The Relative Gain Array of Cayley Graphs

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January 16, 2010

The relative gain array (RGA) is a matrix function which has applications to chemical engineering. When one explores iterates of this function, one of four things will occur. If the input matrix A is singular the $RGA(A) = 0$. If A is nonsingular then in some rare cases, A is fixed by the relative gain array. In other cases, iterates of the function RGA converges to a fixed matrix. And finally, in some cases, iterates of the RGA display chaotic behavior.

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A Cayley graph is a graph with a sharply transitive automorphism group. We explore the RGA of various Cayley graphs. Using both Mathematica and Groups Algorithms and Programming (GAP) we observe the four different behaviors of the RGA. We analyze the defining set S in an attempt to predict the behavior of the relative gain array. We compare the action of the RGA on a Cayley graph with the action of the RGA on the complementary graph. We are especially interested in cases in which either the adjacency algebra of a graph or its complement is closed under the Hadamard product.

Definition

Defn (Cayley Graph): Let G be a finite group. Let S be a subset of G. The Cayley graph, with respect to $\Gamma(G, S)$ has vertex set G and edges (x, sx)

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Lemma: For all $x \in V$, the degree of x is equal to the size of the defining set S.

Note: A Cayley graph is *simple* iff S is symmetric (i.e. $S = S^{-1}$ *and 1 is not an element of* S*).*

GAP	Type	Group	Elements (GAP order)
$[8,1]$	Cyclic	C_8	$\{1, x, x^2, x^4, x^3, x^5, x^6, x^7\};$
$[8,2]$	Abelian	$C_4 \times C_2$	$\{1, x, y, x^2, xy, x^3, x^2y, x^3y\};$
$[8,3]$	Dihedral	D_4	$\{1, y, x^3y, x^2, x, x^2y, xy, x^3\};$
$[8,4]$	Quaternion	Q_4	$\{1, x, y, x^2, xy, x^3, x^2y, x^3y\};$
$[8,5]$	Elementary Abelian	C_2^3	$\{1, x, y, z, xy, xz, yz, xyz\};$
$[9,1]$	Cyclic	C_9	$\{1, x, x^3, x^2, x^4, x^6, x^5, x^7, x^8\};$
$[9,2]$	Elementary Abelian	C_3^2	$\{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\};$
$[10,1]$	Dihedral	D_5	$\{1, y, x, x^4y, x^2, x^3y, x^3, x^2y, x^4, xy\};$
$[10,2]$	Cyclic	C_{10}	$\{1, x^5, x^2, x^7, x^4, x^9, x^6, x, x^8, x^3\};$
$[11,1]$	Cyclic	C_{11}	$\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}\};$
$[12,1]$	Quaternion	Q_6	$\{1, y, x^3, x^4, x^3y, x^2y, x, x^2, x^5y, x^4y, x^5, xy\};$
$[12,2]$	Cyclic	C_{12}	$\{1, x^3, x^8, x^6, x^{11}, x^9, x^4, x^2, x^7, x^5, x^{10}, x\};$
$[12,3]$	Alternating	A_4	$\{1, xz^2, xy, y, xyz, z^2, yz^2, x, xz, yz, x^2y, z\};$
$[12,4]$	Dihedral	D_6	$\{1, y, x^3, x^2, x^3y, x^4y, x^5, x^4, xy, x^2y, x, x^5y\};$
$[12,5]$	Elementary Abelian	$C_6 \times C_2$	$\{1, y, x^3, x^4, x^3y, x^4y, x, x^2, xy, x^2y, x^5, x^5y\};$

