CAPE Unit 2
Pure Mathematics
June 2013
Solutions

MathReach
Cape Unit 2 – Pure Mathematics
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1. (a) To calculate the gradient of the curve $\ln(x^2y) - \sin y = 3x - 2y$ at the point (1, 0)

Note: $\ln(x^2y) \equiv 2\ln x + \ln y$. Hence, the equation of the curve can be written as

$$2\ln x + \ln y - \sin y = 3x - 2y$$

Differentiating implicitly:

$2 \cdot \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} - \cos y\frac{dy}{dx} = 3 - 2\frac{dy}{dx}$

$\frac{dy}{dx}\left[\frac{1}{y} + 2 - \cos y\right] = 3 - \frac{2}{x}$

$\frac{dy}{dx}\left[\frac{1+2y-y\cos y}{y}\right] = \frac{3x-2}{x}$

$\frac{dy}{dx} = \frac{3x-2}{x} \cdot \frac{y}{1+2y-y\cos y} = \frac{(3x-2)y}{x[1+2y-y\cos y]}$

At the point (1,0)

$\frac{dy}{dx} = \frac{(3-2)(0)}{1(1+0-0)} = \frac{0}{1} = 0$

(b) Let $f(x, y, \mathrm{z}) = 3yz^2 - e^{4x}\cos 4z - 3y^2 - 4 = 0$

Given: $\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$

To determine $\frac{\partial z}{\partial y}$ in terms of x, y and z

$\frac{\partial f}{\partial y} = 3z^2 - 0 - 6y$ $\qquad$ $\frac{\partial f}{\partial z} = 6yz + 4e^{4x}\sin 4z$

$\therefore \frac{\partial z}{\partial y} = \frac{-[3z^2-6y]}{6yz+4e^{4x}\sin 4z}$

(c) To use de Moivre's theorem to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

De Moivre's theorem: $(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta$

$$[r(\cos\theta + i\sin\theta)]^n \equiv r^n(\cos n\theta + i\sin n\theta)$$

Let $n = 5$

$$[\mathrm{r}(\cos\theta + i\sin\theta)]^5 \equiv r^5(\cos 5\theta + i\sin 5\theta)$$

<u>Note:</u> $(a+b)^5 \equiv a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Let $a = \cos\theta$ and $b = i\sin\theta$

$\therefore (\cos\theta + i\sin\theta)^5 \equiv \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta +$
$5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta \equiv \cos 5\theta + i\sin 5\theta$

$\cos^5\theta + i\,5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta +$
$i\sin^5\theta \equiv \cos 5\theta + i\sin 5\theta$

Equating the real parts of both sides of the identity:

$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Substituting $(1-\cos^2\theta)$ for $\sin^2\theta$:

$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$

$\equiv \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(d) (ii) To write the complex number $Z=(-1+i)^7$ in the form $re^{i\theta}$, where $r=|Z|$ and $\theta = \arg Z$.

Writing $(-1+i)$ in exponential form

$|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\arg(-1+i) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$

$\therefore (-1+i) = \sqrt{2}\, e^{i\frac{3\pi}{4}}$

$\therefore (-1+i)^7 = \left(\sqrt{2}\, e^{i\frac{3\pi}{4}}\right)^7$

$= \left(\sqrt{2}\right)^7 e^{i\left(\frac{21\pi}{4}\right)} = \left(\sqrt{2}\right)^7 e^{i\left(-\frac{3\pi}{4}\right)}$

$= \left(\sqrt{2}\right)^7 e^{i\left(-\frac{3\pi}{4}\right)}$

(iii) Hence to prove that $(-1+i)^7 \equiv -8(1+i)$

$(-1+i)^7 \equiv \left(\sqrt{2}\right)^7 e^{i\left(-\frac{3\pi}{4}\right)}$

$\left(\sqrt{2}\right)^7 e^{i\left(-\frac{3\pi}{4}\right)} \equiv \left(\sqrt{2}\right)^7\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

Note: $e^{i\theta} = (\cos\theta + i\sin\theta)$

$\sqrt{2}^7 e^{i\left(-\frac{3\pi}{4}\right)} \equiv \left(\sqrt{2}\right)^6\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right]$

$\equiv 8(-1-i)$

$\equiv -8(1+i)$

2. (a) (i) To determine $\int \sin x\cos 2x\, dx$

Method 1:

$\sin x\cos 2x = \sin x(\cos^2 x - \sin^2 x)$

$= \sin x\,(2\cos^2 x - 1)$

$\therefore \int \sin x\cos 2x\, dx = \int \sin x\,(2\cos^2 x - 1)\, dx$

$\int \sin x\,(2\cos^2 x-1) dx = \int 2\cos^2 x\sin x\, dx - \int \sin x\, dx$

$\int 2\cos^2 x\sin x\, dx - \int \sin x\, dx = -2\int \cos^2 x\,(-\sin x)\, dx - \int \sin x\, dx$

$= -2\cdot\frac{\cos^3 x}{3} + \cos x + \mathrm{C}$

$= \cos x - 2\frac{\cos^3 x}{3} + C$

Method 2: Reversing the product rule

Let $y = \cos 2x\,(-\cos x)$

$\frac{dy}{dx} = \cos 2x \sin x + (-\cos x)(-2\sin 2x)$

Integrating

$-\cos x \cos 2x = \int \sin x \cos 2x\, dx + 2\int \sin 2x \cos x\, dx$

$\therefore \int \sin x \cos 2x\, dx = -\cos x \cos 2x - 2\int \sin 2x \cos x\, dx$ …..(1)

