CAPE Unit 1
Pure Mathematics
June 2016
Solutions

1. (a) Given: $f(x) = 2x^3 - x^2 + px + q$

$(x+3)$ is a factor of $f(x)$

Remainder = 10 when $f(x)$ is divided by $(x+1)$

(i) To show: $p = -25$ and $q = -12$

$2x^3 - x^2 + px + q \equiv (x+3)Q_1$ $\quad Q_1$ - quotient

Let $x = -3$

$2(-3)^3 - (-3)^2 + p(-3) + q = 0$

$-54 - 9 - 3p + q = 0$

$-3p + q = 63$ ……..**(1)**

$2x^3 - x^2 + px + q \equiv (x+1)Q_2 + 10$ $\quad Q_2$- quotient

Let $x = -1$

$2(-1)^3 - (-1)^2 + p(-1) + q = 0 + 10$

$-2 - 1 - p + q = 10$

$-p + q = 13$…………(**2**)

Subtracting (2) from (1):

$-2p = 50$

$p = -25$

Substituting in (2):

$-(-25) + q = 13$

$25 + q = 13$

$q = -12$

$\therefore p = -25$ and $q = -12$

(ii) Hence to solve $f(x) = 0$

$f(x) = 2x^3 - x^2 + px + q = 0$

$= 2x^3 - x^2 - 25x - 12 = 0$

$\therefore 2x^3 - x^2 - 25x - 12 \equiv (x+3)(Ax^2 + Bx + C)$

From inspection $A = 2, C = -4$

$\therefore 2x^3 - x^2 - 25x - 12 = (x+3)(2x^2 + Bx - 4)$

Equating the terms in x^2 on both sides of the equation

$-x^2 = Bx^2 + 6x^2$

$-1 = B + 6$

$-7 = B$

$\therefore 2x^3 - x^2 - 25x - 12 \equiv (x+3)(2x^2 - 7x - 4) = 0$

$\equiv (x+3)(2x+1)(x-4) = 0$

$\therefore x = -3, -\frac{1}{2}$ or 4

(b) To use mathematical induction to prove that $6^n - 1$ is divisible by 5 for $n \in \mathbb{N}$.
Let $P(n)$ be the proposition that $6^n - 1$ is divisible by 5 for $n \in \mathbb{N}$.

Testing $P(1)$
$6^1 - 1 = 5$
$\therefore P(1)$ is True

Assume $P(k)$ is true
That is, assume $6^k - 1 = 5m \quad m \in \mathbb{N}$
Show $P(k) \Longrightarrow P(k+1)$
$P(k+1)$: $6^{k+1} - 1 = 5r \quad r \in \mathbb{N}$
LHS: $6^{k+1} - 1 = 6.6^k - (6-5)$
$= 6.6^k - 6 + 5$
$= 6(6^k - 1) + 5$
$= 6(5m) + 5$
$= 5[6m + 1]$
$= 5r \quad (6m + 1 = r)$
$\therefore P(k) \Longrightarrow P(k+1)$

Alternate Method:
LHS$= 6^{k+1} - 1 = 6.6^k - 1$ **……..(1)**
Note: We assumed $6^k - 1 = 5m$
$\therefore 6^k = 5m + 1$

Substituting this in (1)
$6^{k+1} - 1 = 6(5m + 1) - 1$
$= 30m + 6 - 1$
$= 30m + 5$
$= 5(6m + 1)$
$= 5r \quad (r = 6m + 1)$
$\therefore P(k) \to P(k+1)$
$\therefore P(1) \to P(2) \quad P(2) \to P(3)$, and so on.
$\therefore P(n)$ is true since $P(1)$ is true.

(c) (i)

p	**q**	**p → q**	**p ∨ q**	**p ∧ q**	**(p ∨ q) → (p ∧ q)**
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	F	F	T

(ii) $p \to q$ and $(p \vee q) \to (p \wedge q)$ are not logically equivalent as they do not have identical truth tables/ values.

