

# Lecture 9 – Recursive Functions

## COSE212: Programming Languages

Jihyeok Park



2024 Fall

- Syntactic Sugar
  - FAE – Removing `val` from FVAE
  - Syntactic Sugar and Desugaring
- Lambda Calculus (LC)
  - Church Encodings
  - Church-Turing Thesis

- Syntactic Sugar
  - FAE – Removing `val` from FVAE
  - Syntactic Sugar and Desugaring
- Lambda Calculus (LC)
  - Church Encodings
  - Church-Turing Thesis
- In this lecture, we will learn **recursion** and **conditionals**.

- Syntactic Sugar
  - FAE – Removing `val` from FVAE
  - Syntactic Sugar and Desugaring
- Lambda Calculus (LC)
  - Church Encodings
  - Church-Turing Thesis
- In this lecture, we will learn **recursion** and **conditionals**.
- **RFAE – FAE with recursive functions**
  - Concrete and Abstract Syntax
  - Interpreter and Natural Semantics

## 1. Recursion and Conditionals

- Recursion in F1VAE

- Recursion in FAE

## 2. Recursion without New Syntax in FAE

- `mkRec`: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

- Concrete Syntax

- Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

- Interpreter and Natural Semantics

- Arithmetic Comparison Operators

- Conditionals

- Recursive Function Definitions

## 1. Recursion and Conditionals

Recursion in F1VAE

Recursion in FAE

## 2. Recursion without New Syntax in FAE

mkRec: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

Concrete Syntax

Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

Interpreter and Natural Semantics

Arithmetic Comparison Operators

Conditionals

Recursive Function Definitions

A **recursive function** is a function that calls itself, and it is useful for **iterative processes** on **inductive data structures**.

A **recursive function** is a function that calls itself, and it is useful for **iterative processes** on **inductive data structures**.

Let's define a **recursive function** `sum` that computes the sum of integers from 1 to  $n$  in Scala:

```
/* Scala */  
def sum(n: Int): Int =  
  if (n < 1) 0           // base case  
  else n + sum(n - 1)    // recursive case  
  
sum(10) // 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 55
```



A **recursive function** is a function that calls itself, and it is useful for **iterative processes** on **inductive data structures**.

Let's define a **recursive function** `sum` that computes the sum of integers from 1 to  $n$  in Scala:

```
/* Scala */  
def sum(n: Int): Int =  
  if (n < 1) 0           // base case  
  else n + sum(n - 1)    // recursive case  
  
sum(10) // 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 55
```

For recursive functions, we need **conditionals** to define 1) **base cases** and 2) **recursive cases**.

Most programming languages support **recursive functions**:

- Scala

```
def sum(n: Int): Int = if (n < 1) 0 else n + sum(n - 1)
```

- C++

```
int sum(int n) { return n < 1 ? 0 : n + sum(n - 1); }
```

- Python

```
def sum(n): return 0 if n < 1 else n + sum(n - 1)
```

- Rust

```
fn sum(n: i32) -> i32 { if n < 1 {0} else {n + sum(n-1)} }
```

- ...

The F1VAE language already supports **recursive functions**:

```
/* F1VAE */  
def sum(n) = n + sum(n + -1);  
sum(10)
```

Why?

The F1VAE language already supports **recursive functions**:

```
/* F1VAE */  
def sum(n) = n + sum(n + -1);  
sum(10)
```

Why? The **function environment**  $\Lambda$  stores all the function definitions before evaluating the expressions.

$$\Lambda = [\text{sum} \mapsto \text{def sum}(n) = n + \text{sum}(n + -1)]$$

We can lookup and invoke the function `sum` in its body.

The F1VAE language already supports **recursive functions**:

```
/* F1VAE */  
def sum(n) = n + sum(n + -1);  
sum(10)
```

Why? The **function environment**  $\Lambda$  stores all the function definitions before evaluating the expressions.

$$\Lambda = [\text{sum} \mapsto \text{def sum}(n) = n + \text{sum}(n + -1)]$$

We can lookup and invoke the function `sum` in its body.

However, is it enough to support recursive functions?

The F1VAE language already supports **recursive functions**:

```
/* F1VAE */  
def sum(n) = n + sum(n + -1);  
sum(10)
```

Why? The **function environment**  $\Lambda$  stores all the function definitions before evaluating the expressions.

$$\Lambda = [\text{sum} \mapsto \text{def sum}(n) = n + \text{sum}(n + -1)]$$

We can lookup and invoke the function `sum` in its body.

