CAPE Unit 2
Pure Mathematics
June 2016
Solutions

1. (a) Given: $ax^2 + bx + c = 0 \quad a, b, c \in \mathbb{R}$, has complex roots $\alpha = 1 - 3i,\ \beta$

(i) To calculate $(\alpha + \beta)$ and (α, β)

Note: The roots of a quadratic equation occur in conjugates since $a, b, c \in \mathbb{R}$

$\therefore\ \beta = 1 + 3i$

$\therefore \alpha + \beta = 1 + 3i + 1 - 3i$

$= 2$

$\alpha\beta = (1 + 3i)(1 - 3i)$

$= 1 - 9i^2$

$= 10$

(ii) Hence, to show that an equation with roots $\frac{1}{\alpha-2}$ and $\frac{1}{\beta-2}$ as given by $10x^2 + 2x + 1 = 0$

Sum of Roots

$$\frac{1}{\alpha - 2} + \frac{1}{\beta - 2}$$

$$= \frac{\beta - 2 + \alpha - 2}{(\alpha - 2)(\beta - 2)} = \frac{\alpha + \beta - 4}{\alpha\beta - 2\alpha - 2\beta + 4}$$

$$= \frac{\alpha + \beta - 4}{\alpha\beta - 2(\alpha + \beta) + 4}$$

$$= \frac{2 - 4}{10 - 2(2) + 4} = -\frac{2}{10} = -\frac{1}{5}$$

Product of Roots

$$\frac{1}{\alpha - 2} \cdot \frac{1}{\beta - 2} = \frac{1}{\alpha\beta - 2(\alpha + \beta) + 4}$$

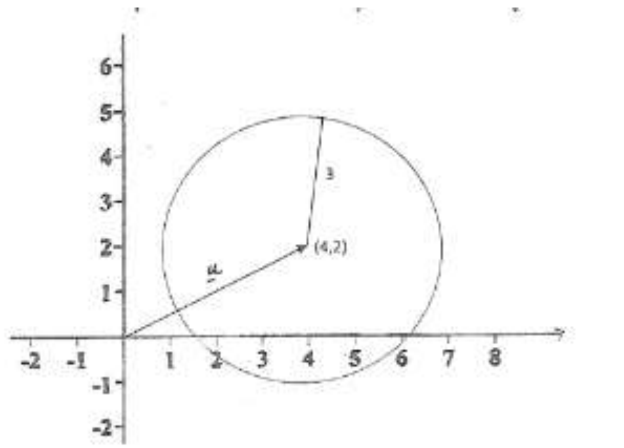
$$\frac{1}{10 - 2(2) + 4} = \frac{1}{10}$$

$\therefore$ Equation is $x^2 - \left(-\frac{1}{5}\right)x + \frac{1}{10} = 0$

Multiplying by 10: $10x^2 + 2x + 1 = 0$

(b) Given: $4 + 2i = u$ and $v=1 + 2\sqrt{2}i$

(i) To complete to complete the Argand diagram to illustrate u



(iii) To calculate the modulus and principal argument of $z = \left(\frac{u}{v}\right)^5$

$$\frac{u}{v} = \frac{4+2i}{1+2\sqrt{2}i} \cdot \frac{1-2\sqrt{2}i}{1-2\sqrt{2}i}$$

$$= \frac{4 - 8\sqrt{2}i + 2i - 4\sqrt{2}i^2}{1 - 8i^2}$$

$$= \frac{(4+4\sqrt{2}) + (2-8\sqrt{2})i}{9} = \frac{4+4\sqrt{2}}{9} + \frac{2-8\sqrt{2}i}{9}$$

$$\text{Modulus} = \sqrt{\left(\frac{4+4\sqrt{2}}{9}\right)^2 + \left(\frac{2-8\sqrt{2}}{9}\right)^2}$$

$$= \sqrt{\frac{16 + 32\sqrt{2} + 32 + 4 - 32\sqrt{2} + 128}{81}}$$

$$= \sqrt{\frac{180}{81}}$$

$$= \frac{\sqrt{36} \times 5}{\sqrt{81}} = \frac{6}{9}\sqrt{5} = \frac{2}{3}\sqrt{5}$$

$$\therefore \text{ Modulus of } \left(\frac{u}{v}\right)^5 = 5\left(\frac{2}{3}\sqrt{5}\right)$$

$$= 7.45 \text{ to 3 sig. figures}$$

$$\text{Argument} = \tan^{-1}\left(\frac{\frac{2-8\sqrt{2}}{9}}{\frac{4+4\sqrt{2}}{9}}\right)$$

$$= \tan^{-1}\left(\frac{1-4\sqrt{2}}{2+2\sqrt{2}}\right)$$

$$= \tan^{-1}\frac{-4.6568\ldots}{4.8284\ldots} = -0.767r \text{ to 3 sig. figures}$$

$$\text{Argument of } \left(\frac{u}{v}\right)^5 = 5(-0.767)$$

$$= -3.836$$

Note: The principal argument must fall in one of the intervals $0 \leq \theta \leq \pi$ or $0 \leq \theta < \pi$

