Geometric Superconvergence in the Surface Eigenvalue Problem

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1. The Surface Eigenvalue Problem

Let γ be an n-dimensional surface without boundary in $\mathbb{R}^{n+1}$ and let (u,λ) weakly solve $-\Delta_\gamma u = \lambda u$: Find $(u,\lambda) \in H_0^1(\gamma) \times \mathbb{R}^+$ s.t. $\int_\gamma u d\sigma = 0$ and

$$a(u,v) = \lambda m(u,v) \qquad \forall v \in H_0^1(\gamma)$$

with bilinear form and L_2 inner product:

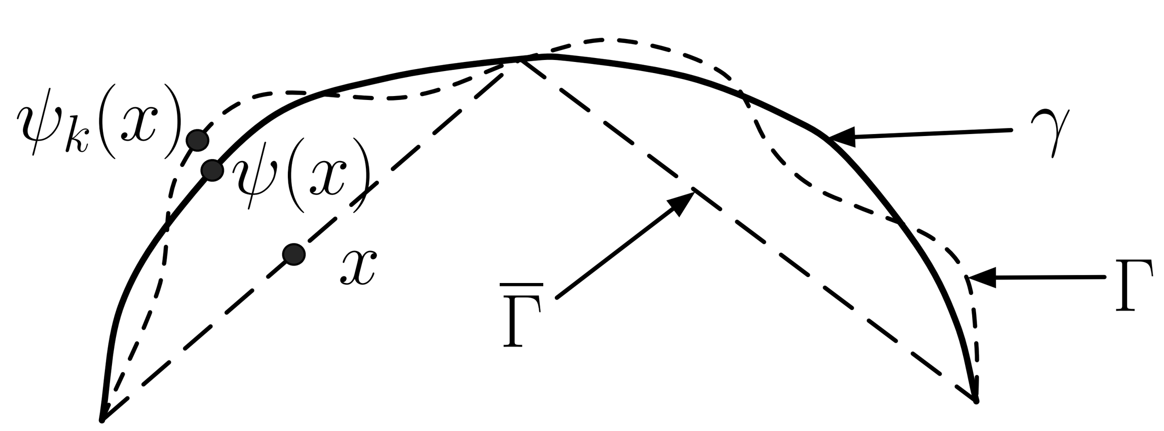
$$a(u,v) := \int_\gamma \nabla_\gamma u \nabla_\gamma v \, d\sigma, \qquad m(u,v) := \int_\gamma uv \, d\sigma.$$

Goal: Estimate the approximation error of surface finite element solutions to the eigenvalue problem.

2. Surface Finite Elements

Distance Function: If γ is a C^2 closed surface, then γ is the zero level-set of a signed distance function $d(x)$.

Closest Point Projection onto γ: $\psi(x) = x - d(x)\nu(x)$, where $\vec{\nu} = \nabla d$ is the unit normal vector.



Discrete Surface: $\overline{\Gamma}$ is a polyhedron with shape-regular triangular faces of diameter h having vertices on γ.

Polynomial Surface: $\Gamma = \psi_k(\overline{\Gamma})$ with ψ_k a degree-k polynomial interpolant of ψ on each face of $\overline{\Gamma}$. The distance between γ and Γ is $O(h^{k+1})$.

Finite Element Space: $\mathbb{V}$ is the piecewise degree-r polynomials defined on $\overline{\Gamma}$ and lifted to Γ.

