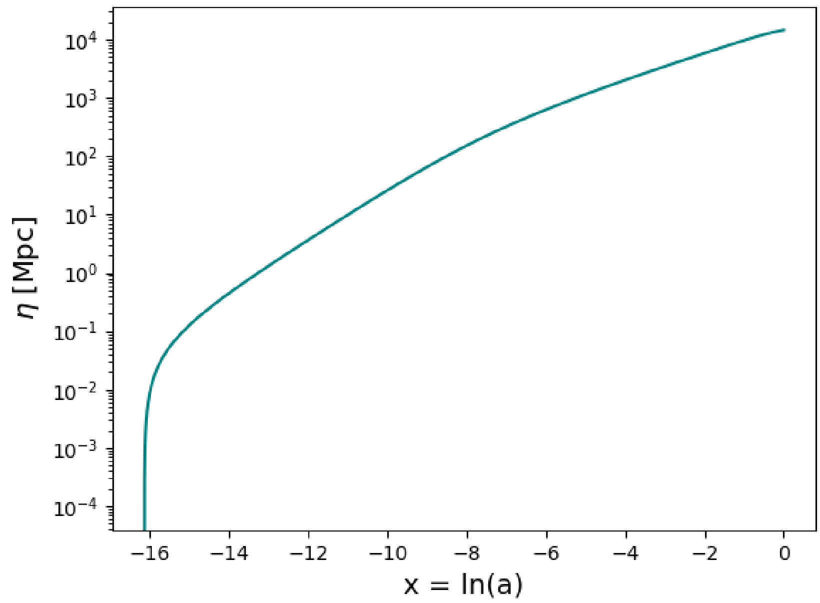

Figure 2: The evolution of conformal time, η, as a function of logarithm of the scale factor, $x = \ln(a)$.

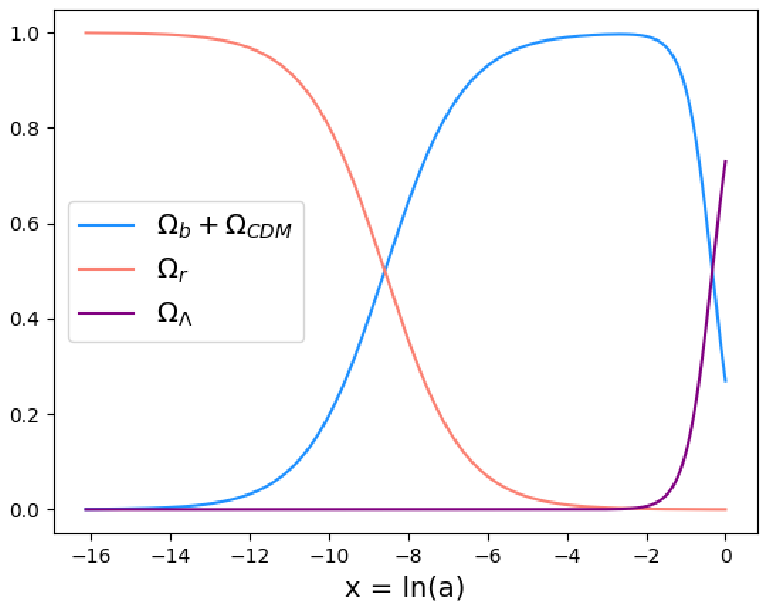

Figure 3: Evolution of relative density parameters as a function of logarithm of the scale factor, $x = \ln(a)$. $\Omega_b + \Omega_{\mathrm{CDM}}$ corresponds to the sum of contributions from baryons and dark matter, Ω_r corresponds to radiation and Ω_Λ to dark energy.

BIBLIOGRAPHY

Callin, P. (2006) *How to calculate the CMB spectrum.* Retrieved from: https://arxiv.org/pdf/astro-ph/0606683.pdf

Winther, H. (2020) *Overview: Milestone I.* Retrieved from: http://folk.uio.no/hansw/AST5220/notes/milestone1.html#Theory