CAPE Unit 1
Pure Mathematics
June 2014
Solutions

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p	q	r	$p \to q$	$r \to q$	$(p \to q) \wedge (r \to q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

[5 marks]

b. Given $y \oplus x = y^3 + x^3 + ay^2 + ax^2 - 5y - 5x + 16,\ a \in \mathbb{N};\ \ x, y \in \mathbb{R}^+$

i. To state, with reason, if $\oplus$ is commutative

$y \oplus x = (y^3 + x^3) + a(y^2 + x^2) - 5(y + x) + 16$

$x \oplus y = (x^3 + y^3) + a(x^2 + y^2) - 5(x + y) + 16$

$x^2,\ y^2, x^3,\ y^3 \in \mathbb{R}^+$ since multiplication is closed in $\mathbb{R}^+$

$y^3 + x^3 = x^3 + y^3$ since addition is commutative in $\mathbb{R}^+$

$y^2 + x^2 = x^2 + y^2$ since addition is commutative in $\mathbb{R}^+$

$y + x = y + x$ since addition is commutative in $\mathbb{R}^+$

$\therefore (y^3 + x^3) + a(y^2 + x^2) - 5(y + x) + 16 = (x^3 + y^3) + a(x^2 + y^2) - 5(x + y) + 16$

$y \oplus x = x \oplus y$

Therefore $\oplus$ is commutative. [3 marks]

ii. Given $f(x) = 2 \oplus x$

$(x - 1)$ is a factor of $f(x)$

To find:

a. the value of a.

$f(x) = 2 \oplus x = 2^3 + x^3 + a(2)^2 + ax^2 - 5(2) - 5x + 16$

$= 8 + x^3 + 4a + ax^2 - 10 - 5x + 16$

$= x^3 + ax^2 - 5x + 4a + 14$

Since $(x - 1)$ is a factor:

$x^3 + ax^2 - 5x + 4a + 14 \equiv (x - 1)(Q)$

Let $x = 1$

$1 + a - 5 + 4a + 14 = 0$

$5a + 10 = 0$

$a = -2$ [4 marks]

b. To factorize $f(x)$ completely

$f(x) = x^3 - 2x^2 - 5x - 8 + 14$

$\quad\;\; = x^3 - 2x^2 - 5x + 6$

$\therefore x^3 - 2x^2 - 5x + 6 \equiv (x-1)(Ax^2 + Bx + C)$

From inspection $A = 1, C = -6$

$\therefore x^3 - 2x^2 - 5x + 6 \equiv (x-1)(x^2 + Bx - 6)$

Equating the terms in x on both sides of the equation:

$-5x = -6x - Bx$

$-5 = -6 - B$

$B = -1$

$\therefore x^3 - 2x^2 - 5x + 6 \equiv (x-1)(x^2 - x - 6)$

$\therefore x^3 - 2x^2 - 5x + 6 \equiv (x-1)(x+2)(x-3)$ [3 marks]

c. To use math induction to prove that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n}{3}(4n^2 - 1)$

Let $P(n)$ be the proposition that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n}{3}(4n^2 - 1)$

Testing $P(1)$

LHS $= 1^2 = 1$

RHS $= \frac{1}{3}(4(1)^2 - 1) = 1$

$\therefore$ LHS = RHS and $P(1)$ is true

Assume that $P(k)$ is true

$\therefore 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k}{3}(4k^2 - 1)$ ……. (1)

Show that c; that is, if the proposition is true for $n = k$,
it must be true for $n = k + 1$

$P(k+1)$:

$$\underbrace{1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2} + [2(k+1) - 1]^2 = \frac{k+1}{3}(4(k+1)^2 - 1) \ldots (2)$$

$$\frac{k}{3}(4k^2 - 1) + [2(k+1) - 1]^2 = \frac{k+1}{3}[4(k^2 + 2k + 1) - 1]$$

$$\frac{k}{3}(4k^2 - 1) + (2k+1)^2 = \frac{k+1}{3}[4k^2 + 8k + 4 - 1]$$

$$\frac{k(4k^2-1)}{3} + \frac{3(2k+1)^2}{3} = \frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$\frac{4k^3-k+3(4k^2+4k+1)}{3} = \frac{4k^3+12k^2+11k+3}{3}$$

$$\frac{4k^3-k+12k^2+12k+3}{3} = \frac{4k^3+12k^2+11k+3}{3}$$

$$\frac{4k^3+12k^2+11k+3}{3} = \frac{4k^3+12k^2+11k+3}{3}$$

Therefore $P(k) \Rightarrow P(k+1)$

That is, if the proposition is true for $n = k$ it must be true for $n = k + 1$.

