COSE436

Lecture 19: Curves & Surfaces

Nov 19, 2024

Won-Ki Jeong
(wkjeong@korea.ac.kr)

KOREA UNIVERSITY

Outlines

- Curves and surfaces representation
- Continuity
- Curves and surfaces

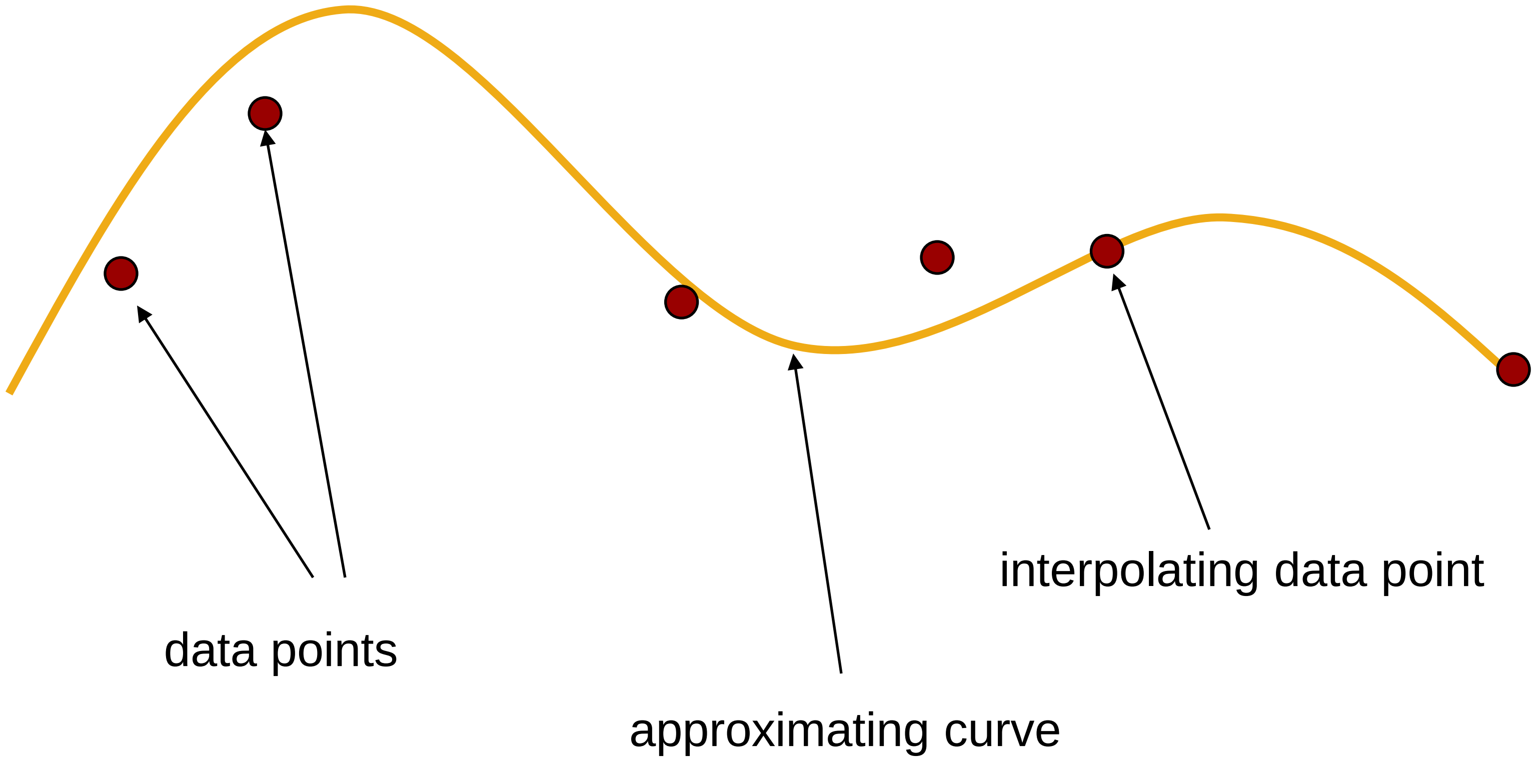
Outlines

- Curves and surfaces representation
- Continuity
- Curves and surfaces

Curves and Surfaces

- So far, we used flat entities such as lines and flat polygons
  - Fit well with graphics hardware
  - Mathematically simple
- Real world objects are not flat entities
  - Need curves and curved surfaces
  - May only have need at the application level
  - Implementation can render them approximately with flat primitives

Modeling with Curves



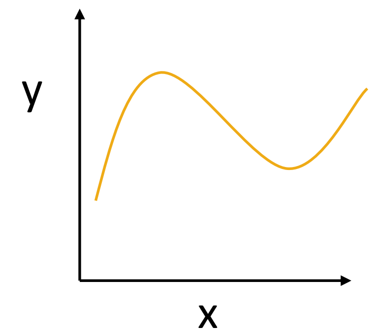
What Makes a Good Representation?

- There are many ways to represent curves and surfaces
- Want a representation that is
  - Smooth
  - Easy to evaluate
  - Local control, stable
  - Interpolation vs. approximation
  - Derivatives are well defined

Explicit Representation

- Most familiar form of curve in 2D

$$y = f(x)$$

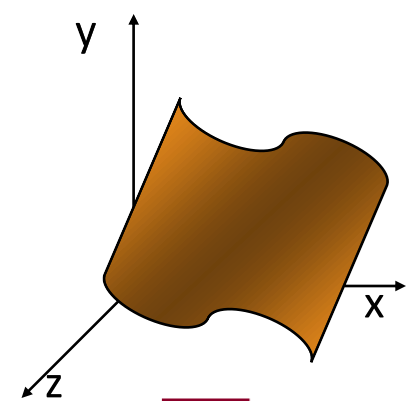


- Cannot represent all curves
  - Vertical lines, Circles
- Extension to 3D curve
  - $y = f(x),\ z = g(x)$

independent variable

dependent variable

- 3D surface
  - $z = f(x, y)$



Implicit Representation

- Implicit form: $f(x, y, z, \ldots) = 0$
- Two dimensional lines

$$f(x, y) = ax + by + c = 0$$

- Two dimensional circles

$$f(x, y) = x^2 + y^2 - r^2 = 0$$

- Three dimensional planes

$$f(x, y, z) = ax + by + cz + d = 0$$

Implicit Representation

- Implicit 3D curve: Intersection between two implicit surfaces
  - Collection of all (x,y,z) satisfying the two implicit equations

$$f(x,y,z)=0$$

$$g(x,y,z)=0$$

- In general, we cannot solve for points that satisfy implicit equation
  - We can test if point is on the surface / curve

Implicit Algebraic Surfaces

- Sum of polynomials

$$f(x,y,z)=\sum_i\sum_j\sum_k x^i y^j z^k = 0$$

- Example: quadric surface (i+j+k <= 2)

$$f(x,y,z)=a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{23}yz+2a_{13}xz+b_1x+b_2y+b_3z+c=0$$

$$p^T A p + b^T p + c = 0,\ \text{where } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

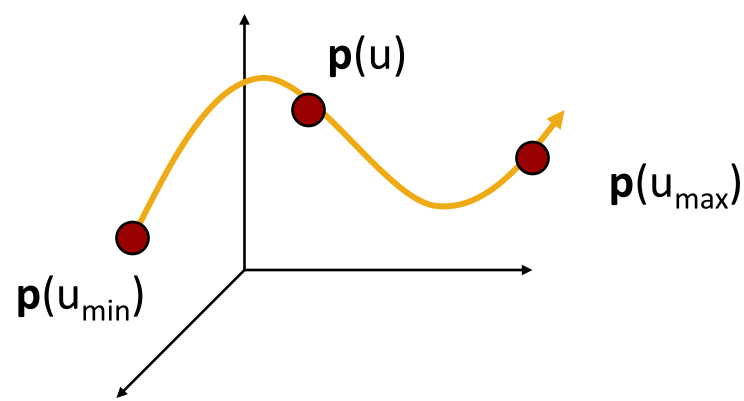
Parametric Curves

- Separate equation for each spatial variable

$$x = x(u)$$
$$y = y(u)$$
$$z = z(u)$$

$\mathbf{p}(u)=[x(u), y(u), z(u)]^T$

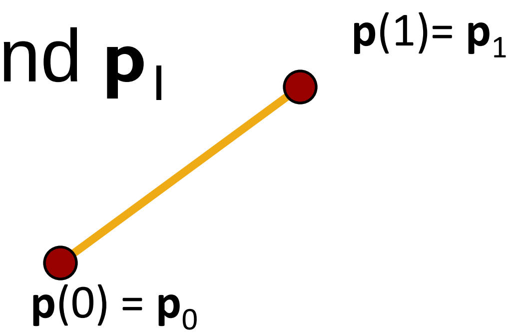
- For $u_{min} \leq u \leq u_{max}$ we trace out a curve in two or three dimensions



