Chain Rule Assignment

① Given $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

$\Rightarrow$ If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then $x^T = [x_1, x_2, \ldots\ldots x_d]$

$x^T x = [x_1^2 + x_2^2 + \ldots\ldots + x_d^2]$

Applying chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}(\log_e(1+z)) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + \ldots\ldots + x_d^2)$$

$$= \frac{1}{1+z} \cdot 1 \left(2x_1 + 2x_2 + \ldots\ldots + 2x_d\right)$$

$$= \frac{2}{1+z}(x_1 + x_2 + \ldots\ldots + x_d)$$

$$\therefore \frac{df}{dx} = \frac{2}{1+z} \sum_{i=1}^{d} x_i .$$

② $f(z) = e^{-\frac{z}{2}}$; where $z = g(y), g(y) = y^T S^{-1} y, y = h(x), h(x) = x - \mu$

$\Rightarrow$ Using chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

here, $\frac{df}{dz} = \frac{d}{dz}\left(e^{-\frac{z}{2}}\right)$

$$= -\frac{e^{-\frac{z}{2}}}{2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} \left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)$$

$$= y^T S^{-1} + S^{-1} y$$

$$\frac{dy}{dx} = \frac{d}{dx}(x - \mu)$$

$$= 1$$

$$\therefore \frac{df}{dx} = -\frac{e^{-\frac{z}{2}}}{2}\left(y^{T}S^{-1} + S^{-1}y\right) \cdot 1$$

$$= -\frac{e^{-\frac{z}{2}}}{2S}\left(y^{T} + y\right)$$