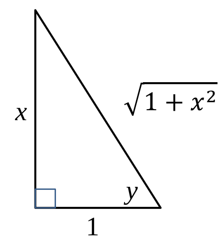
CAPE Unit 2
Pure Mathematics
June 2014
Solutions

1 a) i To differentiate: $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Note to students:

1. If $y = \ln f(x)$ then
$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
2. If $y = \tan^{-1}(x)$ then
$\tan y = x$
$\sec^2 y \frac{dy}{dx} = 1$
$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$
$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$
3. If $y = \tan^{-1} f(x)$ then
$\frac{dy}{dx} = \frac{f'(x)}{1+(f(x))^2}$



$$\cos y = \frac{1}{\sqrt{1+x^2}}$$

$$y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$$

To differentiate the above expression, apply the rule for $\ln(f(x))$ as shown above; apply the product rule to $x\tan^{-1}\left(\frac{x}{2}\right)$; and, in doing the latter, apply the rule for $\tan^{-1} f(x)$. This is done below.

$$\frac{dy}{dx} = \frac{2x}{x^2+4} - \left[\left[x \cdot \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}\right] + \left[\tan^{-1}\left(\frac{x}{2}\right)\right] \cdot 1\right]$$

$$= \frac{2x}{x^2+4} - \frac{\frac{1}{2}x}{\frac{4+x^2}{4}} - \tan^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{2x}{x^2+4} - \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right)$$

$$= -\tan^{-1}\left(\frac{x}{2}\right)$$

(5 marks)

ii Given $x = a\cos^3 t$; $y = a\sin^3 t$

To show: that the tangent at $P(x, y)$ is: $y\cos t + x\sin t = a\sin t\cos t$

$\frac{dy}{dt} = 3a\sin^2 t\cos t$

$\frac{dx}{dt} = -3a\cos^2 t\sin t$

$\frac{dy}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = 0 - \frac{\sin t}{\cos t}$

Let the equation of the tangent be $y = mx + c$

$m = \frac{dy}{dx} = -\frac{\sin t}{\cos t}$

Substituting $a\cos^3 t$ for x and $a\sin^3 t$ for y,

$a\sin^3 t = -\frac{\sin t}{\cos t} \cdot a\cos^3 t + c$

$a\sin^3 t = -\sin t \cdot a\cos^2 t + c$

$c = a\sin^3 t + a\sin t \cdot a\cos^2 t$
$c = a\sin t(\sin^2 t + \cos^2 t)$
$c = a\sin t$
$\therefore y = -\frac{\sin t}{\cos t}x + a\sin t$
$y\cos t + x\sin t = a\sin t\cos t$ (7 marks)

b) i) Given: $x^2 + 3x + 9 = 0$ has roots α and β.
To determine the nature of the roots
$a = 1, b = 3, c = 9$
$\therefore b^2 - 4ac = 9 - 36 = -27$
Hence the roots are complex. (2 marks)

ii) To express α and β in the form $re^{i\theta}$ where r is the modulus and θ is the argument $-\pi < \theta \le \pi$

$x = \frac{-3\pm\sqrt{-27}}{2} = \frac{-3\pm\sqrt{27}\sqrt{-1}}{2}$
$x = \frac{-3\pm3\sqrt{3}i}{2}$
$\alpha = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ or $\beta = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$
$|\alpha| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{\sqrt{27}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = 3$
$|\beta| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-\sqrt{27}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = 3$
Finding $\arg\alpha$
$\tan^{-1}\alpha = -\frac{\sqrt{27}}{2} \div \frac{3}{2} = -\sqrt{3}$
$\alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$
$\therefore \alpha = 3e^{i\frac{2\pi}{3}}$
Finding $\arg\beta$
$\tan^{-1}\beta = -\frac{\sqrt{27}}{2} \div -\frac{3}{2} = \sqrt{3}$
$\beta = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$
$\therefore \beta = 3e^{i\left(-\frac{2\pi}{3}\right)}$ (4 marks)

iii) To use deMoivre's theorem or otherwise to compute $\alpha^3 + \beta^3$

$\alpha^3 = \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)^3 = \left[(3)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right]^3$
$= (3)^3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$
$= 27\,(\cos\theta + i\sin\theta)^3$ where $\theta = \frac{2\pi}{3}$
$= 27(\cos 3\theta + i\sin 3\theta)$ from deMoivre's theorem
$= 27\,(\cos 2\pi + i\sin 2\pi)$
$= 27\,(1 + 0) = 27$

