Sections 4.3 and 4.4 : The Chain Rule and Derivatives of Exponential and Logarithmic Functions

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Note: Some of these figures come from your Webassign practice and your textbook *Calculus: Applications & Technology*, 3rd ed., by Tomastik.

Question: *How do we differentiate complicated functions that are compositions of simpler functions?*

In Section 4.1 we learned how to differentiate simple functions, then in Sections 4.1 and 4.2 we learned derivative rules for when we multiply the simple functions by constants, add simple functions, subtract simple functions, multiply simple functions, and divide simple functions.

If we just stick with arithmetic operations, this is as far as we can go in creating functions. However, functions can also be composed ($h(x) = f(g(x))$). This is how we create the most interesting functions and it's a good way to think about functions in general since any function $f(x)$ can be written as $f(x) = f(g(x))$, if we choose $g(x) = x$.

Recall Composition of Functions: Given two functions $f(x)$ and $g(x)$, we write their composition as $f(g(x))$.

Example: Identify the two simple functions composed to create $h(x) = e^{x^2}$.

Example: Identify the two simple functions composed to create $h(x) = \sqrt{5-x}$.

Example: Identify the three simple functions composed to create $h(x) = e^{e^{\sqrt{x}}}$.

Chain Rule: If both $f'(x)$ and $g'(x)$ exist, then $\frac{d}{dx}f(g(x))$ exists and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Recipe for Performing Chain Rule Successfully on a Function:

1. Identify composition in terms of simple functions you know how to differentiate.
2. Perform Chain Rule (Number of compositions will determine number of times chain rule is used.)

Example: Calculate the derivative of $h(x) = x^2$ using the chain rule.

Example: Calculate the derivative of $h(x) = \sqrt{\ln(x)}, x > 0$ using the chain rule.

Example: Calculate the derivative of $h(x) = (x^2 + 5)^3$ using the chain rule.

Special Cases of the Chain Rule:

Derivative of $e^{f(x)}$:

$$\frac{d}{dx}[e^{f(x)}] = f'(x) \cdot e^{f(x)}$$

Example: Calculate the derivative of $h(x) = e^{e^{\sqrt{x}}}$.

Derivative of $a^{f(x)}$:

$$\frac{d}{dx}[a^{f(x)}] = f'(x) \cdot a^{f(x)} \ln(a)$$

Example: Calculate the derivative of $h(x) = 5^{\sqrt[3]{x}}$ using the chain rule.

Derivative of Logarithmic Function:

$$\frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)}$$

Example: Calculate the derivative of $h(x) = \ln(x^6)$.