The angle -3.836 radians is not in either of these two and thus must be converted to an equal angle which falls in one of them. -3.836 falls in the second quadrant since $-3.836 = \pi + 0.6944\ldots$

Therefore, the principal argument is $\pi - 0.6944 = 2.45r$ to 3 sig. fig

The answer could also have been found by add 2π to -3.836.

Alternate method for finding the argument:

Note: $\arg\left(\frac{u}{v}\right) = \arg u - \arg v$

$$= \tan^{-1}\left(\frac{2}{4}\right) - \tan^{-1}\left(\frac{2\sqrt{2}}{1}\right)$$

$$= 0.464 - 1.23$$

$$= -0.766$$

$$\arg\left(\frac{u}{v}\right)^5 = 5(-0.766)$$

$$= -3.83 + 2\pi$$

$$= 2.45 \text{ radians to 3 sig. figs.}$$

(c) Given: $x = 4\cos t$ and $y = 3\sin 2t, \quad 0 \leq t \leq \pi$

To determine: The x-coordinates of the two stationary points of f

$y = 3\sin 2t \qquad x = 4\cos t$

$\frac{dy}{dt} = 6\cos 2t \qquad \frac{dx}{dt} = -4\sin t$

$$\frac{dy}{dx} = \frac{6\cos 2t}{-4\sin t} = \frac{6\,(\cos^2 t - \sin^2 t)}{-4\sin t}$$

At a stationary point $\frac{dy}{dx} = 0$

$\therefore 6\cos 2t = 0$

$\cos 2t = 0$

$$\therefore 2t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

x- coordinates are:

$x = 4\cos\frac{\pi}{4}$ $\quad$ $x = 4\cos\frac{3\pi}{4}$

$= 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$ $\quad$ $= 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

2. (a) Given: $w(x, y) = \ln\left|\frac{2x+y}{x-10}\right|$

To determine: $\frac{\partial w}{\partial x}$

$w(x, y) = \ln|2x + y| - \ln|x - 10|$

$$\frac{\partial w}{\partial x} = \frac{2}{2x + y} - \frac{1}{x - 10}$$

(b) To determine $\int e^{2x} \sin e^x$

Note: You can use the formula $\int u\frac{dv}{dx}dx = uv - \int u\frac{du}{dx}dx$ or you can use the Reverse of the Product Rule. Both are actually the same.

Thinking through the Product Rule. Which function if I apply the product rule to it would give me one expression being $e^{2x}\sin x$? By trial and error this is:

$y = e^x(-\cos e^x)$

$\frac{dy}{dx} = e^{2x}\sin e^x + (-\cos e^x)e^x$

Integrating both sides

$\int \frac{dy}{dx}dx = \int e^{2x}\sin e^x dx - \int e^x \cos e^x dx$

$\int \frac{dy}{dx}dx = y = e^x(-\cos e^x)$

$\therefore e^x(-\cos e^x) + \int e^x \cos e^x dx = \int e^{2x}\sin e^x dx$

$\int e^{2x}\sin e^x dx = -e^x \cos e^x + \int e^x \cos e^x dx$

Finding $\int e^x \cos e^x dx$

$y = \sin e^x$

$\frac{dy}{dx} = (\cos e^x)(e^x) \Rightarrow \int e^x \cos e^x dx = y \sin e^x + C$

$\therefore \int e^{2x}\sin e^x dx = -e^x \cos e^x + \sin e^x + c$

(c) (i) Given: $f(x) = \frac{x^2+2x+3}{(x-1)(x^2+1)}$ for $2 \leq x \leq 5$

To use the trapezium rule with three intervals to estimate the area bounded by

$y = 0, x = 2$ and $x = 5$

$f(2) = \frac{2^2+2(2)+3}{(2-1)(2^2+1)} = \frac{4+4+3}{1(5)} = \frac{11}{5}$

$f(3) = \frac{3^2+2(3)+3}{(3-1)(3^2+1)} = \frac{9+6+3}{(2)(10)} = \frac{18}{20}$

$f(4) = \frac{4^2+2(4)+3}{(4-1)(4^2+1)} = \frac{16+8+3}{(3)(17)} = \frac{27}{51}$