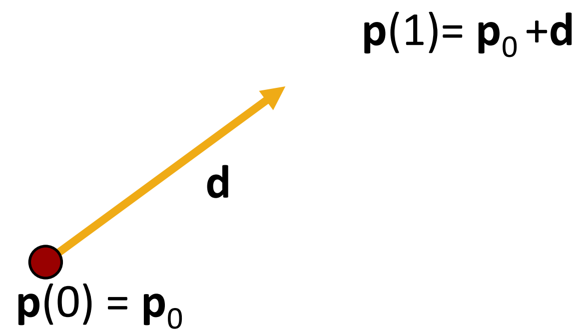
Parametric Lines

- We can normalize u to be over the interval (0,1)

- Line connecting two points $\mathbf{p}_0$ and $\mathbf{p}_1$
  - $\mathbf{p}(u)=(1-u)\mathbf{p}_0+u\mathbf{p}_1$



- Ray from $\mathbf{p}_0$ in the direction $\mathbf{d}$
  - $\mathbf{p}(u)=\mathbf{p}_0+u\mathbf{d}$

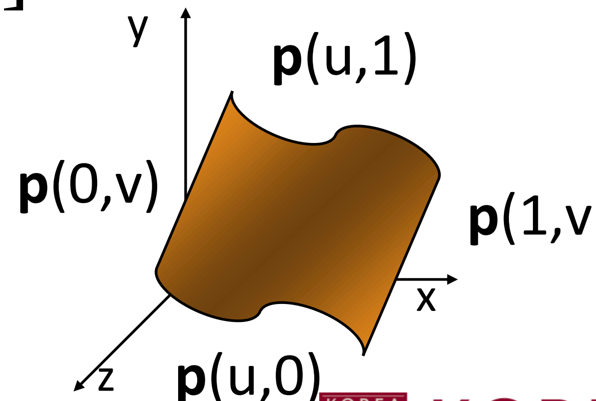


Parametric Surfaces

- Surfaces require 2 parameters

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

$$\mathbf{p}(u,v) = [x(u,v), y(u,v), z(u,v)]^T$$

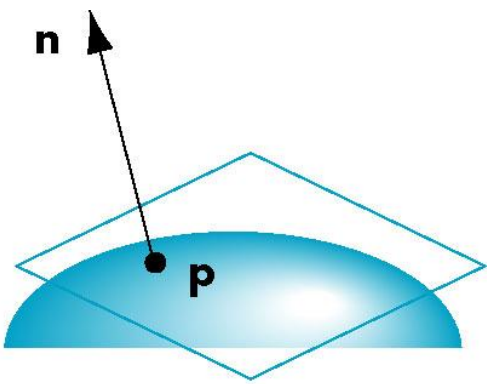


Normals

- We can differentiate with respect to *u* and *v* to obtain the normal at any point **p**

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \partial \mathrm{x}(u,v)/\partial u \\ \partial \mathrm{y}(u,v)/\partial u \\ \partial \mathrm{z}(u,v)/\partial u \end{bmatrix} \qquad \frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \partial \mathrm{x}(u,v)/\partial v \\ \partial \mathrm{y}(u,v)/\partial v \\ \partial \mathrm{z}(u,v)/\partial v \end{bmatrix}$$

$$\mathbf{n} = \frac{\partial \mathbf{p}(u,v)}{\partial u} \times \frac{\partial \mathbf{p}(u,v)}{\partial v}$$



Example

$$\vec{p}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

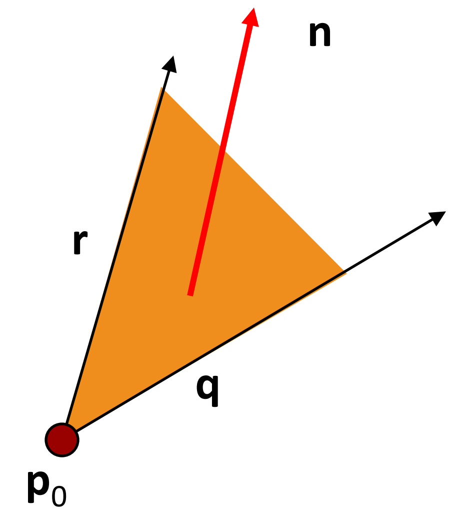
Let $x(u,v) = u + 1, \qquad y(u,v) = v, \qquad z(u,v) = -u^2 + v^2 + 1$

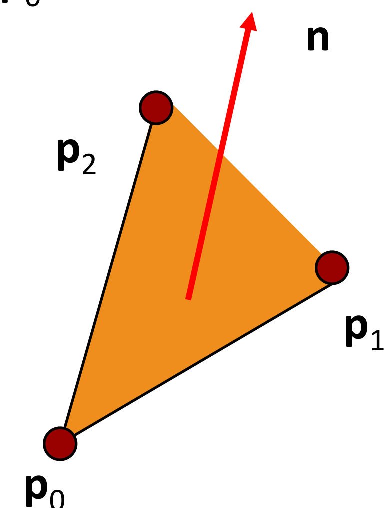
What is the surface normal at (1, -2)?

$$\mathbf{a} \times \mathbf{b} = [a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1]$$

Parametric Planes

- Point-vector form
  - $\mathbf{p}(u,v)=\mathbf{p}_0+u\mathbf{q}+v\mathbf{r}$
  - $\mathbf{n} = \mathbf{q} \times \mathbf{r}$
- Three-point form
  - $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_0$
  - $\mathbf{r} = \mathbf{p}_2 - \mathbf{p}_0$





Parametric Sphere

$$x(q,f) = r\cos q \sin f$$

$$y(q,f) = r\sin q \sin f$$

$$z(q,f) = r\cos f$$

$$0 \pounds q \pounds 2p$$

$$0 \pounds f \pounds p$$

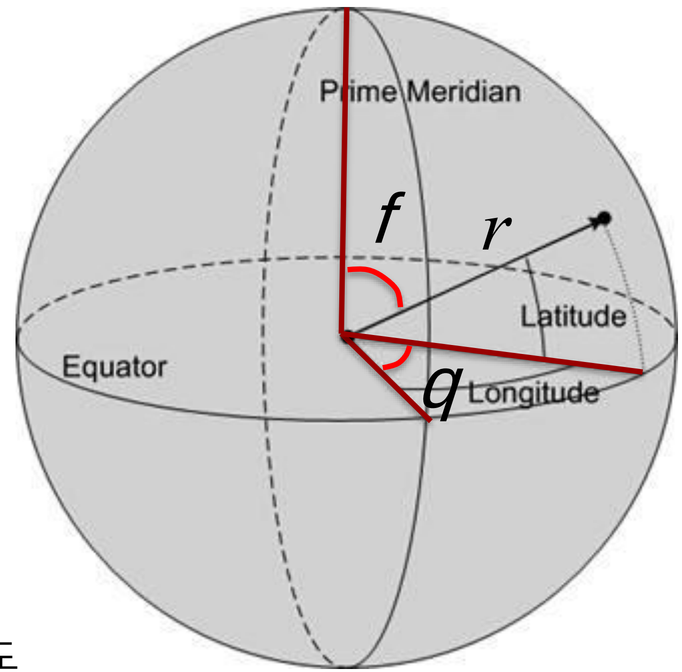


Image Courtesy of Microsoft

Θ constant: circles of constant longitude 경도

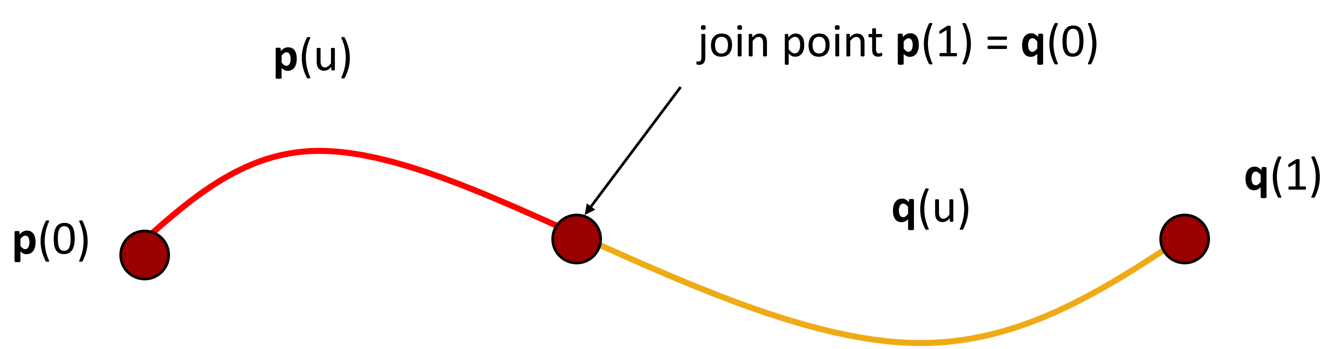
Φ constant: circles of constant latitude 위도