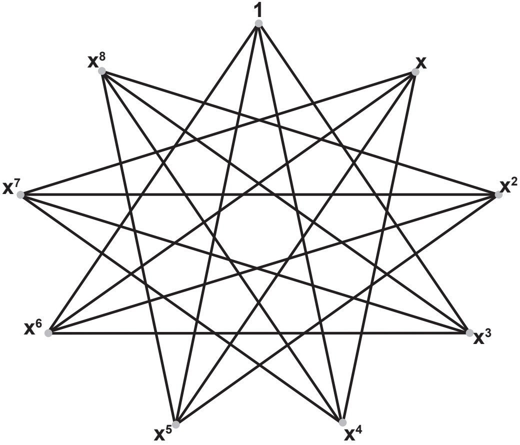

Figure: Graph of $\Gamma([9,1],[3,5,6,7])$ C_9, where $S := \{x^4, x^5, x^3, x^6\}$

Once we have a group and a set, or a picture of a graph, we can also create its adjacency matrix and adjacency diagram.

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \end{array} \\
\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

Figure: The adjacency matrix of $\Gamma(C_9, \{x^4, x^5, x^3, x^6\})$

Definition

Defn(Adjacency Diagram): Let $\Gamma(G, S)$ be a Cayley graph. Define $S_i := \{x \in G : dist(1, x) = i\}$. Partition S_i into d_1 subsets such that $S_1 = \Gamma_1 \cup \Gamma_2 \cup \ldots \Gamma d_1$. Partition S_2 into d_2 subsets such that $S_2 = \Gamma_{d_1+1} \cup \Gamma_{d_1+2} \cup \ldots \Gamma_{d_1+d_2}$. Continue in this manner until all sets S_i are partitioned into a finite number of subsets.

The adjacency diagram of $\Gamma(G, S)$ consists of boxes representing each Γ_i and directed edges between the boxes such that $\Gamma_i \rightarrow \Gamma_j$ if every element in Γ_i is connected to exactly b elements of Γ_j.

Note: An edge on an adjacency diagram from $\Gamma_i \rightarrow \Gamma_j$ without a number signifies that the number of elements in Γ_j connected to elements of Γ_i is 1.

Within each box of an adjacency diagram is a collection of vertices (or a single vertex) that are related to each other and to every other vertex in the graph in the same way. The lines and number give us those relationships. The dot is the starting vertex.

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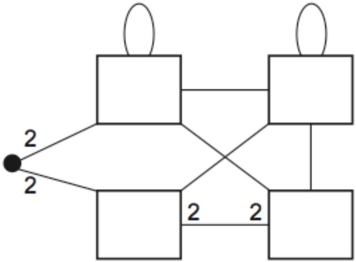


Figure: Adjacency Diagram of $\Gamma(C_9, \{x^4, x^5, x^3, x^6\})$.

For

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$$

then

$$(AB) \in \mathbb{R}^{m \times p}$$

where the elements of A,B are given by

$$(AB)_{i,j} = \sum_{r=1}^{n} A_{i,r} B_{r,j}$$

for each pair i and j with $1 \leq i \leq m$ and $1 \leq j \leq p$.

For

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n}$$

then

$$(A \circ B) \in \mathbb{R}^{m \times n}$$

where the elements of $A \circ B$ are given by

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}$$

Properties of the Hadamard Product:

1. The Hadamard product is a submatrix of the Kronecker product.
2. The Hadamard product is commutative.

Relative Gain Array

Defn(Relative Gain Array):

Given the adjacency matrix of a Cayley graph, A:

$$\Phi(A) := A \circ (A^{-1})^T$$

Note: Matrix multiplication is using the Hadamard product.

Properties of the Relative Gain Array:

1. The Relative Gain Array function returns a matrix with a row sum of 1.
2. We define Φ-rank to be the number of unique non-zero elements in the matrix returned by $\Phi(A)$.
3. When P is a permutation matrix:
   - $\Phi P = P$
   - $\Phi(\frac{1}{2}P_1 + \frac{1}{2}P_2) = \frac{1}{2}P_1 + \frac{1}{2}P_2$
4. $\Phi(cA) = \Phi(A)$, where c is a constant.