Integrating further:

$\int \sin 2x \cos x\, dx$

Let $y = \sin x \sin 2x$

$\frac{dy}{dx} = \sin 2x \cos x + \sin x(2\cos 2x)$

Integrating:

$\sin x \sin 2x = \int \sin 2x \cos x\, dx + 2\int \sin x \cos 2x\, dx$

$\int \sin 2x \cos x\, dx = \sin x \sin 2x - 2\int \sin x \cos 2x\, dx$ …..(2)

Substituting in (1):

$\int \sin x \cos 2x\, dx = -\cos x \cos 2x - 2\left(\sin x \sin 2x - 2\int \sin x \cos 2x\, dx\right)$

$\int \sin x \cos 2x\, dx = -\cos x \cos 2x - 2\sin x \sin 2x + 4\int \sin x \cos 2x\, dx$

$\therefore 3\int \sin x \cos 2x\, dx = \cos x \cos 2x + 2\sin x \sin 2x$

$\int \sin x \cos 2x\, dx = \frac{1}{3}\cos x \cos 2x + \frac{2}{3}\sin x \sin 2x + C$

Method 3: Integration by parts using the formula $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = \sin x$ and let $\frac{dv}{dx} = \cos 2x$

$\frac{du}{dx} = \cos x$ and $v = \frac{1}{2}\sin 2x$

$\therefore \int \sin x \cos 2x\, dx = \frac{1}{2}\sin x \sin 2x - \int \frac{1}{2}\cos x \sin 2x\, dx$

$\int \sin x \cos 2x\, dx = \frac{1}{2}\sin x \sin 2x - \frac{1}{2}\int \cos x \sin 2x\, dx$ ……...(1)

Integrating further:

$\int \cos x \sin 2x\, dx$

Let $u = \cos x$ and let $\frac{dv}{dx} = \sin 2x$

$\frac{du}{dx} = -\sin x$ $v = -\frac{1}{2}\cos 2x$

$\therefore \int \cos x \sin 2x\, dx = -\frac{1}{2}\cos x \cos 2x - \int \frac{1}{2}\sin x \cos 2x\, dx$ …..(2)

Substituting in (1):

$\therefore \int \sin x \cos 2x\, dx \equiv \frac{1}{2}\sin x \sin 2x - \frac{1}{2}\left[-\frac{1}{2}\cos x \cos 2x - \frac{1}{2}\int \sin x \cos 2x\, dx\right]$

$\int \sin x \cos 2x\, dx \equiv \frac{1}{2}\sin x \sin 2x + \frac{1}{4}\cos x \cos 2x + \frac{1}{4}\int \sin x \cos 2x\, dx$

$$\frac{3}{4}\int \sin x \cos 2x \, dx \equiv \frac{1}{2}\sin x \sin 2x + \frac{1}{4}\cos x \cos 2x$$

$$\int \sin x \cos 2x \, dx = \frac{2}{3}\sin x \sin 2x + \frac{1}{3}\cos x \cos 2x + c$$

Method 4: Using trig identities in sin x only

$\int \sin x \cos 2x \, dx$

Note: $\cos 2x = \cos^2 x - \sin^2 x$

$\qquad = 1 - 2\sin^2 x$

$\therefore \int \sin x \cos 2x \, dx \ \equiv \int \sin x(1 - 2\sin^2 x)\, dx$

$\equiv \int (\sin x - 2\sin^3 x)\, dx$

$\equiv \int \sin x \, dx - 2\int \sin^3 x \, dx$

Note: $\sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$\therefore -2\int \sin^3 x \, dx \equiv -2\int (\frac{3}{4}\sin x - \frac{1}{4}\sin 3x) dx$

$\qquad = -2\left[\frac{3}{4}(-\cos x) + \frac{1}{4}\cdot\frac{1}{3}\cdot\cos 3x\right]$

$\qquad = +\frac{3}{2}\cos x - \frac{1}{6}\cos 3x$

$\therefore \int \sin x \cos 2x \, dx \ \equiv \int \sin x \, dx - 2\int \sin^3 x \, dx$

$\qquad \equiv \ -\cos x + \frac{3}{2}\cos x - \frac{1}{6}\cos 3x + c$

$\qquad = \frac{1}{2}\cos x - \frac{1}{6}\cos 3x + c$

(You can use trigonometric identities to show that all four answers above are the same.

(ii) Hence to calculate $\int_0^{\frac{\pi}{2}} \sin x \cos 2x \, dx$

From Method 3:

$\int_0^{\frac{\pi}{2}} \sin x \cos 2x \, dx = \left[\frac{2}{3}\sin x \sin 2x + \frac{1}{3}\cos x \cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{2}{3}\sin\frac{\pi}{2}\sin \pi + \frac{1}{3}\cos\frac{\pi}{2}\cos \pi\right] - \left[\frac{2}{3}\sin 0 \sin 0 + \frac{1}{3}\cos 0 \cos 0\right]$

$= \left[\frac{2}{3}(1)(0) + \frac{1}{3}(0)(-1)\right] - \left[\frac{2}{3}(0)(0) + \frac{1}{3}(1)(1)\right]$

