CAPE Unit 1
Pure Mathematics
June 2015
Solutions

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CAPE Unit 1 June 2015 - Solutions

1. (a) (i) inverse: $\sim p \rightarrow \sim q$

contrapositive $\sim q \rightarrow \sim p$ [2 marks]

(ii).

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

[4 marks]

The compound statements $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are logically equivalent since they have the same truth values [2 marks]

(b) (i) $x^3 + px^2 - x + q \equiv (x-5)Q_1$

Let $x = 5$

$125 + 25p - 5 + q = 0$

$25p + q = -120$ ....(1)

$x^3 + px^2 - x + q \equiv (x-1)Q_2 + 24$

Let $x = 1$

$1 + p - 1 + q = 24$

$p + q = 24$ ……………(2)

From (2): $q = 24 - p$

Substituting in (1):

$25p + (24 - p) = -120$

$25p - p = -144$

$24p = -144$

$\boldsymbol{p = -6}$

Substituting -6 for p in (2):

$-6 + q = 24$

$\boldsymbol{q = 30}$

$\boldsymbol{p = -6, q = 30}$ **[4 marks]**

(ii) To factorize $x^3 + px^2 - x + q$

$x^3 + px^2 - x + q \equiv x^3 - 6x^2 - x + 30$

$x^3 - 6x^2 - x + 30 \equiv (x-5)(Ax^2 + Bx + C)$

From inspection $a = 1, c = -6$

$x^3 - 6x^2 - x + 30 \equiv (x-5)(x^2 + Bx - 6)$

Equating the terms in x^2 on both sides of the equation:

$-6x^2 = Bx^2 - 5x^2$

$$-6 = B - 5$$
$$B = -1$$
$$x^3 - 6x^2 - x + 30 \equiv (x-5)(x^2 - x - 6)$$
$$x^3 - 6x^2 - x + 30 \equiv (x-5)(x-3)(x+2)$$

[5 marks]

c. **Given that $\boldsymbol{S(n) = 5 + 5^2 + 5^3 + \cdots + 5^n}$**
To use math induction to prove that $\boldsymbol{4S(n) = 5^{n+1} - 5}$

Let $P(n)$ be the proposition that $4S(n) = 5^{n+1} - 5$

Testing $P(1)$

LHS= $S(1) = 5^1$

$4S(1) = 4(5) = 20$

RHS = $5^{1+1} - 5 = 25 - 5 = 20$

$\therefore 4S(1) = 5^{1+1} - 5$

$P(1)$ is true

Assume that $P(k)$ is true

That is, assume $\therefore 4S(k) = 5^{k+1} - 5$

$$4(\mathbf{5} + \mathbf{5^2} + \mathbf{5^3} + \cdots + \mathbf{5^n}) = 5^{k+1} - 5 \quad \text{.....(1)}$$

Show that $P(k) \to P(k+1)$;

$P(k+1)$: $4S(k+1) = 5^{k+2} - 5$

$S(k+1) = \mathbf{5} + \mathbf{5^2} + \mathbf{5^3} + \cdots + \mathbf{5^k} + \mathbf{5^{k+1}}$

$4S(k+1) = \mathbf{4}(\mathbf{5} + \mathbf{5^2} + \mathbf{5^3} + \cdots + \mathbf{5^k} + \mathbf{5^{k+1}})$

$P(k+1)$: $4(\mathbf{5} + \mathbf{5^2} + \mathbf{5^3} + \cdots + \mathbf{5^k} + \mathbf{5^{k+1}}) = 5^{k+2} - 5$

$LHS = 4\underbrace{(\mathbf{5} + \mathbf{5^2} + \mathbf{5^3} + \cdots + \mathbf{5^k})} + 4 \cdot \mathbf{5^{k+1}}$

$= \quad 5^{k+1} - 5 \quad + 4 \cdot \mathbf{5^{k+1}}$ (see equation (1) above)

$= 5^{k+1} + 4 \cdot 5^{k+1} - 5$

$= 5 \cdot 5^{k+1} - 5$

$= 5^{k+2} - 5 \equiv RHS$

Therefore $P(k) \to P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$, then $P(n)$ is true for all positive integers n.