Fixed: $\Gamma(C_4 \times C_2, \{x, y, x^3\})$

$$A = \begin{bmatrix} 0, & 1, & 1, & 0, & 0, & 1, & 0, & 0 \\ 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0, & 1, & 0, & 0, & 1 \\ 0, & 1, & 0, & 0, & 0, & 1, & 1, & 0 \\ 0, & 1, & 1, & 0, & 0, & 0, & 1, & 0 \\ 1, & 0, & 0, & 1, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 1, & 1, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0, & 0, & 1, & 1, & 0 \end{bmatrix}$$

Figure: Adjacency Matrix

$$\Phi(A) = \begin{bmatrix} 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & 0, & 0 \\ \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & 0 \\ \frac{1}{3}, & 0, & 0, & 0, & \frac{1}{3}, & 0, & 0, & \frac{1}{3} \\ 0, & \frac{1}{3}, & 0, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & 0 \\ 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & 0, & \frac{1}{3}, & 0 \\ \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & 0, & 0, & 0, & \frac{1}{3} \\ 0, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & \frac{1}{3} \\ 0, & 0, & \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & \frac{1}{3}, & 0 \end{bmatrix}$$

Figure: $\Phi(A)$

$$\text{Fixed}: \Gamma(C_4 \times C_2, \{x, y, x^3\})$$

$$A = \begin{bmatrix} 0, & 1, & 1, & 0, & 0, & 1, & 0, & 0 \\ 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0, & 1, & 0, & 0, & 1 \\ 0, & 1, & 0, & 0, & 0, & 1, & 1, & 0 \\ 0, & 1, & 1, & 0, & 0, & 0, & 1, & 0 \\ 1, & 0, & 0, & 1, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 1, & 1, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0, & 0, & 1, & 1, & 0 \end{bmatrix}$$

$$A[1] = \begin{bmatrix} 0, & 1, & 1, & 0, & 0, & 1, & 0, & 0 \end{bmatrix}$$

$$\Phi(A)[1] = \begin{bmatrix} 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & 0, & 0 \end{bmatrix}$$

$$\Phi^2(A)[1] = \begin{bmatrix} 0, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & \frac{1}{3}, & 0, & 0 \end{bmatrix}$$

Convergence : $\Gamma(C_9, \{x, x^2, x^7, x\})$

$$A = \begin{bmatrix} 0, & 1, & 0, & 1, & 0, & 0, & 0, & 1, & 1 \\ 1, & 0, & 1, & 1, & 0, & 0, & 0, & 0, & 1 \\ 0, & 1, & 0, & 1, & 1, & 0, & 1, & 0, & 0 \\ 1, & 1, & 1, & 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 1, & 0, & 1, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0, & 1, & 1, & 1 \\ 0, & 0, & 1, & 0, & 1, & 1, & 0, & 1, & 0 \\ 1, & 0, & 0, & 0, & 0, & 1, & 1, & 0, & 1 \\ 1, & 1, & 0, & 0, & 0, & 1, & 0, & 1, & 0 \end{bmatrix}$$

Figure: Adjacency Matrix

$$\Phi(A) = \begin{bmatrix} 0, & \frac{3}{4}, & 0, & \frac{-1}{4}, & 0, & 0, & 0, & \frac{-1}{4}, & \frac{3}{4} \\ \frac{3}{4}, & 0, & \frac{-1}{4}, & \frac{3}{4}, & 0, & 0, & 0, & 0, & \frac{-1}{4} \\ 0, & \frac{-1}{4}, & 0, & \frac{3}{4}, & \frac{3}{4}, & 0, & \frac{-1}{4}, & 0, & 0 \\ \frac{-1}{4}, & \frac{3}{4}, & \frac{3}{4}, & 0, & \frac{-1}{4}, & 0, & 0, & 0, & 0 \\ 0, & 0, & \frac{3}{4}, & \frac{-1}{4}, & 0, & \frac{-1}{4}, & \frac{3}{4}, & 0, & 0 \\ 0, & 0, & 0, & 0, & \frac{-1}{4}, & 0, & \frac{3}{4}, & \frac{3}{4}, & \frac{-1}{4} \\ 0, & 0, & \frac{-1}{4}, & 0, & \frac{3}{4}, & \frac{3}{4}, & 0, & \frac{-1}{4}, & 0 \\ \frac{-1}{4}, & 0, & 0, & 0, & 0, & \frac{3}{4}, & \frac{-1}{4}, & 0, & \frac{3}{4} \\ \frac{3}{4}, & \frac{-1}{4}, & 0, & 0, & 0, & \frac{-1}{4}, & 0, & \frac{3}{4}, & 0 \end{bmatrix}$$