$= 0 - \frac{1}{3}$

$= -\frac{1}{3}$

From Method 4:

$\int_0^{\frac{\pi}{2}} \sin x \cos 2x \, dx = \left[\frac{1}{2}\cos x - \frac{1}{6}\cos 3x\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{1}{2}\cos\frac{\pi}{2} - \frac{1}{6}\cos\frac{3\pi}{2}\right] - \left[\frac{1}{2}\cos 0 - \frac{1}{6}\cos 0\right]$

$= \left[\frac{1}{2}(0) - \frac{1}{6}(0)\right] - \left[\frac{1}{2}(1) - \frac{1}{6}(1)\right]$

$= 0 - \frac{1}{2} + \frac{1}{6} = -\frac{2}{6} = -\frac{1}{3}$

$= -\frac{1}{3}$

(b) Let $f(x) = x|x| = \begin{cases} x^2; x \geq 0 \\ -x^2; x < 0 \end{cases}$

To use the trapezium rule with four intervals to calculate the area between $f(x)$ and the x-axis for the domain $0.75 \leq x \leq 2.25$

Treating each of the four intervals as a trapezium, the required area is:

$A = \frac{1}{2}\,[\,f(-0.75) + f(0)]0.75 + \frac{1}{2}[\,f(0) + f\,(0.75)]0.75$

$\quad + \frac{1}{2}[f(0.75) + f\,(1.50)]0.75 + \frac{1}{2}[\,f(1.50) + f(\,2.25)]0.75$

$= \frac{1}{2}(\,0.75)\,[\,f(-0.75) + f(0) + f(0) + f(0.75) + f(0.75) + f(1.50) + f(1.50) + f(2.25)]$

$= \frac{1}{2}(0.75)[\,2\,f(0) + 3\,f(0.75) + 2\,f(1.5) + f(2.25)]$

Note $f\,(-0.75) = f(0.75)$

$f(0) = 0$

$f(0.75) = (0.75)^2 = 0.5625$

$f(1.5) = (1.5)^2 = 2.25$

$f(2.25) = (2.25)^2 = 5.0625$

$f(-0.75) = (-0.75)^2 = 0.5625$

$\therefore A = \frac{1}{2}(0.75)[\,2(0) + 3\,(0.5625) + 2(2.25) + 5.0625]$

$A = \frac{1}{2}(0.75)[0 + 1.6875 + 4.50 + 5.0625]$

$A = \frac{1}{2}(0.75)(11.25)$

$A = 4.21875$ unit^2

$A = 4.22$ unit^2 to 3 sig. fig.

(c) (ii) To show $\frac{2x^2+4}{(x^2+4)^2} = \frac{2}{x^2+4} - \frac{4}{(x^2+4)^2}$

$\frac{2x^2+4}{(x^2+4)^2} = \frac{A}{x^2+4} + \frac{B}{(x^2+4)^2}$

Note: From what we are asked to show we know that there will be constants only in the numerator.

$\therefore\ 2x^2 + 4 \equiv A(x^2 + 4) + B$

$2x^2 + 4 \equiv Ax^2 + 4A + B$

Equating the coefficients of corresponding terms on both sides of the equation:

$2 = A$

$4 = 4A + B$

Solving simultaneously:

$4 = 8 + B$

$B = -4$

Or

$$\frac{2x^2+4}{(x^2+4)^2} \equiv \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$2x^2 + 4 \equiv (Ax + B)(x^2 + 4) + Cx + D$$

$$2x^2 + 4 \equiv Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$$2A = 0$$

$$2 = B$$

$$0 + 4A = C$$

$$0 + 0 = C$$

$$4 = 4B + D$$

$$4 = 8 + D$$

$$D = -4$$

$$\therefore \frac{2x^2+4}{(x^2+4)^2} \equiv \frac{2}{x^2+4} - \frac{4}{(x^2+4)^2}$$

(iii) Hence, to find $\int \frac{2x^2+4}{(x^2+4)^2} dx$ using the substitution: $x = 2\tan\theta$

$$\frac{2x^2+4}{(x^2+4)^2} = \frac{2}{x^2+4} - \frac{4}{(x^2+4)^2}$$

$$\int \frac{2x^2+4}{(x^2+4)^2} dx = \int \frac{2}{x^2+4} dx - \int \frac{4}{(x^2+4)^2} dx$$

Let $x = 2\tan\theta$

$$dx = 2\sec^2\theta \, d\theta$$

$$\therefore \int \frac{2}{x^2+4} dx \equiv \int \frac{2}{4\tan^2\theta+4} \cdot 2\sec^2\theta \, d\theta$$

$$\int \frac{2}{x^2+4} dx = \frac{4}{4}\int \frac{1}{\tan^2\theta+1} \cdot \sec^2\theta \, d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int d\theta = \theta$$

$$= \tan^{-1}\frac{x}{2}$$

$$\int \frac{4}{(x^2+4)^2} dx = \int \frac{4}{(4\tan^2\theta+4)^2} \cdot 2\sec^2\theta \, d\theta$$

$$= \int \frac{8\sec^2\theta}{4^2(\tan^2\theta+1)^2} \, d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{sec^4\theta} d\theta$$

$$= \frac{1}{2}\int \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]$$

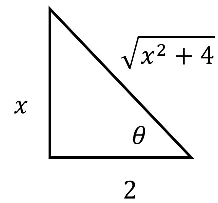
$$\therefore \int \frac{4}{(x^2+4)^2} \, dx = \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta$$