2. (a) To solve $\log_2(10-x)+\log_2 x=4$

$\log_2(10-x)+\log_2 x=4$

$\therefore \log_2[(10-x)(x)]=4$

$\therefore 2^4=(10-x)(x)$

$16=10x-x^2$

$x^2-10x+16=0$

$(x-2)(x-8)=0$

$x=2,8$

(b) Given: $f(x)=\frac{x+3}{x-1} \qquad x\neq 1$

To determine if *f* is bijective

Note: To be bijective *f* must be both 1- 1 and onto.

Let $f(a)=f(b)$

$f(a)=\frac{a+3}{a-1} \qquad f(b)=\frac{b+3}{b-1}$

$\therefore \ \frac{a+3}{a-1}=\frac{b+3}{b-1}$

$\therefore (a+3)(b-1)=(a-1)(b+3)$

$ab-a+3b-3=ab+3a-b-3$

$-a+3b=-b+3a$

$-a-3a=-b-3b$

$-4a=-4b$

$a=b$

$\therefore f(a)=f(b)\Longrightarrow a=b$

$\therefore f(x) is\ 1-1$ (injective)

Let $f(x)=m$

$\frac{x+3}{x-1}=m$

$x+3=m(x-1)$

$x+3=mx-m$

$mx-x=m+3$

$\frac{x(m-1)}{m-1}=\frac{m+3}{m-1}$

$x=\frac{m+3}{m-1}$

$\therefore$ if $m=1$ there is no corresponding value for *x*.

Hence$f(x)$ is not onto unless the co-domain is restricted to $\mathbb{R}$, $y\neq 1$.

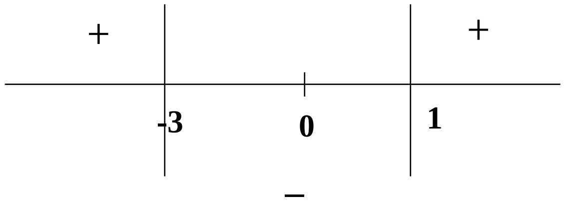
<u>Method 2</u>

Given $f(x)=\frac{x+3}{x-1}$

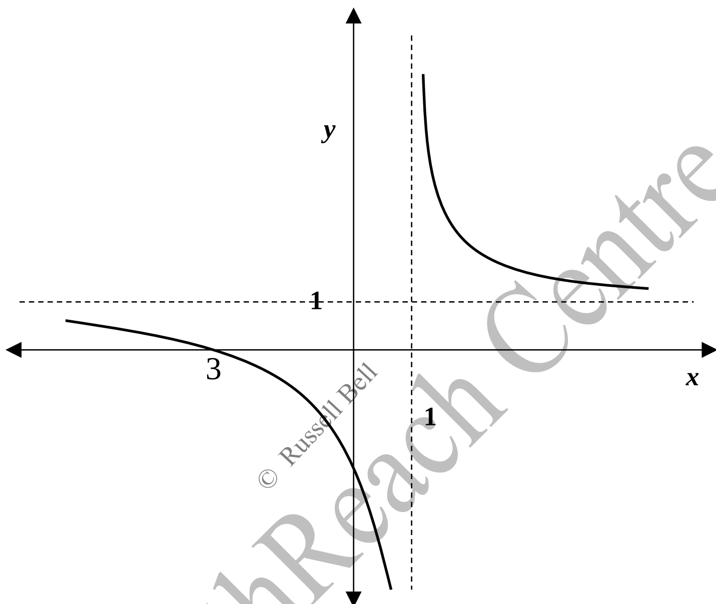
From inspection, the graph of $f(x)$

- Crosses the *x*-axis at -3
- Has a horizontal asymptote at $y=1$
- Is undefined when $x=1$; has a vertical asymptote when $x=1$
- Is equal to 1 at extreme values, $\frac{x}{x}$