Similarly,

$\beta^3 = \left(-\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)^3 = \left[(3)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]^3$

$= (3)^3 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

$= 27\,(\cos\theta + i\sin\theta)^3$ where $\theta = -\frac{2\pi}{3}$

$= 27(\cos 3\theta + i\sin 3\theta)$ from deMoivre's theorem

$= 27\,(\cos(-2\pi) + i\sin(-2\pi))$

$= 27\,(1 + 0) = 27$

$= 27(1 + 0) = 27$

$\therefore \alpha^3 + \beta^3 = 27 = 27 = 54$

Or

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha + \beta = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i + \left(-\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)$

$\alpha + \beta = 2\left(-\frac{3}{2}\right) = -3$

$\alpha\beta = \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)\left(-\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)$

$= \left(-\frac{3}{2}\right)^2 - \left(\frac{3\sqrt{3}}{2}i\right)^2 = \frac{9}{4} + \frac{27}{4} = 9$

$\therefore \alpha^3 + \beta^3 = (-3)^3 - 3(9)(-3)$

$\alpha^3 + \beta^3 = -27 + 81 = 54$ (4 marks)

iv) Hence to obtain the quadratic equation with roots α^3 and β^3

The required equation is:

$x^2 - (\alpha^3 + \beta^3)x + \alpha^3\beta^3 = 0$

By substitution:

$x^2 - 54x + 729 = 0$ (3 marks)

2. Let $F_n(x) = \int (\ln x)^n dx$

a. i. To show that $F_n(x) = x(\ln x)^n - nF_{n-1}(x)$

Reversing the product rule

Let $y = x(\ln x)^n$

$\frac{dy}{dx} = xn\,(\ln x)^{n-1} \cdot \frac{1}{x} + (\ln x)^n\,(1)$

Integrating

$x(\ln x)^n = \int n(\ln x)^{n-1}dx + \int (\ln x)^n\,dx$

$\int (\ln x)^n\,dx = x(\ln x)^n - \int n(\ln x)^{n-1}dx$

$F_n(x) = x(\ln x)^n - nF_{n-1}(x)$

Or, use the formula for integration by parts

let $u = (\ln x)^n$

$\frac{du}{dx} = n(\ln x)^{n-1} \cdot \left(\frac{1}{x}\right)$

And let $\frac{dv}{dx} = 1$

Then $v = x$

$\therefore \int (\ln x)^n dx = x(\ln x)^n - \int x\, n(\ln x)^{n-1} \cdot \left(\frac{1}{x}\right) dx$

$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1}\, dx$

$\therefore F_n(x) = x(\ln x)^n - \mathrm{n}F_{n-1}(x)$ (3 marks)

ii. Hence, or otherwise, to show that: $F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\,(\ln 2) - 6$

$F_3(x) = x(\ln x)^3 - 3F_2(x)$

$F_2(x) = x(\ln x)^2 - 2F_1(x)$

$\therefore F_3(x) = x(\ln x)^3 - 3[x(\ln x)^2 - 2F_1(x)]$

$F_3(x) = x(\ln x)^3 - 3x(\ln x)^2 + 6F_1(x)$

$F_1(x) = x(\ln x)^1 - 1F_0(x)$

$\therefore F_3(x) = x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x)^1 - 6F_0(x)$

$F_0(x) = \int (\ln x)^0 dx = \int 1 dx = x$

$\therefore F_3(x) = x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x)^1 - 6(x)$