Note: $\theta = \tan^{-1}\frac{x}{2}$

$$\therefore \int \frac{2}{x^2+4}dx - \int \frac{4}{(x^2+4)^2}dx = \tan^{-1}\frac{x}{2} - \frac{1}{4}\tan^{-1}\frac{x}{2} - \frac{1}{8}\sin 2\theta + c$$

$$\int \frac{2}{x^2+4}dx - \int \frac{4}{(x^2+4)^2}dx = \frac{3}{4}\tan^{-1}\frac{x}{2} - \frac{1}{8}\sin 2\theta + c$$

Note: since $x = 2\tan\theta$, then $\tan\theta = \frac{x}{2}$, so:



$$\therefore \sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$\cos\theta = \frac{2}{\sqrt{x^2+4}}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\therefore \sin 2\theta = \frac{4x}{x^2+4}$$

Substituting in the above equation:

$$\therefore \int \frac{2x^2+4}{(x^2+4)^2}dx = \frac{3}{4}\tan^{-1}\frac{x}{2} - \frac{1}{8}\cdot\frac{4x}{x^2+4} + c$$

$$= \frac{3}{4}\tan^{-1}\frac{x}{2} - \frac{x}{2(x^2+4)} + c$$

3. (a) Given: $\{a_n\}$ defined by $a_1 = 1,\ a_{n+1} = 4 + 2\sqrt[3]{a_n}$

To use math induction to prove that $1 \le a_n \le 8$ for all $n \epsilon \mathbb{Z}^+$

Let $P(n)$ be the proposition that given $\{a_n\}$ with $a_1 = 1,\ a_n = 4 + 2\sqrt[3]{a_n},$ then $1 \le a_n \le 8$ for all $n \in \mathbb{Z}^+$

Testing $P(1)$:

$a_1 = 1$ (given)

$1 \le 1 \le 8$

$\therefore P(1)$ is true

Assume $P(k)$ is true: That is, assume

$1 \le a_k \le 8$ for all $k \epsilon \mathbb{Z}^+$………(1)

Show $P(k) \Longrightarrow P(k+1)$

$P(k+1)$: $1 \le a_{k+1} \le 8$ for all $k \epsilon \mathbb{Z}^+$

$a_{k+1} = 4 + 2\sqrt[3]{a_k}$ (Given)

So we now need to show that

$1 \le 4 + 2\sqrt[3]{a_k} \le 8$ …..(2)

The minimum value of a_{k+1} occurs when $a_k = 1$. This is

$a_{k+1} = 4 + 2\sqrt[3]{1} = 6$

The maximum value of a_{k+1} occurs when $a_k = 8$. This is

$a_{k+1} = 4 + 2\sqrt[3]{8} = 8$

$\therefore 1 \le a_{k+1} \le 8$

Alternate methoid

Subtracting 4 from each part of the inequality:

$-3 \le 2\sqrt[3]{a_k} \le 4$

Dividing each part of the inequality by 2:

$-\frac{3}{2} \le \sqrt[3]{a_k} \le 2$ ….. (3)

So this is what we now need to prove.

But it has been assumed that $1 \le a_k \le 8$ ….. (1)

Substituting the smallest and largest value from (1) in (3):

Substituting 1 for a_k:

$\frac{-3}{2} \le \sqrt[3]{1} \le 2$

$\frac{-3}{2} \le 1 \le 2$ ….which is valid (true)

Substituting 8 for a_k

$\frac{-3}{2} \leq \sqrt[3]{8} \leq 2$

$\frac{-3}{2} \leq 2 \leq 2$ ….which is valid

$\therefore -\frac{3}{2} \leq \sqrt[3]{a_k} \leq 2$ for all $k \epsilon \mathbb{Z}^+$

Hence $P(k+1)$ is valid.

Thus $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$, then $P(n)$ is true. for all $k \epsilon \mathbb{Z}^+$

(b) Given $k > 0$ and $f(k) = \frac{1}{k^2}$

(ii) To show

a) $f(k) - f(k+1) = \frac{2k+1}{k^2(k+1)^2}$

$f(k) = \frac{1}{k^2}; f(k+1) = \frac{1}{(k+1)^2}$

$\therefore f(k) - f(k+1) = \frac{1}{k^2} - \frac{1}{(k+1)^2}$

$= \frac{(k+1)^2 - k^2}{k^2(k+1)^2}$

$\equiv \frac{k^2+2k+1-k^2}{k^2(k+1)^2}$

$\equiv \frac{2k+1}{k^2(k+1)^2}$

b) $\sum_{k=1}^{n}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = 1 - \frac{1}{(n+1)^2}$

$\sum_{k=1}^{n}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \left[\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right] + \left[\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right] + \left[\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right] + \cdots + \left[\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right]$

$= \frac{1}{1} - \frac{1}{(n+1)^2}$

$= 1 - \frac{1}{(n+1)^2}$

(iii) Hence to prove that

$\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} = 1$

$\frac{2k+1}{k^2(k+1)^2} \equiv \frac{1}{k^2} - \frac{1}{(k+1)^2}$ (already shown)

$\therefore \sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} \equiv \sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$

$\sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \lim_{k\to\infty} \sum_{k=1}^{n}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$

$\sum_{k=1}^{n}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = 1 - \frac{1}{(n+1)^2}$

