Geometric Superconvergence in FEM Approximation of Eigenvalues of the Laplace-Beltrami Operator

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joint work with

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Laplace-Beltrami Eigenvalue Problem

- ***Eigenvalue Problem:*** Given a D-dimensional smooth closed surface γ in $\mathbb{R}^{D+1}$, find (u, λ) such that:

  $$-\Delta_\gamma u = \lambda u,$$

  where Δ_γ is the Laplace-Beltrami operator for γ.

- ***Focus of talk:*** Approximation of the eigenvalues

Weak Formulation on Surfaces

- ***Bilinear form on $H^1(\gamma)$ and the $L^2(\gamma)$ inner product on γ:***

$$a(u,v) := \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v \; d\sigma, \qquad m(u,v) := \int_\gamma uv \; d\sigma$$

- ***Weak eigenvalue problem:*** Find $(u,\lambda) \in H^1(\gamma) \times \mathbb{R}^+$ s.t. $\int_\gamma u \; d\sigma = 0$ and

$$a(u,v) = \lambda m(u,v) \qquad \forall v \in H^1(\gamma).$$

Surface Properties

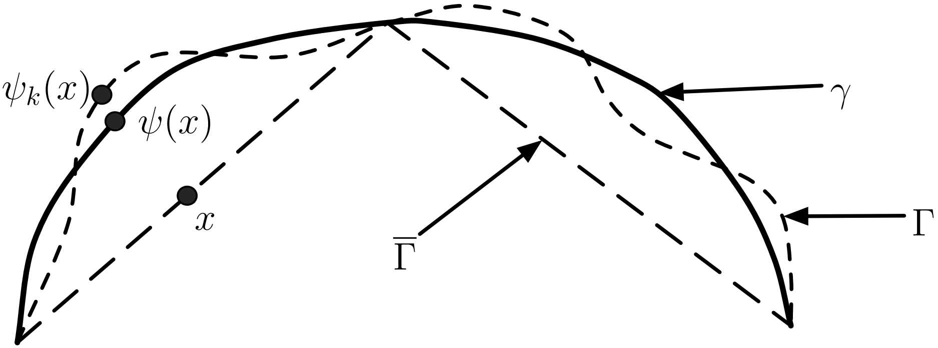
- If γ is a C^2 closed surface, then γ is the zero level-set of a signed distance function d.
- For x lying in a sufficiently small tubular neighborhood $\mathcal{N}$ we have:
  - ***Distance function:*** $d(x) = \operatorname{dist}(x, \gamma)$.
  - ***Normal vector:*** $\vec{\nu} = \nabla d$ satisfies $|\vec{\nu}| = 1$; $\vec{\nu}$ is a unit normal on γ.
  - ***Closest point (orthogonal) projection onto*** γ**:**

$$\psi(x) = x - d(x)\vec{\nu}(x).$$

Approximate Surface Construction

- ***Base discrete surface***: $\overline{\Gamma}$ is a polyhedron with shape-regular triangular faces of diameter h having vertices on γ.

- ***Polynomial surface approximation***: $\Gamma = \psi_k(\overline{\Gamma})$ with ψ_k a degree-k Lagrange interpolation of ψ on each face of $\overline{\Gamma}$.



The Surface Finite Element Method (SFEM)

- ***Base discrete surface***: $\overline{\Gamma}$ is a polyhedron with shape-regular triangular faces of diameter h having vertices on γ.
- ***Polynomial surface approximation***: $\Gamma = \psi_k(\overline{\Gamma})$ with ψ_k a degree-k Lagrange interpolation of ψ on each face of $\overline{\Gamma}$.
- ***Finite element space***: $\mathbb{V}_h$ is the piecewise degree-r polynomials defined on $\overline{\Gamma}$ and lifted to Γ.
- ***Bilinear form and the $L^2(\Gamma)$ inner product on Γ***:

$$A(U,V) := \int_\Gamma \nabla_\Gamma U \nabla_\Gamma V \, d\Sigma, \qquad M(U,V) := \int_\Gamma UV \, d\Sigma.$$

- ***Finite element eigenvalue problem***: Find $(U,\Lambda) \in \mathbb{V}_h \times \mathbb{R}^+$ s.t. $\int_\Gamma U \, d\Sigma = 0$ and

$$A(U,V) = \Lambda M(U,V) \qquad \forall V \in \mathbb{V}_h.$$

Typical SFEM A Priori Errors

- ***Geometric consistency errors*:**

$$|a(u,v) - A(u,v)| \lesssim h^{k+1}$$
$$|m(u,v) - M(u,v)| \lesssim h^{k+1}$$

- ***Source problem bound*:** The SFEM solution, u_h, to $-\Delta_\gamma u = f$ satisfies:

$$\|u - u_h\|_{H^1(\gamma)} \lesssim h^r + h^{k+1},$$
$$\|u - u_h\|_{L_2(\gamma)} \lesssim h^{r+1} + h^{k+1}.$$

