COSE436

Lecture 3: Geometry

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Outline

- Basic geometry
- Coordinate systems

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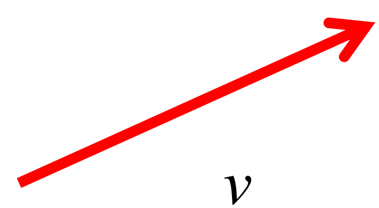
- Basic geometry
- Coordinate systems

Geometry

- Study of shape, size, relative position of figures, and the properties of space
- In graphics and visualization, 3D geometry is commonly used
- Minimum set of primitives
  - Scalars
  - Vectors
  - Points

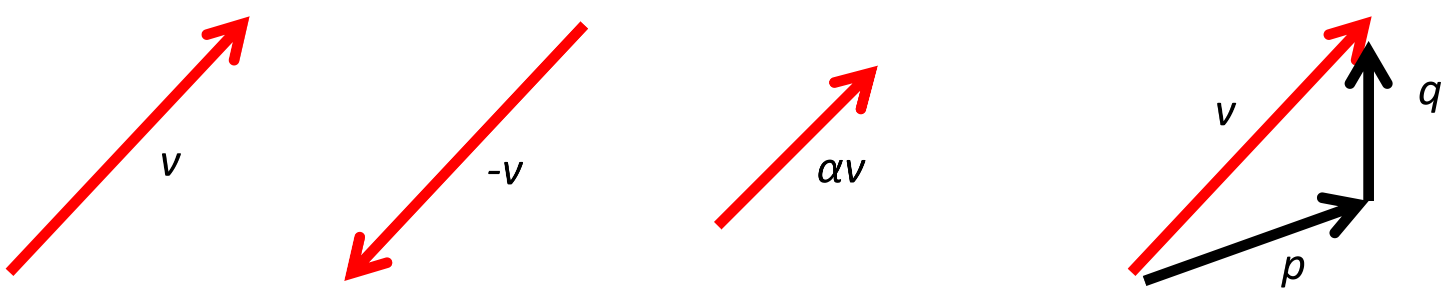
Scalars and Vectors

- Scalar
  - Single number representing *magnitude*
  - Real numbers in linear algebra (and in this class)
  - Length, area, size, volume…
- Vector (n-tuple)
  - *n*-tuple of scalar $(a_1, a_2, \cdots, a_n)$ representing *direction* and *magnitude*
  - Displacement, velocity…



Vector Operations

- Inverse
- Scalar multiplication
- Vector addition
- Zero vector



Vector Space

- A set of vector *V* on which two operations are defined
  - Vector addition

$$u = v + w$$

  - Scalar multiplication

$$u = \alpha v$$

  - Generalization

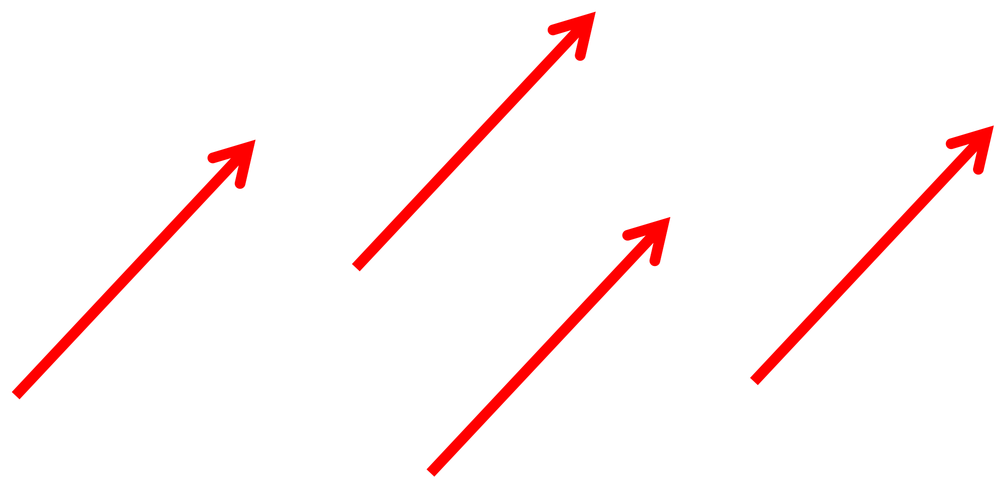
$$u = \alpha v + \beta w + \ldots$$

Vector Space Axioms

- Vector addition is associative and commutative
  - $(u+v)+w=u+(v+w),\ u+v=v+u$
- Vector addition has a unique identity element
  - *0* vector
- Each vector has an additive inverse
  - $v+(-v)=v-v=0$
- Scalar multiplication has an identity element
  - 1
- $\alpha(u+v)=\alpha u+\alpha v,\ \alpha(\beta v)=(\alpha\beta)v$