$\therefore F_3(2) = 2(\ln 2)^3 - 3(2)(\ln 2)^2 + 6(2)(\ln 2)^1 - 6(2)$

$\therefore F_3(2) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12 \ln 2 - 12$

$\therefore F_3(1) = 1(\ln 1)^3 - 3(1)(\ln 1)^2 + 6(1)(\ln 1)^1 - 6(1)$

$\therefore F_3(1) = 0 - 0 + 0 - 6 = -6$

$\therefore F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12 \ln 2 - 12 - (-6)$

$\therefore F_3(2) - F_3(1) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12 \ln 2 - 6$

(7 marks)

b. i To decompose $\frac{y^2+2y+1}{y^4+2y^2+1}$ into partial fraction, and in doing,

To show $\frac{y^2+2y+1}{y^4+2y^2+1} = \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}$

Note: $y^4 + 2y^2 + 1 = (y^2+1)^2$

Let $\frac{y^2+2y+1}{y^4+2y^2+1} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}$

$y^2 + 2y + 1 \equiv (Ay+B)((y^2+1) + Cy + D$

$y^2 + 2y + 1 \equiv Ay^3 + Ay + By^2 + B + Cy + D$

From inspection (equating coefficients of corresponding terms):

$A = 0$

$B = 1$

$A + C = 2$

$\therefore C = 2$

$B + D = 1$

$\therefore D = 0$

$\therefore \frac{y^2+2y+1}{y^4+2y^2+1} = \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}$ (7 marks)

ii. Hence to find $\int_0^1 \frac{y^2+2y+1}{y^4+2y^2+1} dy$

$\int_0^1 \frac{y^2+2y+1}{y^4+2y^2+1} dy = \int_0^1 \frac{1}{y^2+1} dy + \int_0^1 \frac{2y}{(y^2+1)^2} dy$

$= [\tan^{-1} y]_0^1 + \int_0^1 \frac{2y}{(y^2+1)^2} dy$

Finding $\int_0^1 \frac{2y}{(y^2+1)^2} dy$

Method 1

$$\int_0^1 \frac{2y}{(y^2+1)^2} dy \equiv \int_0^1 (2y)(y^2+1)^{-2} dy$$

Recall $\int_a^b f'(x)\left(f(x)\right)^n dx = \left[\frac{\left(f(x)\right)^{n+1}}{n+1}\right]_a^b$

$$\therefore \int_0^1 (2y)(y^2+1)^{-2} dy = \left[\frac{(y^2+1)^{-1}}{-1}\right]_0^1$$

$$= \frac{(1^2+1)^{-1}}{-1} - \frac{(0^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

Method 2

Let $u = y^2 + 1$

$du = 2y\, dy$

When $y = 1, u = 2$

When $y = 0, u = 1$

$\therefore \int_0^1 \frac{2y}{(y^2+1)^2} dy = \int_1^2 u^{-2} du$

$$= [-u^{-1}]_1^2 = \left[-\frac{1}{u}\right]_1^2$$

$$= \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$\therefore \int_0^1 \frac{y^2+2y+1}{y^4+2y^2+1}\,dy = [\tan^{-1} y]_0^1 + \frac{1}{2}$$
$$= \tan^{-1} 1 - \tan^{-1} 0 + \frac{1}{2}$$
$$= \frac{\pi}{4} - 0 + \frac{1}{2}$$
$$= \frac{1}{4}(\pi + 2)$$
(8 marks)

3. a. i. To prove by math induction that for $n \epsilon \mathbb{N}$

$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

Let $P(n)$ be the proposition that: $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

<u>Testing $P(1)$:</u>

$LHS = 1$

$RHS = 2 - \frac{1}{2^{1-1}} = 2 - 1 = 1$

Therefore S_1 is true.