[8 marks]

2. Given: $f: A \to B$; and $g: B \to C$ where f and g are one-to-one and onto

(a) (i) To show that $(g \circ f)$ is one-to-one

Note: A function, $k: X \to Y$, is one to one if $k(a) = k(b) \Rightarrow a = b$ for all $a, b \epsilon X$

Assume $(g \circ f)(a) = (g \circ f)(b)$ for $a, b \epsilon A$

Then $gf(a) = gf(b)$ by definition

$\therefore f(a) = f(b)$ since g is one to one

$\therefore a = b$ since f is one to one

$\therefore (g \circ f)(a) = (g \circ f)(b) \Rightarrow a = b$

Hence, $(g \circ f)$ is one-to-one [4 marks]

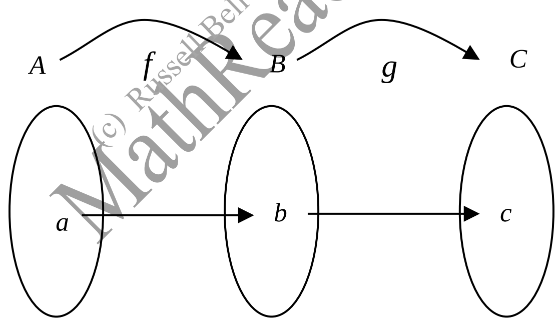
(a) (ii) To show that $(g \circ f)$ is onto

Note: A function $k: X \to Y$ is onto if for all $y \in Y$, there exists an $x \in X$ such that $k(x) = y$

Assume $c \in C$.

Since g is onto, then there must be an element $b \in B$ such that $g(b) = c$

But, since f is onto, there must exist an element $a \in A$ such that $f(a) = b$.



Hence, for all $c \in C$, there must be an $a \in A$ such that $(g \circ f)(a) = c$

Therefore, $(g \circ f)$ is onto. [4 marks]

(b) (i) $3 - \frac{4}{9^x} - \frac{4}{(9^2)^x} = 0$

$9^{2x}(3) - 9^{2x}\left(\frac{4}{9^x}\right) - 9^{2x}\left(\frac{4}{(9^2)^x}\right) = 0$

$9^{2x}(3) - 9^x(4) - 4 = 0$

Let $y = 9^x$

$3y^2 - 4y - 4 = 0$

$(3y + 2)(y - 2) = 0$

$y = -\frac{2}{3}$ or $y = 2$

$\therefore 9^x = 2$

$x \log 9 = \log 2$

$x = \frac{\log 2}{\log 9}$

$x = 0.315$ to three significant figures

Or $9^x = -\frac{2}{3}$

No solution [7 marks]

(ii) To solve $|5x - 6| = x + 5$

Either

$5x - 6 = x + 5$

$5x - x = 5 + 6$

$4x = 11$

$x = \frac{11}{4}$

Or

$-(5x - 6) = x + 5$

$-5x + 6 = x + 5$

$-5 + 6 = x + 5x$

$1 = 6x$

$x = \frac{1}{6}$ [5 marks]

(c) Given: $N = 300 + 5^t$

(i) to determine the number of bacteria present at $t = 0$

$N = 300 + 5^0 = 301$ [1 mark]

(ii) To determine the time required to triple the number of bacteria

$N = 301 \Rightarrow 3N = 903$

Substituting 903 for N in the original equation:

$903 = 300 + 5^t$

$603 = 5^t$

$\log 603 = t \log 5$

$t = \frac{\log 603}{\log 5}$

$t = 3.977$ ... hours

$t = 4$ hours to the nearest whole number [4 marks]

3. (a) (i) To show $\cos 3x = 4\cos^3 x - 3\cos x$

$\cos 3x = \cos\ (2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1)(\cos x) - 2\sin^2 x \cos x$

$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)(\cos x)$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$ [6 marks]