Curve Segments

- After normalizing u, each curve is written

$$\mathbf{p}(u)=[x(u), y(u), z(u)]^T, \quad 0 \le u \le 1$$

- In classical numerical methods, we design a single global curve
- In computer graphics and CAD, it is better to design small connected curve *segments*



KOREA UNIVERSITY

Outlines

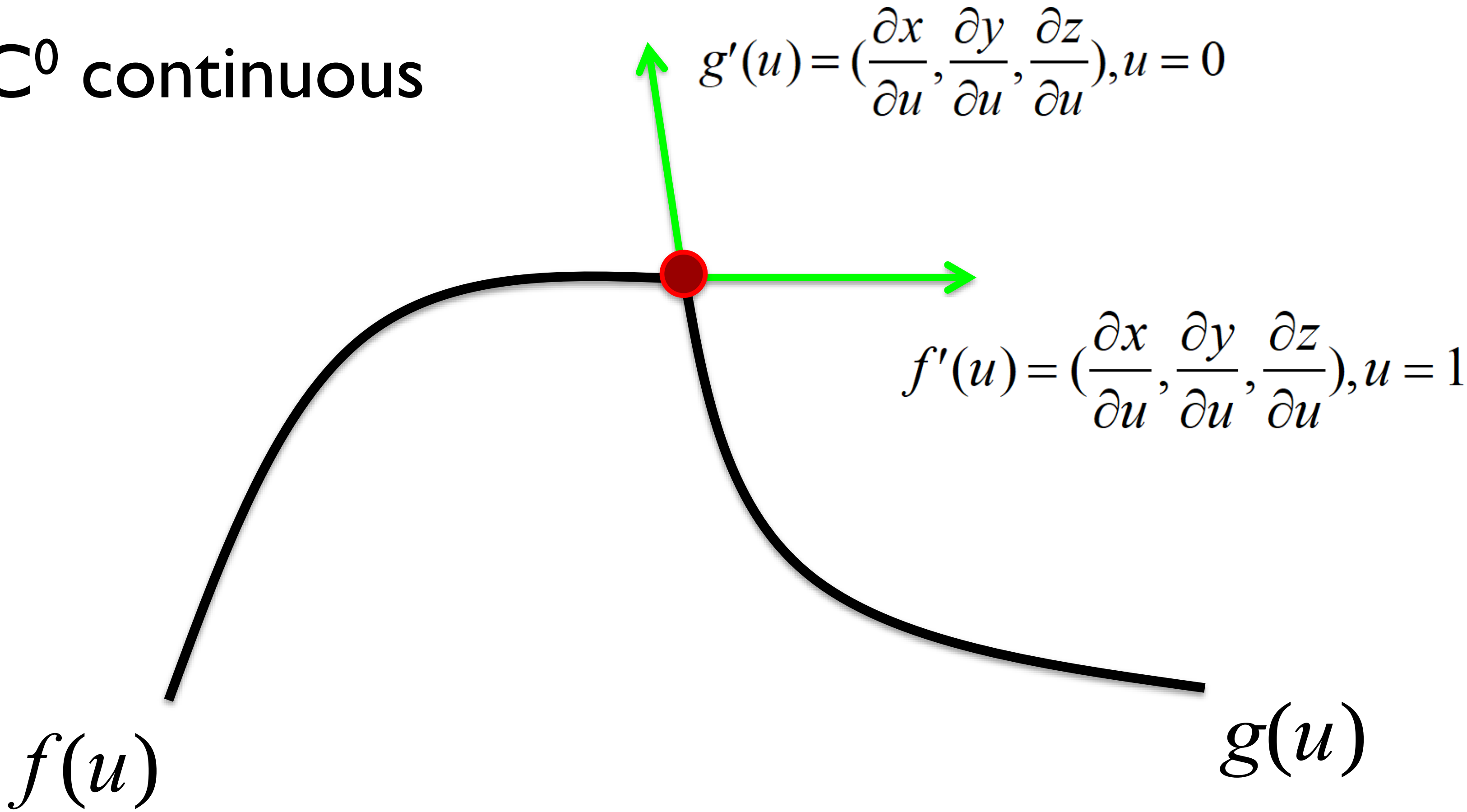
- Curves and surfaces representation
- Continuity
- Curves and surfaces

Continuity at Joint of a Curve

- C^n continuity
  - Parametric continuity
  - *n*-th derivative is equal
  - C^0 : curves are joined (dx/du, dy/du, dz/du)
  - C^1 : C^0 & first derivative (tangent vector) is equal
  - C^2 : C^1 & second derivative is equal
- C^2 continuity
  - Commonly used in computer graphics
  - Tangential vector change (curvature) is continuous