Figure: $\Phi(A)$

Convergence : $\Gamma(C_9, \{x, x^2, x^7, x\})$

$$A[1] = \begin{bmatrix} 0, & 1, & 0, & 1, & 0, & 0, & 0, & 1, & 1 \end{bmatrix}$$

$$\Phi(A)[1] = \begin{bmatrix} 0, & \frac{3}{4}, & 0, & -\frac{1}{4}, & 0, & 0, & 0, & -\frac{1}{4}, & \frac{3}{4} \end{bmatrix}$$

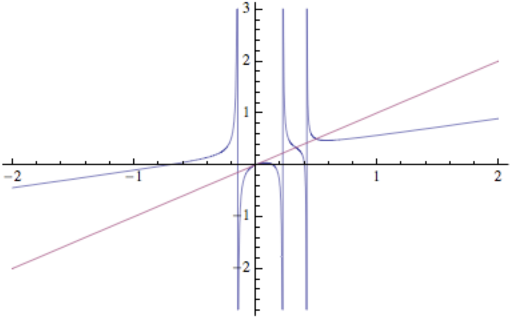
Insert Variables :

$$M[1] = \begin{bmatrix} 0, & r, & 0, & \frac{1}{2}(1-2r), & 0, & 0, & 0, & \frac{1}{2}(1-2r), & r \end{bmatrix}$$

$$\Phi(M)[1] = [0, \tfrac{r(1-8r+4r^2+24r^3)}{1-36r^2+72r^3}, 0, \tfrac{(-1+2r)(1-20r^2+24r^3)}{2-72r^2+144r^3}, 0, 0, 0, \tfrac{(-1+2r)(1-20r^2+24r^3)}{2-72r^2+144r^3}, \tfrac{r(1-8r+4r^2+24r^3)}{1-36r^2+72r^3}]$$

$$\phi(r) = \frac{r(1-8r+4r^2+24r^3)}{1-36r^2+72r^3}$$

```
Plot[{phi, r}, {r, -2, 2}]
```

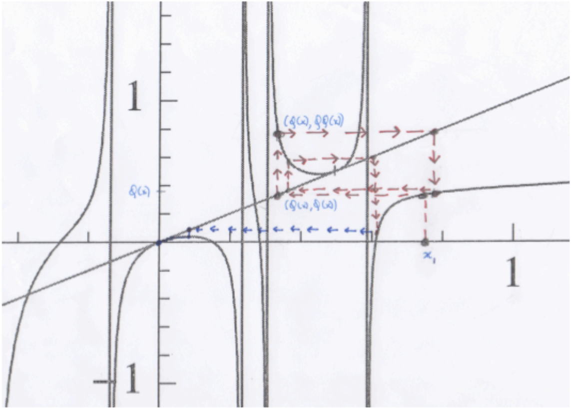



```
Solve[f == 0 , r]
```

$$\left\{\{r \to 0\}, \{r \to 0\}, \left\{r \to \frac{1}{3}\right\}, \left\{r \to \frac{1}{2}\right\}\right\}$$