Vectors Lack Position

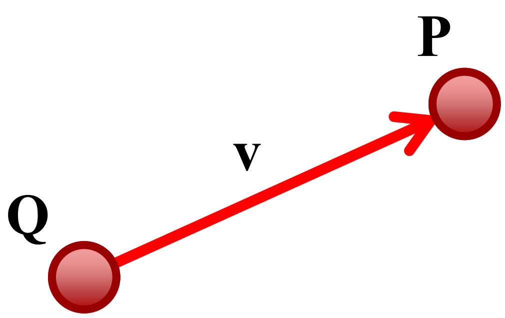
- Identical vectors



- Is vector space enough for graphics/ visualization?

Points

- Location in space
- *n*-tuple (same as vector!)
  – How can we distinguish?
- Operations
  – Point-point subtraction is a vector
  – Equivalent to point-vector addition



$$\mathbf{P} - \mathbf{Q} = \mathbf{v}$$

$$\mathbf{P} = \mathbf{Q} + \mathbf{v}$$

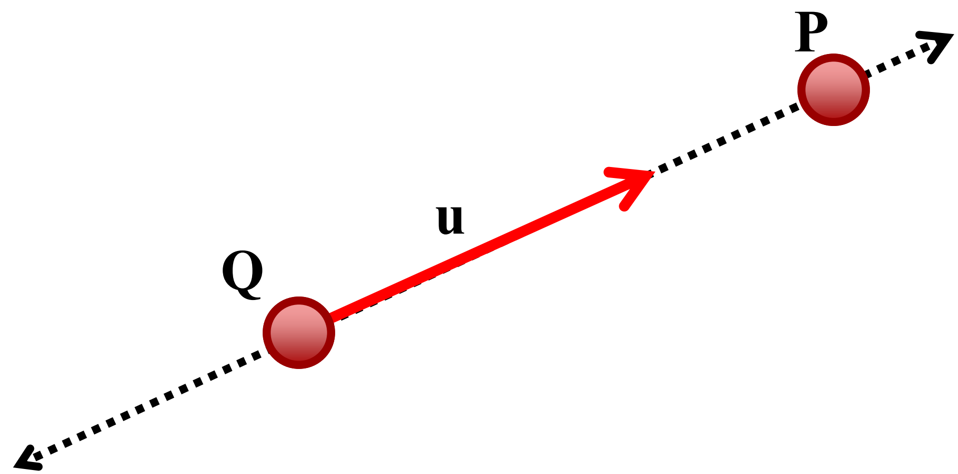
Affine Space

- Vector space + point
- Defined operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition (=point-point subtraction)
- Point-point addition and scalar-point multiplication is not defined
  - However, *affine sum* is defined

Parameteric Line

- Set of all points that pass through **Q** in the direction of the vector **u**

$$\mathbf{P}(\alpha) = \mathbf{Q} + \alpha\mathbf{u}$$

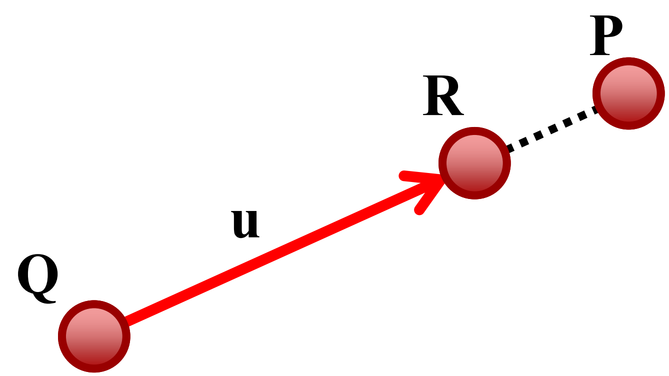


Affine Sum

- Line equation

$$\mathbf{P} = \mathbf{Q} + \alpha\mathbf{u}$$

- Define Affine Sum



$$\mathbf{R} = \mathbf{Q} + \mathbf{u}$$

$$\mathbf{u} = \mathbf{R} - \mathbf{Q}$$

$$\mathbf{P} = \mathbf{Q} + \alpha(\mathbf{R} - \mathbf{Q}) = \alpha\mathbf{R} + (1 - \alpha)\mathbf{Q}$$