Since $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ then $P(n)$ is true $\forall\, n \in \mathbb{Z}^+$

Hence, $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n}{3}(4n^2 - 1)$ [10 marks]

2. a. Given: $f(x) = 2x^2 + 1$

$g(x) = \sqrt{\frac{x-1}{2}}$

where $1 \leq x \leq \infty, x \epsilon \mathbb{R}$

i. To determine, in terms of x,

a. $f^2(x)$

$f^2(x) = 2(2x^2 + 1)^2 + 1$ [3 marks]

b. $f[g(x)]$

$f[g(x)] = 2\left(\sqrt{\frac{x-1}{2}}\right)^2 + 1 = 2\left(\frac{x-1}{2}\right) + 1 = x$ [3 marks]

ii. f and g are inverses of each other since $fg(x) = x$ [1 mark]

b. Given that $a^3 + b^3 + 3a^2b = 5ab^2$

To show that $3\log\left(\frac{a+b}{2}\right) = \log a + 2\log b$

$$a^3 + b^3 + 3a^2b = 5ab^2$$

$$a^3 + b^3 + 3a^2b + 3ab^2 = 5ab^2 + 3ab^2$$

$$(a+b)^3 = 8ab^2$$

$$\frac{(a+b)^3}{8} = ab^2$$

$$\left(\frac{a+b}{2}\right)^3 = ab^2$$

$$\log\left(\frac{a+b}{2}\right)^3 = \log(ab^2)$$

$$3\log\left(\frac{a+b}{2}\right) = \log a + 2\log b$$ [5 marks]

c. i. To solve $e^x + \frac{1}{e^x} - 2 = 0$

let $y = e^x$

by substitution: $y + \frac{1}{y} - 2 = 0$

Multiplying both sides of the equation by y:

$$y^2 - 2y + 1 = 0$$

$$(y-1)(y-1) = 0$$

$$y = 1$$

$$\therefore e^x = 1$$

$$x = 0$$ [4 marks]

ii. To solve $\log_2(x+1) - \log_2(3x+1) = 2$

$$\log_2(x+1) - \log_2(3x+1) = 2$$

$$\log_2 \frac{x+1}{3x+1} = 2$$

Changing to index form

$$2^2 = \frac{x+1}{3x+1}$$

$$4(3x+1) = x + 1$$

$$12x - x = -3$$

$$x = -\frac{3}{11}$$ [4 marks]

d. To show that: $\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{\sqrt{2}+1}{\sqrt{2}-1}=10$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{\left(\sqrt{3}-1\right)^2+(\sqrt{3}+1)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$
$$=\frac{3-2\sqrt{3}+1+3+2\sqrt{3}+1}{2}$$
$$=4$$

$$\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{\left(\sqrt{2}-1\right)^2+(\sqrt{2}+1)^2}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}$$
$$=\frac{2-2\sqrt{2}+1+2+2\sqrt{2}+1}{1}$$
$$=6$$

$\therefore \frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{\sqrt{2}+1}{\sqrt{2}-1}=4+6=10$ [5 marks]

3. (a) (i) To prove: $\frac{\cot y-\cot x}{\cot x+\cot y}=\frac{\sin(x-y)}{\sin(x+y)}$

LHS$=\frac{\frac{\cos y}{\sin y}-\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}-\frac{\cos y}{\sin y}}\equiv\frac{\frac{\sin x\cos y-\sin y\cos x}{\sin x\sin y}}{\frac{\sin y\cos x-\sin x\cos y}{\sin x\sin y}}$

$$\equiv\frac{\sin x\cos y-\sin y\cos x}{\sin x\sin y}\times\frac{\sin x\sin y}{\sin y\cos x-\sin x\cos y}$$