(ii) To solve $\cos 6x - \cos 2x = 0$ for $0 \leq x \leq 2\pi$

$\cos 6x = 4\cos^3 2x - 3\cos 2x$

Substituting $4\cos^3 2x - 3\cos 2x$ for $\cos 6x$ in the given equation:

$4\cos^3 2x - 3\cos 2x - \cos 2x = 0$

$4\cos^3 2x - 4\cos 2x = 0$

$4\cos 2x(\cos^2 2x - 1) = 0$

Either

$4\cos 2x = 0$

$\cos 2x = 0 \qquad 0 \le 2x \le 4\pi$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Or

$\cos^2 2x - 1 = 0$

$\cos^2 2x = 1$

$\cos 2x = \pm 1$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

Answers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ or $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ [9 marks]

(b) (i) To express $f(2\theta) = 3\sin 2\theta + 4\cos 2\theta$ in the form $r\sin\ (2\theta + \alpha)$ $r > 0,\ 0 < \alpha < \frac{\pi}{2}$

$r = \sqrt{3^2 + 4^2} = 5$

$\tan\alpha = \frac{4}{3}$

$\alpha = 53.1°$ to 1 decimal place or 0.927 radians to 3 significant figures

$\therefore 3\sin 2\theta + 4\cos 2\theta = 5\sin(2\theta + 0.927)$ [6 marks]

(ii) Maximum value of $\frac{1}{7-f(\theta)}$ occurs when $7 - f(\theta)$ is a minimum.

Maximum value is $\frac{1}{7-5} = \frac{1}{2}$

Minimum value of $\frac{1}{7-f(\theta)}$ occurs when $7 - f(\theta)$ is a maximum

Minimum value is $\frac{1}{7--5} = \frac{1}{12}$ [4 marks]

4. (a) Given: C_1 and C_2 defined parametrically as:

C_1: $x = \sqrt{10}\cos\theta - 3, \qquad y = \sqrt{10}\sin\theta + 2$

C_2: $x = 4\cos\theta + 3, \qquad y = 4\sin\theta + 2$

(i) To determine the Cartesian equations of C_1 and C_2 in the form $(x-a)^2 + (y-b)^2 = r^2$

C_1

$x = \sqrt{10}\cos\theta - 3 \qquad y = \sqrt{10}\sin\theta + 2$

$\frac{x+3}{\sqrt{10}} = \cos\theta \qquad \frac{y-2}{\sqrt{10}} = \sin\theta$

$\frac{(x+3)^2}{10} = \cos^2 x \qquad \frac{(y-2)^2}{10} = \sin^2\theta$

Adding:

$\frac{(x+3)^2}{10} + \frac{(y-2)^2}{10} = \cos^2 x + \sin^2\theta$

$(x+3)^2 + (y-2)^2 = \left(\sqrt{10}\right)^2$ ……(1)

C_2

$x = 4\cos\theta + 3$ $\quad$ $y = 4\sin\theta + 2$

$\frac{x-3}{4} = \cos\theta$ $\quad$ $\frac{y-2}{4} = \sin\theta$

$\frac{(x-3)^2}{16} = \cos^2 x$ $\quad$ $\frac{(y-2)^2}{16} = \sin^2\theta$

Adding:

$\frac{(x-3)^2}{16} + \frac{(y-2)^2}{16} = \cos^2 x + \sin^2\theta$

$(x-3)^2 + (y-2)^2 = 4^2$ ……(2) [4 marks]

(ii) From (1): $(y-2)^2 = \left(\sqrt{10}\right)^2 - (x+3)^2$

From (2): $(y-2)^2 = 4^2 - (x-3)^2$

$\therefore \left(\sqrt{10}\right)^2 - (x+3)^2 = 4^2 - (x-3)^2$

$10 - (x^2 + 6x + 9) = 16 - (x^2 - 6x + 9)$

$10 - x^2 - 6x - 9 = 16 - x^2 + 6x - 9$

$1 - 6x = 7 + 6x$

$1 - 7 = 6x + 6x$

$-6 = 12x$

$-\frac{1}{2} = x$

Substituting $-\frac{1}{2}$ for x in (1)