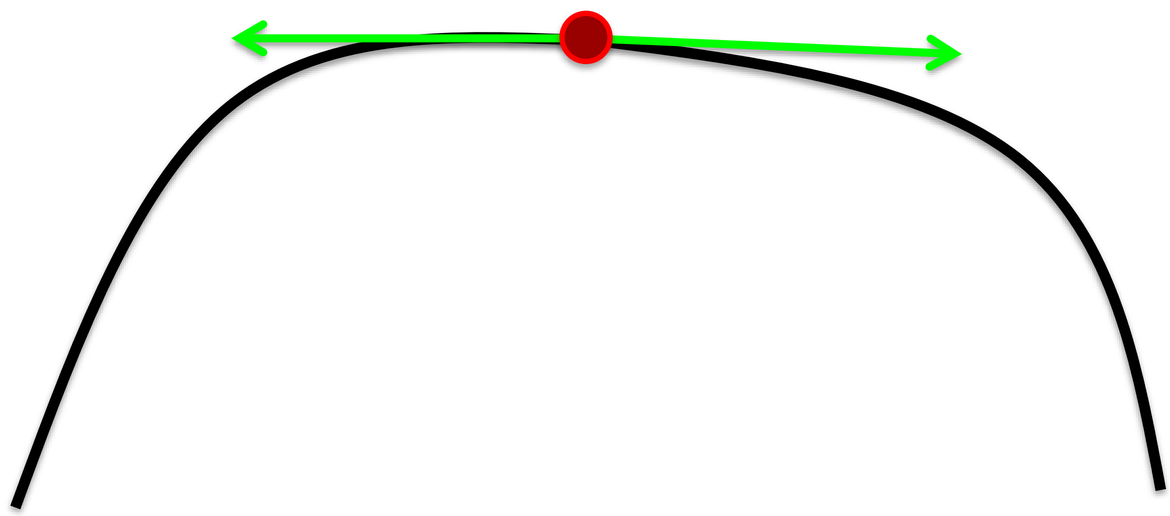
Example

- C^0 continuous



Example

- C^1 continuous



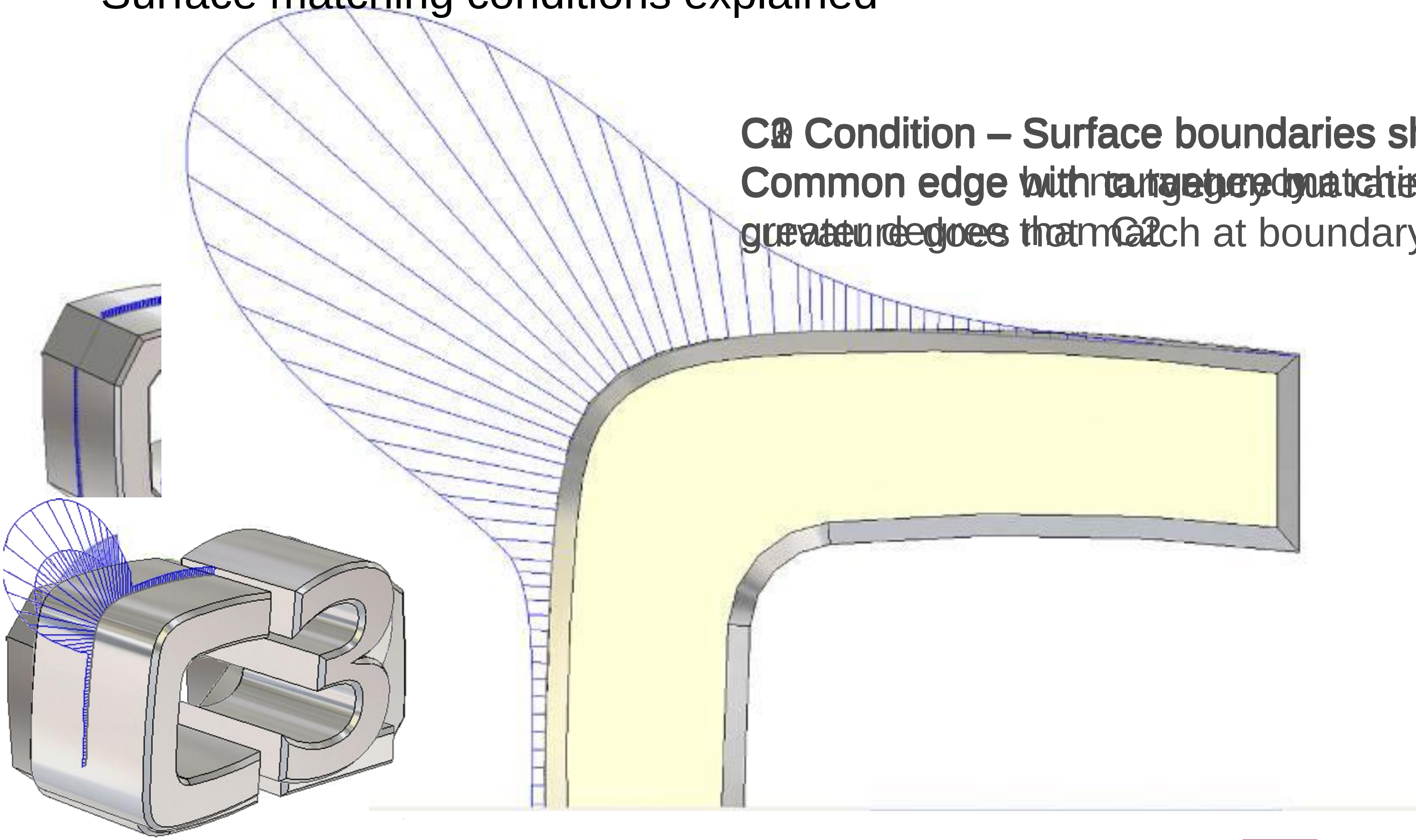
Explanation of C0 thru C3

- Surface matching conditions explained

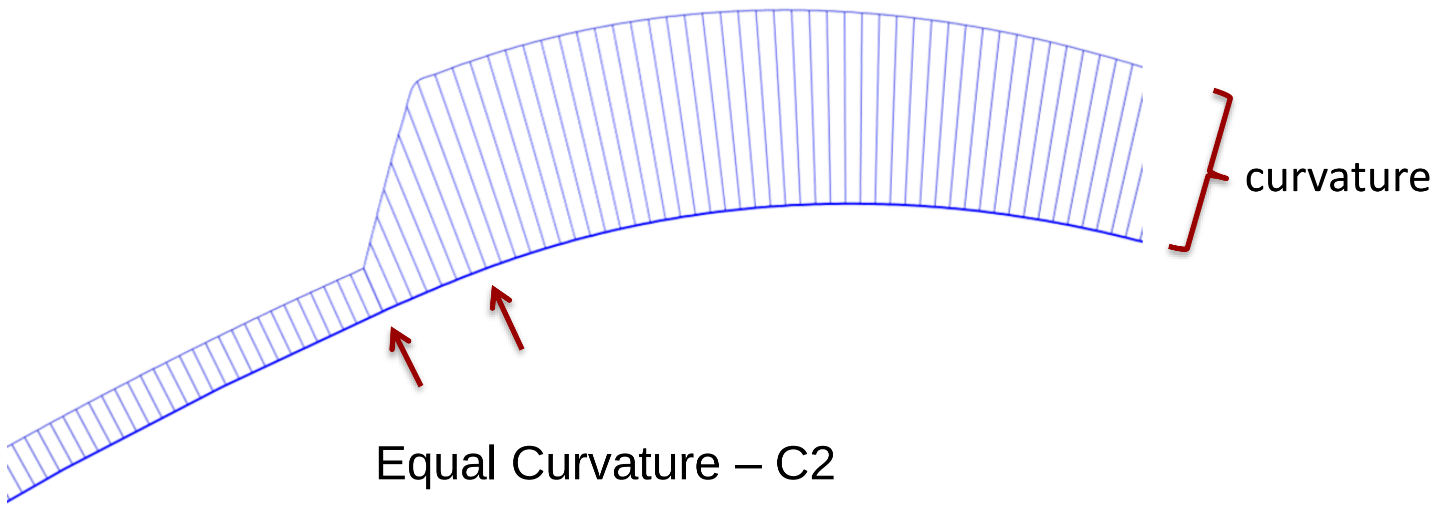
C1 Condition – Surface boundaries share Common edge with tangency but the curvature does not match at boundary

C2 Condition – Surface boundaries share Common edge with curvature matching

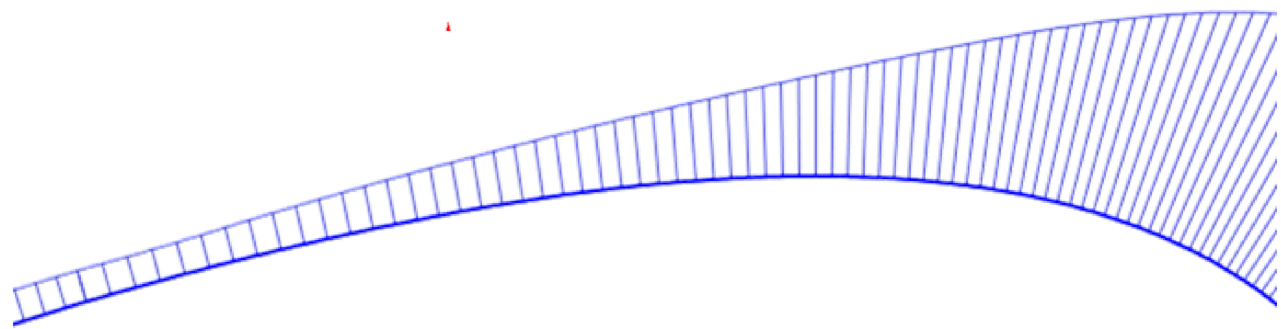
C3 Condition – Surface boundaries share Common edge with curvature matching to greater degree than C2



Curve Continuous

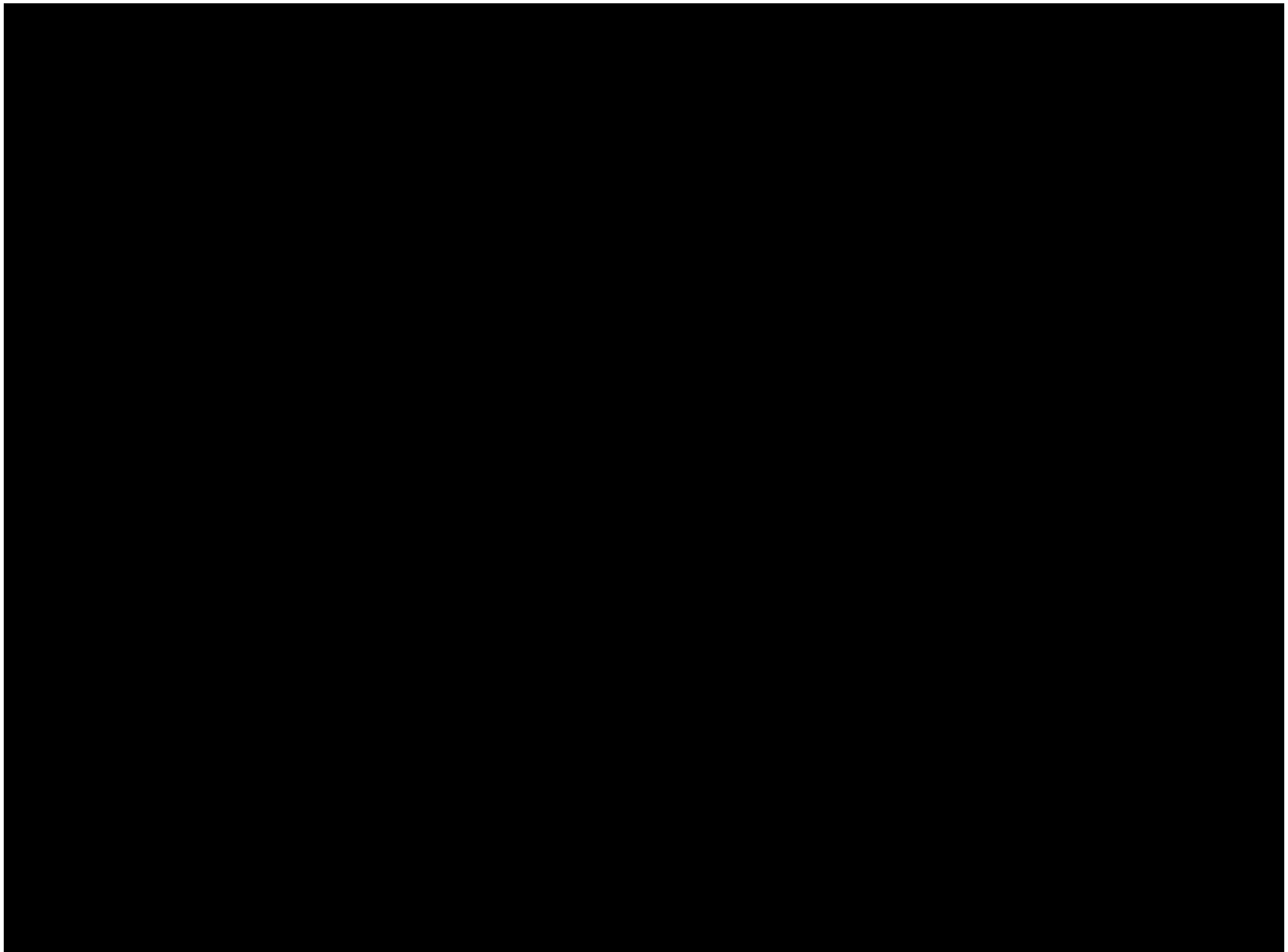


Equal Curvature – C2



Higher Degree – C3>

Animation Path (Parameter = Time)



http://www.youtube.com/watch?v=A3lDRn6jafs

Outlines

- Curves and surfaces representation
- Continuity
- Curves and surfaces
  - Interpolating
  - Hermite
  - Bezier