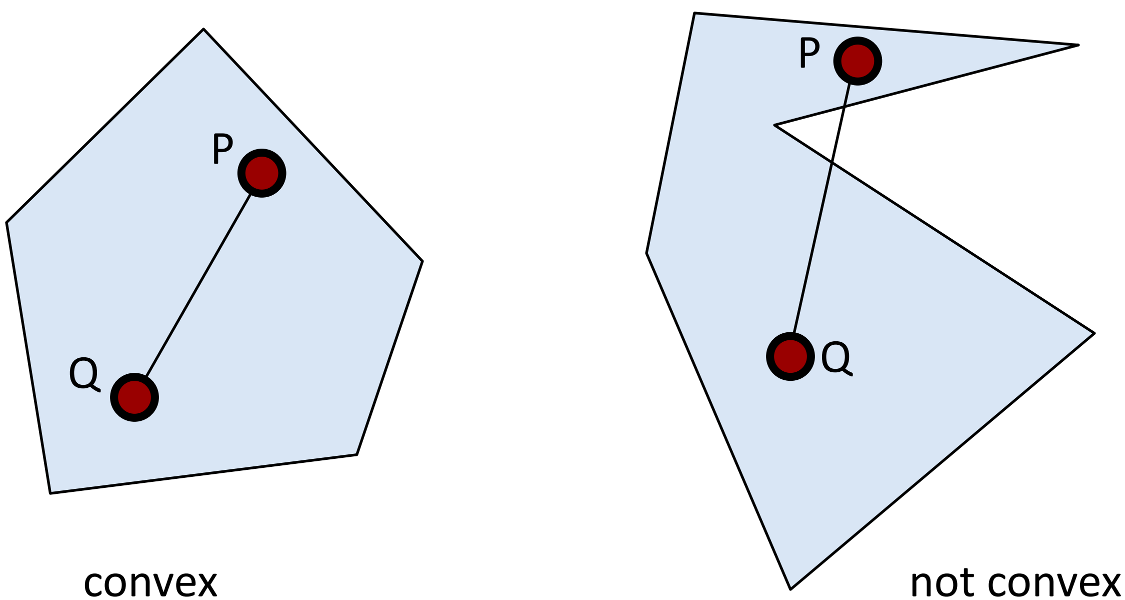
- Generalization

$$\mathbf{P} = \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \cdots + \alpha_n\mathbf{P}_n$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

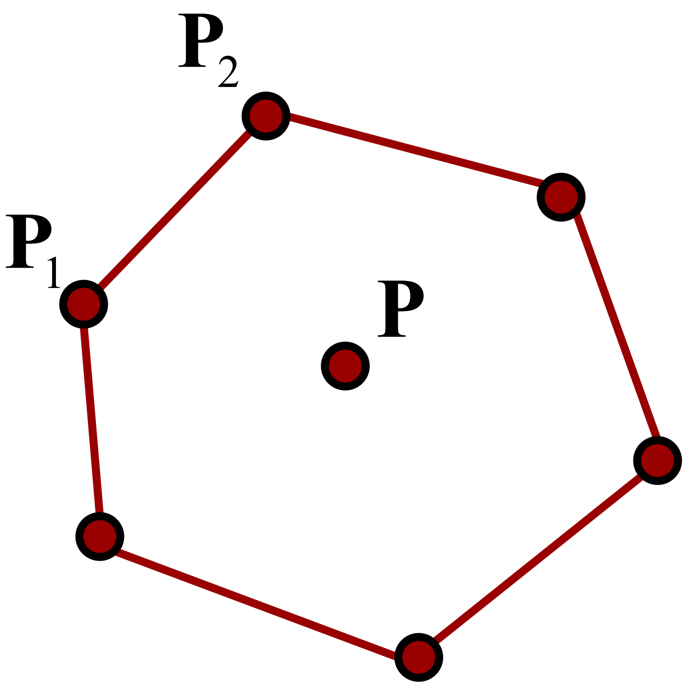
Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



Convex Hull

- Smallest convex object containing points
  - Affine sum with non-negative weights = convex combination