$\therefore \sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \lim_{n\to\infty} 1 - \frac{1}{(n+1)^2} = 1$

(c) To obtain the first four non-zero terms of the Taylor series expansion of $\cos x$ in ascending powers of $(x - \frac{\pi}{4})$

Note: Taylor series expansion

$$f(x) = f(a) + \frac{[f'(a)](x-a)}{1!} + \frac{[f''(a)](x-a)^2}{2!} + \frac{[f'''(a)](x-a)^3}{3!} + \cdots + \frac{[f^n(a)](x-a)^n}{n!} + \cdots$$

Given $f(x) = \cos x\,;\, a = \frac{\pi}{4}$

$f'(x) = -\sin x \qquad \therefore f'(a) = -\sin\frac{\pi}{4}$

$f''(x) = -\cos x \qquad \therefore f''(a) = -\cos\frac{\pi}{4}$

$f'''(x) = \sin x \qquad \therefore f'''(a) = \sin\frac{\pi}{4}$

$f''''(x) = \cos x \qquad \therefore f''''(a) = \cos\frac{\pi}{4}$

$$\therefore \cos x = \cos\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)(-\sin x) + \frac{\left(x-\frac{\pi}{4}\right)^2(-\cos\frac{\pi}{4})}{2!} + \frac{\left(x-\frac{\pi}{4}\right)^3(\sin\frac{\pi}{4})}{3!} + \cdots$$

$$= \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\left(x - \frac{\pi}{4}\right) + \left(-\frac{1}{2\sqrt{2}}\right)\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \cdots$$

(iii) Hence to calculate an approximation of $\cos\left(\frac{\pi}{16}\right)$

$$\therefore \cos\frac{\pi}{16} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{16} - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(\frac{\pi}{16} - \frac{\pi}{4}\right)^2 + \frac{1}{6\sqrt{2}}\left(\frac{\pi}{16} - \frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(-\frac{3\pi}{16}\right) - \frac{1}{2}\left(-\frac{3\pi}{16}\right)^2 + \frac{1}{6}\left(-\frac{3\pi}{16}\right)^3 + \cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{3\pi}{16} - \frac{9\pi^2}{512} - \frac{27\pi^3}{24576} + \cdots\right]$$

$$= \frac{1}{\sqrt{2}}[1.3814949697\ \ldots]$$

$$= 0.976864\ \ldots$$

$$= 0.977 \text{ to 3 d.p.}$$

4. (a) (ii) To obtain the binomial expansion of $\sqrt[4]{1+x}+\sqrt[4]{1-x}$ up to the term in x^2

$$(1+x)^{\frac{1}{4}} \equiv 1+\frac{1}{4}(1)(x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}x^2+\cdots$$

$$=1+\frac{1}{4}x-\frac{3}{32}x^2+\cdots$$

$$(1-x)^{\frac{1}{4}} \equiv 1+\frac{1}{4}(1)(-x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(-x)^2+\cdots$$

$$\equiv 1-\frac{1}{4}x-\frac{3}{32}x^2+\cdots$$

$$\therefore (1+x)^{\frac{1}{4}}+(1-x)^{\frac{1}{4}}=2-2\left(\frac{3}{32}\right)x^2=2-\frac{3}{16}x^2+\cdots$$

(iii) Hence, by letting $x=\frac{1}{16}$, to compute an approximation of $\sqrt[4]{17}+\sqrt[4]{15}$ to four decimal places

$$\sqrt[4]{1+\frac{1}{16}}+\sqrt[4]{1-\frac{1}{16}}=2-\frac{3}{16}\left(\frac{1}{16}\right)^2$$

$$\sqrt[4]{\frac{17}{16}}+\sqrt[4]{\frac{15}{16}}=2-\frac{3}{16}\left(\frac{1}{16}\right)^2$$

$$\frac{\sqrt[4]{17}}{\sqrt[4]{16}}+\frac{\sqrt[4]{15}}{\sqrt[4]{16}}=2-\frac{3}{16}\cdot\left(\frac{1}{16}\right)^2$$

$$\frac{1}{2}\left(\sqrt[4]{17}+\sqrt[4]{15}\right)=2-\frac{3}{16}\left(\frac{1}{16}\right)^2$$

$$\sqrt[4]{17}+\sqrt[4]{15}=4-\frac{3}{8}\left(\frac{1}{16}\right)^2=3.998535156\ \ldots$$

$= 3.9985$ to 4 d.p.

(b) To show that the coefficient of the x^5 term in the product $(x+2)^5(x-2)^4$ is 96

$$(x+2)^5 \equiv x^5+5x^4(2)+10x^3(2)^2+10x^2(2)^3+5x(2)^4+(2)^5$$

$$\equiv x^5+10x^4+40x^3+80x^2+80x+32 \quad \ldots..(1)$$

$$(x-2)^4 \equiv x^4+4x^3(-2)+6x^2(-2)^2+4x\,(-2)^3+(-2)^4$$

$$\equiv x^4-8x^3+24x^2-32x+16 \quad \ldots..(2)$$

Multiplying the terms in (1) by the terms in (2) that will produce a term in x^5:

$x^5(16)+10x^4(-32x)+40x^3(24x^2)+80x^2(-8x^3)+80x(x^4)$

$=x^5[\,16-320+960-640+80]$

$=x^5(96)$

$\therefore$ the coefficient of the x^5 term is 96

(c) (i) To use the intermediate value theorem to prove that $x^3=25$ has at least one root in the interval [2,3].