$\left(-\frac{1}{2} + 3\right)^2 + (y-2)^2 = \left(\sqrt{10}\right)^2$

$\frac{25}{4} + (y-2)^2 = 10$

$(y-2)^2 = 10 - \frac{25}{4}$

$(y-2)^2 = \frac{15}{4}$

$y - 2 = \pm\sqrt{\frac{15}{4}}$

$y = 2 \pm \sqrt{\frac{15}{4}}$

$y = 3.94$ to 2 decimal places or

$y = 0.064$ to 3 decimal places

$x = -\frac{1}{2}, y = 0.064$

$x = -\frac{1}{2}, y = 3.94$ [9 marks]

(b) Let O be the fixed point (0,3) and let Q be the fixed point (5,2).

Then $PO = 2PQ$

$PO = \sqrt{(x-0)^2 + (y-3)^2}$

$PQ = \sqrt{(x-5)^2 + (y-2)^2}$

$$\sqrt{(x-0)^2 + (y-3)^2} = 2\sqrt{(x-5)^2 + (y-2)^2}$$

$$(x-0)^2 + (y-3)^2 = 4[(x-5)^2 + (y-2)^2]$$

$$x^2 + y^2 - 6y + 9 = 4(x^2 - 10x + 25 + y^2 - 4y + 4)$$

$$x^2 + y^2 - 6y + 9 = 4x^2 - 40x + 100 + 4y^2 - 16y + 16$$

$$-107 = 3x^2 - 40x + 3y^2 - 10y$$

$$x^2 - \frac{40}{3}x + y^2 - \frac{10}{3}y = -\frac{107}{3}$$

$$x^2 - \frac{40}{3}x + \left(\frac{40}{6}\right)^2 + y^2 - \frac{10}{3}y + \left(\frac{10}{6}\right)^2 = -\frac{107}{3} + \left(\frac{40}{6}\right)^2 + \left(\frac{10}{6}\right)^2$$

$$\left(x - \frac{40}{6}\right)^2 + \left(y - \frac{10}{6}\right)^2 = \frac{1600+100-1284}{36}$$

$$\left(x - \frac{20}{3}\right)^2 + \left(y - \frac{5}{3}\right)^2 = \frac{416}{36}$$

$$\left(x - \frac{20}{3}\right)^2 + \left(y - \frac{5}{3}\right)^2 = \left(\sqrt{\frac{104}{9}}\right)^2$$

which is a circle centre $\left(\frac{20}{3}, \frac{5}{3}\right)$ and radius $\sqrt{\frac{104}{9}}$. [12 marks]

5. (a)

Given: $f(x) = \begin{cases} \frac{\sin(ax)}{x} & \text{if } x \neq 0 \\ 4 & \text{if } x = 0 \end{cases}$

$f(x)$ is continuous at $x = 0$

To determine the value of a

Since $f(x)$ is continuous at $x = 0$ then, the following three conditions must be true

- $\lim_{x\to 0} f(x) = f(0)$
- $\lim_{x\to 0} \frac{\sin ax}{x} = f(0)$
- $f(0) = 4$ (given)

$\therefore \lim_{x\to 0} \frac{\sin ax}{x} = 4$

$$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} \frac{\sin ax}{ax} \cdot a$$

$$= \lim_{x\to 0} \left(\frac{\sin ax}{ax}\right) \cdot \lim_{x\to 0} a$$

$$.= \quad 1 \quad \cdot \lim_{x\to 0} a$$

$$= \lim_{x\to 0} a$$

$$= a$$

$\therefore a = 4$

(b) $f(x) = \sin 2x$

$f(x+h) = \sin 2(x+h)$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$

$= \lim_{h\to 0} \frac{2 \sin h \cos(2x+h)}{h}$

$= \lim_{h\to 0} \left[\frac{\sin h}{h} \cdot 2\cos(2x+h)\right]$

$= \lim_{h\to 0} \left(\frac{\sin h}{h}\right) \cdot \lim_{h\to 0}[2\cos(2x+h)]$

$= \quad 1 \quad \cdot \quad 2\cos 2x$

$= 2\cos 2x$ [6 marks]