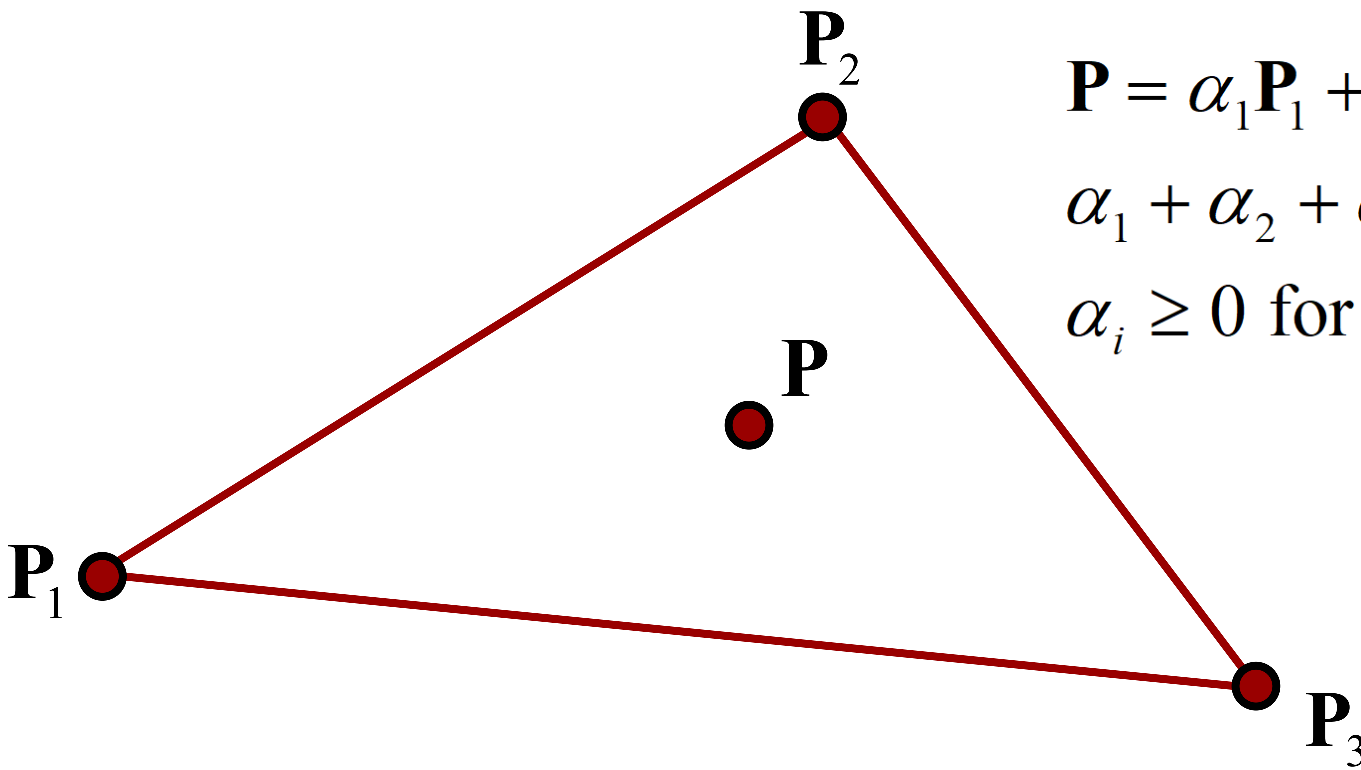
$$\mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_n \mathbf{P}_n$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

$$\alpha_i \geq 0 \text{ for } i = 1, \ldots, n$$

Example: Triangle

- Triangle is a convex hull of three points



$$\mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \alpha_3 \mathbf{P}_3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

