Note: Some of these figures come from your Webassign practice and your textbook *Calculus: Applications & Technology*, 3rd ed., by Tomastik.

Question: *If the derivative helps us find mins and maxes of a function, what can we learn from looking at the derivative of the derivative?*

In Section 5.1 we saw that looking at the zeros of the derivative allowed us to locate critical points which were candidates for local mins and maxes. We also saw that looking at how the sign of the derivative changes as we cross these critical points allows us to classify the critical points as the locations of either mins, maxes, or neither. This second step involves looking how the derivative changes. Since derivatives are measurements of how a function is changing, maybe we can gain some insight by looking at the derivative of the derivative. This quantitiy is commonly called the second derivative.

Practical Definition of Second Derivative: Given a function $y = f(x)$, the second derivative, denoted by $f''(x)$, is defined to be the derivative of the first derivative. Thus,

$$f''(x) = \frac{d}{dx}[f'(x)]$$

Example: Calculate the second derivative of $f(x) = x^2$

Example: Calculate the second derivative of $f(x) = 5x$

Example: Calculate the second derivative of $f(x) = e^x$

<u>Limit Definition of Second Derivative:</u> The second derivative of $f(x)$ is

$$f''(x) = \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h}.$$

<u>Recall:</u> If the limit of a function exists, then it is equal to it's left and right limits.

<u>Right limit of Second Derivative:</u> For $h > 0$, we have the following interpretation of the second derivative:

$$f''(x) = \lim_{h\to 0^+} \frac{\overbrace{\frac{f(x+h)-f(x)}{h}}^{\text{slope of secant line to right of x}} - \overbrace{\frac{f(x)-f(x-h)}{h}}^{\text{slope of secant line to left of x}}}{h}$$

Example: If at a point x, $\overbrace{\frac{f(x+h)-f(x)}{h}}^{\text{slope of secant line to right of x}} > \overbrace{\frac{f(x)-f(x-h)}{h}}^{\text{slope of secant line to left of x}}$ for small $h > 0$, then what is the sign of $f''(x)$? Is $f'(x)$ increasing or decreasing?

Example: Draw two different scenarios where $\overbrace{\frac{f(x+h)-f(x)}{h}}^{\text{slope of secant line to right of x}} > \overbrace{\frac{f(x)-f(x-h)}{h}}^{\text{slope of secant line to left of x}}$ for small $h > 0$ could happen.

Example: If at a point x, $\overbrace{\frac{f(x+h)-f(x)}{h}}^{\text{slope of secant line to right of x}} < \overbrace{\frac{f(x)-f(x-h)}{h}}^{\text{slope of secant line to left of x}}$ for small $h > 0$, then what is the sign of $f''(x)$? Is $f'(x)$ increasing or decreasing?

Example: Draw two different scenarios where $\overbrace{\frac{f(x+h)-f(x)}{h}}^{\text{slope of secant line to right of x}} < \overbrace{\frac{f(x)-f(x-h)}{h}}^{\text{slope of secant line to left of x}}$ for small $h > 0$ could happen.

Concave Up and Down:

1. We say that the graph of f is concave up on (a,b) if $f'(x)$ is increasing on (a,b).
2. We say that the graph of f is concave down on (a,b) if $f'(x)$ is decreasing on (a,b).

Test for Concavity:

1. If $f''(x) > 0$ on (a,b), then the graph of f is concave up on (a,b).
2. If $f''(x) < 0$ on (a,b), then the graph of f is concave down on (a,b).

Note: Just as the critical points where the derivative equaled zero or didn't exist marked where the function changed from increasing to decreasing or vice versa, the zeros of the second derivative and points where it doesn't exist have special meaning. They mark where the concavity changes.

Inflection Point: A point $(c, f(c))$ on a graph is an inflection point and c is an inflection value if $f(c)$ is defined and the concavity of the graph of f changes at $(c, f(c))$.

Intuitively: The Inflection points are just the critical points of the derivative function.

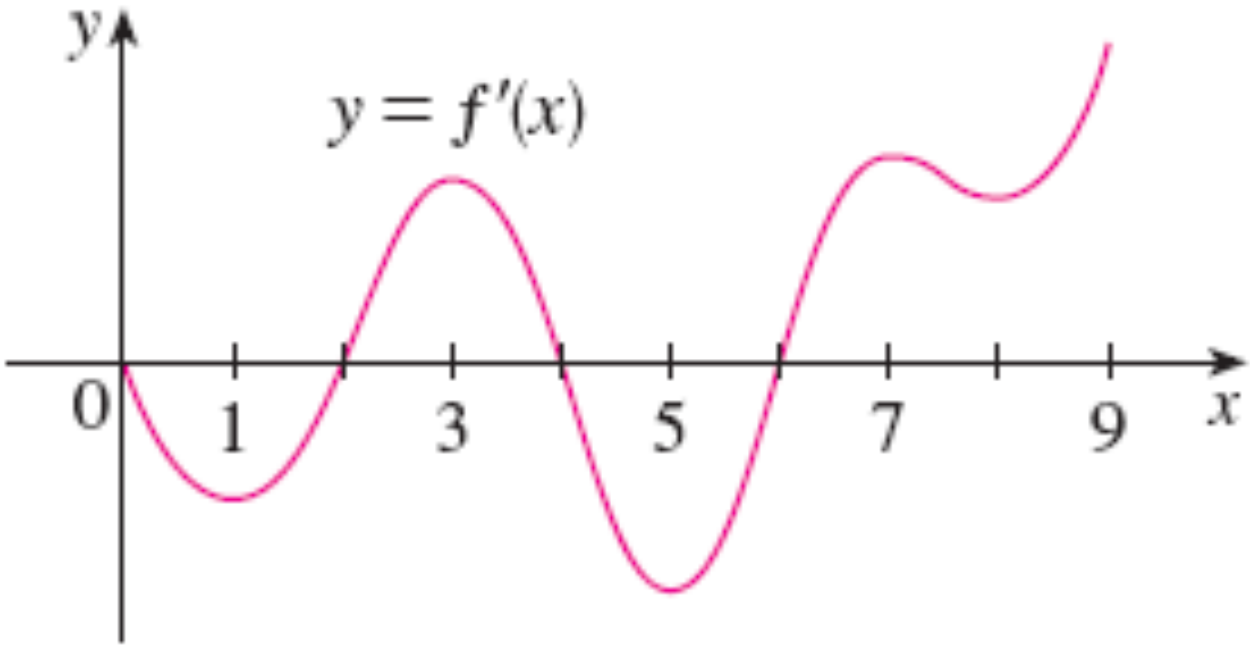
Example: Calculate the inflection points of $f(x) = x^3$. Sketch $f(x)$. Label the concave up and concave down regions.

Example: Sketch $h(x) = 5 - x^2$. Let $x = 0$. Label the left secant line region and the sign of it's slope. Label the right secant line region and the sign of it's slope. Based on your labels, what should be the sign of $f''(x)$?

<u>Second Derivative Test:</u> Suppose that f is defined on (a,b), $f'(c) = 0$, and $c \in (a,b)$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

Example: Given the graph of $f'(x)$, label the critical points and where f is increasing and decreasing.



Example: Given the graph of $f'(x)$, label the inflection points, the concavity of each region, and the local mins/maxes.