COSE 382 HW 2 Solutions

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1. There are 100 seats in a class room, but 110 students are registered for the class. Each student will show up for the class with the probability 0.9, independently. Find the probability that there will be enough seats for the students showing up for the class.

Solution:

$$\text{Let } x \sim \text{Bin}(110, 0.9)$$

$$p(x \leq 100) = \sum_{k=0}^{100} \binom{110}{k} 0.9^k (0.1)^{110-k} \approx 0.671$$

2. Let X be the number of Heads in 10 fair coin tosses.

(a) Find the conditional PMF of X, given that the first two tosses both land Heads.

(b) Find the conditional PMF of X, given that at least two tosses land Heads.

Solution:

a)

$$\begin{aligned} P\left(X=k \mid X_1=4, X_2=H\right) &= P(Y=k-2) \\ &= \binom{8}{k-2}\left(\frac{1}{2}\right)^{k-2}\left(\frac{1}{2}\right)^{8-(k-2)} = \binom{8}{k-2}\left(\frac{1}{2}\right)^{8} \end{aligned}$$

where Y is the number of Heads after two trials.

b)

$$\begin{aligned} P(X=k \mid X \geq 2) &= \frac{P(X=k, X \geq 2)}{p(X \geq 2)} \\ &= \frac{P(X=k)}{1-P(x=0)-P(x=1)} = \frac{\binom{10}{k}\left(\frac{1}{2}\right)^{10}}{1-\left(\frac{1}{2}\right)^{10}-10\left(\frac{1}{2}\right)^{10}} \\ &= \frac{1}{1013}\binom{10}{k} \quad k=2,3,4\ldots \end{aligned}$$

3. Consider n i.i.d. r.v.s. $X_1, \cdots, X_n$, where $X_i \sim \text{Bern(p)}$. Show that the conditional PMF of $(X_1, X_2, \cdots, X_n)$ given a number of successes,

$$P(X_1 = a_1, X_2 = a_2, \cdots, X_n = a_n | \sum_{i=1}^n X_i = k), \quad \text{where } a_i \in \{0, 1\}$$

is uniform.

Solution:

$$P\left(X_1=a_1, \cdots X_n=a_n \mid \sum x_i=k\right) = \frac{P\left(X_1=a_1, \ldots X_n=a_n, \sum x_i=k\right)}{P(X=k)}$$

$$= \frac{P\left(X_1=a_1, \cdots X_n=a_n\right)}{P(X=k)} = \frac{p^k(1-p)^{n-k}}{\binom{n}{k} p^k(1-p)^{n-k}}$$

$$= \frac{1}{\binom{n}{k}}$$

4. There are 100 prizes, with one worth \$1, one worth \$2, . . . , and one worth \$100. There are 100 boxes, each of which contains one of the prizes. You get 5 prizes by picking random boxes one at a time, without replacement. Find the PMF of how much your most valuable prize is worth.

Solution:
Let X be the most valuable prize among 5 picked boxes. Then $X \in \{5, 6, 7, \cdots 100\}$.

$$P(X = k) = \frac{\binom{k-1}{4}}{\binom{100}{5}}$$

where $\binom{k-1}{4}$ represents the number of possible ways of picking 4 boxes among $k-1$ boxes (which has less prize than k)

5. Let F_1 and F_2 be CDFs, $0 < p < 1$, and $F(x) = pF_1(x) + (1-p)F_2(x)$ for all x.

(a) Show directly that F has the properties of a valid CDF.

(b) Consider creating an r.v. in the following way. Flip a coin with probability p of Heads. If the coin lands Heads, generate an r.v. according to F_1; if the coin lands Tails, generate an r.v. according to F_2. Show that the r.v. obtained in this way has CDF F.

Solution:
a) $F(x) = pF_1(x) + (1-p)F_2(x)$

$$\begin{aligned} F(0) &= p0 + (1-p)_0 = 0 \\ F(1) &= p + (1-p) = 1 \end{aligned}$$

$F_1(x)$ and F_2 are increasing function, so does F.
b)

$$\begin{aligned} P(X \leq x) &= P\left(X_1 \leq x\right) p + (1-p) P\left(X_2 \leq x\right) \\ &= pF_1(x) + (1-p)F_2(x) \end{aligned}$$

6. Let X, Y, Z be discrete r.v.s such that X and Y have the same conditional distribution given Z, i.e., for all a and z we have

$$P(X = a|Z = z) = P(Y = a|Z = z).$$

Show that X and Y have the same distribution.

Solution:

$$\begin{aligned} P(X = a) &= \sum_z P(X = a \mid Z = z) P(Z = z) \\ &= \sum_z P(Y = a \mid Z = z) P(Z = z) \\ &= P(Y = a) \quad \text{for } \forall a. \end{aligned}$$

7. If $X \sim \text{HGeom}(w, b, n)$, what is the distribution of $n - X$?

Solution:

$$n - X \sim \text{HGem}(b, w, n)$$

8. For x and y binary digits (0 or 1), let

$$x \oplus y = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(this operation is called exclusive or (often abbreviated to XOR), or addition mod 2).

(a) Let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(1/2)$, independently. What is the distribution of $X \oplus Y$?

(b) Is $X \oplus Y$ independent of X? Is $X \oplus Y$ independent of Y ? Be sure to consider both the case $p = 1/2$ and the case $p \neq 1/2$.

Solution:
a)

$$\begin{aligned} P(X \oplus Y = 1) &= P(X \oplus Y = 1 \mid X = 1)P(X = 1) + P(X + Y = 1 \mid X = 0)P(x = 0) \\ &= P(Y = 0)P(X = 1) + P(Y = 1)P(X = 0) \\ &= p/2 + (1 - p)/2 = 1/2 \\ &X \oplus Y \sim \text{Bern}(1/2) \end{aligned}$$

b) Let $Z = X \oplus Y$

$$P(Z = z \mid Z = x) = \frac{1}{2} \quad \text{for} \quad \forall z, x \in \{0, 1\}$$

Therefore, Z and X are independent.