<u>Assume $P(k)$ is true</u>

That is assume: $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^{k-1}} = 2 - \frac{1}{2^{k-1}}$

<u>Show $P(k) \Rightarrow P(k+1)$</u>

$P(k+1)$: $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^{k-1}} + \frac{1}{2^k} = 2 - \frac{1}{2^k}$

LHS $= \left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \frac{1}{2^{k-1}}\right] + \frac{1}{2^k}$

$= \left[2 - \frac{1}{2^{k-1}}\right] + \frac{1}{2^k}$

$= 2 - \frac{1}{2^{k-1}} + \frac{1}{2^k}$

$= 2 - \frac{2}{2} \cdot \frac{1}{2^{k-1}} + \frac{1}{2^k}$

$= 2 - \frac{2}{2^k} + \frac{1}{2^k}$

$= 2 + \frac{1}{2^k}(-2 + 1)$

$= 2 - \frac{1}{2^k} \equiv RHS$

$\therefore P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ then $P(n)$ is true for $n \epsilon \mathbb{N}$.

ii. Hence to find $\lim_{n\to\infty} S_n$

$S_n = 2 - \frac{1}{2^{n-1}}$

$= 2 - \frac{2}{2^n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{2}{2^n}$

$\lim_{n\to\infty} S_n = 2 - 0 = 2$

b) To find Maclaurin series for

$f(x) = (1+x)^2 \sin x$

Up to an including the term in x^3

$f(x) = (1+x)^2 \sin x$

$f'(x) = (1+x)^2 \cos x + (\sin x)\,(2)(1+x)$

$f'(0) = (1+0)^2 \cos 0 + (\sin 0)\,(2)(1+0)$

$f'^{(0)} = 1(1) + (0)2(1) = 1$

$f^2(x) = -(1+x)^2 \sin x + (\cos x)(2)(1+x) + 2(\sin x)1 + (1+x)(2)(\cos x)$

$\quad = (\sin x)[-(1+x)^2 + 2] + \cos x[2(1+x) + 2(1+x)]$

$\quad = [2-(1+x)^2] \sin x + 4(1+x)\cos x$

$f^2(0) = (2-1)(0) + 4(1)(1) = 4$

$f^3(x) = [2-(1+x)^2]\cos x + (\sin x)\left(-2(1+x)\right) + 4(1+x)\,(-\sin x) + (\cos x)4$

$f^3(x) = [2-(1+x)^2 + 4]\cos x + (\sin x)\,(-2-2x-4-4x)$

$f^3(x) = [6-(1+x)^2]\cos x + (-6-6x)(\sin x)$

$f^3(0) = [6-(1+0)^2]\cos 0 + (-6-6(0))(\sin 0)$

$f^3(0) = (5)(1) + 0 = 5$

$\therefore f(x) = 0 + (1)x + \frac{(4)x^2}{2!} + \frac{5x^3}{3!} + \ldots$

$f(x) = 0 + x + 2x^2 + \frac{5}{6}x^3 + \ldots$

4. a. i Given $(2x+3)^{20}$

To show: $\frac{ax^6}{bx^7} = \frac{3}{4x}$

Where *a* and *b* are the coefficients of the terms in x^6 and x^7 respectively.

$(2x+3)^{20} = (2x)^{20} + {}^{20}C_1(2x)^{19}(3) + {}^{20}C_2(2x)^{18}(3)^2 + \ldots + (3)^{20}$

Note that the general term is ${}^{20}C_n(2x)^{20-n}(3)^n$

For the term in x^6, $20 - n = 6$

$n = 14$

${}^{20}C_{14}(2x)^{20-14}(3)^{14} = {}^{20}C_{14}(2x)^6(3)^{14}$

For the term in x^7, $20 - n = 7$

$n = 13$

${}^{20}C_{13}(2x)^{20-13}(3)^{13} = {}^{20}C_{13}(2x)^7(3)^{13}$

Therefore the ratio of the terms is:

$$\frac{{}^{20}C_{14}(2x)^6(3)^{14}}{{}^{20}C_{13}(2x)^7(3)^{13}} = \frac{{}^{20}C_{14}}{{}^{20}C_{13}} \cdot \frac{1}{2x} \cdot 3 = \frac{3}{4x}$$

