CAPE Unit 2
Pure Mathematics
June 2015
Solutions

1. (a) Given: $z_1 = 1 + (7 - 4\sqrt{3})i$

$z_2 = \sqrt{3} + 3i$

$z_3 = -2 + 2i$

(i) To express $\frac{z_3}{z_2}$ in the form $x + iy$

$$\frac{-2+2i}{\sqrt{3}+3i} = \frac{-2+2i}{\sqrt{3}+3i} \cdot \frac{\sqrt{3}-3i}{\sqrt{3}-3i}$$
$$= \frac{-2\sqrt{3}+6i+2\sqrt{3}i-6i^2}{3-9i^2}$$
$$= \frac{-2\sqrt{3}+6+(6+2\sqrt{3})i}{3+9}$$
$$= \frac{-2\sqrt{3}+6}{12} + \frac{6+2\sqrt{3}}{12}i$$
$$= \frac{3-\sqrt{3}}{6} + \frac{3+\sqrt{3}}{6}i$$

(ii) Given: $\arg w = \arg z_3 - [\arg z_1 + \arg z_2]$; $|z_1| = 1$, $\arg z_1 = \frac{\pi}{12}$

To rewrite $w = \frac{z_3}{z_1 z_2}$ in the form $re^{i\theta}$ where $r = |w|$ and $\theta = w$

$$\arg z_3 = \tan^{-1}\left(-\frac{2}{2}\right) = \frac{3\pi}{4}$$
$$\arg z_2 = \tan^{-1}\left(\frac{3}{\sqrt{3}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{3}}{i}\right) = \frac{\pi}{3}$$
$$\arg w = \arg z_3 - [\arg z_1 + \arg z_2]$$
$$\arg w = \frac{3\pi}{4} - \left[\frac{\pi}{12} + \frac{\pi}{3}\right]$$
$$= \frac{9\pi}{12} - \left[\frac{\pi}{12} + \frac{4\pi}{12}\right]$$
$$= \frac{4\pi}{12}$$
$$= \frac{\pi}{3}$$

$$w = \frac{z_3}{z_1 z_2}$$

$\lvert z_3\rvert = \sqrt{4+4} = \sqrt{8}$	$\lvert z_2\rvert = \sqrt{3+9} = \sqrt{12}$	$\lvert z_1\rvert = 1$
$= 2\sqrt{2}$	$= 2\sqrt{3}$	$z_1 = re^{i\theta}$
$z_3 = re^{i\theta}$	$z_2 = re^{i\theta}$	$= e^{i\frac{\pi}{12}}$
$= 2\sqrt{2}\, e^{i\frac{3\pi}{4}}$	$= 2\sqrt{3}e^{i\frac{\pi}{3}}$	

$$|w| = \frac{2\sqrt{2}}{2\sqrt{3}\cdot 1} = \sqrt{\frac{2}{3}}$$
$$\arg w = \frac{\pi}{3}$$
$$\therefore w = \sqrt{\frac{2}{3}}e^{i\frac{\pi}{3}}$$

(b) Given: $v = x + iy$

$v^2 = 2 + i$

To show: $x^2 = \frac{2+\sqrt{5}}{2}$

$v^2 = (x+iy)^2 = 2 + i$

$x^2 + 2xyi + i^2y^2 = 2 + i$

$x^2 - y^2 + 2xyi \quad = 2 + i$

$\therefore x^2 - y^2 = 2$ ….(1)

$2xy = 1$ ….(2)

From 2: $y = \frac{1}{2x}$

Substituting in 1:

$x^2 - \left(\frac{1}{2x}\right)^2 = 2$

$x^2 - \frac{1}{4x^2} = 2$

$4x^2(x^2) - 1 = 2(4x^2)$

$4x^4 - 8x^2 - 1 = 0$

Let $p = x^2$

$4p^2 - 8p - 1 = 0$

$p = \frac{+8 \pm \sqrt{64-4(4)(-1)}}{2(4)}$

$p = \frac{8 \pm \sqrt{80}}{8}$

$p = \frac{8 \pm 4\sqrt{5}}{8}$

$p = 1 \pm \frac{1}{2}\sqrt{5}$

$p = \frac{2+\sqrt{5}}{2}$

$p = \frac{2+\sqrt{5}}{2}$

$x^2 = \frac{2+\sqrt{5}}{2}$

(c) Given: $x = \frac{e^{-t}}{\sqrt{1-t^2}}$; $y = \sin^{-1} t$

(i) To show: $\frac{dy}{dx} = \frac{e^t(1-t^2)}{t^2+t-1}$

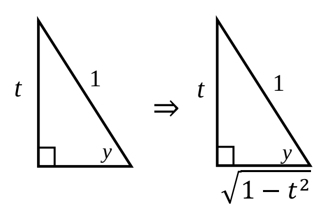
$\sin^{-1} t = y$

$\sin y = t$

$\cos y \frac{dy}{dt} = 1$

$\frac{dy}{dt} = \frac{1}{\cos y}$

$\frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$



$\sin y = t \Rightarrow \cos y = \frac{1}{\sqrt{1-t^2}}$

$x = (e^{-t})\,(1-t^2)^{-\frac{1}{2}}$

Applying the Product Rule

$\frac{dx}{dt} = e^{-t}\left(-\frac{1}{2}\right)(1-t^2)^{-\frac{3}{2}}(-2t) + (1-t^2)^{-\frac{1}{2}}(-e^{-t})$

Removing the common factor $e^{-t}(1-t^2)^{-\frac{3}{2}}$

$\frac{dx}{dt} = e^{-t}(1-t^2)^{-\frac{3}{2}}\left(t + (1-t^2)(-1)\right)$

$= \frac{(t^2+t-1)}{e^t(1-t^2)^{\frac{3}{2}}}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\frac{dy}{dx} = \frac{1}{(1-t^2)^{\frac{1}{2}}} \times \frac{e^t(1-t^2)^{\frac{3}{2}}}{t^2+t-1}$

$\frac{dy}{dx} = \frac{(1-t^2)e^t}{(t^2+t-1)}$

(ii) To show *f* has no stationary value

For a stationary value to exist, $\frac{dy}{dx} = 0$. Hence, $(1-t^2)e^t = 0$

$e^t \neq 0$ for all values of t.

$(1-t^2)e^t = 0 \quad iff \quad t = \pm 1$ (iff means if and only if)

However $-1 < t \leq 0.5$

$\therefore t \neq \pm 1$

$\therefore$ *f* has no stationary value.

2. (a) Given: $4x^2 + 3xy^2 + 7x + 3y = 0$

(i) To use implicit differentiation to show that $\frac{dy}{dx} = \frac{8x+3y^2+7}{3(1+2xy)}$

Differentiating implicitly

$8x + 3x \cdot 2y\frac{dy}{dx} + y^2(3) + 7 + 3\frac{dy}{dx} = 0$

$\frac{dy}{dx}(6xy+3) = -8x - 3y^2 - 7$

$\frac{dy}{dx} = \frac{-8x-3y^2-7}{3(2xy+1)} = -\frac{8x+3y^2+7}{3(1+2xy)}$

(Note: There was an error in the question)

(ii) Given: $f(x,y) = 4x^2 + 3xy^2 + 7x + 3y$

To show: $6\,\frac{\partial f(x,y)}{\partial y} - 10 = \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)\left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right) + \frac{\partial^2 f(x,y)}{\partial x^2}$

Finding the partial derivatives:

$f(x,y) = 4x^2 + 3xy^2 + 7x + 3y$

$\frac{\partial f(x,y)}{\partial y} = 0 + 6xy + 0 + 3$

$= 6xy + 3$

$$6\frac{\partial f(x,y)}{\partial y} = 6(6xy+3)$$
$$= 36xy + 18$$
$$\therefore \text{LHS} \;= 36\,xy + 18 - 10$$
$$= 36xy + 8$$

