CHME 2310
Fall 2024
L14

July 31, 2025

Reminder: Scalar Operator

Material Derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla$$

Apply operator on $\boldsymbol{V}$ to obtain $\boldsymbol{a}$:

$$\boldsymbol{a} = \frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V}$$

Local Derivative:

"I'm sitting at point A." How does flow in my vicinity change with time?

$$\left(\text{If steady flow, } \frac{\partial \boldsymbol{V}}{\partial t} = 0\right)$$

Convective Derivative:

"I am a molecule of fluid moving with the fluid."

$$\boldsymbol{V} \cdot \nabla$$

Quantifies degree to which $\boldsymbol{V}$ is aligned with "change".

Change of Quantity

The quantity being operated on in this case: velocity.

We can apply $\frac{D}{Dt}$ on other quantities:

$$\rho : \quad \frac{D\rho}{Dt} \text{ appears in continuity for compressible flows}$$

$$T, C \text{ (concentration)} : \quad \frac{DT}{Dt} \text{ and } \frac{DC}{Dt} \text{ (Transport II!)}$$

Next Module: Potential Flows

Stream Function

$$\Psi \quad \text{"psi"}$$

Potential Function

$$\Phi \quad \text{"phi"}$$

We will investigate potential flows in 2-dimensions:

$$\boldsymbol{V} = V_x(x,y)\hat{i} + V_y(x,y)\hat{j}$$

or

$$\boldsymbol{V} = V_r(r,\theta)\hat{e}_r + V_\theta(r,\theta)\hat{e}_\theta$$

Ψ and Φ are scalars:

$$\Psi(x,y) \quad \Phi(x,y)$$

(depend on x and y (2D only))

Set $\Psi =$ constant: solve for Ψ to get an equation for a streamline.

Set $\Phi =$ constant: solve for Φ to get an equation for an equipotential line.

Definitions

$$\boldsymbol{V} = V_x \hat{i}_x + V_y \hat{i}_y$$

$$\Psi \quad \Phi$$

$$V_x \equiv \frac{\partial \Psi}{\partial y}$$

$$V_y \equiv -\frac{\partial \Psi}{\partial x}$$

$$V_x \equiv \frac{\partial \Phi}{\partial x}$$

$$V_y \equiv \frac{\partial \Phi}{\partial y}$$

Not every $\boldsymbol{V}$ has a Ψ and/or Φ. Often we ask: is $\boldsymbol{V}$ a potential flow? Or can $\boldsymbol{V}$ be described using Φ?

Example:

$$\boldsymbol{V} = (2-x)\hat{i}_x + (3+y)\hat{i}_y$$

- Q: Does Ψ exist? What is it?
- Q: Does Φ exist? What is it?

Solution

For $V_x = 2 - x$:

$$2 - x = \frac{\partial \Psi}{\partial y} \Rightarrow \Psi(x,y) = 2y - xy + C(x)$$

For $V_y = 3 + y$:

$$3 + y = -\frac{\partial \Psi}{\partial x} \Rightarrow \Psi(x,y) = -3x - \frac{y^2}{2} + C(y)$$

$$\Psi(x,y) = 2y - xy + C(x) \quad \text{and} \quad \Psi(x,y) = -3x - \frac{y^2}{2} + C(y)$$

Is there a $\Psi(x,y)$?

Yes.

$$c(x) \quad \& \quad c(y) \quad \text{can accommodate terms from the other answer.}$$

- Term that depends on both x and y must be the same in both $\int$ results.

- Term that is $c(x)$ from the $\int dy$ calculation is found in the $\int dx$ calculation.
- Term that is $c(y)$ from the $\int dx$ integration is found from the $\int dy$ integration.

$$\Psi(x, y) = -3x + 2y - xy + C$$

where C is a "true constant" $\rightarrow$ just a number.

Let's draw streamline associated with Ψ.

Set:

$$\Psi(x, y) = -3x + 2y - xy + C$$

Alternatively, if $C = 1$:

$$\Psi(x, y) = -3x + 2y - xy + 1$$

Solve for y:

$$2y - xy - 3x - 1 = 0$$

Extra credit: plot 3 lines of constant Ψ.

Shortcut to "Is there a Ψ?"

$$V_x = \frac{\partial \Psi}{\partial y} \quad V_y = -\frac{\partial \Psi}{\partial x}$$

Expand:

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}$$

Use Ψ instead:

$$= \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \Psi}{\partial x}\right) = \frac{\partial^2 \Psi}{\partial y \partial x} - \frac{\partial^2 \Psi}{\partial x \partial y}$$

If Ψ is continuous and "well-behaved," derivative order does not matter $\left(\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}\right)$.

Thus,

$$\vec{\nabla} \cdot \vec{V} = 0$$

Shortcut: If $\vec{V}$ is for an incompressible fluid, and if $\vec{\nabla} \cdot \vec{V} = 0$, then $\Psi(x, y)$ must exist.

Is there a Φ? If so, what is it?

$$\vec{V} = (2-x)\hat{i}_x + (3+y)\hat{i}_y$$

For Φ:

$$\frac{\partial \Phi}{\partial x} = 2 - x \quad \Rightarrow \quad \Phi = 2x - \frac{x^2}{2} + C(y)$$

$$\frac{\partial \Phi}{\partial y} = 3 + y \quad \Rightarrow \quad \Phi = 3y + \frac{y^2}{2} + C(x)$$

Matching terms:

$$C(y) = 3y + \frac{y^2}{2}$$

$$C(x) = 2x - \frac{x^2}{2}$$

There is no term with both x and y.

$$\Phi(x,y) = 2x - \frac{x^2}{2} + 3y + \frac{y^2}{2} + C$$

Lines of constant Φ are equipotential lines.

If there is no Φ, there will be a problem or inconsistency if the mixed term that depends on both (x,y) exists:

$$\int dy \to \text{some term } f(x,y)$$

$$\int dx \to \text{some other term } g(x,y)$$

where $f(x,y) \neq g(x,y)$.

Potential Flow Condition

$\vec{V}$ is a potential flow if there is a scalar ϕ such that

$$\vec{V} = \nabla \phi$$

which implies

$$V_x = \frac{\partial \phi}{\partial x}$$

$$V_y = \frac{\partial \phi}{\partial y}$$