$f(5) = \frac{5^2+2(5)+3}{(5-1)(5^2+1)} = \frac{25+10+3}{(4)(26)} = \frac{38}{104}$

Area $\simeq= \frac{1}{2}[f(2) + f(3)]1 + \frac{1}{2}[f(3) + f(4)]1 + \frac{1}{2}[f(4) + f(5)]1$

$= \frac{1}{2}\left(\frac{11}{5} + \frac{9}{10}\right) + \frac{1}{2}\left(\frac{9}{10} + \frac{9}{17}\right) + \frac{1}{2}\left(\frac{9}{17} + \frac{19}{52}\right)$

$= 1\frac{11}{20} + \frac{243}{340} + \frac{791}{1768}$

$= \frac{4795}{1768}$

$= 2.71210 \ldots = 2.71$ to 3 sig. figs.

(ii) To use partial fractions to show that $f(x) = \frac{3}{x-1} - \frac{2x}{x^2+1}$

$\frac{x^2+2x+3}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+c}{x^2+1}$

$x^2 + 2x + 3 = A(x^2 + 1) + (Bx + c)(x - 1)$

$x^2 + 2x + 3 = Ax^2 + A + Bx^2 - Bx + +Cx - c$

Let $x = 1$

$1^2 + 2(1) + 3 = A + A + 0$

$2A = 6$

$A = 3$

Equating the terms in x^2 on both sides of the equation:

$1 = A + B$

$1 = 3 + B$

$B = -2$

Equating the terms independent of x on both sides of the equation

$3 = A - C$

$3 = 3 - C$

$C = 0$

$\therefore \frac{x^2+2x+3}{(x-1)(x^2+1)} \equiv \frac{3}{x-1} - \frac{2x}{x^2+1}$

(iii) Hence, to integrate: $\int_2^5 f(x)dx$

$$\int_2^5 \frac{x^2+2x+3}{(x-1)(x^2+1)}dx = 3\int_2^5 \frac{1}{x-1}dx - \int_2^5 \frac{2x}{x^2+1}dx$$

$= 3\ln|x-1| - \ln|x^2+1|$

$= \left[\ln\left|\frac{(x-1)^3}{x^2+1}\right|\right]_2^5$

$= \ln\left(\frac{64}{26}\right) - \ln\left(\frac{1}{5}\right)$

$= \ln\left(\frac{64}{26}\cdot\frac{5}{1}\right)$

$= \ln\frac{160}{13}$

$= 2.51$ to 2 d.p.

3. (a) Given: $U_{n+1} = U_{n-1} + x(U_n)'$ $\quad u_1 = 1, u_2 = x$

To find $(U_9)'$

Let $n = 9$

$U_{9+1} = U_{9-1} + x(U_9)'$

$U_{10} = U_8 + x(U_9)'$

$34x + 1 = 13x + 1 + x\,(U_9)'$

$21x = x(U_9)'$

$21 = (U_9)'$

OR working each element in the sequence…

$U_1 = 1$

$U_2 = x$

$U_3 = U_{2+1} = U_{2-1} + x(U_2)'$

$= U_1 + x(1)$

$= 1 + x$

$U_4 = U_{3+1} = U_{3-1} + x\,(U_3)'$

$= U_2 + x(1)$

$= x + x$

$= 2x$

$U_5 = U_{4+1} = U_{4-1} + x(U_4)'$

$= U_3 + x(2)$

$= 1 + x + 2x$

$= 1 + 3x$

$U_6 = U_{5+1} = U_{5-1} + x(U_5)'^1$

$= U_4 + x(3)$

$= 2x + 3x$

$5x$

$U_7 = U_{6+1} = U_{6-1} + x(U_6)'$

$= U_5 + x(5)$

$= 1 + 3x + 5x$

$= 1 + 8x$

$U_8 = U_{7+1} = U_{7-1} + x(U_7)'$

$= U_6 + x(8)$

$= 5x + 8x$

$= 13x$

$U_9 = U_{8+1} = U_{8-1} + x(U_8)'$

$= U_7 + x(13)$

$= 1 + 8x + 13x$

$= 1 + 21x$

$\therefore (U_9)' = 21$

(b) (i) Given $S_n = \sum_{r=1}^{n} r(r-1)$

To show $S_n = \frac{n(n^2-1)}{3}$

$r(r-1) = r^2 - r$

$\therefore \sum_{r=1}^{n} r(r-1) \equiv \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$