Bilinear Form and the L_2 Inner Product on Γ:

$$A(U,V) := \int_\Gamma \nabla_\Gamma U \nabla_\Gamma V \, d\Sigma, \qquad M(U,V) := \int_\Gamma UV \, d\Sigma.$$

Finite Element Eigenvalue Problem: Find $(U,\Lambda) \in \mathbb{V} \times \mathbb{R}^+$ s.t. $\int_\Gamma U \, d\Sigma = 0$ and

$$A(U,V) = \Lambda M(U,V) \qquad \forall V \in \mathbb{V}.$$

3. Error Analysis

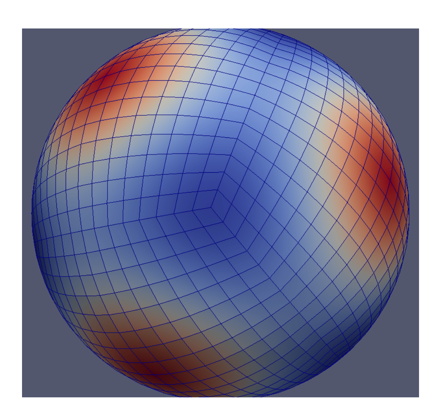
Theorem 1 (Eigenvalue Bound) *Let λ be an eigenvalue of the surface eigenvalue problem and let (U,Λ) be a surface FEM eigenpair associated with λ. Define $\boldsymbol{P}_\lambda$ to be the projection onto the set of eigenfunctions associated with λ using the L_2 inner product $m(\cdot,\cdot)$. Then*

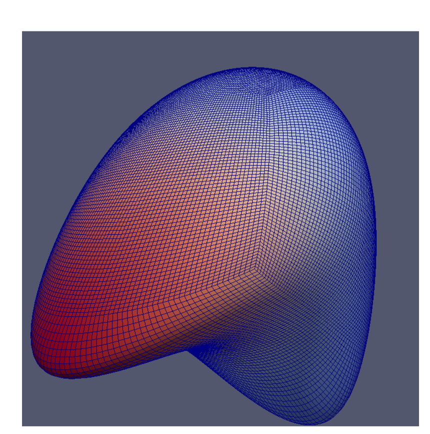
$$|\lambda_j - \Lambda_j| \le \underbrace{\|\boldsymbol{P}_\lambda U - U\|_a^2}_{O(h^{2r})+O(h^{2k+2})} + \lambda \underbrace{\|\boldsymbol{P}_\lambda U - U\|_m^2}_{O(h^{2r+2})+O(h^{2k+2})}$$
$$+ \Lambda \underbrace{|m(U,U) - M(U,U)|}_{\text{Geometric}} + \underbrace{|A(U,U) - a(U,U)|}_{\text{Geometric}}.$$