Checking the value of x in these interval



$x < -3$: when $x = -4$ $f(x) > 0$
$-3 < x < 1$: when $x = 0$ $f(x)\ O$
$x > 1$: when $x = 2$ $f(x) > 0$



(c) Given $2x^3 - 5x^2 + 4x + 6 = 0$ has roots α, β, γ

(i) To state the values of

- $\alpha + \beta + \gamma$
- $\alpha\beta + \alpha\gamma + \beta\gamma$
- $\alpha\beta\gamma$

$$2x^3 - 5x^2 + 4x + 6 = 0$$
$$x^3 - \frac{5}{2}x^2 + 2x + 3 = 0$$
$$\therefore\ \alpha + \beta + \gamma = \frac{5}{2}$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = 2$$
$$\alpha\beta\gamma = -3$$

(ii) Hence, to determine an equation with integer coefficients which has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

Note to student:

The equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ is:

$$x^3 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)x^2 + \left(\frac{1}{\alpha^2}\cdot\frac{1}{\beta^2} + \frac{1}{\alpha^2}\cdot\frac{1}{\gamma^2} + \frac{1}{\beta^2}\cdot\frac{1}{\gamma^2}\right)x - \left(\frac{1}{\alpha^2}\cdot\frac{1}{\beta^2}\cdot\frac{1}{\gamma^2}\right) = 0$$

Or,

$$x^3 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)x^2 + \left(\frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\gamma)^2} + \frac{1}{(\beta\gamma)^2}\right)x - \left(\frac{1}{(\alpha\beta\gamma)^2}\right) = 0$$

Finding: $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{(\alpha\beta\gamma)^2}$

$\frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{(\alpha\beta\gamma)^2} = \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(-3)^2}$

$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{2^2-2(-3)\left(\frac{5}{2}\right)}{9}$

$= \frac{4+15}{9}$

$= \frac{19}{9}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{19}{9}$

Finding: $\frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\gamma)^2} + \frac{1}{(\beta\gamma)^2}$

$\frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\gamma)^2} + \frac{1}{(\beta\gamma)^2} = \frac{\gamma^2+\beta^2+\alpha^2}{\alpha^2\beta^2\gamma^2}$

$\frac{\gamma^2+\beta^2+\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)}{(\alpha\beta\gamma)^2}$

$\therefore \frac{1}{(\alpha\beta)^2} + \frac{1}{(\alpha\gamma)^2} + \frac{1}{(\beta\gamma)^2} = \frac{\left(\frac{5}{2}\right)^2-2(2)}{(-3)^2}$

$= \frac{\frac{25}{4}-\frac{16}{4}}{9}$

$= \frac{\frac{9}{4}}{9}$

$= \frac{1}{4}$

Finding $\left(\frac{1}{(\alpha\beta\gamma)^2}\right)$

$\frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{(-3)^2} = \frac{1}{9}$

Therefore the equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ is:

$\therefore x^3 - \frac{19}{9}x^2 + \frac{1}{4}x - \frac{1}{9} = 0$

3. (a) (i) To show $\sec^2\theta = \frac{\text{cosec}\,\theta}{\text{cosec}\,\theta - \sin\theta}$

$LHS = \frac{1}{\cos^2\theta}$

$RHS = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin\theta}{1}}$

$RHS = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}}$

$RHS = \frac{\frac{1}{\sin\theta}}{\frac{1-\sin^2\theta}{\sin\theta}}$

$RHS = \frac{\frac{1}{\sin\theta}}{\frac{\cos^2\theta}{\sin\theta}} = \frac{1}{\sin\theta} \div \frac{\cos^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta} \times \frac{\sin\theta}{\cos^2\theta}$

$= \frac{1}{\cos^2\theta} = LHS$

$\therefore \sec^2\theta = \frac{\text{cosec}\,\theta}{\text{cosec}\,\theta - \sin\theta}$

