COSE 382 HW 5 Solutions

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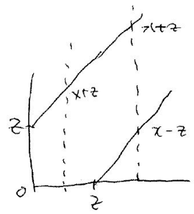
1. Let X and Y be i.i.d. Expo(1). Find the CDF and PDF of $Z = |X - Y|$.

Solution:

$$Z = |X - Y| = \begin{cases} X - Y & \text{for } X \geq Y \\ Y - X & \text{for } X < Y \end{cases}$$



$$\begin{aligned} p(Z \leq z) &= \int_0^z \int_0^{x+z} e^{-(x+y)} dy dx + \int_z^\infty \int_{y-z}^{x+z} e^{-(x+y)} dy dx \\ &= \int_0^z \left(e^{-x} - e^{-2x-z}\right) dx + \int_z^\infty \left(e^{-2x+z} - e^{-2x-z}\right) dx \\ &= 1 - e^{-z} - \frac{1}{2}e^{-z} + \frac{1}{2}e^{-z} = 1 - e^{-z} \\ f_Z(z) &= \frac{d}{dz}F(z) = e^{-z} \quad \text{for } z \geq 0 \end{aligned}$$

2. A stick of length L (a positive constant) is broken at a uniformly random point X. Given that $X = x$, another breakpoint Y is chosen uniformly on the interval $[0, x]$.

(a) Find the joint PDF of X and Y. Be sure to specify the support.

(b) Find the marginal distribution of Y.

(c) Find the conditional PDF of X given Y = y.

Solution:

a)

$$\begin{aligned} f_{X,Y}(x, y) = f_X(x) \cdot f_{Y|X}(y \mid x) &= \begin{cases} \frac{1}{L} \cdot \frac{1}{x} & 0 < x < L, 0 < y < x \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{Lx} & 0 < y < x < L \\ 0 & \text{else} \end{cases} \end{aligned}$$

b)

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_y^L \frac{1}{Lx} dx = \frac{\ln L - \ln y}{L} = \frac{1}{L} \ln \frac{L}{y} \quad \text{for } 0 < y < L, \text{ else } 0$$

c)

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{x \ln \frac{L}{y}} \quad \text{for } y < x < L, \text{ else } 0$$

3. Let X and Y have joint PDF

$$f_{X,Y}(x,y) = cxy, \text{ for } 0 < x < y < 1.$$

(a) Find c to make this a valid joint PDF.

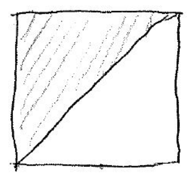
(b) Are X and Y independent?

(c) Find the marginal PDFs of X and Y.

(d) Find the conditional PDF of Y given $X = x$.

Solution:

$$f_{XY}(x,y) = \begin{cases} cxy & 0 < x < y < 1 \\ 0 & \text{else} \end{cases}$$



a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = c\int_0^1 y\left(\int_0^y xdx\right)dy = c\int_0^1 \frac{y^3}{2}dy = c\frac{1}{8}$$

Hence, $c = 8$.

b) Not independent since edt $f_{XY}(x,y)$ is non-zero over $0 < x < y < 1$

c)

$$f_X(x) = \int_x^1 8xydy = 8x\int_x^1 ydy = 4x - 4x^3$$
$$f_Y(y) = \int_0^y 8xydx = 8y\int_0^y xdx = 4y^3$$
$$f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y)$$

d)

$$f_{y|x}(y \mid x) = \frac{8xy}{4x - 4x^3} = \frac{2y}{1 - x^2} \text{ for } 0 < x < y < 1$$

4. Let (X, Y) be a uniformly random point in the triangle in the plane with vertices $(0,0), (0,1), (1,0)$.

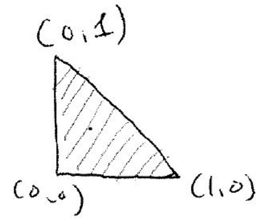
(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of X.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathrm{Cov}(X, Y)$

Solution:



a)

$$f_{xy}(x, y) = \begin{cases} 2 & x + y \leq 1, 0 \leq x, y \\ 0 & \text{else} \end{cases}$$

b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^{1-x} 2dy = 2(1 - x) \text{ for } 0 \leq x \leq 1$$

c)

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1 - y)} = \frac{1}{1 - y}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_0^{1-y} 2dx = 2(1 - y)$$

d)

$$E(Y) = E(X) = \int_0^1 2x(1 - x) dx = \left[2\left(\frac{x^3}{2} - \frac{x^3}{3}\right)\right]_0^1 = \frac{1}{3}$$

$$E(XY) = \int_0^1 \int_0^{1-y} 2xydxdy = \int_0^1 y(1 - y)^2 dy$$

$$= \int_0^1 \left(y^3 - 2y^2 + y\right) dy = \frac{1}{12},$$

$$\mathrm{cov}(XY) = E(XY) - E(X)E(Y) = \frac{1}{12} - \frac{1}{3^2} = -\frac{1}{36}$$