Figure: Graph of $\phi(r)$ vs. $y = x$

$$\text{Chaos}: \Gamma(C_{11}, \{x, x^2, x^9, x^{10}\})$$



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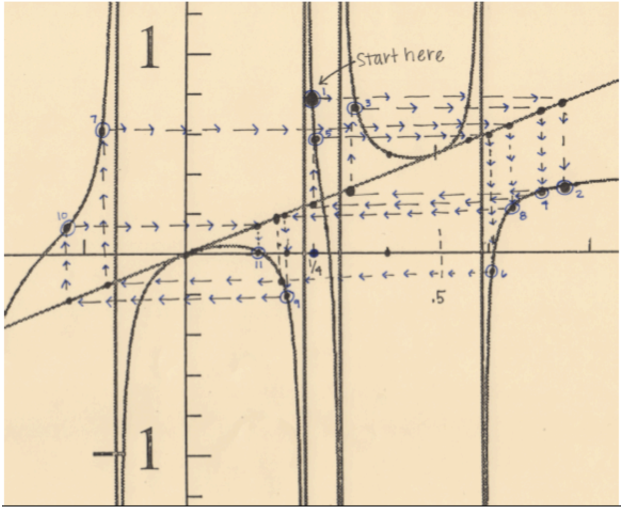


Definition

Defn (Complement Graph) : Let G be a finite group with vertex set V. Suppose S is the defining set of a Cayley graph. The complement of the Cayley graph $\Gamma(G,S)$ is $\Gamma(G,S')$ where $S' = V \setminus (S \cup \{1\})$.

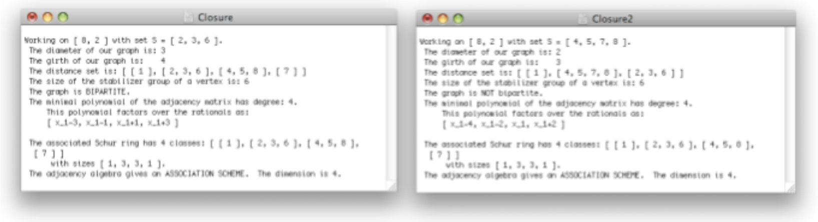


Figure: Analyze Cayley graph

From this program we can view similarities between this graph and its complement.

We can also see that each have an adjacency algebra that is closed under the Hadamard product.

Closure

```
Working on [ 8, 2 ] with set S = [ 2, 3, 6 ].
 The diameter of our graph is: 3
 The girth of our graph is:    4
 The distance set is: [ [ 1 ], [ 2, 3, 6 ], [ 4, 5, 8 ], [ 7 ] ]
 The size of the stabilizer group of a vertex is: 6
 The graph is BIPARTITE.
 The minimal polynomial of the adjacency matrix has degree: 4.
     This polynomial factors over the rationals as:
     [ x_1-3, x_1-1, x_1+1, x_1+3 ]

 The associated Schur ring has 4 classes: [ [ 1 ], [ 2, 3, 6 ], [ 4, 5, 8 ],
  [ 7 ] ]
      with sizes [ 1, 3, 3, 1 ].
 The adjacency algebra gives an ASSOCIATION SCHEME.  The dimension is 4.
```

Closure2

```
Working on [ 8, 2 ] with set S = [ 4, 5, 7, 8 ].
 The diameter of our graph is: 2
 The girth of our graph is:    3
 The distance set is: [ [ 1 ], [ 4, 5, 7, 8 ], [ 2, 3, 6 ] ]
 The size of the stabilizer group of a vertex is: 6
 The graph is NOT bipartite.
 The minimal polynomial of the adjacency matrix has degree: 4.
     This polynomial factors over the rationals as:
     [ x_1-4, x_1-2, x_1, x_1+2 ]

 The associated Schur ring has 4 classes: [ [ 1 ], [ 2, 3, 6 ], [ 4, 5, 8 ],
  [ 7 ] ]
      with sizes [ 1, 3, 3, 1 ].
 The adjacency algebra gives an ASSOCIATION SCHEME.  The dimension is 4.
```

What is GAP?