(ii) Hence, to solve $\frac{\text{cosec}\,\theta}{\text{cosec}\,\theta - \sin\theta} = \frac{4}{3}$

$\frac{\text{cosec}\,\theta}{\text{cosec}\,\theta - \sin\theta} = \sec^2\theta$

$= \frac{1}{\cos^2\theta} = \frac{4}{3}$

$\therefore \cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ (reference angle)

$\therefore \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

Or,

$\theta = \frac{\pi}{6}, \frac{6\pi}{6} - \frac{\pi}{6}, \frac{6\pi}{6} + \frac{\pi}{6}, \frac{12\pi}{6} - \frac{\pi}{6}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) (i) Given $f(\theta) = \sin\theta + \cos\theta$

To express $f(\theta)$ in the form $r\sin(\theta + \alpha)$

$\gamma = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\alpha = \frac{1}{1} \rightarrow \alpha = \frac{\pi}{4}$ or 45°

$\therefore f(\theta) = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

(ii) To find the maximum value of f

- Maximum value of f occurs when $\sin\left(\theta + \frac{\pi}{4}\right)$ is a maximum
- Maximum value of $\sin\left(\theta + \frac{\pi}{4}\right)$ is 1

$\therefore$ Maximum value of $f(\theta) = \sqrt{2}(1) = \sqrt{2}$

To find the smallest non-negative value of θ at which the maximum value of $f(\theta)$ occurs

$$\sqrt{2} = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

Or $\sin\left(\theta + \frac{\pi}{4}\right) = 1 \qquad 0 \le \theta \le \frac{\pi}{2}$

$\therefore\ \theta + \frac{\pi}{4} = \frac{\pi}{2} \qquad$ (Note $\sin\left(\frac{\pi}{2}\right) = 1$)

$$\theta = \frac{\pi}{2} - \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}$$

(c) To prove $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$

LHS= $\tan(A + B + C) = \tan[(A + B) + C]$

$$= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right]\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C(1 - \tan A \tan B)}{1 - \tan A \tan B}}{1 - \left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right]\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C - \tan A \tan B tan C)}{1 - \tan A \tan B}}{1 - \left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right]\tan C}$$

$$= \frac{\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B}}{\frac{(1 - \tan A \tan B)}{1 - \tan A \tan B} - \frac{(\tan A + \tan B)\tan C}{1 - \tan A \tan B}}$$

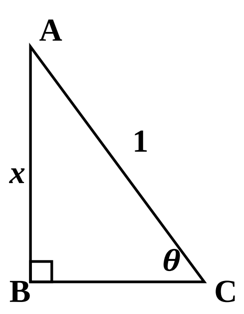
$$= \frac{\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B}}{\frac{(1 - \tan A \tan B) - (\tan A + \tan B)\tan C}{1 - \tan A \tan B}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

4. (a) (i) Given: $\sin\theta = x$

To show: $\tan\theta = \frac{x}{\sqrt{1 - x^2}}$

Method 1:



In triangle ABC: $\sin\theta = x$

Using Pythagoras' theorem:

$(AC)^2 = (AB)^2 + (BC)^2$

$1 = x^2 + (BC)^2$

$1 - x^2 = (BC)^2$

$BC = \sqrt{1 - x^2}$

$\therefore \tan\theta = \frac{x}{\sqrt{1 - x^2}}$

Method 2:

$\sin^2\theta + \cos^2\theta = 1$

$x^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - x^2$

$\cos\theta = \sqrt{1-x^2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$

(ii) Hence, to determine the Cartesian equation of the curve defined parametrically by