$\equiv\frac{\sin(x-y)}{\sin(x+y)}$ (4 marks)

(ii) Hence or otherwise to solve:

$\frac{\cot y-\cot x}{\cot x+\cot y}=1 \quad 0\le y\le 2\pi$

when $\sin x=\frac{1}{2} \quad 0\le x\le\frac{\pi}{2}$

Note to student: $\sin x=\frac{1}{2}\Rightarrow\cos x=\frac{\sqrt{3}}{2}$

$$\frac{\cot y-\cot x}{\cot x+\cot y}\equiv\frac{\sin(x-y)}{\sin(x+y)}$$
$$\equiv\frac{\sin x\cos y-\sin y\cos x}{\sin x\cos y+\sin y\cos x}$$
$$\equiv\frac{\frac{1}{2}\cos y-\frac{\sqrt{3}}{2}\sin y}{\frac{1}{2}\cos y+\frac{\sqrt{3}}{2}\sin y}$$

$$\therefore \frac{\frac{1}{2}\cos y-\frac{\sqrt{3}}{2}\sin y}{\frac{1}{2}\cos y+\frac{\sqrt{3}}{2}\sin y}=1$$

$$\frac{1}{2}\cos y-\frac{\sqrt{3}}{2}\sin y=\frac{1}{2}\cos y+\frac{\sqrt{3}}{2}\sin y$$
$$0=2\frac{\sqrt{3}}{2}\sin y$$
$$0=\sqrt{3}\sin y$$
$$\sin y=0° \quad 0\le y\le 2\pi$$

$\therefore y=0,\pi,2\pi$ (8 marks)

3. (b) (i) To express $f(\theta) = 3\sin 2\theta + 4\cos 2\theta$ in the form $r\sin(2\theta + \alpha)$ where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$

Note: $a\sin x + b\cos x = R\ sin\ (x + \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$

$\therefore r = \sqrt{3^2 + 4^2} = 5$

$\tan\alpha = \frac{4}{3}$

$\alpha = 53.1°$ to 1 decimal place, or

$\alpha = 0.927$ radians to 3 significant figures

Note: Generally, you should give your answer in the measure they used for the range. Since the question used radians, it might be better to give your answer in radians

Hence, $3\sin 2\theta + 4\cos 2\theta = 5\sin(2\theta + 0.927)$ (4 marks)

(ii) (a) Hence or otherwise, to determine the value of θ for $0 \leq \theta \leq 2\pi$ radians at which $f(\theta)$is a minimum.

Note: $f(\theta)$ is a minimum when $\sin(2\theta + 0.927) = -1$

$\sin(2\theta + 0.927) = -1$

$2\theta + 0.927 = \frac{3\pi}{2}$

$\theta = \left(\frac{3\pi}{2} - 0.927\right) \div 2$

$\theta = 1.89$ radians to 2 decimal places (4 marks)

(b) To determine the maximum and minimum values of $\frac{1}{7-f(\theta)}$

Note; The maximum value of $f(\theta)$ occurs when $\sin(2\theta + 0.927) = 1$

Therefore the maximum value of $f(\theta)$ is $5(1) = 5$

Similarly, the minimum value of $f(\theta)$ occurs when $\sin(2\theta + 0.927) = -1$

Therefore the minimum value of $f(\theta)$ is $5(-1) = -5$

That is, $-5 \leq f(\theta) \leq 5$

The maximum and minimum values of $\frac{1}{7-f(\theta)}$ depend on these two values of $f(\theta)$

Substituting 5 and -5 for $f(\theta)$ in $\frac{1}{7-f(\theta)}$:

$\frac{1}{7-5} = \frac{1}{2}$

$\frac{1}{7--5} = \frac{1}{12}$

Hence the maximum and minimum values of $\frac{1}{7-f(\theta)}$ are $\frac{1}{2}$ and $\frac{1}{12}$ respectively. (5 marks)

4. (a) Given: L_1 and L_2 are diameters of a circle C

Equation of L_1: $x - y + 1 = 0$

Equation of L_2: $x + y - 5 = 0$

(i) To show: The coordinates of the centre of the circle (where L_1 and L_2 meet is (2,3)

Solving simultaneously

Adding both equations $2x - 4 = 0$

$$x = 2$$

Substituting in : $x - y + 1 = 0$

$$2 - y + 1 = 0$$

$$y = 3$$

Therefore the centre is (2,3) (3 marks)

(ii) Given further A, B are endpoints of L_1, with $A(1,2)$ and L_1 and L_2 bisect each other.

