COSE 382 HW 4

Date: 2024. 10. 09 **Due:** 2024. 10. 16

1. Let F be the CDF of a continuous r.v., and $f = F'$ be the PDF.

   (a) Show that g defined by $g(x) = 2F(x)f(x)$ is also a valid PDF.

   (b) Show that h defined by $h(x) = \frac{1}{2}f(-x) + \frac{1}{2}f(x)$ is also a valid PDF.

2. A stick of length 1 is broken at a uniformly random point, yielding two pieces. Let X and Y be the lengths of the shorter and longer pieces, respectively, and let $R = X/Y$ be the ratio of the lengths X and Y.

   (a) Find the CDF and PDF of R.

   (b) Find the expected value of R.

3. Let $U \sim \text{Unif}(0,1)$. As a function of U, create an r.v. X with CDF $F(x) = 1 - e^{-x^3}$ for $x > 0$.

4. Let $Z \sim N(0,1)$. Define $X = Z \cdot I_{Z>0}$, where $I_{Z>0}$ is the indicator r.v. for $\{Z > 0\}$. Find $E(X)$ and $\text{Var}(X)$.

5. Let T be the time until a radioactive particle decays, and suppose (as is often done in physics and chemistry) that $T \sim \text{Expo}(\lambda)$.

   (a) The half-life of the particle is the time at which there is a 50% chance that the particle has decayed. Find the half-life of the particle.

   (b) Now consider n radioactive particles, with i.i.d. times until decay $T_1, \cdots, T_n \sim \text{Expo}(\lambda)$. Let L be the first time at which one of the particles decays. Find the CDF of L. Also, find $E(L)$ and $\text{Var}(L)$.

   (c) Continuing (b), find the mean and variance of $M = \max(T_1, \cdots, T_n)$, the last time at which one of the particles decays.

6. Let $U_1, U_2, \cdots, U_{60}$ be i.i.d. Unif(0, 1) and $X = U_1 + U_2 + \cdots + U_{60}$. Find the MGF of X.

7. Let X and Y be i.i.d. Expo(1), and $L = X - Y$. The Laplace distribution has PDF

$$f(x) = \frac{1}{2}e^{-|x|}$$

for all real x. Use MGFs to show that the distribution of L is Laplace.

8. Let $W = X^2+Y^2$, with X, Y i.i.d. $\mathcal{N}(0,1)$. The MGF of X^2 turns out to be $(1-2t)^{-1/2}$ for $t < 1/2$.

(a) Find the MGF of W.

(b) From the MGF of W, identify the distribution of W.

9. Let $Y = X^3$, with $X \sim \text{Expo}(1)$

(a) Find $P(Y > s+t|Y > s)$ for $s, t > 0$. Does Y have the memoryless property?

(b) Find the mean and variance of Y , and the n-th moment $E(Y^n)$ for $n = 1, 2, \cdots$.

(c) Determine whether or not the MGF of Y exists.