Parametric Polynomial Curves

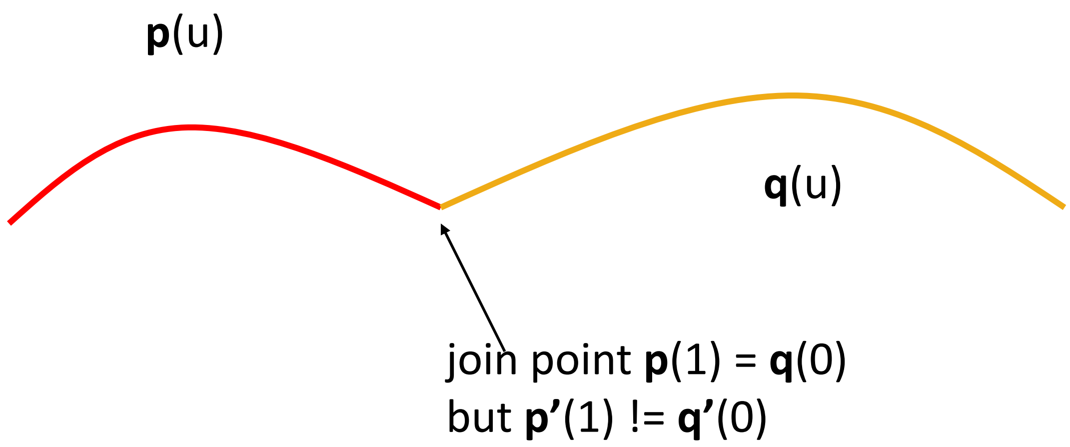
$$x(u) = \sum_{i=0}^{N} c_{xi} u^i \quad y(u) = \sum_{j=0}^{M} c_{yj} u^j \quad z(u) = \sum_{k=0}^{L} c_{zk} u^k$$

- If N=M=K, we need to determine 3(N+1) coefficients
- Equivalently we need 3(N+1) independent conditions
- Noting that the curves for x, y and z are independent, we can define each independently in an identical manner
- We will use the form where p can be any of x, y, z

$$\mathrm{p}(u) = \sum_{k=0}^{L} c_k u^k$$

Why Polynomials?

- Easy to evaluate
- Continuous and differentiable everywhere
  - Must worry about continuity at join points including continuity of derivatives



Cubic Parametric Polynomials

- N=M=L=3, gives balance between ease of evaluation and flexibility in design

$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

- 4 coefficients to determine for each of x, y and z
- Seek 4 independent conditions for various values of *u* resulting in 4 equations in 4 unknowns for each of *x*, *y* and *z*
  - Conditions are a mixture of continuity requirements at the join points and conditions for fitting the data

Cubic Parametric Polynomial Surfaces

$$\mathbf{p}(u,v)=[x(u,v), y(u,v), z(u,v)]^T$$

where

$$p(u,v)=\sum_{i=0}^{3}\sum_{j=0}^{3}c_{ij}u^i v^j$$

p is any of x, y or z

Need 48 coefficients (3 independent sets of 16) to determine a surface patch

Matrix-Vector Form of Curve

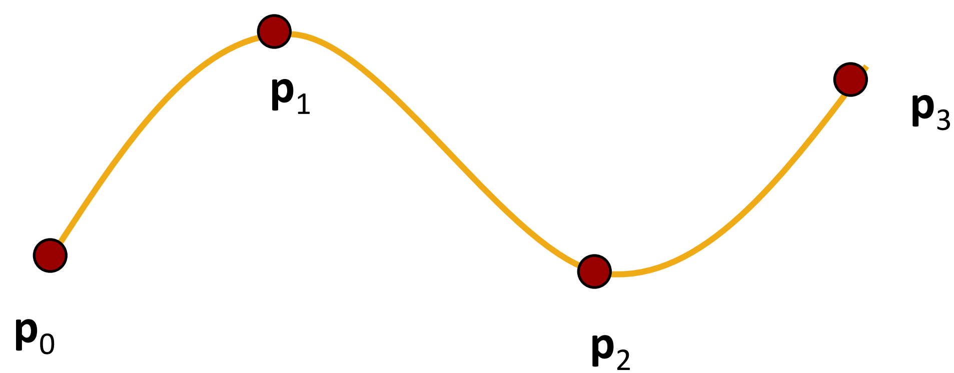
$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

define $\quad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$

then $\quad \mathrm{p}(u) = \mathbf{u}^T \mathbf{c} = \mathbf{c}^T \mathbf{u}$

Interpolating Curve

- Given four data (control) points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$
- determine cubic $\mathbf{p}(u)$ which passes through them
- Must find $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$



Interpolation Equations

Apply the interpolating conditions at u=0, 1/3, 2/3, 1

$p_0=p(0)=c_0$
$p_1=p(1/3)=c_0+(1/3)c_1+(1/3)^2c_2+(1/3)^3c_2$
$p_2=p(2/3)=c_0+(2/3)c_1+(2/3)^2c_2+(2/3)^3c_2$
$p_3=p(1)=c_0+c_1+c_2+c_2$

or in matrix form with $\mathbf{p} = [p_0\ p_1\ p_2\ p_3]^T$

p=Ac

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Interpolation Matrix

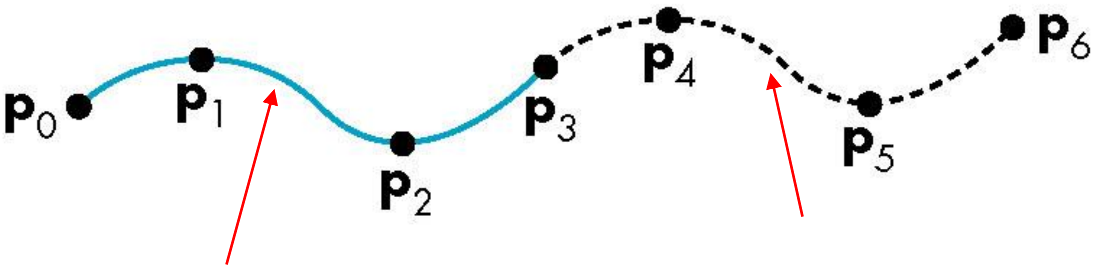
- Solving for **c** we find the *interpolation matrix*

$$\mathbf{M}_I = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix}$$

$$\mathbf{c} = \mathbf{M}_I \mathbf{p}$$

Note that $\mathbf{M}_I$ does not depend on input data and can be used for each segment in x, y, and z

Interpolating Multiple Segments



use $\mathbf{p} = [p_0\ p_1\ p_2\ p_3]^T$

use $\mathbf{p} = [p_3\ p_4\ p_5\ p_6]^T$

Get C^0 continuity at join points but not C^1 continuity of derivatives

Blending Functions

Rewriting the equation for p(u)

$$p(u)=\mathbf{u}^T\mathbf{c}=\mathbf{u}^T\mathbf{M}_I\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$$

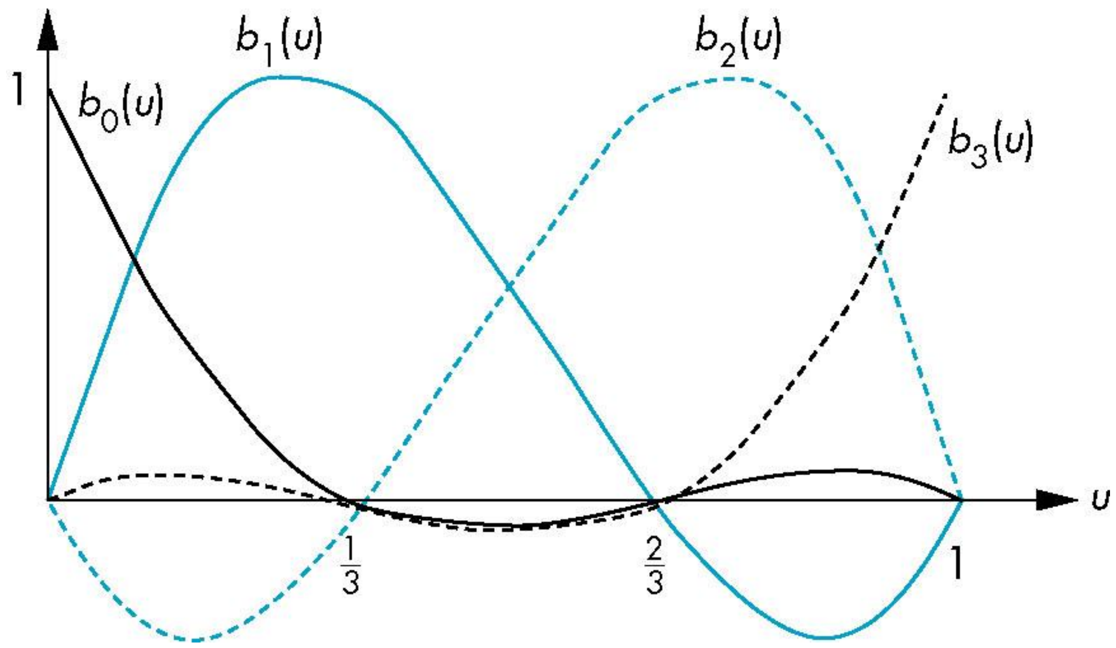
where $\mathbf{b(u) = [b_0(u)\ b_1(u)\ b_2(u)\ b_3(u)]^T}$ is an array of *blending polynomials* such that
$\mathbf{p(u) = b_0(u)p_0 + b_1(u)p_1 + b_2(u)p_2 + b_3(u)p_3}$