ii a To determine the first three terms of $(1+2x)^{10}$

$(1+2x)^{10} = 1^{10} + {}^{10}C_1 1^9(2x) + {}^{10}C_2 1^8(2x)^2 + \ldots$

$= 1 + 20x + 180x^2 + \ldots$

Hence, to estimate $(1.01)^{10}$

Let $2x = 0.01$

$x = 0.005$

$[1 + 2(0.005)x]^{10} = 1 + 20(0.005) + 180(0.005)^2 + \ldots$

$= 1 + 0.1 + 0.0045 + \ldots$

$\cong 1.1045$

b) To show $\frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} = \frac{(n+1)!}{(n-r+1)!r!}$

The LCM is $(n-r+1)!\,r!$

Rewriting the terms on the LHS with the LCM as denominator:

$$\frac{n!}{(n-r)!r!} = \frac{(n-r+1)n!}{(n-r+1)(n-r)!r!} = \frac{(n-r+1)n!}{(n-r+1)!r!}$$

$$\frac{n!}{(n-r+1)!(r-1)!} = \frac{r\,n!}{(n-r+1)!r(r-1)!} = \frac{r\,n!}{(n-r+1)!r!}$$

$$\text{LHS} = \frac{(n-r+1)n!}{(n-r+1)!r!} + \frac{r\,n!}{(n-r+1)!r!} = \frac{(n-r+1)n! + r\,n!}{(n-r+1)!r!}$$

$$= \frac{n!(n-r+1+r)}{(n-r+1)!r!}$$

$$= \frac{n!(n+1)}{(n-r+1)!r!}$$

$$= \frac{(n+1)!}{(n-r+1)!r!} = \text{RHS}$$

c) i To show that $f(x) = -x^3 + 3x + 4$ has a root in the interval [1,3].

$f(x)$ is continuous function since all polynomials are continuous.

$f(x) = -x^3 + 3x + 4$

$f(1) = -(1)^3 + 3(1) + 4 = 6$

$f(3) = -(3)^3 + 3(3) + 4 = -14$

Since there is a change in sign and the function is continuous, then there must exist a value $1 < a < 3$ such that $f(a) = 0$

Hence there must be at least one root in the interval [1,3].

ii. By taking $x_1 = 2.1$, to use the Newton Raphson method to obtain a second approximation, x_2 in the interval [1,3].

Note: The Newton-Raphson's formula is:

$x_{n+1} = x_n - \frac{f(x_1)}{f'(x_1)}$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f(x_2)}$

$x_1 = 2.1$

$f(x_1) = -(2.1)^3 + 3(2.1) + 4 = 1.039$

$f'(x) = -3x^2 + 3$

$f'(2.1) = -3(2.1)^2 + 3 = -10.23$

$\therefore x_2 = 2.1 + \frac{1.039}{10.23} = 2.20156 \ldots$

$= 2.20$ to three significant figures

5. Given: 5 teams are to meet at a round table

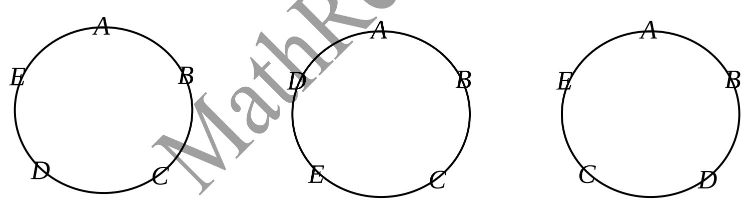
Each team consists of two members and one leader

a) (i) To determine the number of different seating arrangements possible if each team sits together with the leader in the centre

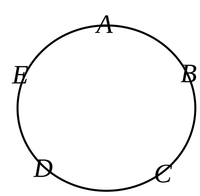
$\frac{5!}{5} \cdot 2^5 = 768$

Explanation

There are two considerations here. The first is to determine the number of ways of arranging 5 objects in a circle. The answer to this is $(5-1)!$ or $\frac{5!}{5}$. Here are three of these 24 possibilities



