A Brief Introduction to Finite Elements

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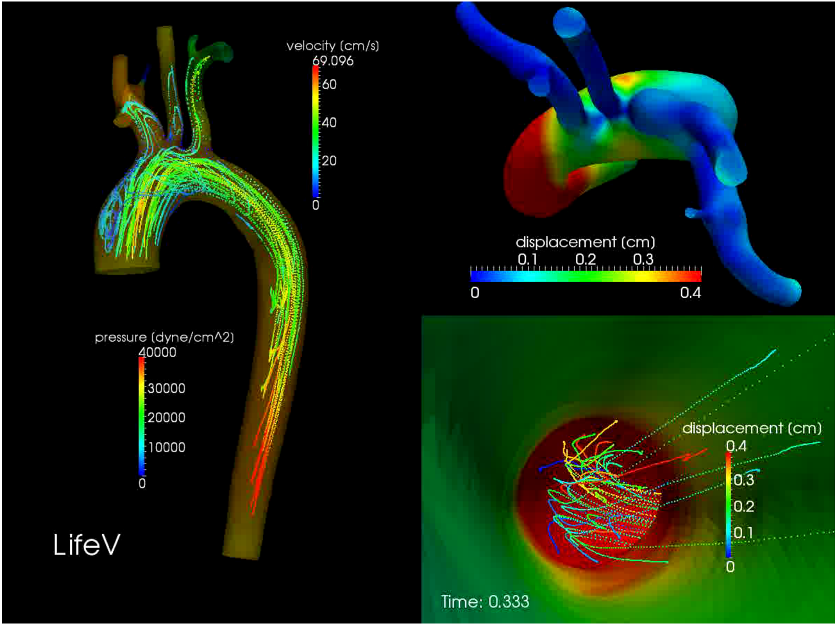
Impressive Looking Picture



A Simple Model Problem

Find a function u such that:

$$-\Delta u := -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

Here:

- Ω is a bounded open set in $\mathbb{R}^2$.
- $\partial\Omega$ is the boundary of Ω.

Weak Forms

Definitions:

1. $L_2(\Omega) = \{u : \int_\Omega u^2 dx < \infty\}$
2. $H_0^1(\Omega) = \{u \in L_2(\Omega) : u = 0 \text{ on } \partial\Omega, \int_\Omega |\nabla u|^2 dx < \infty\}$
3. $\|u\|_{L_2(\Omega)} = \sqrt{\int_\Omega u^2 dx}, \qquad \|u\|_{H_0^1(\Omega)} = \sqrt{\|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2}$

Deriving Weak Form:

1. Multiply PDE by test functions from $H_0^1(\Omega)$ and integrate:
$$\int_\Omega (-\Delta u)v = \int_\Omega fv$$
2. Integrate by parts:
$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$
3. **Weak Form:** Find $u \in H_0^1(\Omega)$ such that
$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall v \in H_0^1(\Omega)$$

Galerkin Method: Converting the infinite to the finite

The weak form can be thought of as an infinite-dimensional system of linear equations:

- Let $\{\psi_i\}_{i=1}^{\infty}$ be a basis for $H_0^1(\Omega)$
- Rewrite u as:

$$u(x) = \sum_{i=1}^{\infty} \alpha_i \psi_i(x)$$

- Find $\{\alpha_i\}_{i=1}^{\infty}$ such that

$$\int_\Omega \nabla \sum_{i=1}^{\infty} \alpha_i \psi_i \cdot \nabla \psi_j = \int_\Omega f \psi_j, \qquad j = 1, 2, ...$$

- **Galerkin Method:** Use only a finite subset of the basis. Find $\{\alpha_i\}_{i=1}^{N}$ such that

$$\sum_{i=1}^{N} \alpha_i \left[\int_\Omega \nabla \psi_i \cdot \nabla \psi_j \right] = \int_\Omega f \psi_j, \qquad j = 1, 2, ..., N$$

Big Linear Algebra Problem

$$A = \begin{pmatrix} \int_\Omega \nabla\psi_1 \cdot \nabla\psi_1 & \cdots & \int_\Omega \nabla\psi_N \cdot \nabla\psi_1 \\ \vdots & \ddots & \vdots \\ \int_\Omega \nabla\psi_1 \cdot \nabla\psi_N & \cdots & \int_\Omega \nabla\psi_N \cdot \nabla\psi_N \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} \int_\Omega f\psi_1 \\ \vdots \\ \int_\Omega f\psi_N \end{pmatrix}$$

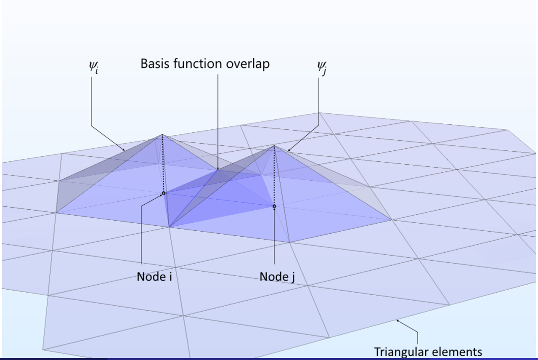
- **Linear Algebra Problem:** Solve $A\vec{\alpha} = \vec{b}$ for $\vec{\alpha}$
- **Trouble:** N is usually large. For some problems N can be $\mathcal{O}(10^9)$. We can't possibly solve this by hand.
- **Solution:** Numerical linear algebra to the rescue!
- **Remark:** Numerical Linear Algebra can solve systems quickly when the matrix A is sparse, meaning most of the elements of A are zero and the number of nonzero elements is $\mathcal{O}(N)$. (Keep in mind A has N^2 elements.)