Note: Students should know:

$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

$\therefore \sum_{r=1}^{n} r(r-1) \equiv \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$

$\sum_{r=1}^{n} r(r-1) = \frac{n(n+1)}{2}\left[\frac{2n+1}{3} - \frac{3}{3}\right]$

$= \frac{n(n+1)(2n-2)}{6}$

$$= \frac{n(n+1)(n-1)}{3} = \frac{n(n^2-1)}{3}$$

(ii) Hence to evaluate $\sum_{r=10}^{20} r(r-1)$

$$\sum_{r=10}^{20} r(r-1) = \sum_{r=1}^{20} r(r-1) - \sum_{r=1}^{9} r(r-1)$$

$$= \frac{20(20^2-1)}{3} - \frac{9(9^2-1)}{3}$$

$$= \frac{7980}{3} - \frac{720}{3} = \frac{7260}{3}$$

$$= 2420$$

(c) Given: ${}^{n}P_r = \frac{n!}{(n-r)!}$

To show: $\frac{{}^{2r}P_r\, {}^{n}P_r}{(2r)!}$ = binomial coefficient ${}^{n}C_r$

Note: ${}^{n}C_r = \frac{n!}{(n-r)!r!}$

$${}^{2r}P_r = \frac{(2r)!}{(2r-r)!} = \frac{2r!}{r!}$$

$${}^{n}P_r = \frac{n!}{n-r!}$$

$$\therefore \left(\frac{{}^{2r}P_r \cdot {}^{n}P_r)}{(2r)!}\right. = \frac{2r!}{r!} \cdot \frac{n!}{(n-r)!} \cdot \frac{1}{(2r)!} = \frac{n!}{(n-r)!r!} = {}^{n}C_r$$

(i) To determine the coefficient of the term in x^3 in the binomial expansion of $(3x+2)^5$

$$(3x+2)^5 = (3x)^5 + {}^{5}C_1(3x)^4(2) + {}^{5}C_2(3x)^3(2)^2 + \cdots$$

$${}^{5}C_2 = 10$$

$$\therefore\ {}^{5}C_2(3x)^3(2)^2 = (10)(27x^3)(4)$$

$$= 1080x^3$$

$\therefore$ the coefficient of the x^3 term is: 1080

4. (a) Given: $f(x) = \sqrt[6]{4x^2+4x+1}$ for $-1 < x < 1$

(i) To show $f(x) = (1+2x)^{\frac{1}{3}}$

$$4x^2+4x+1 \equiv (2x+1)^2$$

$$\equiv (1+2x)^2$$

$$\therefore \sqrt[6]{4x^2+4x+1} \equiv \sqrt[6]{(1+2x)^2}$$

$$\equiv (1+2x)^{\frac{2}{6}}$$

$$\equiv (1+2x)^{\frac{1}{3}}$$

Given: $(1+x)^k$ as $1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\cdots \qquad k\in\mathbb{R} \quad -1<x<1$

(ii) To determine the series expansion of $f(x)$ up to and including the term in x^4

$$f(x)=(1+2x)^{\frac{1}{3}}$$

$$=1+\frac{1}{3}(2x)+\frac{\left(\frac{1}{3}\right)\left(-1+\frac{1}{3}\right)(2x)^2}{2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)(2x)^3}{6}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)(2x)^4}{24}$$

$$=1+\frac{2}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)4x^2}{2}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)8x^3}{6}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)16x^4}{24}+\cdots$$

$$=1+\frac{2}{3}x-\frac{4}{9}x^2+\frac{40}{81}x^3-\frac{160}{243}x^4+\cdots$$

(iii) Hence to approximate $f(0.4)$ to 2 d.p.

$$f(0.4)=1+\frac{2}{3}(0.4)-\frac{4}{9}(0.4)^2+\frac{40}{81}(0.4)^3-\frac{160}{243}(0.4)^4$$

$$=1+0.266-0.0711+0.03160-0.01685+\cdots$$

$$=1.21 \text{ to 2 d.p.}$$

(b) Given: $h(x)=x^3+x-1$ [0,1]

(i) To show $h(x)=0$ has a root in the interval; [0,1]

$h(o)=0^3+0-1=-1$

$h(1)=1^3+1-1=1$

$h(x)$ is a polynomial and hence is a continuous function. Since there is a sign change going from $h(0)$ to $h(1)$ and the function is continuous, then there must be at least one root of the equation $h(x)=0$ in the interval [0,1].