$$\alpha_i \geq 0 \text{ for } i = 1, 2, 3$$

Euclidean Space

- Affine space + inner product
  - Measure of size/length
- Inner product

$$u \cdot v = v \cdot u$$: commutative

$$(\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w$$: linearity

$$v \cdot v > 0 \text{ if } v \neq 0$$: positive definite

$$0 \cdot 0 = 0$$

- Length of a vector

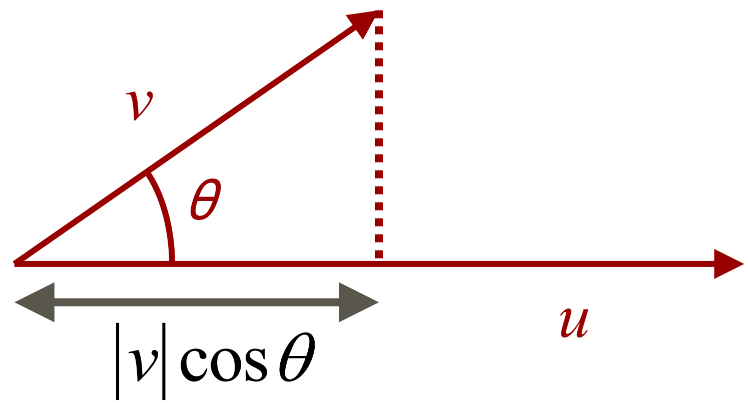
$$|v| = \sqrt{v \cdot v}$$

Dot and Cross Product

- Dot product

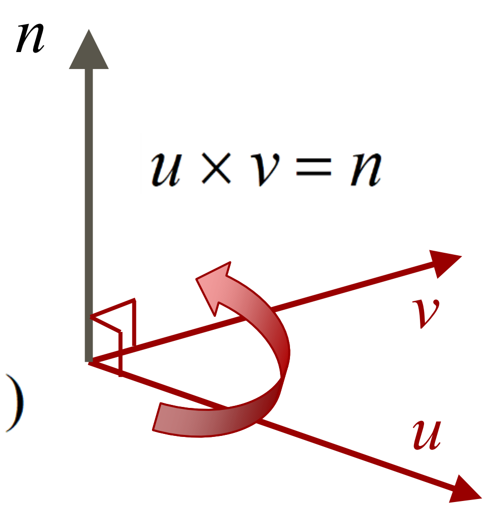
$$u \cdot v = |u||v|\cos\theta$$

$$\frac{u \cdot v}{|u|} = |v|\cos\theta$$



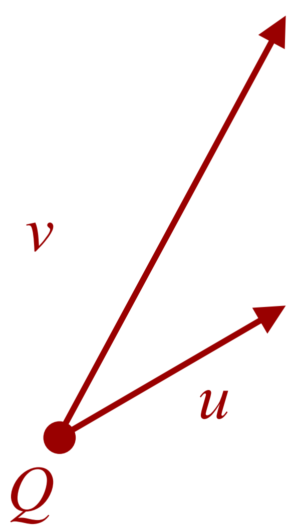
- Cross product
  - Right-hand system

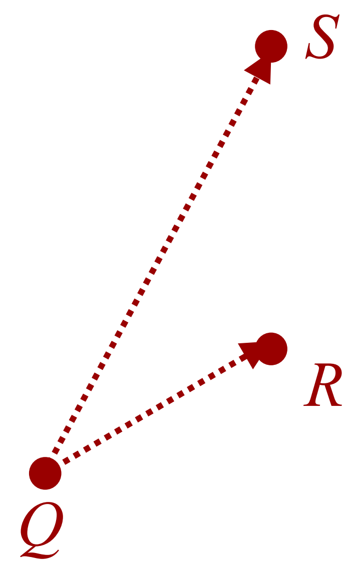
$$(x_1, y_1, z_1) \times (x_2, y_2, z_2)$$
$$= (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1)$$