To determine the coordinates of B

Let B be (x, y). Then

$\frac{x+1}{2} = 2 \;\Rightarrow x = 3$

$\frac{y+2}{2} = 3 \;\Rightarrow y = 4$

Therefore B has coordinates (3,4) (3 marks)

(iii) Given further that a point moves such that its distance from C(2,3) is always $\sqrt{2}$ units.

To determine the locus of p

The locus of P is a circle centre (2,3) and radius $\sqrt{2}$.

The equation of the circle is: $(x-2)^2 + (y-3)^2 = 2$ (3 marks)

4. (b) Given: Parametric equations of a curve, S are

$x = \frac{1}{1+t}$, $y = \frac{t}{1-t^2}$

To determine the Cartesian equation of S.

$x = \frac{1}{1+t}$ Given

Multiplying both sides of the equation by $(1 + t)$

$(1 + t)x = (1 + t)\frac{1}{1+t}$

$x + tx = 1$

$tx = 1 - x$

$t = \frac{1-x}{x}$

Substituting this in $y = \frac{t}{1-t^2}$

$$y = \frac{\frac{1-x}{x}}{1-\left(\frac{1-x}{x}\right)^2}$$

$$y = \frac{\frac{1-x}{x}}{\frac{x^2-(1-x)^2}{x^2}}$$

$$y = \frac{\frac{1-x}{x}}{\frac{x^2-1+2x-x^2}{x^2}} = \frac{1-x}{x} \div \frac{x^2-1+2x-x^2}{x^2}$$

$$y = \frac{1-x}{x} \times \frac{x^2}{2x-1}$$

$$y = \frac{x(1-x)}{2x-1}$$

or, $2xy - y - x + x^2 = 0$

$x^2 - x + 2xy - y = 0$ (6 marks)

4. (c) Given: $P(3,-2,1)$, $Q(-1,\lambda,5)$, $R(2,1,-4)$ are the vertices of a triangle

(i) To express $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RP}$ in the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

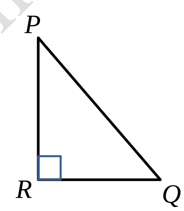
$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ \lambda \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 2+\lambda \\ 4 \end{pmatrix} = -4\boldsymbol{i} + (2+\lambda)\boldsymbol{j} + 4\boldsymbol{k}$$

$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR} = \begin{pmatrix} 1 \\ -\lambda \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1-\lambda \\ -9 \end{pmatrix} = 3\boldsymbol{i} + (1-\lambda)\boldsymbol{j} - 9\boldsymbol{k}$$

$$\overrightarrow{RP} = \overrightarrow{RO} + \overrightarrow{OP} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$$ (4 marks)

(ii) Hence, find λ, given triangle PQR is a right angled triangle with PQ the hypotenuse.

Note: The right angle is at R since PQ is the hypotenuse



$\overrightarrow{RP} \cdot \overrightarrow{RQ} = 0$

$$\overrightarrow{RQ} = \overrightarrow{RO} + \overrightarrow{OQ} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ \lambda \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ \lambda-1 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ \lambda-1 \\ 9 \end{pmatrix} = -3 - 3\lambda + 3 + 45 = 0$$

$-3\lambda = -45$

$\lambda = 15$ (6 marks)

5. (a) Given:

$$f(x) = \begin{cases} ax + 2 & x < 3 \\ ax^2 & x \geq 3 \end{cases}$$

(i) To find the value of a if $f(x)$ is continuous at $x = 3$

$\lim_{x \to 3^+} f(x) = (a)(3^2)$

$= 9a$

$\lim_{x \to 3^-} f(x) = (a)(3) + 2$

$= 3a + 2$

If $f(x)$ is continuous at $x = 3$ then

$9a = 3a + 2$

$a = \frac{1}{3}$ (4 marks)

(ii) Given $g(x) = \frac{x^2+2}{bx^2+x+4}$, and $\lim_{x \to 1} 2g(x) = \lim_{x \to 0} g(x)$