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

- http://www.gap-system.org

Figure: GAP

GAP Kernel

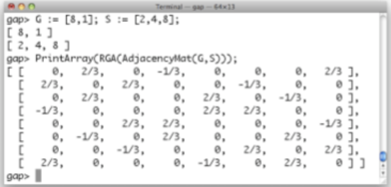
Compiled in C, GAP's kernel implements:

1. the GAP language,
2. a command-line interface to run GAP programs,
3. memory management, and
4. time-saving algorithms for operations and data types.

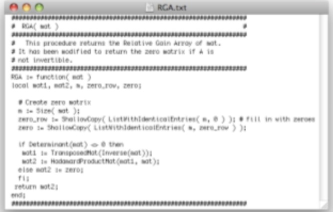
All other functions are written in the GAP language. Packages are commonly written in the GAP language, but some have standalones. Some packages interface with other systems.

Requirements and Availability

1. The GAP system will run on UNIX-based or recent Windows and MacOS platforms, although some packages require UNIX.
2. The computer must have a reasonable amount of RAM and disk space.
3. The current version, GAP 4, can be obtained at no cost at http://www.gap-system.org



```
gap> G := [8,1]; S := [2,4,8];
[ 8, 1 ]
[ 2, 4, 8 ]
gap> PrintArray(RGA(AdjacencyMat(G,S)));
[ [     0,   2/3,     0,  -1/3,     0,     0,     0,   2/3 ],
  [   2/3,     0,   2/3,     0,     0,  -1/3,     0,     0 ],
  [     0,   2/3,     0,     0,   2/3,     0,  -1/3,     0 ],
  [  -1/3,     0,     0,     0,   2/3,   2/3,     0,     0 ],
  [     0,     0,   2/3,   2/3,     0,     0,     0,  -1/3 ],
  [     0,  -1/3,     0,   2/3,     0,     0,   2/3,     0 ],
  [     0,     0,  -1/3,     0,     0,   2/3,     0,   2/3 ],
  [   2/3,     0,     0,     0,  -1/3,     0,   2/3,     0 ] ]
gap>
```



```
##################################################################
#  RGA( mat )                                                    #
##################################################################
#   This procedure returns the Relative Gain Array of mat.
# It has been modified to return the zero matrix if A is
# not invertible.
##################################################################
RGA := function( mat )
local mat1, mat2, m, zero_row, zero;

  # Create zero matrix
  m := Size( mat );
  zero_row := ShallowCopy( ListWithIdenticalEntries( m, 0 ) ); # fill in with zeroes
  zero := ShallowCopy( ListWithIdenticalEntries( m, zero_row ) );

  if Determinant(mat) <> 0 then
   mat1 := TransposedMat(Inverse(mat));
   mat2 := HadamardProductMat(mat1, mat);
  else mat2 := zero;
  fi;
 return mat2;
end;
##################################################################
```

Figure: RGA Function in GAP

Terminal — gap — 64×13

```
gap> G := [8,1]; S := [2,4,8];
[ 8, 1 ]
[ 2, 4, 8 ]
gap> PrintArray(RGA(AdjacencyMat(G,S)));
[ [     0,   2/3,     0,  -1/3,     0,     0,     0,   2/3 ],
  [   2/3,     0,   2/3,     0,     0,  -1/3,     0,     0 ],
  [     0,   2/3,     0,     0,   2/3,     0,  -1/3,     0 ],
  [  -1/3,     0,     0,     0,   2/3,   2/3,     0,     0 ],
  [     0,     0,   2/3,   2/3,     0,     0,     0,  -1/3 ],
  [     0,  -1/3,     0,   2/3,     0,     0,   2/3,     0 ],
  [     0,     0,  -1/3,     0,     0,   2/3,     0,   2/3 ],
  [   2/3,     0,     0,     0,  -1/3,     0,   2/3,     0 ] ]
gap> 
```