Evaluating the RHS:

$$\frac{\partial^2 f(x,y)}{\partial y^2} = 6x$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = 6y$$
$$\frac{\partial f(x,y)}{\partial x} = 8x + 3y^2 + 7$$
$$\frac{\partial^2 f(x,y)}{\partial x^2} = 8$$

$$\therefore \frac{\partial^2 f(x,y)}{\partial y^2}\cdot\frac{\partial^2 f(x,y)}{\partial y \partial x} + 8 = 6x \cdot 6y = 36xy + 8$$
$$\therefore \text{RHS} = 36xy + 8$$

$$\therefore 6\,\frac{\partial f(x,y)}{\partial y} - 10 = \left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)\left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right) + \frac{\partial^2 f(x,y)}{\partial x^2}$$

(b) Given: $f(x) = \frac{18x^2+13}{9x^2+4}$ $\;-2 \leq x \leq 2$

(i) To express $f(x)$ in the form $a + \frac{b}{9x^2+4}$ $\;a, b \in \mathbb{R}$

$$\begin{array}{r|l} & \;2 \\ 9x^2+4 & \;18x^2+13 \\ & \;\underline{18x^2+8} \\ & \;\;\;\;\;\;\;\;5 \end{array}$$

$$\therefore 18x^2 + 13 = 2 + \frac{5}{9x^2+4}$$

(ii) Given $f(x)$ symmetric about the y-axis

To Evaluate $\int_{-2}^{2} f(x)dx$

$$\int_{-2}^{2} f(x)dx = 2\int_0^2 f(x)dx$$
$$\int_0^2 f(x)dx = \int_0^2 2dx + \frac{5}{4}\int_0^2 \frac{1}{\frac{9}{4}x^2+1}dx$$
$$= \int_0^2 2dx + \frac{5}{4}\int_0^2 \frac{1}{\left(\frac{3}{2}x\right)^2+1}dx$$
$$= \int_0^2 2dx + \frac{5}{4}\cdot\frac{2}{3}\int_0^2 \frac{\frac{3}{2}}{\left(\frac{3}{2}x\right)^2+1}dx$$
$$= [2x]_0^2 + \frac{5}{6}\left[\tan^{-1}\left(\frac{3}{2}x\right)\right]_0^2$$
$$= 4 + \frac{5}{6}[\tan^{-1}(3) - \tan^{-1}(0)]$$
$$= 4 + \frac{5}{6}[1.249 \ldots]$$
$$= 5.5040 \ldots$$

$\therefore 2\int_0^2 f(x)dx = 10.0817 \ldots = 10.1$ to 3 significant figures

(c) (i) To show $\int h^n \ln h \, dh = \frac{h^{n+1}}{(n+1)^2}[-1 + (n+1)\ln h] + c$

Reversing the Product Rule

$y = \ln h \cdot \frac{h^{n+1}}{n+1}$

$\frac{dy}{dh} = \ln h \cdot \frac{(n+1)h^n}{n+1} + \frac{h^{n+1}}{n+1} \cdot \frac{1}{h}$

$\int \frac{dy}{dh} dh = \int \ln h \cdot h^n dh + \int \frac{h^n}{n+1} dh$

$\int h^n \ln h \; dh = y - \int \frac{h^n}{n+1} dh$

$= \ln h \cdot \frac{h^{n+1}}{n+1} - \frac{h^{n+1}}{(n+1)^2} + c$

$= \frac{h^{n+1}}{(n+1)^2}[(n+1)\ln h - 1] + c$

(ii) Hence to find $\int \sin^2 x \cos x \ln(\sin x) dx$

Let $h(x) = \sin x$

$\frac{dh}{dx} = \cos x$

$dh = \cos x \; dx$

$\therefore \int \sin^2 x \cos x \ln(\sin x) \, dx = \int h^2 \ln h \, dh$

$\int h^2 \ln h \, dh = \frac{\sin^3 x}{9}[3\ln(\sin x) - 1] + c$

$\therefore \int \sin^2 x \cos x \ln(\sin x) \, dx = \frac{\sin^3 x}{9}[3\ln(\sin x) - 1] + c$

3. (a) Given: $T_n = \frac{2n+1}{\sqrt{n^2+1}}$

(i) To determine $\lim_{n\to\infty} T_n$

$\lim_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} \equiv \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{1}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}$

$\equiv \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}} = \frac{2}{1} = 2$

(ii) To show that $T_4 = \frac{9}{4}\left[1 + \frac{1}{16}\right]^{-\frac{1}{2}}$