$\left.\begin{array}{r} y = \tan 2t \\ x = \sin t \end{array}\right\} 0 < t < \frac{\pi}{2}$

$$\tan 2t = \frac{2\tan t}{1-\tan^2 t}$$

$$y = \tan 2t$$

$$y = \frac{2\tan t}{1-\tan^2 t}$$

$$y = \frac{2\frac{x}{\sqrt{1-x^2}}}{1-\frac{x^2}{1-x^2}}$$

$$y = \frac{\frac{2x}{\sqrt{1-x^2}}}{\frac{1-x^2-x^2}{1-x^2}}$$

$$y = \frac{2x}{\sqrt{1-x^2}} \cdot \frac{1-x^2}{1-2x^2}$$

$$y = \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

$\therefore$ Cartesian equation is $y = \frac{2x\sqrt{1-x^2}}{1-2x^2}$

Or

$\sin 2t = 2\sin t\cos t = 2x \cdot \frac{\sqrt{1-x^2}}{1}$

$\cos 2t = \cos^2 - \sin^2 t$

$= \left(\frac{\sqrt{1-x^2}}{1}\right)^2 - x^2 = 1 - x^2 - x^2$

$= 1 - 2x^2$

$\tan 2t = \frac{\sin 2t}{\cos 2t} = \frac{2x\sqrt{1-x^2}}{1-2x^2} = y$

$\therefore$ Cartesian equation is $y = \frac{2x\sqrt{1-x^2}}{1-2x^2}$

(b) Given $\underline{\mathbf{u}} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ $\underline{\mathbf{v}} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ Two proposition vectors in $\mathbb{R}^3$

(i) To calculate $|\underline{u}|, |\underline{v}|$

$|\underline{\mathbf{u}}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$ units

$|\underline{\mathbf{v}}| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{4 + 1 + 25} = \sqrt{30}$ units

(ii) To find $\cos\theta$ where θ is the angle between $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ in $\mathbb{R}^3$.

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = |\underline{\mathbf{u}}|\,|\underline{\mathbf{v}}| \cos\theta$

$\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \sqrt{14}\sqrt{30}\cos\theta$

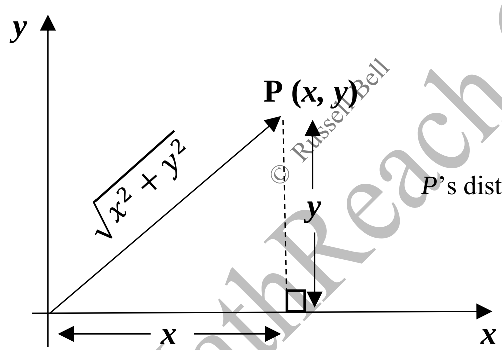
$2 - 3 + 10 = \sqrt{14}\sqrt{30}\cos\theta$

$\therefore \frac{9}{\sqrt{420}} = \cos\theta$

(c) $P(x, y)$ moves such that its distance from the *x*- axis is $\frac{1}{2}$ its distance from the origin.

Note: *P*'s distance from the *x*- axis is *y* - the *y*- coordinate of *P*.

P's distance from the origin is $\sqrt{x^2 + y^2}$



$\therefore y = \frac{1}{2}\sqrt{x^2 + y^2}$

$y^2 = \frac{1}{4}\,(x^2 + y^2)$

$y^2 = \frac{1}{4}x^2 + \frac{1}{4}y^2$

$\frac{3}{4}y^2 = \frac{1}{4}x^2$

$y^2 = \frac{4}{3} \cdot \frac{1}{4}x^2$

$y^2 = \frac{1}{3}x^2$

(d) Given: Line *L* has equation $2x + y + 3 = 0$**……..(1)**

Circle C has equation $x^2 + y^2 = 9$**……….(2)**

To find: The points of intersection

From (1) $y = -2x - 3$

Substituting in (2) $x^2 + (-2x - 3)^2 = 9$

$x^2 + 4x^2 + 12x + 9 = 9$

$5x^2 + 12x = 0$

$x(5x+12)=0$

$x=0$ or $\frac{-12}{5}$

Substituting in (1)