However, is it enough to support recursive functions?

**No!** We need **conditionals** to define 1) **base cases** and 2) **recursive cases** for recursive functions. The above example causes an **infinite loop**.

If we only add **conditionals** to F1VAE, we can define recursive functions in F1VAE without any more extensions for recursion.

Programs  $\mathbb{P} \ni p ::= f^* e$  (Program)

Function Definitions  $\mathbb{F} \ni f ::= \text{def } x(x) = e$  (FunDef)

Expressions  $\mathbb{E} \ni e ::= \dots$

|  $e < e$  (Lt)

|  $\text{if } (e) e \text{ else } e$  (If)

Values  $\mathbb{V} \ni v ::= n \mid \bar{b} \mid \langle \lambda x.e, \sigma \rangle$

Function Environments  $\Lambda \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{F}$  (FEnv)

Boolean  $b \in \mathbb{B} = \{\text{true}, \text{false}\}$  (Boolean)

```
/* F1VAE + conditionals */
def sum(n) = if (n < 1) 0 else n + sum(n + -1);
sum(10) // 55
```

$\Lambda = [\text{sum} \mapsto \text{def sum}(n) = \text{if } (n < 1) 0 \text{ else } n + \text{sum}(n + -1)]$

```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

What happens if we add **conditionals** to FAE? Is the following FAE expression a recursive function?



```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

What happens if we add **conditionals** to FAE? Is the following FAE expression a recursive function? **No!** `sum` is a **free identifier**! Why?

```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

What happens if we add **conditionals** to FAE? Is the following FAE expression a recursive function? **No!** `sum` is a **free identifier**! Why?

We use **static scoping** for function definitions in FAE. At the definition site, the variable `sum` is not defined in the environment.

```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

What happens if we add **conditionals** to FAE? Is the following FAE expression a recursive function? **No!** `sum` is a **free identifier**! Why?

We use **static scoping** for function definitions in FAE. At the definition site, the variable `sum` is not defined in the environment.

Then, how to support recursive functions in FAE? There are two ways:

- 1 **Without new syntax** – using `mkRec` to define recursive functions
- 2 **With new syntax** – extending FAE with recursive function definitions

## 1. Recursion and Conditionals

Recursion in F1VAE

Recursion in FAE

## 2. Recursion without New Syntax in FAE

**mkRec**: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

Concrete Syntax

Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

Interpreter and Natural Semantics

Arithmetic Comparison Operators

Conditionals

Recursive Function Definitions

```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

How to let `sum` know itself in its body?

```
/* FAE + conditionals */  
val sum = n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
};  
sum(10)
```

How to let `sum` know itself in its body?

Let's pass the function as an argument to itself!

```
/* FAE + conditionals */  
val sumX = sumY => {  
  n => {  
    if (n < 1) 0  
    else n + sumY(sumY)(n + -1)  
  }  
};  
sumX(sumX)(10)
```

```
/* FAE + conditionals */  
val sumX = sumY => {  
  n => {  
    if (n < 1) 0  
    else n + sumY(sumY)(n + -1)  
  }  
};  
sumX(sumX)(10)
```

However, it is annoying to always pass the function to itself!



```
/* FAE + conditionals */  
val sumX = sumY => {  
  n => {  
    if (n < 1) 0  
    else n + sumY(sumY)(n + -1)  
  }  
};  
sumX(sumX)(10)
```

However, it is annoying to always pass the function to itself!

Let's wrap this to get sum back!

```
/* FAE + conditionals */  
val sum = n => {  
  val sumX = sumY => {  
    n => {  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)(n)  
};  
sum(10)
```

<sup>1</sup>[https://en.wikipedia.org/wiki/Lambda\\_calculus#%CE%B7-reduction](https://en.wikipedia.org/wiki/Lambda_calculus#%CE%B7-reduction)

```
/* FAE + conditionals */  
val sum = n => {  
  val sumX = sumY => {  
    n => {  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)(n)  
};  
sum(10)
```

We can simplify this using  $\eta$ -**reduction**<sup>1</sup>:

$$e \quad \equiv \quad \lambda x.e(x) \quad \text{only if } x \text{ is NOT FREE in } e.$$

<sup>1</sup>[https://en.wikipedia.org/wiki/Lambda\\_calculus#%CE%B7-reduction](https://en.wikipedia.org/wiki/Lambda_calculus#%CE%B7-reduction)

```
/* FAE + conditionals */  
val sum = {  
  val sumX = sumY => {  
    n => { // ALMOST the same as the original body  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

```
/* FAE + conditionals */  
val sum = {  
  val sumX = sumY => {  
    n => { // ALMOST the same as the original body  
      if (n < 1) 0  
      else n + sumY(sumY)(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

The function body is almost the same as the original version except that we need to call the function as `sumY(sumY)` instead of `sum`.