The second fact to consider is that within each team, there are two possible arrangements. For example for team A the possibilities are $A_1A_cA_2$ and $A_2A_cA_1$. So, for example, for the arrangement shown below



Each team can be arranged in 2 ways. This means that this one basic arrangement can produce

$$2 \times 2 \times 2 \times 2 \times 2 = 32$$

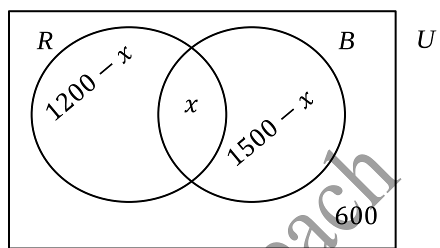
different arrangements. Hence, using the multiplication principle, the total number of arrangements is

$$\frac{5!}{5} \times 32 = 768$$

ii Given Individuals were asked to select one colour, two colours or no colour
600 chose no colour
80% used a colour

a. If 40% used red and 50% used blue
To calculate the probability that an individual used both colours
Note:
80% used colours, hence 20% used no colour. Therefore
$20\% = 600$
$100\% = 3000$
$40\% = 1200$
$50\% = 1500$

Using a Venn diagram



$(1200 - x) + x + (1500 - x) + 600 = 3000$
$3300 - x = 3000$
$x = 300$
Therefore $P(R \cap B) = \frac{300}{3000} = .1$

b To find: $n(U)$
$n(U) = 3000$ see above

b. Given: $A = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{pmatrix}$

i. To determine the range of values of x for which A^{-1} exists
Note: A^{-1} exists so long as $|A| \neq 0$

$$|A| = 1\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$$
$= -2 - x(-4) - 1(3)$
$= -2 + 4x - 3$
$= 4x - 5$
$4x - 5 = 0, \quad x = \frac{5}{4}$
Therefore A^{-1} exists for $\{x : x \epsilon \mathbb{R}, x \neq \frac{5}{4}\}$
Recall, $A^{-1} = \frac{1}{\det A} Adj\ A$; hence if $\det A = 0$, the inverse does not exist.

ii. Given that $|AB| = -21$

To show that $x = 3$

$$AB = \begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2x & 3x+1 & 4x+3 \\ 5 & 8 & 19 \\ 4 & 7 & 14 \end{pmatrix}$$

$$|AB| = 2x\begin{vmatrix} 8 & 19 \\ 7 & 14 \end{vmatrix} - (3x+1)\begin{vmatrix} 5 & 19 \\ 4 & 14 \end{vmatrix} + (4x+3)\begin{vmatrix} 5 & 8 \\ 4 & 7 \end{vmatrix}$$

$$= 2x(-21) - (3x+1)(-6) + (4x+3)(3)$$

$$= -42x + 18x + 6 + 12x + 9$$

$$-12x + 15$$

$\therefore -12x + 15 = -21$

$-12x = -36$

$x = 3$

iii. Hence to obtain A^{-1}

$|A| = 4x - 5$

$= 4(3) - 5 = 7$

Finding the matrix of cofactors

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

$$A_{11} = +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2\,,\ A_{12} = -\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} = 4\,,\ A_{13} = +\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3$$

$$A_{21} = -\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 1\,,\ A_{22} = +\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2\,,\ A_{23} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$$

$$A_{31} = +\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6\,,\ A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5\,,\ A_{33} = +\begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} = -9$$

Co-factor matrix is: $\begin{pmatrix} -2 & 4 & 3 \\ 1 & 2 & 5 \\ 6 & -5 & -9 \end{pmatrix}$

$Adj\ A =$ Transpose of the co-factor matrix $= \begin{pmatrix} -2 & 1 & 6 \\ 4 & 2 & -5 \\ 3 & 5 & -9 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|}Adj\ A = \frac{1}{7}\begin{pmatrix} -2 & 1 & 6 \\ 4 & 2 & -5 \\ 3 & 5 & -9 \end{pmatrix}$$

6 a. i. To show that the general solution of

$y' + y \tan x = \sec x$

is $y = \sin x + C \cos x$

Note: The question is in the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.