$T_4 = \frac{2(4)+1}{\sqrt{4^2+1}} = \frac{9}{\sqrt{16+1}}$

$= \frac{9}{\sqrt{\left(1+\frac{1}{16}\right)16}}$

$= \frac{9}{4} \cdot \frac{1}{\left(1+\frac{1}{16}\right)^{\frac{1}{2}}}$

$= \frac{9}{4} \cdot \left(1 + \frac{1}{16}\right)^{-\frac{1}{2}}$

(iii) To approximate the value of T_4 for terms up to x^3 in the binomial expansion

$\frac{9}{4}\left(1 + \frac{1}{16}\right)^{-\frac{1}{2}} \equiv \frac{9}{4}\left[1^{-\frac{1}{2}} + {}^{-\frac{1}{2}}C_1 \, 1 \left(\frac{1}{16}\right) + {}^{-\frac{1}{2}}C_2 \, 1 \left(\frac{1}{16}\right)^2 + {}^{-\frac{1}{2}}C_3 \, 1 \left(\frac{1}{16}\right)^3 + \cdots\right]$

$= \frac{9}{4}\left[1 - \frac{1}{2}\left(\frac{1}{16}\right) + \frac{-\frac{1}{2}\times-\frac{3}{2}}{2\times1}\left(\frac{1}{16}\right)^2 + \frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{3\times2\times1}\left(\frac{1}{16}\right)^3 + \cdots\right]$

$= \frac{9}{4}\left[1 - \frac{1}{32} + \frac{3}{8}\cdot\frac{1}{16^2} + - \frac{5}{16}\cdot\frac{1}{16^3} + \cdots\right]$

$= \frac{9}{4}[1 - 0.03125 + 0.0014648 \ldots - 0.00007629 \ldots]$

$= 2.182811648 \ldots$

$= 2.18$ to 2 d.p.

(b) Given: $2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \cdots$

(i) To express the n[th] partial sum S_n in sigma notation.

Note:

n =	1	2	3	4	5
Numerator =	2	3	4	5	6
Denominator =	1	2^2	3^2	4^2	5^2

$S_n = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$

(ii) Hence, given $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$

Show that S_n diverges as $n \to \infty$

$\sum_{n=1}^{\infty} \frac{n+1}{n^2} \equiv \sum_{n=1}^{\infty} \frac{n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n} + \frac{\pi^2}{6}$

Which diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Showing that $\sum_1^{\infty} \frac{1}{n}$ diverges

$$\sum_1^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

Here is a centuries-old proof that this series, called the harmonic series, diverges.
Let us group the terms as shown below

$$\sum_1^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots\right) + \cdots$$

Use your calculator to show that the sum of the first two sets of bracketed fractions is greater than ½. Further, if we include enough fractions after $\frac{1}{9}$ we can get a sum greater than ½. And we can continue this process without bound.
Hence the series can be re-written as

$$\sum_1^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(> \frac{1}{2}\right) + \left(> \frac{1}{2}\right) + \left(> \frac{1}{2}\right) + \cdots$$

Although the sum of this series grows slowly, there is no number beyond which it cannot get to and surpass if we sum enough terms. Hence the series diverges.

(c) To use math induction to prove $\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$

Let $P(n)$ be the proposition that $\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$

Testing P1

$LHS = \sum_{r=1}^{1} \mathrm{r}(\mathrm{r}-1) = 1(0) = 0$

$RHS = \frac{1(1^2-1)}{3} = 0$

$\therefore P(1)$ is true

Assume $P(k)$ is true

$\sum_{r=1}^{k} (r)(r-1) = \frac{k(k^2-1)}{3}$

Show $P(k) \Rightarrow P(k+1)$

$P(k+1): \sum_{r=1}^{k} r(r-1) + \sum_{r=k+1}^{k+1} r(r-1) = \frac{(k+1)[(k+1)^2-1]}{3}$

$$\begin{aligned}
LHS &= \frac{k(k^2-1)}{3} + (k+1)(k) \\
&= \frac{k(k^2-1)}{3} + \frac{3(k+1)(k)}{3} \\
&= \frac{(k+1)[k(k-1)+3k]}{3} \\
&= \frac{(k+1)(k^2+2k)}{3} \\
&= \frac{(k+1)[(k+1)^2-1]}{3} = RHS
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

$\therefore P(1) \Rightarrow P(2) \Rightarrow P(3)$ and so on.