```
/* FAE + conditionals */
val sum = {
  val sumX = sumY => {
    n => { // ALMOST the same as the original body
      if (n < 1) 0
      else n + sumY(sumY)(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)
```

The function body is almost the same as the original version except that we need to call the function as `sumY(sumY)` instead of `sum`.

Let's define a variable `sum` to be `sumY(sumY)`!

```
/* FAE + conditionals */
val sum = {
  val sumX = sumY => {
    val sum = sumY(sumY);
    n => { // EXACTLY the same as the original body
      if (n < 1) 0
      else n + sum(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)
```

```
/* FAE + conditionals */  
val sum = {  
  val sumX = sumY => {  
    val sum = sumY(sumY); // INFINITE LOOP  
    n => { // EXACTLY the same as the original body  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

Unfortunately, this is an **infinite loop**!



```
/* FAE + conditionals */
val sum = {
  val sumX = sumY => {
    val sum = sumY(sumY); // INFINITE LOOP
    n => { // EXACTLY the same as the original body
      if (n < 1) 0
      else n + sum(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)
```

Unfortunately, this is an **infinite loop**!

We need to **delay** the evaluation of `sum` using the  $\eta$ -**expansion**:

$$e \quad \equiv \quad \lambda x.e(x) \quad \text{only if } x \text{ is } \mathbf{NOT \text{ FREE}} \text{ in } e.$$

```
/* FAE + conditionals */
val sum = {
  val sumX = sumY => {
    val sum = x => sumY(sumY)(x);
    n => { // EXACTLY the same as the original body
      if (n < 1) 0
      else n + sum(n + -1)
    }
  };
  sumX(sumX)
};
sum(10)
```

```
/* FAE + conditionals */  
val sum = {  
  val sumX = sumY => {  
    val sum = x => sumY(sumY)(x);  
    n => { // EXACTLY the same as the original body  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

Do we need to do this for every recursive function?

```
/* FAE + conditionals */  
val sum = {  
  val sumX = sumY => {  
    val sum = x => sumY(sumY)(x);  
    n => { // EXACTLY the same as the original body  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  sumX(sumX)  
};  
sum(10)
```

Do we need to do this for every recursive function?

To avoid such boilerplate code, let's define a helper function `mkRec`!

```
/* FAE + conditionals */  
val sum = {  
  val fX = fY => {  
    val sum = x => fY(fY)(x);  
    n => {  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  fX(fX)  
};  
sum(10)
```

First, we rename `sumX` and `sumY` to `fX` and `fY`, respectively.

```
/* FAE + conditionals */  
val sum = {  
  val fX = fY => {  
    val sum = x => fY(fY)(x);  
    n => {  
      if (n < 1) 0  
      else n + sum(n + -1)  
    }  
  };  
  fX(fX)  
};  
sum(10)
```

Then, let's desugar the inside variable definition `sum`.

```
/* FAE + conditionals */  
val sum = {  
  val fX = fY => {  
    (sum => n => {  
      if (n < 1) 0  
      else n + sum(n + -1)  
    })(x => fY(fY)(x))  
  };  
  fX(fX)  
};  
sum(10)
```

```
/* FAE + conditionals */  
val sum = {  
  val fX = fY => {  
    (sum => n => {  
      if (n < 1) 0  
      else n + sum(n + -1)  
    })(x => fY(fY)(x))  
  };  
  fX(fX)  
};  
sum(10)
```

Finally, let's define a helper function `mkRec` that takes a body of a recursive function and returns a recursive function.



```
/* FAE + conditionals */  
val mkRec = body => {  
  val fX = fY => body(x => fY(fY)(x))  
  fX(fX)  
};  
val sum = mkRec(sum => n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
});  
sum(10)
```

```
/* FAE + conditionals */  
val mkRec = body => {  
  val fX = fY => body(x => fY(fY)(x))  
  fX(fX)  
};  
val sum = mkRec(sum => n => {  
  if (n < 1) 0  
  else n + sum(n + -1)  
});  
sum(10)
```

Now, we can also define other recursive functions using `mkRec`.