Note

$\sin C - \sin D \equiv 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$

Let $C = 2(x+h) = 2x + 2h$; let $D = 2x$

$\therefore \sin(2x+2h) - \sin 2x \equiv 2\sin\frac{2x+2h-2x}{2}\cos\frac{2x+2h+2x}{2}$

$\equiv 2\sin h\cos(2x+h)$

(c) Given: $y = \frac{2x}{\sqrt{1+x^2}}$

(i) To show: $x\frac{dy}{dx} = \frac{y}{1+x^2}$

$$\frac{dy}{dx} = \frac{\left(\sqrt{1+x^2}\right)(2) - 2x\left(\frac{1}{2}\right)(1+x^2)^{-\frac{1}{2}}(2x)}{1+x^2}$$

$$= \frac{2(1+x^2)^{-\frac{1}{2}}(1+x^2-x^2)}{1+x^2}$$

$$= \frac{2(1+x^2)^{-\frac{1}{2}}}{1+x^2}$$

$$= \frac{2}{(1+x^2)^{\frac{3}{2}}}$$

$$x\frac{dy}{dx} = \frac{2x}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{y}{1+x^2} = \frac{2x}{\sqrt{1+x^2}} \cdot \frac{1}{1+x^2}$$

$$\frac{y}{1+x^2} = \frac{2x}{(1+x^2)^{\frac{3}{2}}}$$

$$\therefore x\frac{dy}{dx} = \frac{y}{1+x^2}$$

[7 marks]

(ii) To show: $\frac{d^2y}{dx^2} + \frac{3y}{(1+x^2)^2} = 0$

$$\frac{dy}{dx} = \frac{2}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^{\frac{3}{2}}(0) - 2\left(\frac{3}{2}\right)(1+x^2)^{\frac{1}{2}}(2x)}{(1+x^2)^3}$$

$$= \frac{-6x}{(1+x^2)^{\frac{5}{2}}}$$

$$\frac{3y}{(1+x^2)^2} = \frac{6x}{(1+x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1+x^2)^2}$$

$$= \frac{6x}{(1+x^2)^{\frac{5}{2}}}$$

$$\therefore \frac{d^2y}{dx^2} + \frac{3y}{(1+x^2)^2} = \frac{-6x}{(1+x^2)^{\frac{5}{2}}} + \frac{6x}{(1+x^2)^{\frac{5}{2}}}$$

$$= 0$$

[8 marks]

6. (a) Given: $y = 3x - 7$ …..(1)

$y + x = 9$ …..(2)

$3y = x + 3$ …..(3)

(i) from (1) and (2): $y = 3x - 7$

$y = 9 - x$

$\therefore 3x - 7 = 9 - x$

$4x = 16$

$x = 4$

Substituting in (1): $y = 3(4) - 7$

$y = 5$

Point $A(4,5)$

from (1) and (3): $y = 3x - 7$

$y = \frac{x}{3} + 1$

$\therefore 3x - 7 = \frac{x}{3} + 1$

$9x - 21 = x + 3$

$8x = 24$

$x = 3$

Substituting in (1): $y = 3(3) - 7$

$y = 2$

Point $B(3,2)$

from (2) and (3): $y = 9 - x$

$y = \frac{x}{3} + 1$

$\therefore 9 - x = \frac{x}{3} + 1$

$27 - 3x = x + 3$

$24 = 4x$

$x = 6$

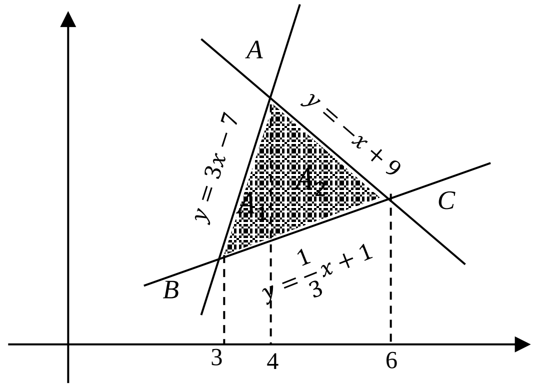
Substituting in (2): $y = 9 - 6$

$y = 3$

Point $C(6,3)$ [5 marks]