Let $f(x)=x^3-25$

1. $f(x)=x^3-25$ is a continuous function since it is a polynomial and all polynomials are continuous functions.

2. $f(2) = 8 - 25 = -17$

$f(3) = 27 - 25 = 2$

Since there is a sign change from $f(2)$ to $f(3)$ and $f(x)$ is continuous, then there must exist a value "a" such that $f(a) = 0$, which means that "a" would be a root to the equation:

$x^3 - 25 = 0$ or

$x^3 = 25$

(ii) Given the table below up to the row $n = 4$, which shows the result of four iterations in the estimation of the root of $f(x) = x^3 - 25 = 0$ using interval bisection.

P_n is the estimation of the root for the n[th] iteration.

To complete the table to obtain an approximation of the root of $x^3 = 25$ correct to 2 decimal places.

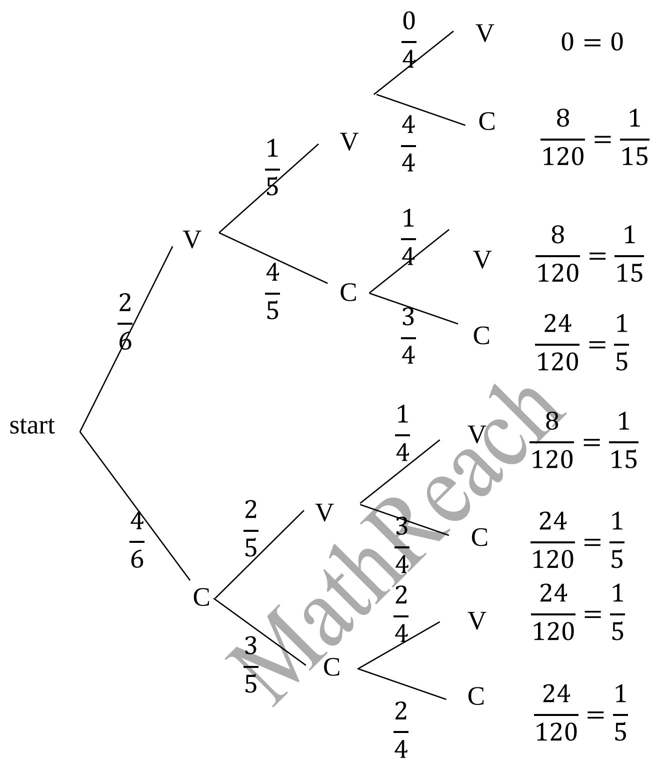
n	a_n	b_n	P_n	$f(P_n)$
1	$2 -$	$3 +$	$2.5 -$	-9.375
2	$2.5 -$	$3 +$	$2.75 -$	-4.2031
3	$2.75 -$	$3 +$	$2.875 -$	-1.2363
4	$2.875 -$	$3 +$	$2.9375 +$	0.3474
5	$2.875 -$	$2.9375 +$	$2.90625 -$	-0.45297
6	$2.90625 -$	$2.9375 +$	$2.921875 -$	-0.054920
7	$2.921875 -$	$2.9375 +$	$2.9296875 +$	0.1457095
8	$2.921875 -$	$2.9296875 +$	$2.92578125 +$	0.0452607
9	$2.921875 -$	$2.92578125 +$	$2.923828125 -$	-0.00486
10	$2.923828125 -$	$2.92578125 +$	$2.924804688 +$	0.02019

$\therefore$ The root lies between 2.9238 ... and 2.9248 ...

Hence, to two decimal places, the root is 2.92.

5. (a) Given: Three letters from BRIDGE are selected without replacement (one at a time). When a letter is selected it is classified as a vowel V or a consonant C.

To use a tree diagram to show the possible outcomes (vowels or consonants) of the three selections.



(b) (ii) Given: augmented matrix of a system of equations in x, y and z

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -5 & 1 & 1 & 2 \\ 1 & -5 & 3 & 3 \end{array}\right)$$

To determine if the system is consistent

Note: The system is consistent if it has at least one solution

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -5 & 1 & 1 & 2 \\ 1 & -5 & 3 & 3 \end{array}\right)$$

$5\,R1 + R2 \rightarrow \text{R2}$

$R1 - R3 \rightarrow \text{R3}$

$R2$: $\quad 5 \;\; 5 \; -5 \;\; 5$

$\underline{+(-5 \;\; 1 \;\;\; 1 \;\;\; 2)}$

$0 \;\; 6 - 4 \;\; 7$

$R3$: $\quad 1 \;\;\; 1 - 1 \;\;\; 1$

$\underline{-(1 \; -5 \;\;\; 3 \;\;\; 3)}$

$0 \;\;\; 6 \; -4 \; -2$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 6 & -4 & 7 \\ 0 & 6 & -4 & -2 \end{array}\right)$$

$R2 \rightarrow R2 \div 6$

$R3 \rightarrow R2 - R3$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{6} \\ 0 & 0 & 0 & 9 \end{array}\right)$$

At this point, it can be concluded that the system is inconsistent as the last row, R3 is showing that $0x + 0y + 0z = 9$ which is a contradiction. Hence, the system is inconsistent system has no solution.