```
/* FAE + conditionals */
val mkRec = body => {
  val fX = fY => body(x => fY(fY)(x))
  fX(fX)
};
val sum = mkRec(sum => n => {
  if (n < 1) 0
  else n + sum(n + -1)
});
sum(10)
```

Now, we can also define other recursive functions using `mkRec`. For example, the following recursive function `fac` computes the factorial:

```
/* FAE + conditionals */
val mkRec = ...;
val fac = mkRec(fac => n => if (n < 1) 1 else n * fac(n + -1));
fac(5) // 5 * 4 * 3 * 2 * 1 = 120
```

## 1. Recursion and Conditionals

Recursion in F1VAE

Recursion in FAE

## 2. Recursion without New Syntax in FAE

mkRec: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

Concrete Syntax

Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

Interpreter and Natural Semantics

Arithmetic Comparison Operators

Conditionals

Recursive Function Definitions

The second way to support recursive functions in FAE is to extend FAE with **recursive function definitions**.

The second way to support recursive functions in FAE is to extend FAE with **recursive function definitions**.

RFAE is an extension of FAE with **recursion** and **conditionals**.

```
/* RFAE */  
def sum(n) = if (n < 1) 0 else n + sum(n - 1);  
sum(10) // 55
```

The second way to support recursive functions in FAE is to extend FAE with **recursive function definitions**.

RFAE is an extension of FAE with **recursion** and **conditionals**.

```
/* RFAE */  
def sum(n) = if (n < 1) 0 else n + sum(n + -1);  
sum(10) // 55
```

For RFAE, we need to extend **expressions** of FAE with

- ① **arithmetic comparison operators**
- ② **conditionals**
- ③ **recursive function definitions**

```
/* RFAE */  
def sum(n) = if (n < 1) 0 else n + sum(n + -1);  
sum(10) // 55
```

A **recursive function definition** consists of four parts:

- a function name
- a parameter name
- a function body expression
- a scope expression



```
/* RFAE */  
def sum(n) = if (n < 1) 0 else n + sum(n + -1);  
sum(10) // 55
```

A **recursive function definition** consists of four parts:

- a function name
- a parameter name
- a function body expression
- a scope expression

Note that a **recursive function definition** is also an expression can be used in any place where an expression is expected:

```
/* RFAE */  
2 * {  
  def sum(n) = if (n < 1) 0 else n + sum(n + -1);  
  sum(10) // 55  
} + 1 // 2 * 55 + 1 = 111
```

```
// expressions
<expr> ::= ...
    | <expr> "<" <expr>
    | "if" "(" <expr> ")" <expr> "else" <expr>
    | "def" <id> "(" <id> ")" "=" <expr> ";" <expr>
```

For RFAE, we need to extend **expressions** of FAE with

- ① **arithmetic comparison operators**
- ② **conditionals**
- ③ **recursive function definitions**

Let's define the **abstract syntax** of RFAE in BNF:

Expressions  $\mathbb{E} \ni e ::= \dots$

$e < e$	(Lt)
---------	------

$\text{if } (e) \ e \ \text{else } e$	(If)
---------------------------------------	------

$\text{def } x(x) = e; \ e$	(Rec)
-----------------------------	-------

Let's define the **abstract syntax** of RFAE in BNF:

Expressions  $\mathbb{E} \ni e ::= \dots$

$e < e$	(Lt)
$\text{if } (e) \ e \ \text{else } e$	(If)
$\text{def } x(x) = e; \ e$	(Rec)

```
enum Expr:
    ...
    // less-than
    case Lt(left: Expr, right: Expr)
    // conditionals
    case If(cond: Expr, thenExpr: Expr, elseExpr: Expr)
    // recursive function definition
    case Rec(name: String, param: String, body: Expr, scope: Expr)
```

## 1. Recursion and Conditionals

Recursion in F1VAE

Recursion in FAE

## 2. Recursion without New Syntax in FAE

mkRec: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

Concrete Syntax

Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

Interpreter and Natural Semantics

Arithmetic Comparison Operators

Conditionals

Recursive Function Definitions

Now, let's 1) implement the **interpreter**:

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** for **recursive function definitions** and other new cases.

$$\sigma \vdash e \Rightarrow v$$

Expressions  $\mathbb{E} \ni e ::= \dots$

$e < e$	(Lt)
<b>if</b> ( $e$ ) $e$ <b>else</b> $e$	(If)
<b>def</b> $x(x) = e$ ; $e$	(Rec)

Values  $\mathbb{V} \ni v ::= n \mid b \mid \langle \lambda x.e, \sigma \rangle$