(ii) Given: A figure/region bounded by the lines $y = 3x - 7$; $y + x = 9$; $3y = x + 3$

To use integration to determine the area of the region



$$A_1 = \int_3^4 \left((3x - 7) - \left(\frac{1}{3}x + 1\right)\right) dx = \int_3^4 \left(\frac{8}{3}x - 8\right) dx$$

$$= \left[\frac{8}{3} \cdot \frac{x^2}{2} - 8x\right]_3^4$$

$$= \left(\frac{4}{3}(16) - 32\right) - \left(\frac{4}{3}(9) - 24\right)$$

$$= \frac{64}{3} - \frac{96}{3} - \frac{36}{3} + \frac{72}{3}$$

$$A_1 = \frac{4}{3} \text{ unit}^2$$

$$A_2 = \int_4^6 \left((9 - x) - \left(\frac{1}{3}x + 1\right)\right) dx = \int_4^6 \left(8 - \frac{4}{3}x\right) dx$$

$$= \left[8x - \frac{4x^2}{6}\right]_4^6$$

$$= \left[8x - \frac{2x^2}{3}\right]_4^6$$

$$= (48 - 24) - \left(32 - \frac{32}{3}\right)$$

$$= 24 - 32 + \frac{32}{3}$$

$$= \frac{-24 + 32}{3}$$

$$A_2 = \frac{8}{3} \text{ unit}^2$$

Area of region = area A_1 + area A_2

Area of region $= \frac{4}{3} + \frac{8}{3} = 4$ unit^2 [6 marks]

(b) Given: $\frac{dy}{dx} = 3x^2 + 8x - 3$

(i) $y = x^3 + 4x^2 - 3x + c$

Substituting $(0, -6)$

$c = -6$ (the y-intercept)

$y = x^3 + 4x^2 - 3x - 6$ [3 marks]

(ii) $\frac{dy}{dx} = 3x^2 + 8x - 3 = 0$

$(3x - 1)(x + 3) = 0$

$x = \frac{1}{3}$ or $x = -3$

$\frac{d^2y}{dx^2} = 6x + 8$

$x = \frac{1}{3},: \frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) + 8 = 10$

Therefore at $x = \frac{1}{3},$ there is a local minimum and at $x = -3$ there is a local maximum.

When $x = \frac{1}{3},$

$y = \left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) - 6$

$y = -6\frac{14}{27}$

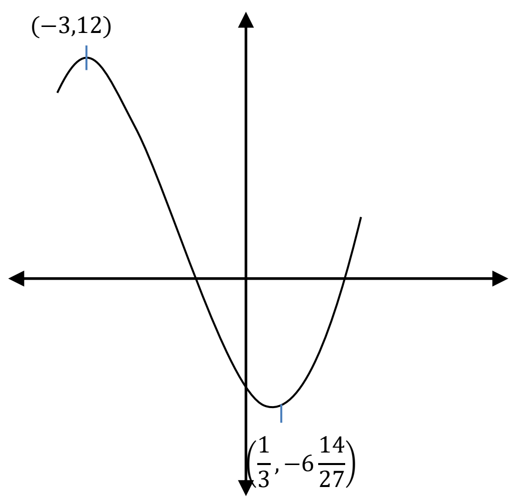
$\left(\frac{1}{3}, -6\frac{14}{27}\right)$ coordinates of minimum point

When $x = -3$

$y = (-3)^3 + 4(-3)^2 - 3(-3) - 6$

$y = 12$

$(-3,12)$ coordinates of maximum point [8 marks]



[3 marks]

END OF EXAM
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