$2x+y+3=0$

$0+y+3=0$

$y=-3$

$\therefore$ a point of intersection is $(0,-3)$

$2x+y+3=0$

$2\left(\frac{-12}{5}\right)+y+3=0$

$\frac{-24}{5}+y+\frac{15}{5}=0$

$y=\frac{9}{5}$

$\therefore$ a point of intersection is $\left(\frac{-12}{5},\frac{9}{5}\right)$

Answers: (0, -3), $\left(-\frac{12}{5},\frac{9}{5}\right)$

5. (a) To use an appropriate substitution to find $\int (x+1)^{\frac{1}{3}}dx$

Let $u=x+1$

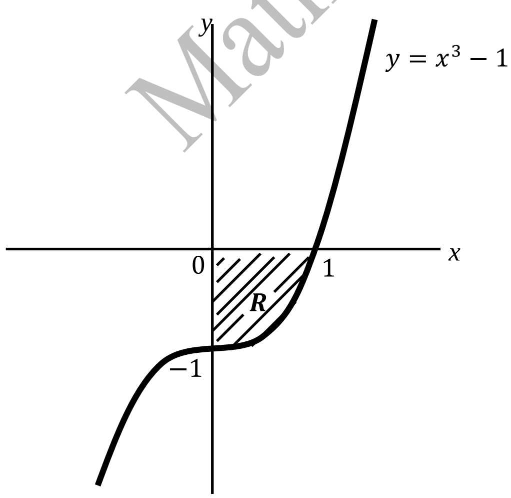
$du=dx$

$\therefore \int (x+1)^{\frac{1}{3}}dx=\int n^{\frac{1}{3}}\,du$

$=\frac{3}{4}u^{\frac{4}{3}}+c$

$=\frac{3}{4}(x+1)^{\frac{4}{3}}+c$

(b) Given



To calculate the volume of the solid that results from rotating R about the y-axis

$v=\int_a^b \pi x^2 dy$

Finding an expressing for x^2

$y=x^3-1$

$y + 1 = x^3$

$(y+1)^{\frac{1}{3}} = x$

$(y+1)^{\frac{2}{3}} = x^2$

$\therefore v = \pi \int_{-1}^{0} (y+1)^{\frac{2}{3}} dy$

$\quad = \pi \left[\frac{3}{5}(y+1)^{\frac{5}{3}}\right]_{-1}^{0}$

$\quad = \pi \left[\frac{3}{5}(0+1)^{\frac{5}{3}} - \frac{3}{5}(-1+1)^{\frac{5}{3}}\right]$

$\quad = \pi \cdot \frac{3}{5}\ [1-0]$

$\quad = \frac{3\pi}{5}\,\text{unit}^3$

(c) Given $\int_0^a f(x)dx = \int_0^a f(a-x)dx \qquad a > 0$

To show: $\int_0^1 \frac{e^x}{e^x + e^{1-x}} dx = \frac{1}{2}$

$\int_0^1 \frac{e^x}{e^x+e^{1-x}} dx = \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^{1-(1-x)}} dx$ …..(1) since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Simplifying the RHS of equation (1):

$\int_0^1 \frac{e^{1-x}}{e^{1-x}+e^{1-(1-x)}} dx = \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^x} dx$

Substituting in (1):

$\int_0^1 \frac{e^x}{e^x+e^{1-x}} dx = \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^x} dx$ …..(2)

$\therefore \int_0^1 \frac{e^x}{e^x+e^{1-x}} dx + \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^{1-(1-x)}} dx = \int_a^b \frac{e^x}{e^x+e^{1-x}} dx + \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^x} dx$ …..(3)

Using $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b [f(x)+g(x)]dx$ to add the integrals on the RHS of (3)

$\therefore \int_a^b \frac{e^x}{e^x+e^{1-x}} dx + \int_0^1 \frac{e^{1-x}}{e^{1-x}+e^x} dx = \int_0^1 \frac{e^x+e^{1-x}}{e^x+e^{1-x}} dx$