$$b_0(u) = -4.5(u-1/3)(u-2/3)(u-1)$$
$$b_1(u) = 13.5u\ (u-2/3)(u-1)$$
$$b_2(u) = -13.5u\ (u-1/3)(u-1)$$
$$b_3(u) = 4.5u\ (u-1/3)(u-2/3)$$

compute p for a given u as blending of interpolating points!

Blending Functions

- These functions are not smooth (up/down)
  - Hence the interpolation polynomial is not smooth

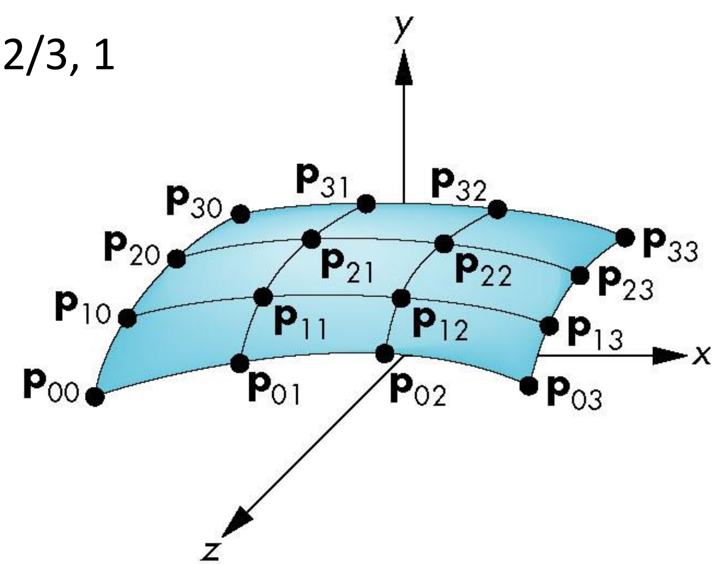


Interpolating Patch

$$p(u,v) = \sum_{i=o}^{3} \sum_{j=0}^{3} c_{ij} u^i v^j$$

Need 16 conditions to determine the 16 coefficients c_{ij}

Choose at u,v = 0, 1/3, 2/3, 1



Matrix Form

- Define $\mathbf{v} = [1\ v\ v^2\ v^3]^T$, $\mathbf{C} = [c_{ij}]$, $\mathbf{P} = [p_{ij}]$

$$p(u,v) = \mathbf{u}^T\mathbf{C}\mathbf{v}$$

- If we observe that for constant u, we obtain interpolating curve in v (and vice versa)

$$\mathbf{C}=\mathbf{M}_I\mathbf{P}\mathbf{M}_I^T$$

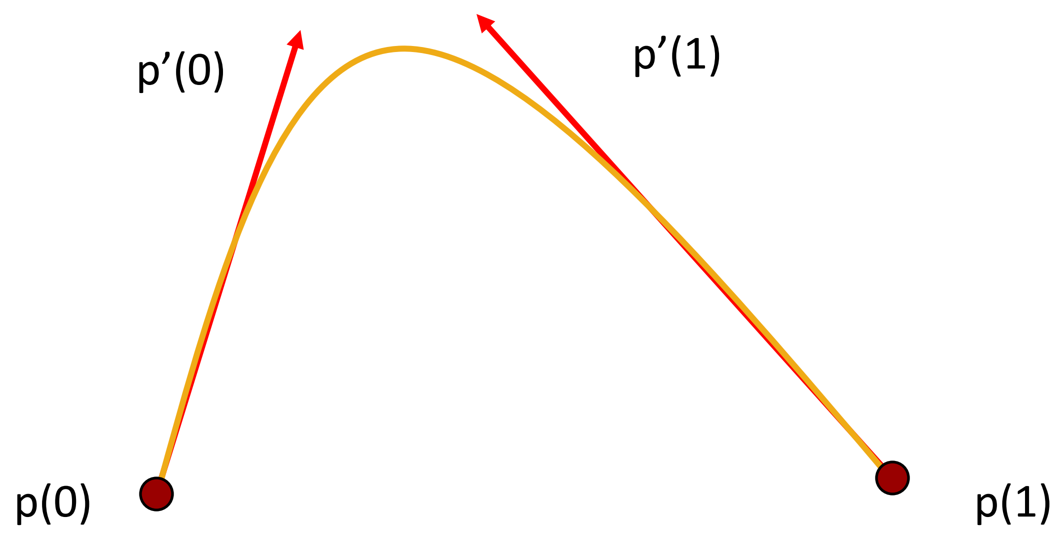
$$p(u,v) = u^T\mathbf{M}_I\mathbf{P}\mathbf{M}_I^T v = \mathbf{b}(u)^T\mathbf{P}\mathbf{b}(v)$$

$$p(u,v) = \sum_{i=o}^{3} \sum_{j=0}^{3} \underbrace{b_i(u)\, b_j(v)}_{\text{blending patch}}\, p_{ij}$$

Other Types of Curves and Surfaces

- How can we get around the limitations of the interpolating form
  - Lack of smoothness
  - Discontinuous derivatives at join points
- We have four conditions (for cubics) that we can apply to each segment
  - Use them other than for interpolation
  - Need only come close to the data

Hermite Form



- Use two interpolating conditions and two derivative conditions per segment
- Ensures continuity and first derivative continuity between segments

Hermit Form Equations

Interpolating conditions are the same at ends

$$p(0) = p_0 = c_0$$
$$p(1) = p_3 = c_0+c_1+c_2+c_3$$

Differentiating we find $p'(u) = c_1+2uc_2+3u^2c_3$

Evaluating at end points

$$p'(0) = p'_0 = c_1$$
$$p'(1) = p'_3 = c_1+2c_2+3c_3$$

Matrix Form

- We find $\mathbf{c}=\mathbf{M}_H\mathbf{q}$ where $\mathbf{M}_H$ is the Hermite matrix

$$\mathbf{q} = \begin{bmatrix} p_0 \\ p_3 \\ p'_0 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c} \Rightarrow \mathbf{M}_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

Hermit Blending Polynomials

$$p(u) = u^T \mathbf{M_H q} = \mathbf{b}(u)^T \mathbf{q}$$

$$\mathbf{b}(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}$$

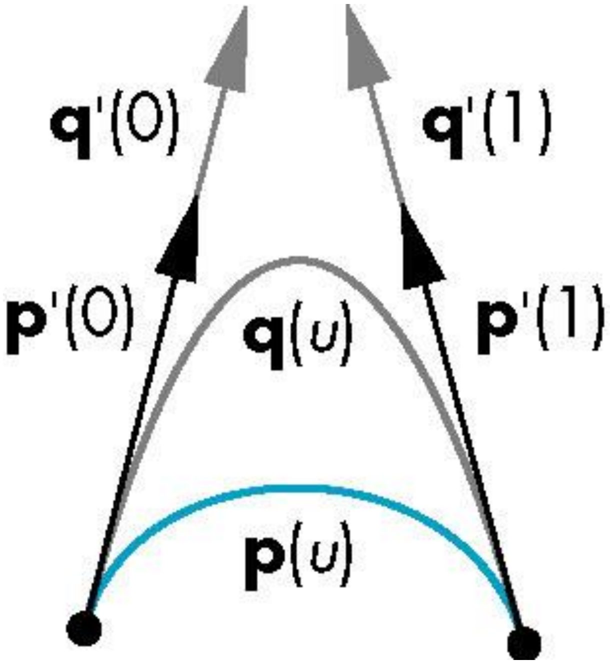
No zeros in [0,1], much smoother than interpolation blending polynomials

Hermit Blending Polynomial

- Although Hermit blending functions are smooth, it is not used directly in Computer Graphics and CAD because <u>we usually have control points rather than derivatives</u>
- However, the Hermite form is the basis of the Bezier form