How do we make A sparse?

- Choose $\{\psi_i\}_{i=1}^N$ so that most of the integrals in A are zero.
- Create basis so that that most elements have nonintersecting support.

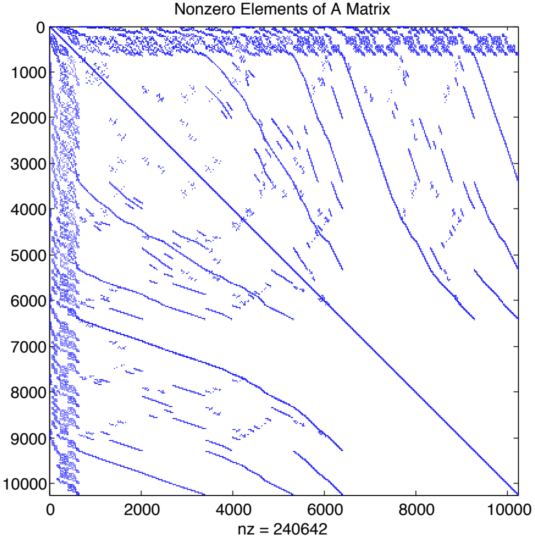
Finite Element Method (FEM):

- Use a mesh of nearly uniformly sized triangles or rectangles.
- Use mesh to create basis functions.
- Piecewise degree-r polynomials on each triangle.



Example of A ($N \approx 10,000$) with approximately $240,000$ nonzero elements

$$\begin{pmatrix} \int_\Omega \nabla\psi_1 \cdot \nabla\psi_1 & \cdots & \int_\Omega \nabla\psi_N \cdot \nabla\psi_1 \\ \vdots & \ddots & \vdots \\ \int_\Omega \nabla\psi_1 \cdot \nabla\psi_N & \cdots & \int_\Omega \nabla\psi_N \cdot \nabla\psi_N \end{pmatrix}$$



Research Questions

- How well does FEM solution, $u_h(x) = \sum_{i=1}^{N} \alpha_i \psi_i(x)$, approximate actual solution, u? In what sense? L_2, L_∞?
- Typical error estimates:

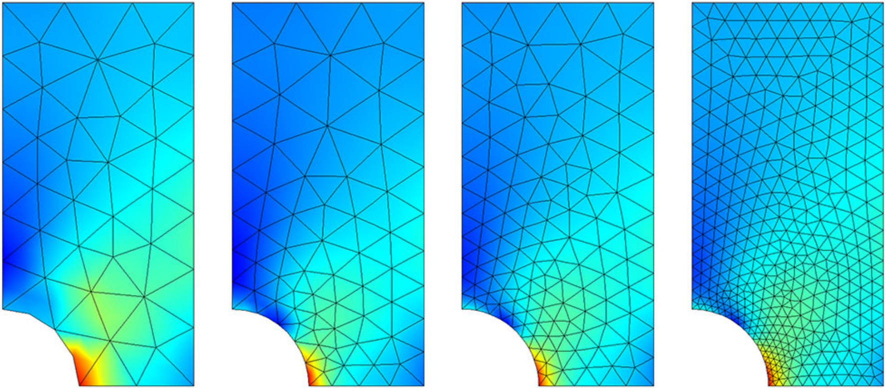
$$\|u - u_h\|_{H^1(\Omega)} \leq ch^r,$$

$$\|u - u_h\|_{L_2(\Omega)} \leq ch^{r+1},$$

where h is the mesh spacing and r is polynomial degree.

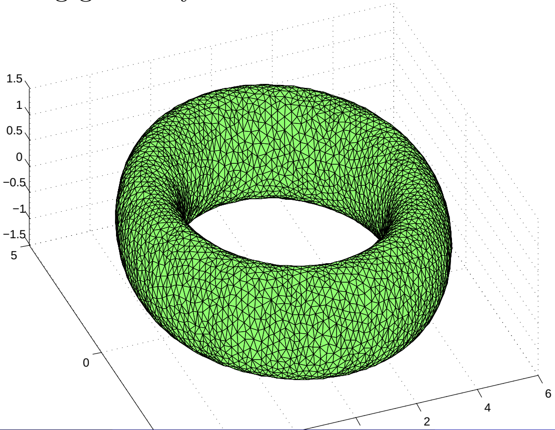
Adaptivity

- Is there a better way of creating the mesh?
- Can we make the computer measure $\|u - u_h\|$ and add basis functions to reduce the error automatically?



Surface FEM

- Solve PDEs on surfaces
- Laplacian becomes Laplace-Beltrami operator, $-\Delta_\gamma u = f$
- Most FEM techniques from Euclidean domains carry over to surfaces.
- Approximating geometry causes headaches



Eigenvalue Problems

- Approximate spectrum of differential operators.
- Solve $-\Delta u = \lambda u$
- Find $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}^+$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv \qquad \forall v \in H_0^1(\Omega)$$

- Linear algebra problem is more difficult.

Surface FEM + Eigenvalue Problems = Shape DNA

- Use Laplace-Beltrami spectrum as an isometry-invariant shape descriptor