Therefore since $P(1)$ is true $P(n)$ is true.

4. (a) Given: $g(x) = e^{3x+1}$

(i) To develop the Maclaurin series expansion for $g(x)$ up to x^4

$f(x) = f(0) + xf^1(0) + \frac{x^2}{2!}f^2(0) + x^3\frac{f^3}{3!}(0) + x^4\frac{f^4}{4!}(0) + \cdots$

$f(x) = e^{3x+1}$

$f(0) = e^1$

$f^1(x) = 3e^{3x+1}$ .... $f^1(0) = 3e^1$

$f^2(x) = 3^2e^{3x+1}$ .... $f^2(0) = 3^2e^1$

$f^3(x) = 3^3e^{3x+1}$ .... $f^3(0) = 3^3e^1$

$f^4(x) = 3^4e^{3x+1}$ .... $f^4(0) = 3^4e^1$

$$\begin{aligned}
e^{3x-1} &= e^1 + x \cdot 3e^1 + \frac{x^2}{2!} \cdot 3^2e^1 + \frac{x^3}{3!} \cdot 3^3e^1 + \frac{x^4}{4!} \cdot 3^4e^1 + \cdots \\
&= e^1\left[1 + 3x + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{3^4x^4}{4!} + \cdots\right] \\
&= e\left[1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \cdots\right]
\end{aligned}$$

(ii) Hence to estimate $g(0.2)$

$g(0.2) = e[1 + 0.6 + 0.18 + 0.036 + 0.0054 + \cdots]$

$= e[1.8214]$
$= 4.951$ to 3 d.p.

(b) (i) Given: $f(x) = x - 3\sin x - 1$

To show at least one root exists in the interval $[-2,0]$

$f(-2) = -2 - 3\sin(-2) - 1 = -0.2721\ldots$

$f(0) = 0 - 3\sin 0 - 1 = -1$

Hence either no root exists in the interval [-2, 0] or there are at least two roots.

Reducing the interval to $[-2,-1]$

$$f(-1) = -1 - 3\sin(-1) - 1 = 0.5244\ldots$$

$\therefore$ at least one root exists in the interval $[-2,-1]$

So, at least one root exists in the interval $[-2,0]$

(ii) To use at least three iterations of the interval bisection method to show that $f(-0.538) \simeq 0$ in the interval $[-0.7,-0.3]$

$f(-0.7) = 0.2326\ldots(+)$
$f(-0.3) = -0.4134\ldots(-)$

$f(-0.5) = -0.0617\ldots(-)$

$f(-0.6) = 0.0939\ldots(+)$

$f(-0.55) = 0.01806\ldots(+)$

$f(-0.525) = -0.02136\ldots(-)$

$f(-0.5375) = -0.0015\ldots \simeq 0$

$\therefore f(-0.538) \simeq 0$

(c) To use Newton-Raphson's method with $x_1 = 5.5$ to approximate the root of $g(x) = \sin 3x$ in the interval [5,6] correct to two decimal places.

$y = \sin 3x$
$y' = 3\cos 3x$

Note: $x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

$= 5.5 - \frac{\sin(16.5)}{3\cos(16.5)}$

$= 5.16211312\ldots$

$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$

$= 5.16221.. - \frac{\sin(3\times 5.162\ldots)}{3\cos(3\times 5.162\ldots)}$

$= 5.2371\ldots$

$x_4 = x_3 - \frac{f(x_3)}{f^1(x_3)}$

$= 5.236\ldots$

Since $x_3 = x_4$ to 2 decimal places, then the root to 2 d.p. is 5.24

5. (a) Given: 3 Females, 7 Males

(i) To select 4 persons

$^{10}C_4 = 210$

(ii) To select at least one female

To select no female : $^{7}C_4 = 35$

∴ to select at least 1 female $210 - 35 = 175$

(b) Given: digits $1, 2\ 3, 4, 5$

(i) To form the greatest positive amount of numbers without repeating any digit

The maximum number of 5-digit numbers = $5! = 120$

The maximum number of 4-digit numbers = $5 \times 4 \times 3 \times 2 = 120$

The maximum number of 3-digit numbers = $5 \times 4 \times 3 = 60$

The maximum number of 2-digit numbers = $5 \times 4 = 20$

The maximum number of 1-digit numbers = 5

Total numbers of numbers = $120 + 120 + 60 + 20 + 5 = 325$

(ii) To determine the probability that a number found is greater than 100

P (number greater than 100)

