Note: Some of these examples and figures come from the textbook *Single Variable Calculus: Concepts & Contexts*, 4th ed., by Stewart and the MATH 131 lecture notes of Aleksanda Sobieska.

# **Question:** *What does a function $f(x)$ do as the x values get closer and closer to a point?*

**Example:** The Heaviside function $H$ is defined by $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$. What happens as the values of $x$ approach 0 from the left? From the right?

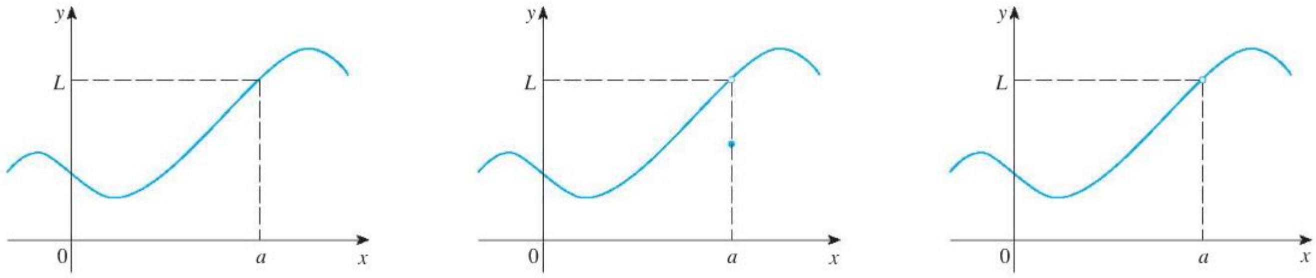
**Definition (Left Limit):** We call $L$ the left limit of $f(x)$ at $a$ if as $x$ approaches $a$ from the left, $f(x)$ approaches $L$, i.e. we can make $f(x)$ arbitrarily close to $L$ (as close to $L$ as we want) for all $x$ sufficiently close to $a$, from the left (without actually letting $x$ be $a$). We write this as

$$\lim_{x \to a^-} f(x) = L.$$

**Definition (Right Limit):** We call $L$ the right limit of $f(x)$ at $a$ if as $x$ approaches $a$ from the right, $f(x)$ approaches $L$, i.e. we can make $f(x)$ arbitrarily close to $L$ (as close to $L$ as we want) for all $x$ sufficiently close to $a$, from the right (without actually letting $x$ be $a$). We write this as

$$\lim_{x \to a^+} f(x) = L.$$

**Example:** What are the left and right limits at $a$ for the three functions below?




(a) (b) (c)

**Definition (Limit):** We call $L$ the limit of $f(x)$ at $a$ if as $x$ approaches $a$, $f(x)$ approaches $L$, i.e. we can make $f(x)$ arbitrarily close to $L$ (as close to $L$ as we want) for all $x$ sufficiently close to $a$, from any direction (without actually letting $x$ be $a$). We write this as

$$\lim_{x \to a} f(x) = L.$$

**Intuitive Definition (Limit):** $f(x)$ gets closer to $L$ as $x$ gets closer to $a$ (from either side of $a$) but $x \neq a$.

**Existence of a Limit:** If the function $f(x)$ is defined near $x = a$ but not necessarily at $x = a$, then

$$\lim_{x \to a} f(x) = L$$

if and only if both the limits

$$\lim_{x \to a^-} f(x) \qquad \text{and} \qquad \lim_{x \to a^+} f(x)$$

exist and are equal to the same number $L$.

**Example:** For the function $f$ whose graph is given, state the value of each quantity, if it exists.

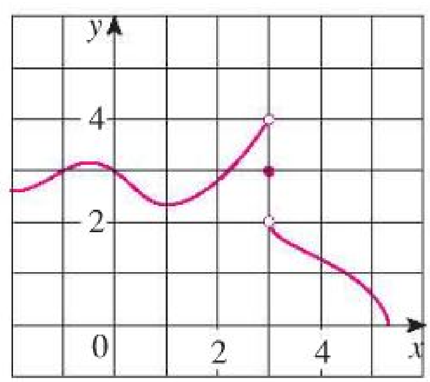
(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) $f(3)$



## Guessing a limit from numerical values

**Example:** Guess the value of $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.

**Example:** Guess the value of $\lim_{x\to 2} g(x)$, where $g(x) = \begin{cases} \frac{x^2+4x-12}{x^2-2x} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$.

**Example:** Estimate the value of the $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

**Example:** Does $\lim_{x\to 0^+} \frac{1}{x}$ exist?

**Definition (Vertical Asymptote):** If as $x \to a^-$ or $x \to a^+$ the function $y = f(x)$ becomes large in magnitude without bound, then the line $x = a$ is called a vertical asymptote.

**The Limit of a Polynomial Function:** If $p(x)$ is any polynomial and $a$ is any number, then $\lim_{x\to a} p(x) = p(a)$.

**Example:** Find $A$ such that $\lim_{x\to 3} x^2 - Ax + 2 = 8$.

**Rules for Limits:** Assume that

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M.$$

Then

**Rule 1:** $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$

**Rule 2:** $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

**Rule 3:** $\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right) = L\cdot M$

**Rule 4:** $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ if $\lim_{x\to a} g(x) = M \neq 0$

**Rule 5:** $\lim_{x\to a}(f(x))^n = L^n$, $n$ any real number, $L^n$ defined , $L \neq 0$

**Example:** What is $\lim\limits_{x\to 0} \dfrac{x^2+5x+6}{x+1}$?

**Example:** What is $\lim_{x\to 1}(x^2-9)^{\frac{1}{3}}$?

<u>**Definition (The Limit of a Function Continuous at a Point):**</u> The function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

<u>**Continuity of Polynomial and Rational Functions:**</u>
• A polynomial function is continuous everywhere.
• A rational function is continuous at every point at which the denominator is not zero.

**Example:** Determine the values where $\dfrac{2x^2+11x}{x^3+6x^2-27x}$ is discontinuous.

<u>**More Rational Function Practice:**</u>

**Example:** Find $\lim_{x\to 0}\dfrac{x^3+x^2-6x}{x}$.

**Example:** Find $\lim_{x\to 0} \frac{x^2+x-6}{x-2}$.

**Example:** Find $\lim_{x\to 1} \frac{5x^2-4x-1}{x-1}$.