Planes

- Defined by a point and two vectors or by three points



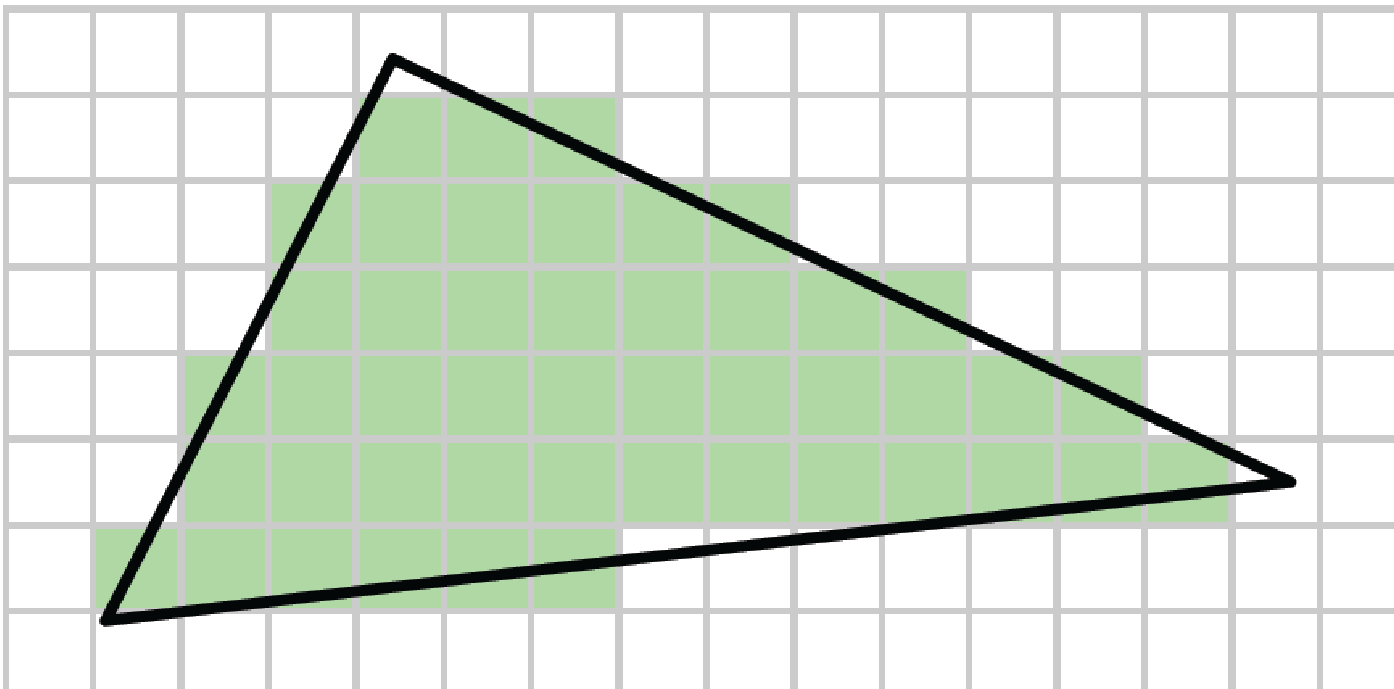


$$P(\alpha, \beta) = Q + \alpha u + \beta v$$

$$P(\alpha, \beta) = Q + \alpha(S - Q) + \beta(R - Q)$$

Rasterization (scan conversion)

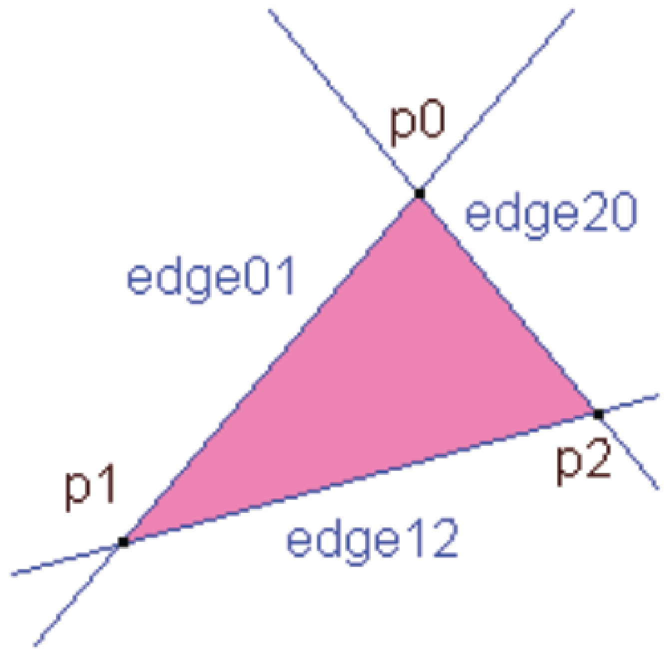
- Creating fragments



Pfister

What Kind of Geometry to Use?

- Triangles? Quads? N-gons?
- Why?