The number of numbers greater than 100 is:

1 digit none

2 digits none

3 digits all $= 60$

4 digits all 120

5 digits all$= 120$

Total$= 60 + 120 + 120 = 300$

∴ $P(> 100) = \frac{300}{325} = \frac{12}{13} = 0.923$ to 3 sig. figures

(c) Given:

$$2x + 3y - z = -3.5$$
$$x - y + 2z = 7$$
$$1.5x + 0y + 3z = 9$$

(i), To rewrite the system as an augmented matrix (ii) & (iii)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -1 & -3.5 \\ 1 & -1 & 2 & 7 \\ 1.5 & 0 & 3 & 9 \end{array}\right)$$

(ii) To use elementary row operations to reduce the system to echelon form

Interchange R_1 and R_2 →above to get:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 2 & 3 & -1 & -3.5 \\ 1.5 & 0 & 3 & 9 \end{array}\right)$$

$R2 - 2R1 \rightarrow$ R2

$$\begin{array}{cccc} 2 & 3 & -1 & -3.5 \\ \underline{-(2} & \underline{-2} & \underline{4} & \underline{14)} \\ 0 & 5 & -5 & -17.5 \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & -5 & -17.5 \\ 1.5 & 0 & 3 & 9 \end{array}\right)$$

$R3 \div 1.5; R2 \div 5$ from above

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & -1 & -3.5 \\ 1 & 0 & 2 & 6 \end{array}\right)$$

$R3 - R1 \rightarrow$ R3 from above

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & -1 & -3.5 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$R3 - R2 \rightarrow$ R3 from above

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & -1 & -3.5 \\ 0 & 0 & 1 & 2.5 \end{array}\right)$$

$\therefore z = 2.5$

$y - 2.5 = -3.5$

$y = -1$

$x - (-1) + 2(2.5) = 7$

$x + 1 + 5 = 7$

$x = 1$

Ans: $x = 1 \quad y = -1 \quad z = 2.5$

(iv) To show that the system has no solutions if $R3$ is changed to $1.5x - 1.5y + 3z = 9$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 2 & 3 & -1 & -3.5 \\ 1 & -1 & 2 & 7 \\ 1.5 & -1.5 & 3 & 9 \end{array}\right)$$

Interchange R1 & R2

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 2 & 3 & -1 & -3.5 \\ 1.5 & -1.5 & 3 & 9 \end{array}\right)$$

$R2 - 2R1 \rightarrow R2$

$$\begin{array}{cccc} 2 & 3 & -1 & -3.5 \\ \underline{-(2} & \underline{-2} & \underline{4} & \underline{14} \\ 0 & 5 & -5 & -17.5 \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 5 & -5 & -17.5 \\ 1.5 & -1.5 & 3 & 9 \end{array}\right)$$

$R2 \div 5: 0$, $R3 \div 1.5$:

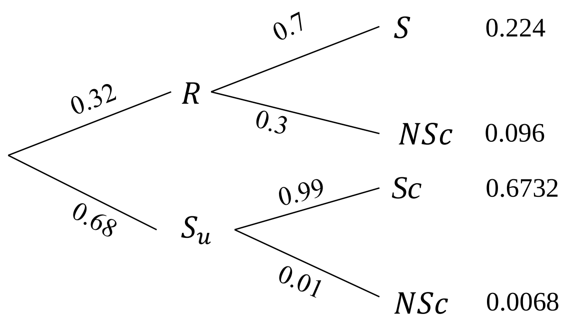
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & -1 & -3.5 \\ 1 & -1 & 2 & 6 \end{array}\right)$$

$R1 - R3 \rightarrow R3$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 1 & -1 & -3.5 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Whatever values we choose for x, y and z, the equation $0 = 1$ cannot be satisfied. This system is inconsistent, that is, it has no solution.