- ***Eigenfunction bound*:** Define $\boldsymbol{P}_\lambda$ to be the projection onto the set of eigenfunctions associated with λ using the L_2 inner product $m(\cdot,\cdot)$. For an SFEM eigenfunction associated with an approximate eigenvalue of λ we have:

$$\|U - \boldsymbol{P}_\lambda U\|_{H^1(\gamma)} \lesssim h^r + h^{k+1},$$
$$\|U - \boldsymbol{P}_\lambda U\|_{L_2(\gamma)} \lesssim h^{r+1} + h^{k+1}.$$

Error Analysis

Theorem (Eigenvalue Bound)

Let λ be an eigenvalue of the surface eigenvalue problem and let (U, Λ) be a surface FEM eigenpair associated with λ. Then

$$|\lambda - \Lambda| \leq \underbrace{\|U - \boldsymbol{P}_\lambda U\|^2_{H^1(\gamma)}}_{O({\color{red}h^{2r}})+O({\color{blue}h^{2k+2}})} + \lambda \underbrace{\|U - \boldsymbol{P}_\lambda U\|^2_{L_2(\gamma)}}_{O({\color{red}h^{2r+2}})+O({\color{blue}h^{2k+2}})}$$

$$+ \Lambda \underbrace{{\color{blue}|m(U,U) - M(U,U)|}}_{Geometric} + \underbrace{{\color{blue}|a(U,U) - A(U,U)|}}_{Geometric}.$$

Eigenvalue Error Bound:

$$|\lambda - \Lambda| \lesssim {\color{red}h^{2r}} + {\color{blue}h^{k+1}}$$

Some Test Shapes

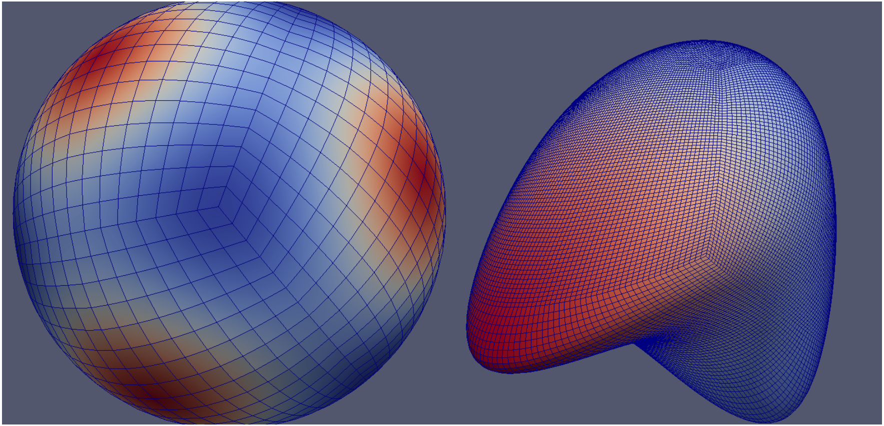
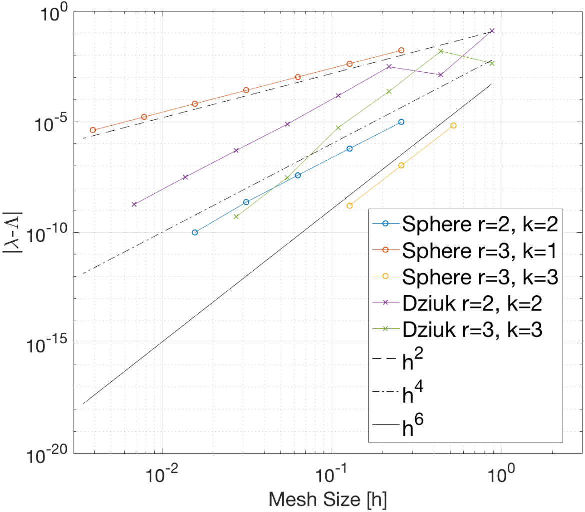