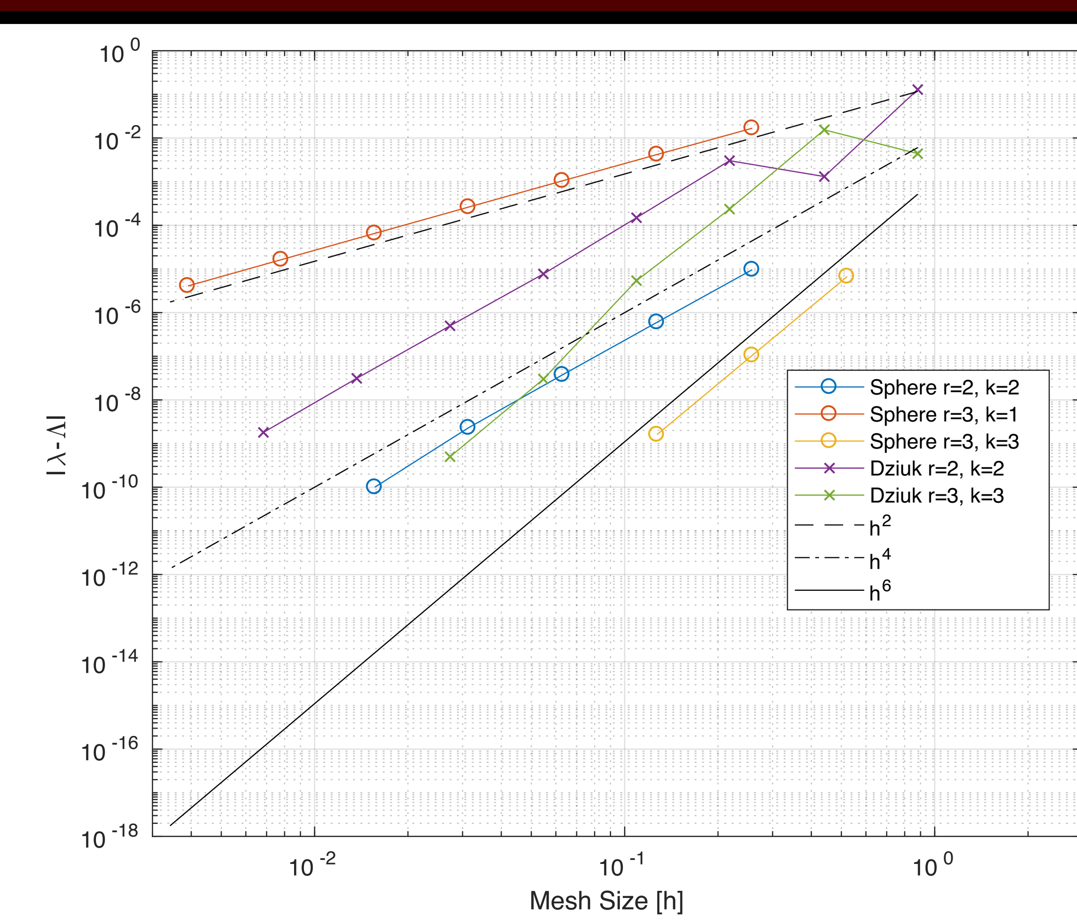
Euclidean Eigenvalue Error: The eigenvalue error for a Euclidean domain is $O(h^{2r})$.

Expected Geometric Consistency Error: Using the standard techniques in [Demlow 2009] and [Dziuk 1988], we expect the geometric terms to be $O(h^{k+1})$. So we expect the surface eigenvalue error to be $O(h^{2r}) + O(h^{k+1})$.

4. Numerical Experiments with Quadrilateral Elements







Experimental Setup: Numerical experiments were run using deal.II with quadrilateral elements and Gauss-Lobatto points for the surface interpolation. The eigenvalues were calculated for various shapes including the sphere and Dziuk surface pictured above. Different combinations of r and k were used to investigate the order of the geometric consistency error.

Strange Behavior: The geometric consistency converges as $O(h^{2k})$ rather than the expected $O(h^{k+1})$.

5. A Closer Look at Geometric Consistency (Exploiting the distance function)

Lemma 2 *Up to terms of order h^{2k+2},*

$$|m(v,v) - M(v,v)| \le \left| \int_\Gamma v^2 d(x) \sum_{i=1}^n \frac{\kappa_i(\psi(x))}{1 + d(x)\kappa_i(\psi(x))} d\Sigma \right|, \tag{1}$$

where $\{\kappa_i\}_{i=1}^n$ are the principal curvatures of the surface.

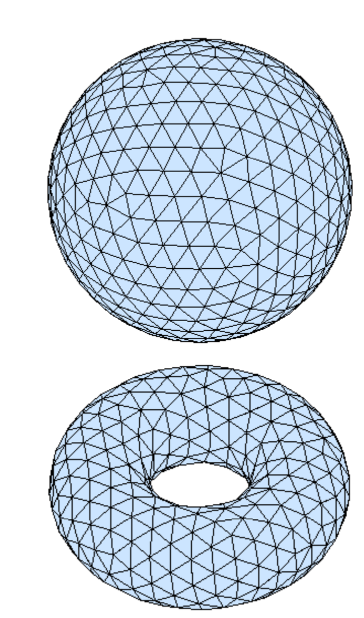
Exploiting the Distance Function in Lemma 2:

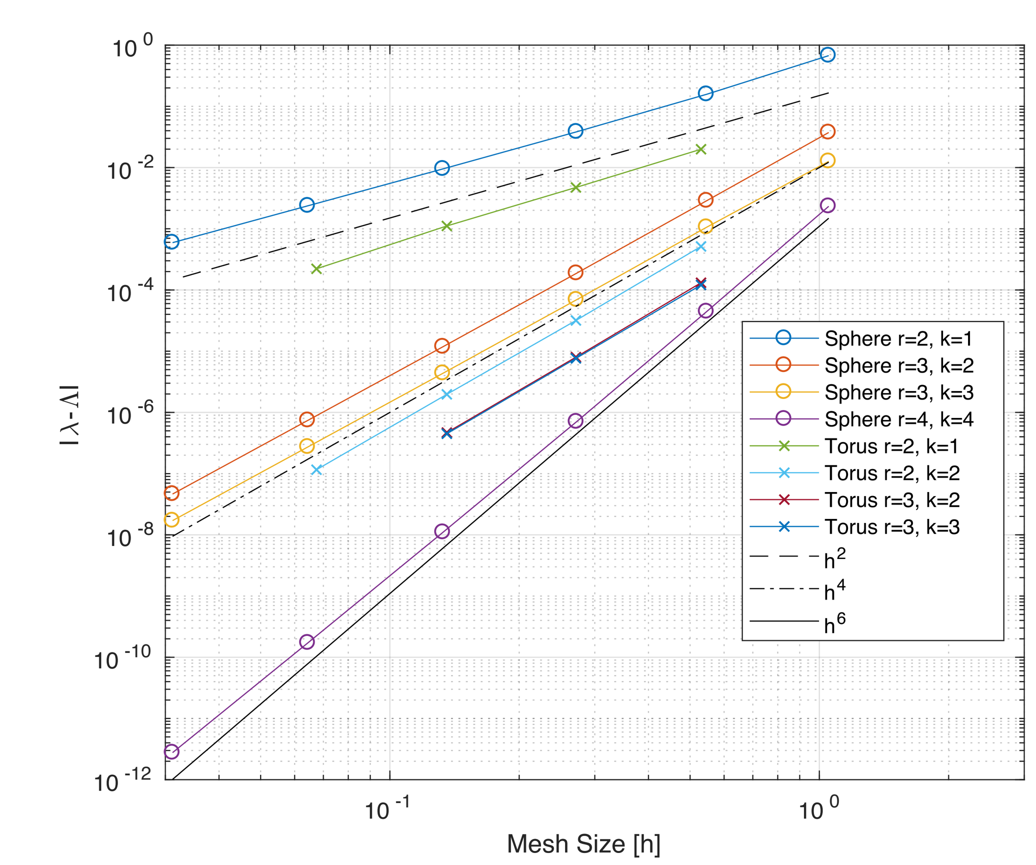
1. **Zeros of the Distance Function:** Restricting the distance function to Γ as in (1), we see that for any face T of Γ, $d(q_i) = 0$ at the interpolation points $\{q_i\}_{j=1}^{k+1}$ used in defining ψ_k.
2. **Subtracting 0:** Subtract a quadrature rule with points at $\{q_i\}_{j=1}^{k+1}$.
3. **Quadrature Error:** Order of geometric consistency error is order of quadrature error.

Bilinear Form Consistency Error: The consistency error associated with $a(\cdot,\cdot)$ can be analyzed in a similar way.

Theorem 3 (Quadrilateral Superconvergence Explained) *The geometric consistency errors in Theorem 1 are bounded by terms with order equal to the order of the quadrature rule that can be associated with the choice of interpolation points used in the construction of Γ.*

6. Unexplained Superconvergence on Triangular Meshes





Geometric Error on Triangular Elements: The geometric error goes as the expected $O(h^{k+1})$ for odd k, but has superconvergence of $O(h^{k+2})$ for even k using standard Lagrange finite elements for interpolation. The Lagrange nodes on triangles do not correspond to an order $k+2$ quadrature rule for even k.