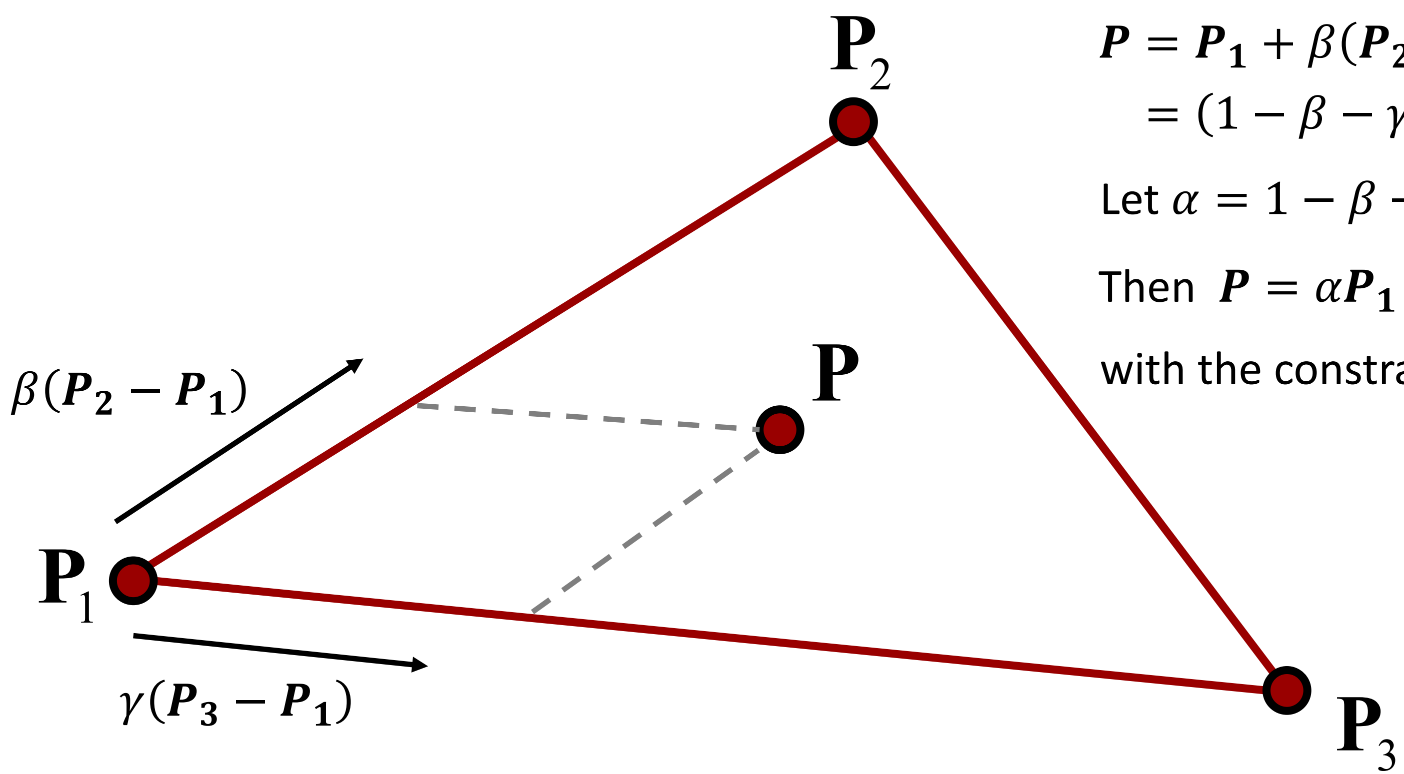


Nice Properties of Triangles

- Mathematically simple
  - 3 points or 3 edges
- Always planar (not true for quads)
- Always convex (not true for quads)
- Easy to rasterize
- Easy to interpolate

Barycentric Coordinate

- Coordinate system defined by triangle



$$\boldsymbol{P} = \boldsymbol{P_1} + \beta(\boldsymbol{P_2} - \boldsymbol{P_1}) + \gamma(\boldsymbol{P_3} - \boldsymbol{P_1})$$
$$= (1 - \beta - \gamma)\boldsymbol{P_1} + \beta\boldsymbol{P_2} + \gamma\boldsymbol{P_3}$$

Let $\alpha = 1 - \beta - r$

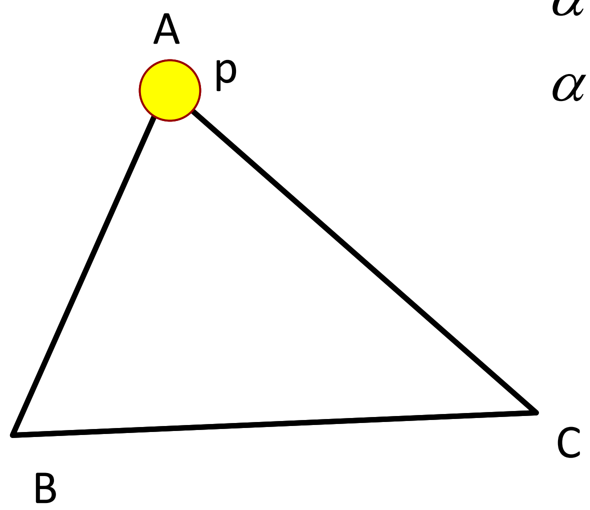
Then $\boldsymbol{P} = \alpha\boldsymbol{P_1} + \beta\boldsymbol{P_2} + \gamma\boldsymbol{P_3}$

with the constraint $\alpha + \beta + r = 1$

Properties on Barycentric Coordinate

- Point p is inside of triangle if $0 < \alpha, \beta, \gamma < 1$
- If only one of barycentric coordinates is 0 then p is on one of the edges of the triangle
- If two are 0 then p is one of the vertices of the triangle
- If one or more barycentric coordinates are less than 0 or greater than 1 then p is outside of the triangle