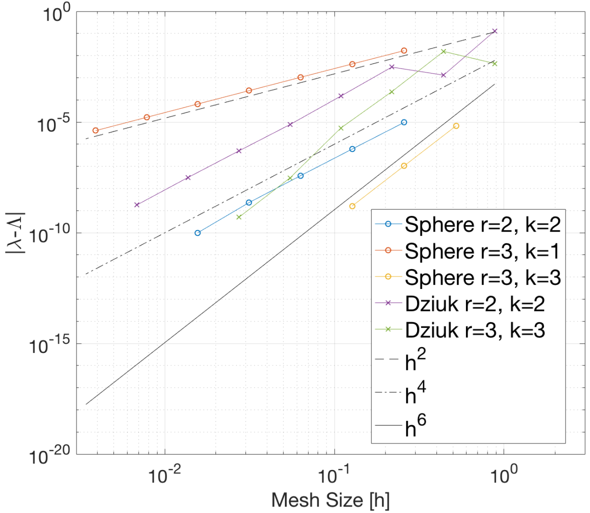
Figure: Sphere and Dziuk surface used in deal.ii computations of eigenvalues.

Numerical Experiments Using Quadrilateral Elements



Looking for : $O(h^{k+1})$.

Numerical Experiments Using Quadrilateral Elements



Strange Behavior: The geometric consistency converges as $O(h^{2k})$ rather than the expected $O(h^{k+1})$.

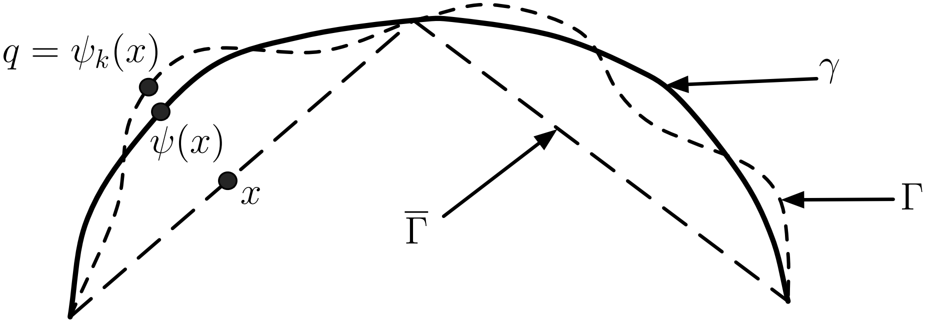
An Explanation of Superconvergence

Lemma

Up to terms of order h^{2k},

$$|m(V,V) - M(V,V)| \leq \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\psi(q))}{1 + d(q)\kappa_i(\psi(q))} \, d\Sigma \right|,$$

where $\{\kappa_i\}_{i=1}^n$ *are the principal curvatures of the surface.*



Geometric Error Acts Like Quadrature Error

- ***Exploit distance function:*** The zeros of $d(q)$, $\{q_j\}_{j=1}^N$, on each face of Γ are the interpolation points used to create Γ.

- ***Create quadrature rule:*** Use the zeros of $d(q)$ to create a quadrature rule:

$$QUAD := \sum_{T \subset \Gamma} \sum_{j=1}^{N} W_j V(q_j)^2 \cancelto{0}{d(q_j)} \sum_{i=1}^{n} \frac{\kappa_i(\psi(q_j))}{1 + d(q_j)\kappa_i(\psi(q_j))} = 0$$

Theorem (Quadrature Error)

Up to terms of order h^{2k},

$$\begin{aligned}
|m(V,V) - M(V,V)| &\leq \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\psi(q))}{1 + d(q)\kappa_i(\psi(q))} \, d\Sigma \right| \\
&= \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\psi(q))}{1 + d(q)\kappa_i(\psi(q))} \, d\Sigma - QUAD \right|.
\end{aligned}$$

Conclusion

Corollary (Convergence in deal.ii Computations)

If degree $-$ k interpolation points based on Gauss-Lobatto quadrature are used in the construction of Γ, U *is the SFEM eigenfunction of* Λ, *and* $\boldsymbol{P}_\lambda U$ *has enough regularity, then*

$$|m(U,U) - M(U,U)| \lesssim h^{2k},$$
$$|a(U,U) - A(U,U)| \lesssim h^{2k},$$

and

$$|\lambda - \Lambda| \lesssim h^{2r} + h^{2k}.$$

- ***Quadrilateral meshes*:** Tensor product of $k+1$ points used in the 1D Gauss-Lobatto quadrature rule yields a quadrature rule exact for degree $2k-1$.
- ***Triangular meshes*:** Our framework also applies to triangular meshes.