(ii) To use the iteration

$$x_{n+1}=\frac{1}{(x_n)^2+1}$$

with initial estimate $x_1=0.7$ to estimate the root of h to 2 d.p.

$$x_2=\frac{1}{(x_1)^2+1}$$

$$=\frac{1}{(0.7)^2+1}=\frac{1}{1.49}$$

$$=0.671140939\ldots$$

$$x_3=\frac{1}{(x_2)^2+1} \text{ etc.}$$

$x_3=0.68945\ldots$

$x_4=0.67780\ldots$

$x_5=0.68520\ldots$

$x_6=0.6805\ldots$

$x_7 = 0.6834$

$\therefore$ estimate of root $= 0.68$ to 2 d.p.

(c) Given: $g(x) = e^{4x-3} - 4$

Initial estimate $x_1 = 1$

To use Newton-Raphson's method with two iterations to approximate the root in the interval [1,2]

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$g(x) = e^{4x-3} - 4$

$g'(x) = 4e^{4x-3}$

$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

$\frac{g(1)}{g'(1)} = \frac{e-4}{4e} \qquad \therefore x_2 = 1 - \frac{e-4}{4e}$

$x_2 = 1 - (-0.117879 \ldots)$

$x_2 = 1 + 0.117879 \ldots$

$x_2 = 1.117879 \ldots$

$$x_3 = x_2 - \frac{g(x_2)}{g'(x_2)}$$

$g(1.117879 \ldots) = 0.355833581 \ldots$

$g'(1.117879 \ldots) = 17.42333433$

$$x_3 = 1.117879 - \frac{0.355833581}{17.42333433}$$

$= 1.097456 \ldots$

$= 1.097$ to 3 d.p.

5. (a) (i) Given: 13 seats and 8 passengers

To determine the number of possible seating arrangements

$= {}^{13}P_8 = {}^{13}C_8 \cdot 8! = \frac{13!}{5!8!} \cdot 8! = 51{,}891{,}840$

(ii) Given: 5 spaces, 8 persons 3 of whom must be together

To determine the number of groups of 5 that can fill the spaces.

Case 1: The 3 are not in the 5 spaces. 1 group possible.

Case 2: The 3 are in the 5 spaces: ${}^{5}C_2 = 10$

$\therefore$ there are 11 possible groups of 5.

(b) Given: Gavin and Alexander are two of 5 batsmen

To find $P(GA)$ that is, Gavin and Alex are the opening pair

Number of ways of selecting Gavin and Alex as opening pair is: $2P2 \times 3P3 = 12$

GA 1 2 3	AG 1 2 3
GA 1 3 2	AG 1 3 2
GA 2 1 3	AG 2 1 3
GA 2 3 1	AG 2 3 1
GA 3 1 2	AG 3 1 2
GA 3 2 1	AG 3 2 1

If arrangement is random, the number of arrangements is: $5P5 = 120$

$\therefore P(AG \text{ or } GA) = \frac{12}{120} = \frac{1}{10}$

(c) Given: $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{pmatrix}$

(i) To find det A

Method 1:

$$\begin{array}{ccccc} 2 & 1 & -1 & 2 & 1 \\ 0 & 4 & 3 & 0 & 4 \\ -1 & 6 & 0 & -1 & 6 \end{array}$$

$= [0 + (-3) + 0] - [4 + 36 + 0]$

$= -3 - 40 = -43$

Method 2:

Using R1

$+(-1)\begin{vmatrix} 0 & 4 \\ -1 & 6 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 6 & 0 \end{vmatrix}$

$+2\begin{vmatrix} 4 & 3 \\ 6 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 4 \\ -1 & 6 \end{vmatrix}$

$2(-18) - 1(3) - 1(4)$

$= -36 - 3 - 4 = -43$

(ii) Hence, or otherwise, to find A^{-1}

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 4 & 6 \\ -1 & 3 & 0 \end{pmatrix}$$

$\begin{vmatrix} 4 & 6 \\ 3 & 0 \end{vmatrix} = -18$ $\quad\begin{vmatrix} 1 & 6 \\ -1 & 0 \end{vmatrix} = +6$ $\quad\begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} = 7$

$\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = 3$ $\quad\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = -1$ $\quad\begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = 6$