RGA.txt

```
##############################################################
#  RGA( mat )                                                #
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   mat1 := TransposedMat(Inverse(mat));
   mat2 := HadamardProductMat(mat1, mat);
  else mat2 := zero;
  fi;
 return mat2;
end;
##############################################################
```
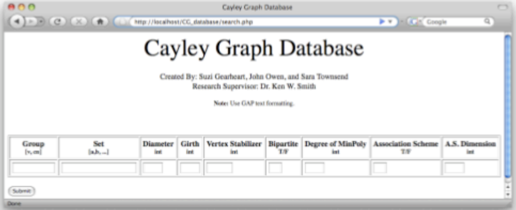
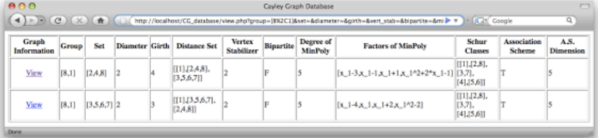

Using the GAP system we created a program to collect information on different properties of over 50,000 Cayley graphs. This data was then put into a searchable internet-accessible database.





Graph Information	Group	Set	Diameter	Girth	Distance Set	Vertex Stabilizer	Bipartite	Degree of MinPoly	Factors of MinPoly	Schur Classes	Association Scheme	A.S. Dimension
View	[8,1]	[2,4,8]	2	4	[[1],[2,4,8],[3,5,6,7]]	2	F	5	[x_1-3,x_1-1,x_1+1,x_1^2+2*x_1-1]	[[1],[2,8],[3,7],[4],[5,6]]	T	5
View	[8,1]	[3,5,6,7]	2	3	[[1],[3,5,6,7],[2,4,8]]	2	F	5	[x_1-4,x_1,x_1+2,x_1^2-2]	[[1],[2,8],[3,7],[4],[5,6]]	T	5

Let S be a defining set:
$S := \{x, y, xz, yz, xz^2, yz^2, y^4, y^4z, y^4z^2\}$

Translate S into group ring notation such that:
$\hat{S} := x + y + xz + yz + xz^2 + yz^2 + y^4 + y^4z + y^4z^2$

Theorem

The eigenvalues of the adjacency matrix A are exactly the number $\chi(\hat{S})$ where χ is a linear character.

Linear Characters

Linear characters are functions from the group to the complex numbers. Our hope is to be able to factor the set S into subsets in such a way that these subsets map to zero and therefore explaining why the determinant is zero.

Example: $\Gamma(D_3 \times C_5, S)$

$S = \{x, xz, xz^2\} \cup \{y, yz, yz^2\} \cup \{y^4, y^4z, y^4z^2\}$

$\hat{S} = x(1 + z + z^2) + y(1 + z + z^2) + y^4(1 + z + z^2)$

We map z to a cube root of unity. $(z \Rightarrow \omega)$

$$\begin{array}{c} (1, z, z^2) \\ \Big\Downarrow \\ 1 + \omega + \omega^2 \end{array}$$

We notice that $1 + \omega + \omega^2$ is a geometric series. Hence, $(1 + \omega + \omega^2) = \frac{1-\omega^3}{1-\omega}$. Using the cube roots of unity $\omega^3 = 1$ we have $\frac{0}{1-\omega} = 0$.

Conjecture 1

If a Cayley graph is of prime order, then its eigenvalues are equal to the number of boxes in its adjacency diagram therefore the adjacency algebra is closed under the Hadamard product.

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The algebra of an adjacency matrix is closed under the Hadamard product if and only if the algebra of its complementary matrix is also closed under the Hadamard product.

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Conjecture 2

The algebra of an adjacency matrix is closed under the Hadamard product if and only if the algebra of its complementary matrix is also closed under the Hadamard product.

Conjecture 3

The Φ-rank of a Cayley graph is less than or equal to the number of boxes in the first neighborhood of the adjacency diagram.

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4. What causes chaos?

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Our thanks goes to:

- Our Advisor, Dr. Ken W. Smith
- Our Graduate Student, Jessica Ellis
- The National Science Foundation
  - (Grant: DMS-0636528)
- Sam Houston State University