(iii) To create a new system- augmented matrix by replacing R3 in (ii) with $1 \;\; -5 \;\; 5 \;\; | \; 3$

To determine whether the solution of the new system is unique

Note: The system is unique if it can be reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

Where $a, b, c \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -5 & 1 & 1 & 2 \\ 1 & -5 & 5 & 3 \end{array}\right)$$

$R2 \to 5R1 + R2$

$R3 \to R1 - R3$

$$\begin{array}{llllll} R2: & 5 & 5 & -5 & 5 \\ & \underline{+(-5} & \underline{1} & \underline{1} & \underline{2)} \\ & 0 & 6 & -4 & 7 \end{array}$$

$$\begin{array}{llllll} R3: & 1 & 1 & -1 & 1 \\ & \underline{-(1} & \underline{-5} & \underline{5} & \underline{3)} \\ & 0 & 6 & -6 & -2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 6 & -4 & 7 \\ 0 & 6 & -6 & -2 \end{array}\right)$$

$R2 \to \frac{1}{6}R2$

$R3 \to \frac{1}{6}R3$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{6} \\ 0 & 1 & -1 & -\frac{1}{3} \end{array}\right)$$

$R3 \to R2 - R3$

$R1 \to R1 - R2$

$$\begin{array}{llllll} R3: & 0 & 1 & -\frac{2}{3} & \frac{7}{6} \\ & -\Big(0 & 1 & -1 & -\frac{1}{3}\Big) \\ \hline & 0 & 0 & \frac{1}{3} & \frac{9}{6} \end{array}$$

$$\begin{array}{llllll} R1: & 1 & 1 & -1 & 1 \\ & -\Big(0 & 1 & -\frac{2}{3} & \frac{7}{6}\Big) \\ \hline & 1 & 0 & -\frac{1}{3} & -\frac{1}{6} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 1 & -\frac{2}{3} & \frac{7}{6} \\ 0 & 0 & \frac{1}{3} & \frac{9}{6} \end{array}\right)$$

$R1 = R1 + R3$

$R2 = R2 + 2\,R3$

$R3 = 3\,R3$

$$R1:\ 1\ \ 0\ -\tfrac{1}{3}\ \ -\tfrac{1}{6}$$
$$+\left(0\ \ 0\ \ \tfrac{1}{3}\ \ \ \tfrac{9}{6}\right)$$
$$1\ \ 0\ \ 0\ \ \ \tfrac{8}{6}$$

$$R2:\ 0\ \ 1\ -\tfrac{2}{3}\ \ \tfrac{7}{6}$$
$$+\left(0\ \ 0\ \ \ \tfrac{2}{3}\ \ \tfrac{18}{6}\right)$$
$$0\ \ 1\ \ \ 0\ \ \tfrac{25}{6}$$

$$R3:\ 0\ \ 0\ \ 1\ \ \tfrac{9}{2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{25}{6} \\ 0 & 0 & 1 & \frac{9}{2} \end{array}\right)$$

Hence, the new system has a unique solution.

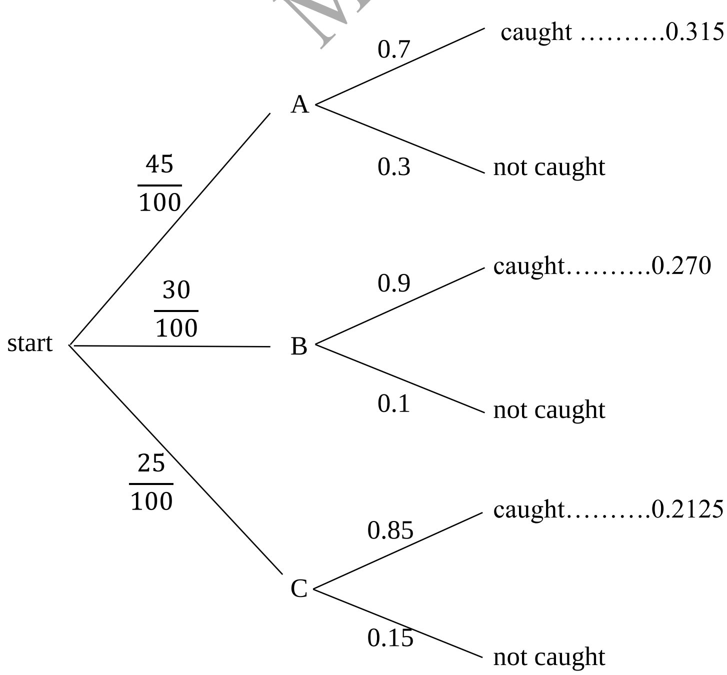
(c) Given: Country *x* has three ports A, B, C

% of travelers using these ports are A- 45%, B- 30%, C- 25%

A traveler has a weapon in his/ her possession, the probability of being caught is 0.7 at A, 0.9 at B, 0.85 at C.

Let the event that the traveler is caught be D, and the event that A, B, C be used be denoted by A, B, C respectively.