6. (a) (i) To construct a tree diagram



(ii) $P(S_c) = 0.224 + 0.6732$

$= 0.8972$

(iii) $P(R/S_c) = \frac{P(R\cap S_c)}{P(S_c)}$

$= \frac{0.224}{0.8972}$

$= 0.2496$ ...

$= 0.25$ to 2 d.p

(b) (i) To show $y + xy + x^2 = 0$ is a solution to $\frac{dy}{dx} = \frac{y-x^2}{x(1+x)}$ …..(1)

$y + xy + x^2 = 0$ …..(2)

$\frac{dy}{dx} + x\frac{dy}{dx} + y(1) + 2x = 0$

$\frac{dy}{dx}(1 + x) + y + 2x = 0$

$\frac{dy}{dx} = \frac{-y-2x}{1+x}$ ……(3)

We now need to show that

$\frac{y-x^2}{x(1+x)} = \frac{-y-2x}{1+x}$ …..(4)

From (2): $y(1 + x) = -x^2$

$y = \frac{-x^2}{1+x}$

Substituting $\frac{-x^2}{1+x}$ for *y* in equation (4)

$$\frac{\frac{-x^2}{1+x}-x^2}{x(1+x)} = \frac{-\left(\frac{-x^2}{1+x}\right)-2x}{1+x}$$

$$\frac{\frac{-x^2}{1+x}-\frac{x^2(1+x)}{1+x}}{x(1+x)} = \frac{-\frac{-x^2}{1+x}-\frac{2x(1+x)}{1+x}}{1+x}$$

$$\frac{-x^2-x^2-x^3}{x(1+x)^2} = \frac{x^2-2x-2x^2}{(1+x)^2}$$

$$\frac{-2x-x^2}{(1+x)^2} = \frac{-2x-x^2}{(1+x)^2}$$

$\therefore y + xy + x^2 = 0$ is a solution for $\frac{dy}{dx} = \frac{y-x^2}{x(1+x)}$

6. (b) (ii) Given: $y'' - 2y = 0$

a) To find the general solution

let $y = e^{mx}$
$y' = me^{mx}$
$y'' = m^2e^{mx}$

$\therefore m^2e^{mx} - 2e^{mx} = 0$
$e^{mx}(m^2 - 2) = 0$
$m = \pm\sqrt{2}$

$\therefore y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x}$ …..(1)

(b) Given: boundary conditions: $y(0) = 1;\ y'\left(\frac{\sqrt{2}}{2}\right) = 0$

To show: $y = \frac{1}{e^2+1}\left(e^{\sqrt{2}x} + e^{2-\sqrt{2}x}\right)$

When $x = 0, y = 1$
Substituting in equation (1)
$1 = Ae^0 + Be^0$
$1 = A(1) + B(1)$
$A + B = 1$ …..(2)

When $x = \frac{\sqrt{2}}{2}, y' = 0$
$y = \sqrt{2}Ae^{\sqrt{2}x} - \sqrt{2}\,Be^{-\sqrt{2}x}$
Substituting the given values
$0 = \sqrt{2}Ae^{\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)} - \sqrt{2}\,Be^{-\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)}$
$0 = \sqrt{2}Ae - \sqrt{2}Be^{-1}$
$0 = Ae - \dfrac{B}{e}$
$0 = Ae^2 - B$
$B = Ae^2$ …..(3)

Substituting in (2)
$A + Ae^2 = 1$
$A(1 + e^2) = 1$
$A = \frac{1}{e^2+1}$ …..(4)

Substituting in (3)
$B = \left(\dfrac{1}{e^2+1}\right)e^2$
$B = \frac{e^2}{e^2+1}$ …..(5)

Substituting (4) and (5) in (1)

$$y = \left(\frac{1}{e^2+1}\right) \cdot e^{\sqrt{2}x} + \left(\frac{e^2}{e^2+1}\right) \cdot e^{-\sqrt{2}x}$$

$$y = \frac{1}{e^2+1}\left(e^{\sqrt{2}x} + e^{2-\sqrt{2}x}\right)$$

END OF TEST