```
enum Value:  
  case NumV(number: BigInt)  
  case BoolV(bool: Boolean)  
  case CloV(param: String, body: Expr, env: Env)
```

```

type BOp[T] = (T, T) => T
type COp[T] = (T, T) => Boolean
def numCOp(op: COp[BigInt], x: String): BOp[Value] =
  case (NumV(l), NumV(r)) => BoolV(op(l, r))
  case (l, r) => error(s"invalid operation: ${l.str} $x ${r.str}")

val numLt: BOp[Value] = numCOp(_ < _, "<")

def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Lt(l, r) => numLt(interp(l, env), interp(r, env))
    
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Lt} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 < e_2 \Rightarrow n_1 < n_2}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case If(c, t, e) => interp(c, env) match
  case BoolV(true)  => interp(t, env)
  case BoolV(false) => interp(e, env)
  case v            => error(s"not a boolean: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{If}_T \frac{\sigma \vdash e_0 \Rightarrow \text{true} \quad \sigma \vdash e_1 \Rightarrow v_1}{\sigma \vdash \text{if } (e_0) \text{ } e_1 \text{ else } e_2 \Rightarrow v_1}$$

$$\text{If}_F \frac{\sigma \vdash e_0 \Rightarrow \text{false} \quad \sigma \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{if } (e_0) \text{ } e_1 \text{ else } e_2 \Rightarrow v_2}$$



```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = ???
  interp(s, newEnv)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Rec} \frac{\sigma' = ??? \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = env + (n -> CloV(p, b, ???))
  interp(s, newEnv)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, ??? \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = env + (n -> CloV(p, b, newEnv)) // not working
  interp(s, newEnv)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, \sigma' \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = env + (n -> CloV(p, b, newEnv)) // not working
  interp(s, newEnv)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, \sigma' \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

While it makes sense in the natural semantics, the above Scala code doesn't work because `newEnv` is not yet defined.

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  val newEnv: Env = env + (n -> CloV(p, b, newEnv)) // not working
  interp(s, newEnv)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, \sigma' \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

While it makes sense in the natural semantics, the above Scala code doesn't work because `newEnv` is not yet defined.

Let's **delay** the evaluation of `newEnv` using the  $\eta$ -**expansion** again:

$$e \quad \equiv \quad \lambda x. e(x) \quad \text{only if } x \text{ is NOT FREE in } e.$$

We augment the closure value with an **environment factory**  $() \Rightarrow \text{Env}$  rather than an **environment**  $(\text{Env})$ :

```
enum Value:
  ...
  case CloV(param: String, body: Expr, env: () => Env)

def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Func(p, b) => CloV(p, b, () => env)
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv() + (p -> interp(e, env)))
    case v                => error(s"not a function: ${v.str}")
  case Rec(n, p, b, s) =>
    val newEnv: Env = env + (n -> CloV(p, b, () => newEnv)) // error
    interp(s, newEnv)
```

It still doesn't work because `newEnv` is not yet defined.

Let's use a **lazy value** (`lazy val`) to delay the evaluation of `newEnv`.

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Rec(n, p, b, s) =>
  lazy val newEnv: Env = env + (n -> CloV(p, b, () => newEnv))
  interp(s, newEnv)
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\text{Rec} \frac{\sigma' = \sigma[x_0 \mapsto \langle \lambda x_1. e_2, \sigma' \rangle] \quad \sigma' \vdash e_3 \Rightarrow v_3}{\sigma \vdash \text{def } x_0(x_1) = e_2; e_3 \Rightarrow v_3}$$

We will learn more about **lazy values** in the later lectures in this course.

<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/rfae>

- Please see above document on GitHub:
  - Implement `interp` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.



## 1. Recursion and Conditionals

- Recursion in F1VAE

- Recursion in FAE

## 2. Recursion without New Syntax in FAE

- mkRec: Helper Function for Recursion

## 3. RFAE – FAE with Recursion and Conditionals

- Concrete Syntax

- Abstract Syntax

## 4. Interpreter and Natural Semantics for RFAE

- Interpreter and Natural Semantics

- Arithmetic Comparison Operators

- Conditionals

- Recursive Function Definitions

- Mutable Data Structures

Jihyeok Park

[jihyeok\\_park@korea.ac.kr](mailto:jihyeok_park@korea.ac.kr)

<https://plrg.korea.ac.kr>