

Note: Some of these examples and figures come from the textbook *Single Variable Calculus: Concepts & Contexts*, 4th ed., by Stewart.

**Question:** *What does a function  $f(x)$  do as the  $x$  values get closer and closer to a point?*

**Example:** The Heaviside function  $H$  is defined by  $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ . What happens as the values of  $x$  approach 0 from the left? From the right?

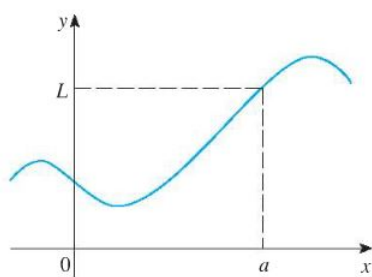
**Definition (Left Limit):** We call  $L$  the left limit of  $f(x)$  at  $a$  if as  $x$  approaches  $a$  from the left,  $f(x)$  approaches  $L$ , i.e. we can make  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we want) for all  $x$  sufficiently close to  $a$ , from the left (without actually letting  $x$  be  $a$ ). We write this as

$$\lim_{x \rightarrow a^-} f(x) = L.$$

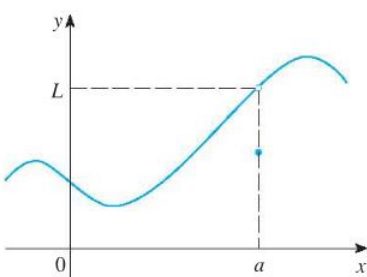
**Definition (Right Limit):** We call  $L$  the right limit of  $f(x)$  at  $a$  if as  $x$  approaches  $a$  from the right,  $f(x)$  approaches  $L$ , i.e. we can make  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we want) for all  $x$  sufficiently close to  $a$ , from the right (without actually letting  $x$  be  $a$ ). We write this as

$$\lim_{x \rightarrow a^+} f(x) = L.$$

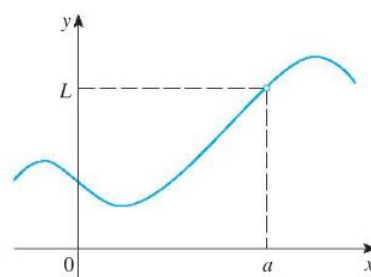
**Example:** What are the left and right limits at  $a$  for the three functions below?



(a)



(b)



(c)

**Definition (Limit):** We call  $L$  the limit of  $f(x)$  at  $a$  if as  $x$  approaches  $a$ ,  $f(x)$  approaches  $L$ , i.e. we can make  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we want) for all  $x$  sufficiently close to  $a$ , from any direction (without actually letting  $x$  be  $a$ ). We write this as

$$\lim_{x \rightarrow a} f(x) = L.$$

**Intuitive Definition (Limit):**  $f(x)$  gets closer to  $L$  as  $x$  gets closer to  $a$  (from either side of  $a$ ) but  $x \neq a$ .

**Existence of a Limit:** If the function  $f(x)$  is defined near  $x = a$  but not necessarily at  $x = a$ , then

$$\lim_{x \rightarrow a} f(x) = L$$

if and only if both the limits

$$\lim_{x \rightarrow a^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x)$$

exist and are equal to the same number  $L$ .

**Example:** For the function  $f$  whose graph is given, state the value of each quantity, if it exists.

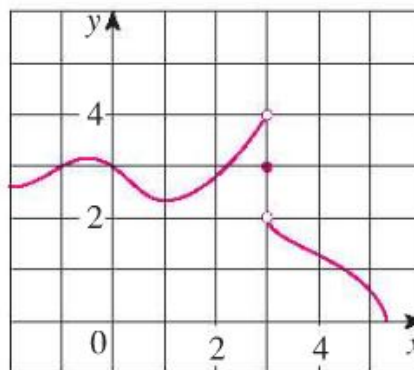
(a)  $\lim_{x \rightarrow 0} f(x)$

(b)  $\lim_{x \rightarrow 3^-} f(x)$

(c)  $\lim_{x \rightarrow 3^+} f(x)$

(d)  $\lim_{x \rightarrow 3} f(x)$

(e)  $f(3)$



### Guessing a limit from numerical values

**Example:** Guess the value of  $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$ .

**Example:** Guess the value of  $\lim_{x \rightarrow 2} g(x)$ , where  $g(x) = \begin{cases} \frac{x^2 + 4x - 12}{x^2 - 2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$ .

**Example:** Estimate the value of the  $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$ .

**Example:** Does  $\lim_{x \rightarrow 0^+} \frac{1}{x}$  exist?

**Definition (Vertical Asymptote):** If as  $x \rightarrow a^-$  or  $x \rightarrow a^+$  the function  $y = f(x)$  becomes large in magnitude without bound, then the line  $x = a$  is called a vertical asymptote.

**The Limit of a Polynomial Function:** If  $p(x)$  is any polynomial and  $a$  is any number, then  $\lim_{x \rightarrow a} p(x) = p(a)$ .

**Example:** Find  $A$  such that  $\lim_{x \rightarrow 3} x^2 - Ax + 2 = 8$ .

**Rules for Limits:** Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M.$$

Then

**Rule 1:**  $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x) = cL$

**Rule 2:**  $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$

**Rule 3:**  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \left( \lim_{x \rightarrow a} f(x) \right) \cdot \left( \lim_{x \rightarrow a} g(x) \right) = L \cdot M$

**Rule 4:**  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$  if  $\lim_{x \rightarrow a} g(x) = M \neq 0$

**Rule 5:**  $\lim_{x \rightarrow a} (f(x))^n = L^n$ ,  $n$  any real number,  $L^n$  defined,  $L \neq 0$

**Example:** What is  $\lim_{x \rightarrow 0} \frac{x^2 + 5x + 6}{x + 1}$ ?

**Example:** What is  $\lim_{x \rightarrow 1} (x^2 - 9)^{\frac{1}{3}}$ ?

**Definition (The Limit of a Function Continuous at a Point):** The function  $f(x)$  is continuous at  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

**Continuity of Polynomial and Rational Functions:**

- A polynomial function is continuous everywhere.
- A rational function is continuous at every point at which the denominator is not zero.

**Example:** Determine the values where  $\frac{2x^2 + 11x}{x^3 + 6x^2 - 27x}$  is discontinuous.

**More Rational Function Practice:**

**Example:** Find  $\lim_{x \rightarrow 0} \frac{x^3 + x^2 - 6x}{x}$ .

**Example:** Find  $\lim_{x \rightarrow 0} \frac{x^2 + x - 6}{x - 2}$ .

**Example:** Find  $\lim_{x \rightarrow 1} \frac{5x^2 - 4x - 1}{x - 1}$ .