Hermit Form Example

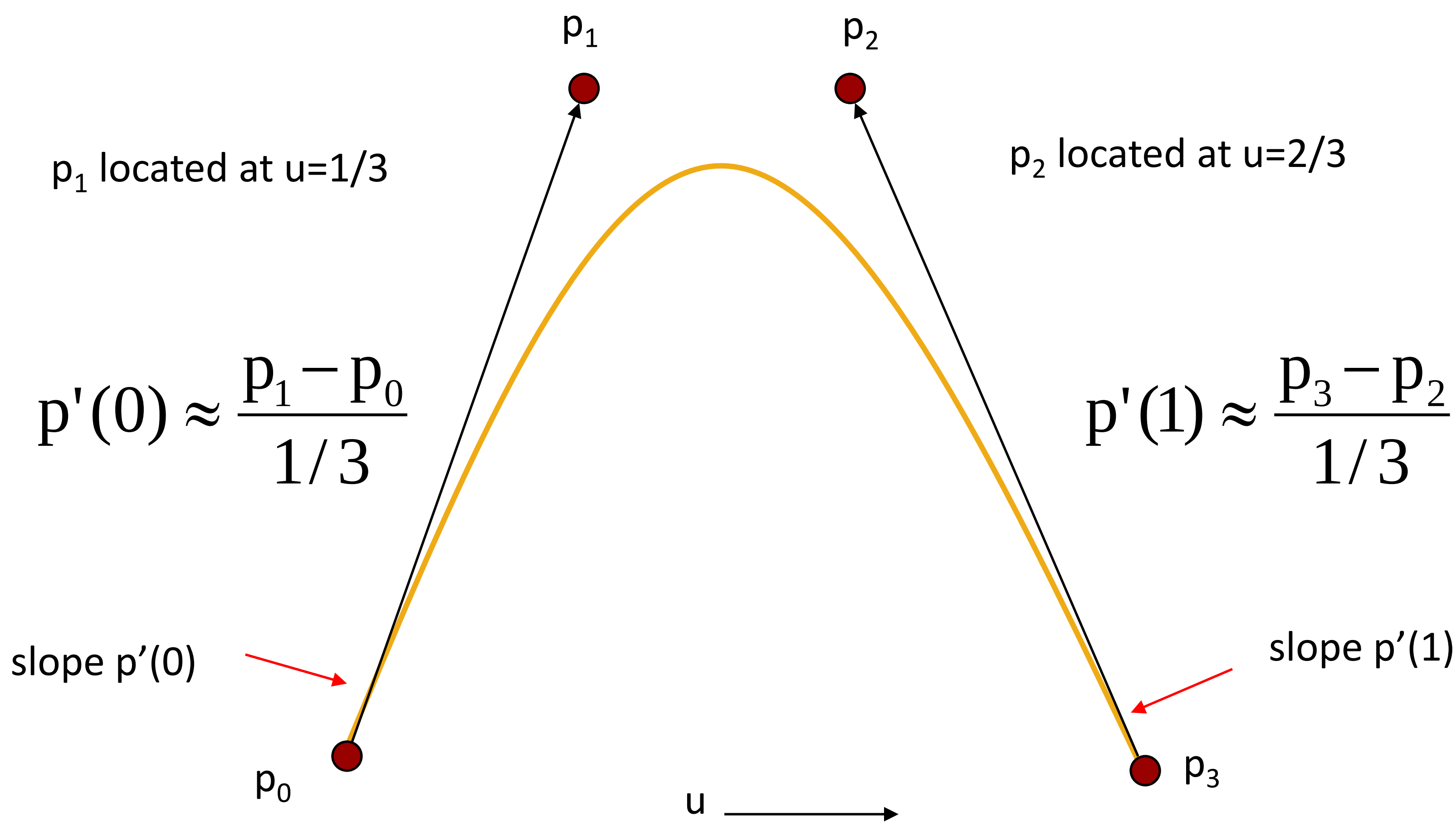
- Here the p and q have the same tangents at the ends of the segment but different derivatives
- Generate different Hermite curves
- This techniques is used in drawing applications



Bézier's Idea

- In graphics and CAD, we do not usually have derivative data
- Bezier suggested using the same 4 data points as with the cubic interpolating curve to approximate the derivatives in the Hermite form

Approximating Derivatives



$$p'(0) \approx \frac{p_1 - p_0}{1/3}$$

$$p'(1) \approx \frac{p_3 - p_2}{1/3}$$

Bézier Equations

- Interpolating conditions are the same

$$p(0) = p_0 = c_0$$
$$p(1) = p_3 = c_0+c_1+c_2+c_3$$

- Approximating derivative conditions

$$p'(0) = (p_1 - p_0) / (1/3) = c_0$$
$$p'(1) = (p_3 - p_2) / (1/3) = c_1+2c_2+3c_3$$

- Solve three linear systems of four equations and four unknowns for $\mathbf{c}=\mathbf{M}_B\mathbf{p}$

Bézier Matrix

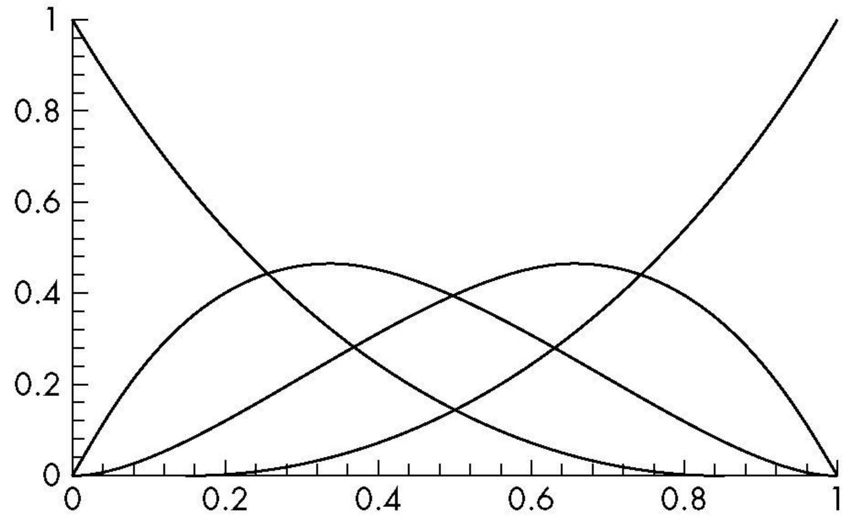
$$\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$p(u) = \mathbf{u}^T\mathbf{c} = \mathbf{u}^T\mathbf{M}_B\mathbf{p} = \mathbf{b}(u)^T\mathbf{p}$$

blending functions

Bézier Blending Functions

$$\mathbf{b}(u) = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}$$



Note that all zeros are at 0 and 1 which forces the functions to be smooth over (0,1)

Bernstein Polynomials

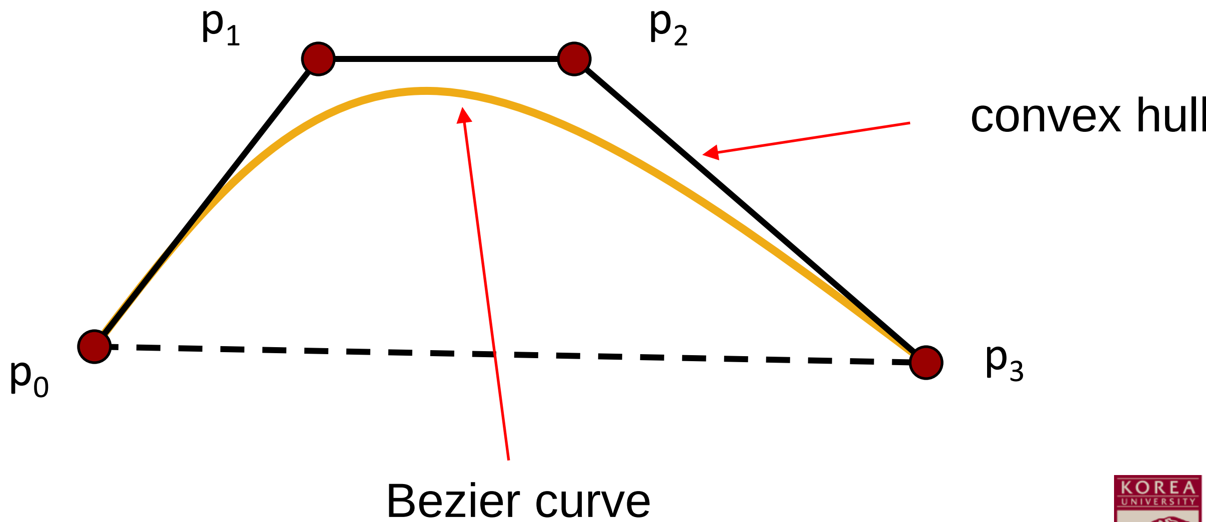
- The blending functions are a special case of the Bernstein polynomials

$$b_{kd}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

- These polynomials give the blending polynomials for any degree Bezier form
  - All zeros at 0 and 1
  - For any degree they all sum to 1 : $\sum_{i=1}^{d} b_{id}(u) = 1$
  - They are all between 0 and 1 inside (0,1)