To find: the value of b

$\lim_{x \to 1} 2g(x) = 2 \lim_{x \to 1} g(x)$

$\lim_{x \to 1} 2g(x) = 2\left[\frac{1^2+2}{b(1^2)+1+4}\right]$

$\lim_{x \to 1} 2g(x) = 2\left[\frac{3}{b+5}\right]$

$\lim_{x \to 1} 2g(x) = \frac{6}{b+5}$

$\lim_{x \to 0} g(x) = \frac{0^2+2}{b(0^2)+0+4}$

$\lim_{x \to 0} g(x) = \frac{2}{4} = \frac{1}{2}$

$\therefore \frac{6}{b+5} = \frac{1}{2}$

$b + 5 = 12$

$b = 7$ (5 marks)

(b) (i) Given $y = \frac{1}{\sqrt{x}}$

To find $\frac{dy}{dx}$ from first principles

$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = \frac{1}{\sqrt{x}}$

$f(x + h) = \frac{1}{\sqrt{x+h}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{x}}{\sqrt{x+h}\sqrt{x}} - \frac{\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{(\sqrt{x+h}\sqrt{x})h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x-(x+h)}{h(\sqrt{x+h}\,\sqrt{x})\cdot(\sqrt{x}+\sqrt{x+h})}$$
$$= \lim_{h \to 0} \frac{-h}{h(\sqrt{x+h}\,\sqrt{x})\cdot(\sqrt{x}+\sqrt{x+h})}$$
$$= \lim_{h \to 0} \frac{-1}{(\sqrt{x+h}\,\sqrt{x})\cdot(\sqrt{x}+\sqrt{x+h})}$$
$$= \frac{-1}{x(2\sqrt{x})}$$
$$= -\frac{1}{2} \cdot \frac{1}{x\sqrt{x}} = -\frac{1}{2x^{\frac{3}{2}}}$$
$$= -\frac{1}{2} x^{-\frac{3}{2}}$$

Note: If $y = x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2} x^{-\frac{3}{2}}$ (8 marks)

(ii) Given: $y = \frac{x}{\sqrt{1+x}}$

To find: $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\sqrt{1+x}(1)-(x)\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}}{1+x}$$
$$\frac{dy}{dx} = \frac{\sqrt{1+x} - \frac{1}{2}x\left(\frac{1}{\sqrt{1+x}}\right)}{1+x}$$
$$\frac{dy}{dx} = \frac{\frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1+x}} - \frac{\frac{1}{2}x}{\sqrt{1+x}}}{1+x}$$
$$\frac{dy}{dx} = \frac{1+x-\frac{1}{2}x}{\sqrt{1+x}} \cdot \frac{1}{1+x}$$
$$\frac{dy}{dx} = \frac{1+\frac{1}{2}x}{(1+x)^{\frac{3}{2}}}$$
(4 marks)

(c) Given: $x = \cos\theta$

$y = \sin\theta$

To find: $\frac{dy}{dx}$ in terms of θ

$$\frac{dy}{d\theta} = \cos\theta$$
$$\frac{dx}{d\theta} = -\sin\theta$$
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$
$$\frac{dy}{dx} = \frac{\cos\theta}{-\sin\theta}$$
$$\frac{dy}{dx} = -\cot\theta$$
(4 marks)

6. (a) Given: Curve passes through the point $(-1,-4)$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

(i) To find

(a) the equation of the curve

$y = \int \frac{dy}{dx} dx = \int (3x^2 - 4x + 1) dx$

$y = x^3 - 2x^2 + x + c$

When $x = -1, y = -4$; substituting

$-4 = (-1)^3 - 2(-1)^2 + (-1) + c$

$-4 = -1 - 2 - 1 + c$

$c = 0$

$\therefore y = x^3 - 2x^2 + x$ (4 marks)

(b) the coordinates of the stationary points and the nature of the stationary points

At a stationary point $\frac{dy}{dx} = 0$

$\therefore 3x^2 - 4x + 1 = 0$

$(3x - 1)(x - 1) = 0$

$x = \frac{1}{3}, 1$

When $x = \frac{1}{3}$, $y = \left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + \frac{1}{3}$

$y = \frac{1}{27} - \frac{2}{9} + \frac{1}{3} = \frac{4}{27}$

When $x = 1$, $y = (1)^3 - 2(1)^2 + 1$

$y = 1 - 2 + 1 = 0$

Hence the coordinates of the stationary points are $\left(\frac{1}{3}, \frac{4}{27}\right)$ and $(1,0)$

From inspection $\left(\frac{1}{3}, \frac{4}{27}\right)$ is the maximum since it has the higher y coordinate and $(1,0)$ the minimum.