(i) To determine the probability that a person using an airport in country *x* is caught



$P(caught) = 0.315 + 0.270 + 0.2125 \ \ = 0.7975$

(ii) To determine the probability that the traveler used airport C given that he/ she has been caught: $P(C|\text{caught}) = \frac{P\ (C \cap \text{caught})}{P(\text{caught})}$

$$= \frac{0.2125}{0.7975} = 0.26645\ ...$$

$$= 0.266 \text{ to 3 sig. fig.}$$

6. (a) (i) $\cos x \frac{dy}{dx} + y \sin x = 2x \cos^2 x$

$\frac{dy}{dx} + \text{y} \tan x = 2x \cos x$

$\therefore$ I. F. $= e^{\int \tan x\ \text{dx}} = e^{\ln|\sec x|} = \sec x$

$P = \sin x, \ \ Q = 2x \cos^2 x$

Note: $\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$

$$= -\int \frac{-\sin x}{\cos x}\ dx = \ -\ln|\cos x|$$

$$= \ln|\cos x|^{-1}$$

$$= \ln|\sec x|$$

Applying the solution: $ye^{\int Pdx} = \int e^{\int Pdx} Qdx$

$$y \sec x = \ \int \sec x \cdot 2x \cos x\ dx$$

$$y \sec x = \ \int 2x\ dx$$

$$= x^2 + c$$

$$y \sec x = x^2 + c$$

$$y = \frac{x^2}{\sec x} + \frac{c}{\sec x}$$

$$y = x^2 \cos x + c \cos x$$

(ii) Given further $y = \frac{15\sqrt{2}\ \pi^2}{32}$ when $x = \frac{\pi}{4}$

To determine 'c' above

$$\frac{15\sqrt{2}\pi^2}{32} = \frac{\pi^2}{16} \cos \frac{\pi}{4} + c \cos \frac{\pi}{4}$$

$$15\ \sqrt{2}\pi^2\ = 2\pi^2 \cdot \frac{1}{\sqrt{2}} + 32\text{c}\ \frac{1}{\sqrt{2}}$$

$$30\pi^2\ = 2\pi^2\ + 32\ c$$

$$\frac{28\pi^2}{32} = c$$

$$c = \ \frac{7\pi^2}{8}$$

(b) (i) Given $y'' + 2y' + 5y = 4\sin 2t$

a) To calculate the roots of the auxiliary equation

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2\pm\sqrt{4-4(1)(5)}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = -1 \pm 2i$

b) Hence, to obtain the complementary function

$y = e^{-1t}[c_1 \cos 2t + c_2 \sin 2t]$

$= c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

(ii) Given: The form of the PI is $n_P(t) = A\cos 2t + B\sin 2t$

To show $A = -\frac{16}{17}$ and $B = \frac{4}{17}$

Let $y = A\cos 2t + B\sin 2t$

$y' = -2A\sin 2t + 2B\cos 2t$

$y'' = -4A\cos 2t - 4B\sin 2t$

Substituting in $y'' + 2y' + 5y = 4\sin 2t$:

$\therefore -4A\cos 2t - 4B\sin 2t + 2[-2A\sin 2t + 2B\cos 2t] + 5[A\cos 2t + B\sin 2t] = 4\sin\ 2t$

$[-4A\cos 2t + 4B\cos 2t + 5A\cos 2t] + [-4B\sin 2t - 4A\sin 2t + 5B\sin 2t] = 4\sin 2t$

$(A + 4B)\cos 2t + (-4A + B)\sin 2t = 4\sin 2t$

Equating coefficients of corresponding terms on both sides of the equation

$\therefore A + 4B = 0 \quad \ldots\ldots(1)$

$-4A + B = 4 \quad \ldots\ldots(2)$

From (1): $A = -4B$

Substituting in (2):

$-4(-4B) + B = 4$

$17B = 4$

$B = \frac{4}{17}$

Substituting in (1):

$A + 4\left(\frac{4}{17}\right) = 0$

$A = \frac{-16}{17}$

(iii) Given further $y(0) = 0.04 \qquad y'(0) = 0$

To obtain the general solution of the differential equation

$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + A\cos 2t + B\sin 2t$

$\therefore y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t - \frac{16}{17}\cos 2t + \frac{4}{17}\sin 2t$

Substituting 0.04 for y and 0 for t

$0.04 = c_1(1)(1) + c_2(1)(0) - \frac{16}{17}(1) + \frac{4}{17}(0)$

$0.04 = c_1 - \frac{16}{17}$

$c_1 = 0.04 + \frac{16}{17} = 0.98117\ldots$

$y = 0.981e^{-t}\cos 2t + c_2 e^{-t}\sin 2t - \frac{16}{17}\cos 2t + \frac{4}{17}\sin 2t$

$\therefore\ y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t - \frac{16}{17}\cos 2t + \frac{4}{17}\sin 2t$

$y' = c_1 e^{-t}(-2\sin 2t) + \cos 2t\left(-c_1 e^{-t}\right) + c_2 e^{-t}(2\cos 2t) + \sin 2t\left(-c_2 e^{-t}\right) + \dfrac{32}{17}\sin 2t + \dfrac{8}{17}\cos 2t$

$0 = c_1(0) + (1)(-c_1)(1) + c_2(1)(2) + 0(-c_2)(1) + \frac{32}{17}(0) + \frac{8}{17}(1)$

$0 = 0 + 1(-0.98117) + 2c_2 + \frac{8}{17}$

$2c_2 = 0.98117 - \frac{8}{17} = 0.5105817 \ldots.$

$c_2 = 0.25529 \ldots$

$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + A\cos 2t + B\sin 2t$

$\therefore\ y = 0.981e^{-t}\cos 2t + 0.255e^{-t}\sin 2t - \frac{16}{17}\cos 2t + \frac{4}{17}\sin 2t$