COSE 382 HW 3

Date: 2024.10. 02 **Due:** 2024. 10.09

1. Let X have PMF

$$P(X = k) = c\frac{p^k}{k}, \quad \text{for } k = 1, 2, \cdots$$

where p is a parameter with $0 < p < 1$ and c is a normalizing constant $c = \dfrac{-1}{\log(1-p)}$, as seen from the Taylor series

$$-\log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots$$

This distribution is called the Logarithmic distribution and has often been used in ecology. Find the mean and variance of X.

2. A couple decides to keep having children until they have at least one boy and at least one girl, and then stop. Assume they never have twins, every birth is independent and with probability 1/2 of a boy. What is the expected number of children?

3. Raindrops are falling at an average rate of 20 drops per square inch per minute. Compute the probability that a 5 inches^2 region has no rain drops in a given 3-second time interval.

4. Alice and Bob have just met, and wonder whether they have a mutual friend. Each has 50 friends, out of 1000 other people who live in their town. Assume that Alice's 50 friends are a random sample of the 1000 people (equally likely to be any 50 of the 1000), and similarly for Bob. Also assume that knowing who Alice's friends are gives no information about who Bob's friends are.

(a) Compute the expected number of mutual friends Alice and Bob have.

(b) Let X be the number of mutual friends they have. Find the PMF of X.

5. Randomly, k distinguishable balls are placed into n distinguishable boxes, with all possibilities equally likely. Find the expected number of empty boxes.

6. Nick and Penny are independently performing independent Bernoulli trials. Nick is flipping a nickel with probability p_1 of Heads and Penny is flipping a penny with probability p_2 of Heads. Let $X_1, X_2, \cdots$ be Nick's results and $Y_1, Y_2, \cdots$ be Penny's results, with $X_i \sim \text{Bern}(p_1)$ and $Y_j \sim \text{Bern}(p_2)$.

(a) Find the distribution and expected value of the first time at which they are simultaneously successful, i.e., the smallest n such that $X_n = Y_n = 1$.

(b) Find the expected time until at least one has a success (including the success).

(c) For $p_1 = p_2$, find the probability that their first successes are simultaneous, and use this to find the probability that Nick's first success precedes Penny's.

7.

(a) Use LOTUS to show that for $X \sim \text{Pois}(\lambda)$ and any function g,

$$E(Xg(X)) = \lambda E(g(X+1)).$$

This is called the *Stein-Chen identity* for the Poisson.

(b) Find the third moment $E(X^3)$ by using the identity from (a).

8.

(a) For $X \sim \text{Pois}(\lambda)$, find $E(e^{tX})$, for a constant t

(b) For $X \sim \text{Geom}(p)$, find $E(e^{tX})$, for a constant t