Convex Hull Property

- The properties of the Bernstein polynomials ensure that all Bezier curves lie in the convex hull of their control points
- Hence, even though we do not interpolate all the data, we cannot be too far away

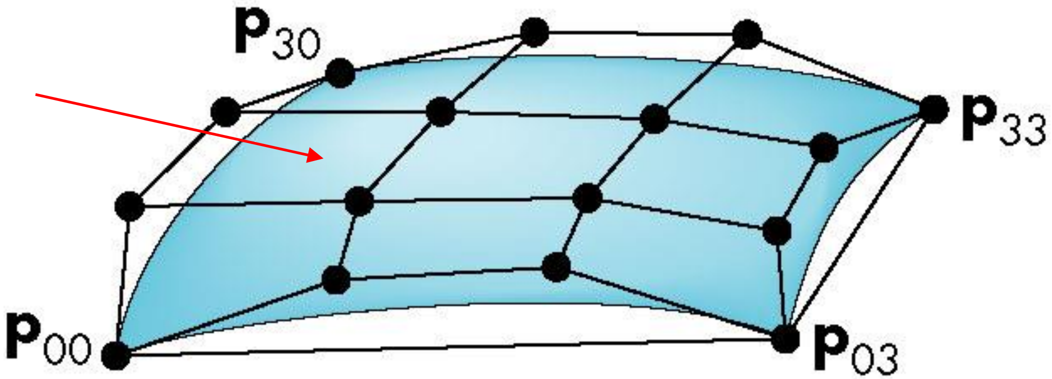


Bézier Patches

- Using same data array **P**=[p_{ij}] as with interpolating form, using bézier blending function

$$p(u,v)=\sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)\, b_j(v)\, p_{ij} = u^T \mathbf{M}_B \mathbf{P} \mathbf{M}_B^T v$$

Patch lies in convex hull



Bézier Curve/Surface Analysis

- Interpolating end points
- C^0 continuous at joint
- C^1 if end line segments are co-linear
- Increasing Bezier degree does not increase continuity at joint (why?)
  - Better to connect lower degree Bezier for local control

Bézier Surface

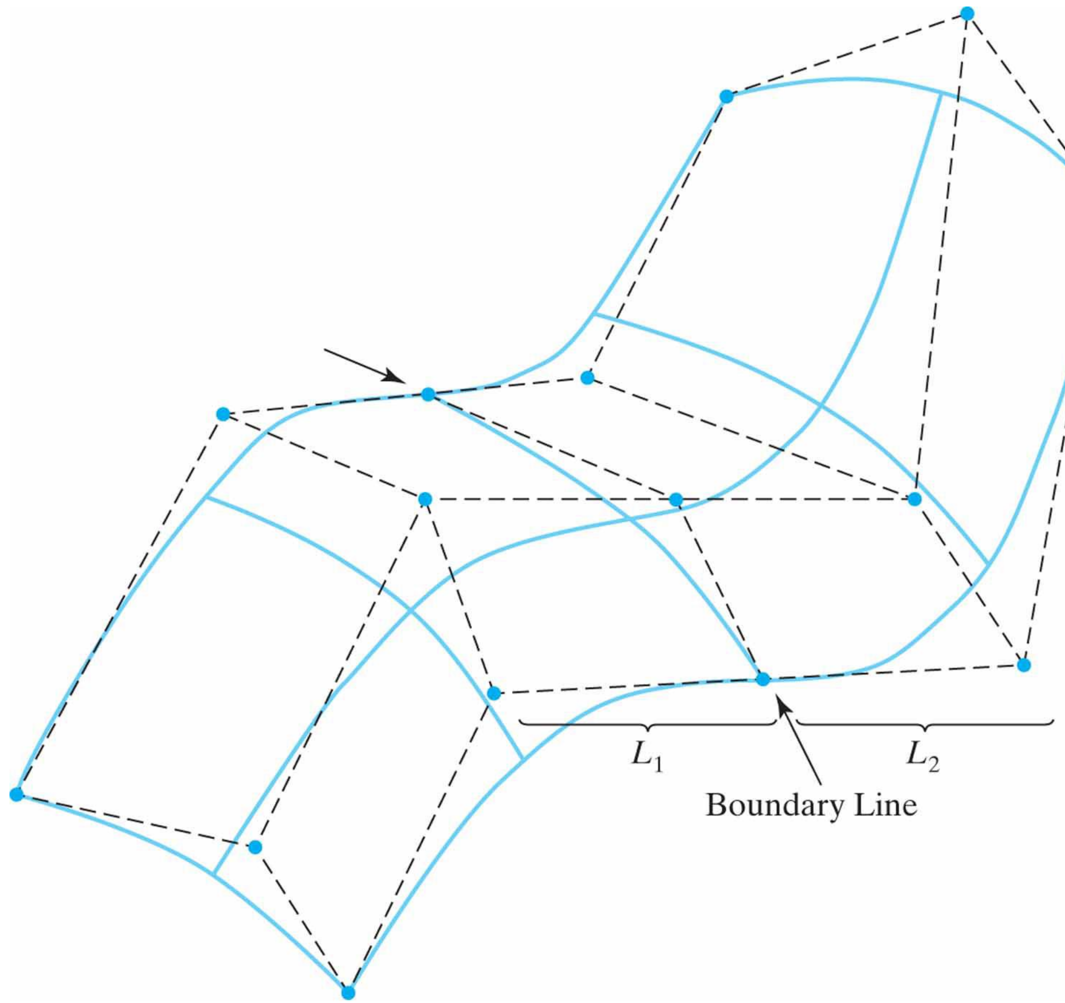
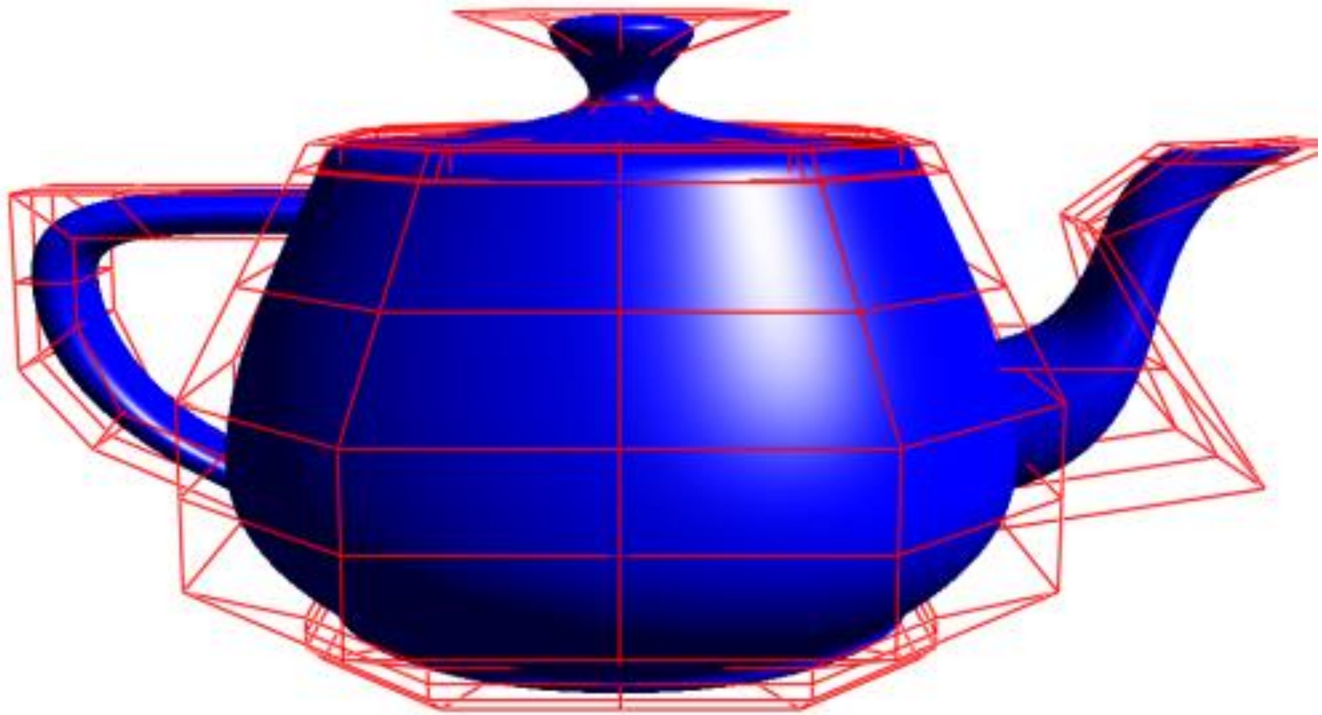


Image courtesy of Hearn

Questions?



Bezier surface rendering of Utah teapot