Or, $\frac{d^2y}{dx^2} = 6x - 4$

When $x = 1$, , $\frac{d^2y}{dx^2} = 6(1) - 4 = 2$ hence, this point is a minimum. Thus $\left(\frac{1}{3}, \frac{4}{27}\right)$ is a maximum. (8 marks)

(ii) To sketch the curve showing the stationary points and the intercepts

Note: the y intercept is 0 since the equation of the curve is $y = x^3 - 2x^2 + x$ (when $x = 0, y = 0$.)

Finding the x-intercepts

$$x^3 - 2x^2 + x = 0$$

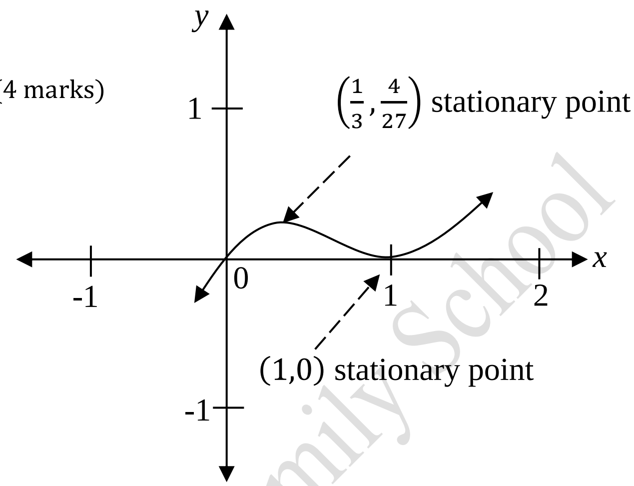
$$x(x^2 - 2x + 1) = 0$$

$$x(x-1)(x-1) = 0$$

Thus the x-intercepts are 0 and 1.

Note: Since there are two equal roots at $x = 1$, then the x-axis is a tangent to this point

(4 marks)



(b) Given: The equation of a curve is $f(x) = 2x\sqrt{1+x^2}$

(i) To evaluate $\int_0^3 f(x)dx$

$$\int_0^3 f(x)dx = \int_0^3 2x\sqrt{1+x^2}dx$$

Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

When $x = 3, u = 1 + 3^2 = 10$

When $x = 0, u = 1 + 0^2 = 1$

$$\therefore \int_0^3 2x\sqrt{1+x^2}dx = \int_1^{10} u^{\frac{1}{2}}du$$

$$\int_1^{10} u^{\frac{1}{2}}du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{10}$$

$$= \left[\frac{2}{3}(10)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left(\left(\sqrt{10}\right)^3 - 1\right)$$

(5 marks)

$= 20.4$ to 3 significant figures.

(ii) To find the volume when $f(x)$ is rotated about the x-axis between the limits $x = 0$ and $x = 2$.

Note; $V = \int_a^b \pi y^2 dx$

$y^2 = [f(x)]^2 = \left[2x\sqrt{1+x^2}\right]^2$

$y^2 = 4x^2(1+x^2) = 4x^2 + 4x^4$

$\therefore V = \int_0^2 \pi(4x^2 + 4x^4)\, dx$

$V = 4\pi \int_0^2 (x^2 + x^4) dx$

$V = 4\pi \left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$

$V = 4\pi \left[\left(\frac{8}{3} + \frac{32}{5}\right) - \left(\frac{0}{3} + \frac{0}{4}\right)\right]$

$V = 4\pi \left[\frac{40}{15} + \frac{96}{15}\right]$

$V = 4\pi \left(\frac{136}{15}\right)$

$V = \frac{544}{15}\pi$ unit^3 (4 marks)