$\begin{vmatrix} 0 & -1 \\ 4 & 6 \end{vmatrix} = 4$ $\quad\begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} = 13$ $\quad\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8$

$$adj\ A = \begin{pmatrix} -18 & -6 & 7 \\ -3 & -1 & -6 \\ 4 & -13 & 8 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{-43}\begin{pmatrix} -18 & -6 & 7 \\ -3 & -1 & -6 \\ 4 & -13 & 8 \end{pmatrix}$$

Checking: $\frac{1}{-43}\begin{pmatrix} -18 & -6 & 7 \\ -3 & -1 & -6 \\ 4 & -13 & 8 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$R1C1 = -36 + 0 - 7 = -43$

$R1C2 = -18 - 24 + 42 = 0$

$R1C3 = 18 - 18 + 0 = 0$

$R2C1 = -6 + 0 + 6 = 0$

$R2C2 = -3 - 4 - 36 = -43$

$R2C3 = 3 - 3 + 0 = 0$

$R3C1 = -4 - 39 + 0 = -43$

$R3C2 = 4 - 52 + 48 = 0$

$R3C3 = -4 - 39 + 0 = -43$

6. (a) Given: two fair coins and a fair die tossed at the same time

(i) To calculate $n(S)$

$n(S) = 2 \times 2 \times 6 = 24$

$S =$	HH1	HH2	HH3	HH4	HH5	HH6
	HT1	HT2	HT3	HT4	HT5	HT6
	TH1	TH2	TH3	TH4	TH5	TH6
	TT1	TT2	TT3	TT4	TT5	TT6

(ii) To find $P(\text{one head})$

$P(\text{one head}) = \frac{2\times1\times6}{24} = \frac{1}{2}$

Or, counting above: $n(\text{one head}) = 12$

$\therefore P(\text{one head}) = \frac{12}{24} = \frac{1}{2}$

(iii) To find $P(\geq 1H \text{ and even } \#)$

$n(\geq 1H \text{ and even } \#) = 9$

$$\therefore P(\geq 1H \text{ and even } \#) = \frac{9}{24} = \frac{3}{8}$$

(b) To determine if $y = C_1x + C_2x^2$ is a solution to $\frac{x^2}{2}y'' - xy' + y = 0$ ----(1)

$y = C_1x + C_2x^2$

$y' = C_1 + 2C_2x$

$y'' = 2C_2$

Substituting in (1)

$$\frac{x^2}{2} \cdot 2C_2 - x(C_1 + 2C_2x) + C_1x + C_2x^2 = 0$$

$$LHS = C_2x^2 - C_1x - 2C_2x^2 + C_1x + C_2x^2$$

$$= C_2x^2 - 2C_2x^2 + C_2x^2 - C_1x + C_1x$$

$$= 0 = RHS$$

$\therefore y = C_1x + C_2x^2$ is a solution to the given differential equation.

(c) (i) Given: $3(x^2 + x)\frac{dy}{dx} = 2y(1 + 2x)$

To show that the general solution is $y = C\sqrt[3]{(x^2 + x)^2} \quad C \in \mathbb{R}$

$$3(x^2 + x)\frac{dy}{dx} = 2y(1 + 2x)$$

Separating the variahbles

$$\frac{dy}{y} = \frac{2(1+2x)}{3(x^2+x)}$$

$$\int \frac{dy}{y} = \int \frac{2}{3}\frac{(1+2x)}{(x^2+x)}dx$$

$$\int \frac{dy}{y} = \frac{2}{3}\int \frac{1+2x}{x^2+x}dx$$

$$\ln y = \frac{2}{3}\ln|x^2+x| + \ln C$$

$$\ln y = \ln(x^2+x)^{\frac{2}{3}} + \ln C$$

$$\ln y = \ln C(x^2+x)^{\frac{2}{3}}$$

$$y = C(x^2+x)^{\frac{2}{3}}$$

$$y = C\sqrt[3]{(x^2+x)^2}$$

(ii) Hence, given $y(1) = 1$

To solve $3(x^2+x)\frac{dy}{dx} = 2y(1+2x)$

Substituting $(1,1)$ in y to find C

$$y = C\sqrt[3]{(x^2+x)^2}$$

$$1 = C\sqrt[3]{(1^2+1)^2}$$

$$1 = C\sqrt[3]{4}$$

$$\frac{1}{\sqrt[3]{4}} = C$$

$$\therefore y = \frac{1}{\sqrt[3]{4}}\sqrt[3]{(x^2+x)^2}$$

$$y = \sqrt[3]{\frac{(x^2+x)^2}{4}}$$