$= \int_0^1 1 dx$

$= [x]_0^1$

$= 1 - 0$

$= 1$

$\therefore 2\int_0^1 \frac{e^x}{e^x+e^{1-x}} dx = 1$

$\therefore \int_0^1 \frac{e^x}{e^x+e^{1-x}} dx = \frac{1}{2}$

(d) Given:

- y_0, population of bacteria a $t = 0$ is 10,000
- Bacteria growth rate: 2% per hour
- $y = f(t)$ where *y* represents the number of bacteria present *t* hours after the initial population was taken

(i) To solve an appropriate differential equation to show $y = 10{,}000e^{0.02t}$

$\frac{dy}{dt}$ represents the growth rate of the bacteria

$\therefore \frac{dy}{dt} = 0.02y$

Separating the variables

$\frac{dy}{y} = 0.02dt$

$\int \frac{dy}{y} = \int 0.02dt$

$\ln y = 0.02t + c$

$y = e^{0.02t+c}$

$y = e^c e^{0.02t}$

Let $A = e^c$

$y = A\, e^{0.02t}$

When $t = 0$ $y = 10{,}000$

$y = Ae^0$

$y = A$

$\therefore A = 10{,}000$

$\therefore y = 10{,}000e^{0.02t}$

(i) To find *t* when $y = 20{,}000$

$20{,}000 = 10{,}000e^{0.02t}$

$2 = e^{0.02t}$

$\ln 2 = \ln e^{0.02t}$

$\ln 2 = 0.02t$

$t = \frac{\ln 2}{0.02}$

$t = 34.657 \ldots \text{hours}$

$t = 34.7 \text{ hours to 1 d.p.}$

6. (a) Given $f(x) = 2x^3 + 5x^2 - x + 12$

To find the equation of the tangent at $x = 3$

$f'(x) = 6x^2 + 10x - 1$

When $x = 3$

$f'(3) = 6(3)^2 + 10(3) - 1$

$= 54 + 30 - 1$

$= 83$

$f(3) = 2(3)^3 + 5(3)^2 - 3 + 12$

$= 54 + 45 - 3 + 12$

$= 108$

Let equation at tangent be $y = mx + c$

Substituting $(3,108)$, $m = 83$

$108 = 83(3) + c$

$108 = 249 + c$

$c = -141$

$\therefore y = 83x - 141$

Therefore, the equation of the tangent at the point where $x = 3$ is $y = 83x - 141$

(b) Given $f(x) = \begin{cases} x^2 + 2x + 3 & x \leq 0 \\ ax + b & x > 0 \end{cases}$

(i) To calculate $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 2x + 3) = 3$ (by direct substitution)

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (ax + b) = b$ (by direct substitution)

(ii) Hence to find the value of a and b such that $f(x)$ is continuous at $x = 0$

If $f(x)$ is continuous at $x = 0$ then

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\therefore b = 3$

In addition, if $f(x)$ is continuous at $x = 0$ then $f(0) = 3$

$f(x) = ax + b$

$f(0) = a(0) + 3$

$f(0) = 3$

Note that these two conditions are satisfied regardless of the value of a.

That is, "a" can be any real number.

(ii) If $b = 3$, to determine a such that $f'(0) = \lim_{t \to 0} \frac{f(0+t) - f(0)}{t}$

$f'(x) = 2x + 2$

$f'(0) = 2$

$\lim_{t \to 0} \frac{f(0+t) - f(0)}{t} = \lim_{t \to 0} \frac{[a(0+t)+b] - [a(0)+b]}{t}$

$= \lim_{t \to 0} \frac{at + b - b}{t}$

$= \lim_{t \to 0} \frac{at}{t} = a$

$\therefore a = 2$

(c) To differentiate $f(x) = \sqrt{x}$ from first principle

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \sqrt{x+h}$$

$$f(x) = \sqrt{x}$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalizing

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h[\sqrt{x+h} + \sqrt{x}]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{1}{2\sqrt{x}}$$