5. A chicken lays a $\mathrm{Pois}(\lambda)$ number N of eggs. Each egg hatches a chick with probability p, independently. Let X be the number which hatch, so $X|N = n \sim \mathrm{Bin}(n, p)$. Find the correlation between N (the number of eggs) and X (the number of eggs which hatch)

Solution:

Let Y be the number of eggs which do not hatch, then $N = X+Y, \quad X \sim \text{Pois}(p\lambda), Y \sim \text{Pois}((1-p)\lambda)$ and X and Y are independent.

$$\begin{aligned}
\text{Cov}(N, X) &= \text{Cov}(X+Y, X) = \text{Cov}(X, X) + \text{Cov}(Y, X) \\
&= \text{Var}(X) = \lambda_p. \\
\text{Corr}(N, X) &= \frac{\lambda p}{\sqrt{\lambda}\sqrt{\lambda p}} = \sqrt{p}.
\end{aligned}$$

6. Let $X = V + W$, $Y = V + Z$, where V, W, Z are i.i.d. $\text{Pois}(\lambda)$.

(a) Find $\text{Cov}(X, Y)$.

(b) Find the conditional joint PMF of X, Y given V, $P(X = x, Y = y|V = v)$.

Solution:

a)
$$\begin{aligned}
\text{Cov}(X, Y) &= \text{Cov}(V, V) + \text{Cov}(V, Z) + \text{Cov}(W, V) + \text{Cov}(W, Z) \\
&= \text{Var}(V) = \lambda
\end{aligned}$$

b)
$$\begin{aligned}
P(X = x, Y = y \mid V = v) &= P(W = x - V, Z = y - V \mid V = v) \\
&= P(W = x - v, Z = y - v) \\
&= P(W = x - v)P(Z = y - v) \\
&= e^{-\lambda}\frac{\lambda^{x-v}}{(x-v)!}e^{-\lambda}\frac{\lambda^{y-v}}{(y-v)!}
\end{aligned}$$

7. Let X and Y be i.i.d. $\mathcal{N}(0, 1)$, and let S be a random sign (1 or -1, with equal probabilities) independent of (X, Y).

(a) Determine whether or not $(X, Y, SX + SY)$ is Multivariate Normal.

(b) Determine whether or not (SX, SY) is Multivariate Normal.

Solution:

a) No, $\quad X + Y + (SX + 5Y) = (1 + S)X + (1 + S)Y = 0$ with probability $\frac{1}{2}$

b) Yes $\quad aSX + bSY = S(aX + bY)$. $Z = aX + bY$ is Normal; $aX + bY \sim N(0, a^2 + b^2)$ and $S \cdot Z$ is Normal! (Example 7.5.2)

8. Consider a two-dimensional jointly Gaussian random vector $\mathbf{X} = [X, Y]^T$ with the mean vector $\mu = [\mu_X \; \mu_Y]^T$ and the covariance matrix $\Sigma = \begin{bmatrix} \sigma_X^2 & Cov(X, Y) \\ Cov(X, Y) & \sigma_Y^2 \end{bmatrix}$. Let the correlation coefficient of X and Y be ρ. Show that the joint pdf given in the matrix form

$$f_{XY}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2|\det\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T\Sigma^{-1}(\mathbf{x} - \mu)\right),$$

for $\mathbf{x} = [x, y]^T$ is equivalent to the following form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right)$$

Solution:
First,

$$\det\Sigma = \sigma_X^2\sigma_Y^2 - Cov(X,Y)^2 = \sigma_X^2\sigma_Y^2(1-\rho^2)$$

Furthermore,

$$\begin{aligned}
\Sigma^{-1} &= \frac{1}{\sigma_X^2\sigma_Y^2(1-\rho^2)}\begin{bmatrix}\sigma_Y^2 & -Cov(X,Y)\\ -Cov(X,Y) & \sigma_X^2\end{bmatrix}\\
(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu) &= \frac{1}{\sigma_X^2\sigma_Y^2(1-\rho^2)}\left[\sigma_Y^2(x-\mu_X)^2 - 2Cov(X,Y)(x-\mu_X)(y-\mu_Y) + \sigma_X^2(y-\mu_Y)^2\right]\\
&= \frac{1}{1-\rho^2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\frac{\rho}{\sigma_X\sigma_Y}(x-\mu_X)(y-\mu_Y) + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]
\end{aligned}$$