$P = \tan x$ and $Q = \sec x$

The integrating factor is: $e^{\int Pdx} = e^{\int \tan x \, dx}$

$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x|$

Therefore the integrating factor is $e^{\ln|\sec x|} = \sec x$

The general solution for all equations of this form is:

$ye^{\int Pdx} = \int Qe^{\int Pdx} \, dx$

$\therefore y \sec x = \int \sec x \cdot \sec x \, dx = \int \sec^2 x \; dx$

$y \sec x = \tan x + c$

$y = \frac{\sin x}{\cos x} \cdot \cos x + c \cos x$

$y = \sin x + c \cos x$

ii. To obtain the particular solution where $y = \frac{2}{\sqrt{2}}$ and $x = \frac{\pi}{4}$

General solution is: $y = \sin x + c \cos x$

Substituting the given values

$\frac{2}{\sqrt{2}} = \sin \frac{\pi}{4} + c \cos \frac{\pi}{4}$

$\frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}} + c \frac{1}{\sqrt{2}}$

Multiplying both sides of the equation by $\sqrt{2}$

$2 = 1 + c$

$c = 1$

Therefore the particular solution is $y = \sin x + \cos x$

b Given: $y'' - 5y' = xe^{5x}$ has a particular solution $y(x) = Ax^2e^{5x} + Bxe^{5x}$

To solve the differential equation

Finding CF

The auxiliary equation is

$m^2 - 5m = 0$

$m(m-5) = 0$

$m = 0, 5$

$\therefore y(x) = C_1e^{0x} + C_2e^{5x} = C_1 + C_2e^{5x}$ - this is the complementary function, *CF*

Finding PI

We are given that a particular solution is: $y = Ax^2e^{5x} + Bxe^{5x}$

So, we only need to find the constants *A* and *B* to get the particular integral. To do this we will find expressions for y' and y'', substitute them in the given differential equation, $y'' - 5y' = xe^{5x}$, and solve for the unknowns *A* and *B*.

<u>Finding y'</u>

$y(x) = Ax^2e^{5x} + Bxe^{5x}$

$y' = (Ax^2)(5)e^{5x} + e^{5x}(2Ax) + (Bx)(5)e^{5x} + e^{5x}(B)$

$\quad = e^{5x}(5Ax^2 + 2Ax + 5Bx + B)$

<u>Finding y''</u>

$y'' = e^{5x}(10Ax + 2A + 5B) + 5e^{5x}(5Ax^2 + 2Ax + 5Bx + B)$

$\quad = e^{5x}(25Ax^2 + 20Ax + 25Bx + 2A + 10B)$

<u>Substituting the expressions for y' and y'' in $y'' - 5y' = xe^{5x}$</u>:

$e^{5x}(25Ax^2 + 20Ax + 25Bx + 2A + 10B) - 5e^{5x}(5Ax^2 + 2Ax + 5Bx + B) = xe^{5x}$

$e^{5x}(25Ax^2 + 20Ax + 25Bx + 2A + 10B - 25Ax^2 - 10Ax - 25Bx - 5B) = xe^{5x}$

$e^{5x}(10Ax + 2A + 5B) = xe^{5x}$

$\therefore 10Axe^{5x} + (2A + 5B)e^{5x} = xe^{5x}$

<u>Equating the coefficients of corresponding terms on both sides of the equation</u>

$10Ax = x$

$10A = 1$

$A = \frac{1}{10}$

$(2A + 5B)e^{5x} = 0e^{5x}$

$2A + 5B = 0$

$B = -\frac{2a}{5} = -\frac{2}{5} \cdot \frac{1}{10} = -\frac{1}{25}$

Therefore *PI* is $y(x) = \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}$

Solution: $\boldsymbol{y = CF + PI}$

Therefore the solution of the equation $y'' - 5y' = xe^{5x}$ is

$y = C_1 + C_2e^{5x} + \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}$

Baby stuff!!!