Examples



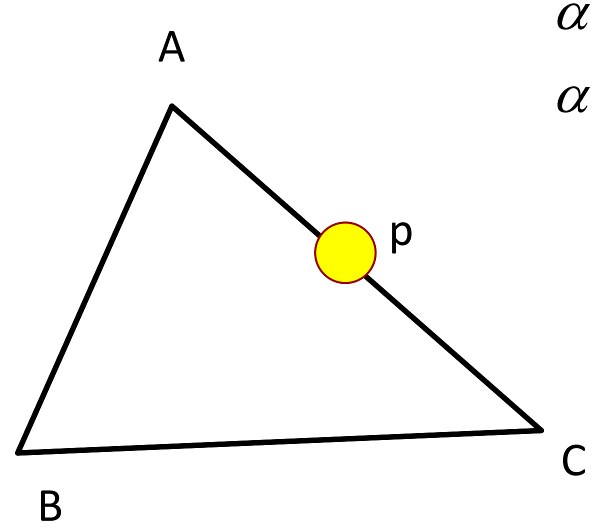
$$p = \alpha A + \beta B + \gamma C$$

$$\alpha + \beta + \gamma = 1$$

$$\alpha = 1, \beta = \gamma = 0$$

Examples

- Blending A & C



$$p = \alpha A + \beta B + \gamma C$$

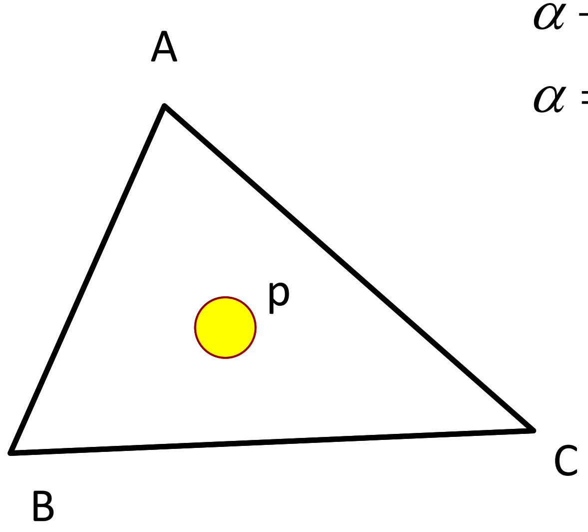
$$\alpha + \beta + \gamma = 1$$

$$\alpha = \gamma = 0.5, \beta = 0$$

Examples

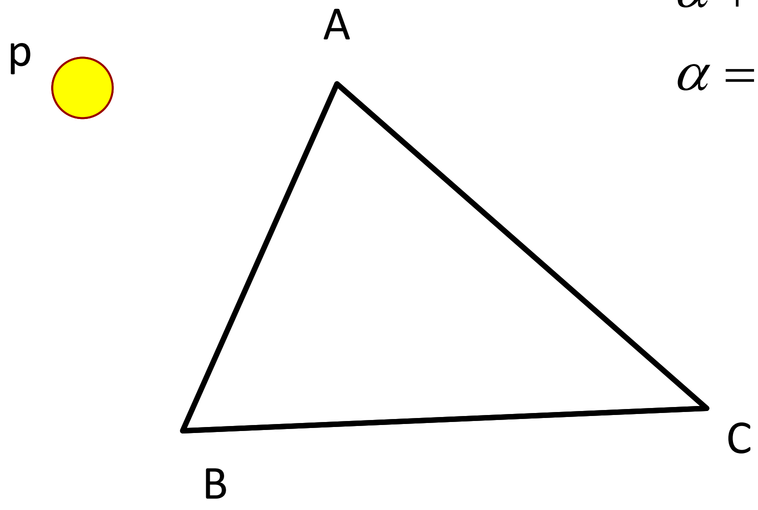
- Blending A, B & C, Inside

$$p = \alpha A + \beta B + \gamma C$$
$$\alpha + \beta + \gamma = 1$$
$$\alpha = \beta = \gamma = 0.333...$$



Examples

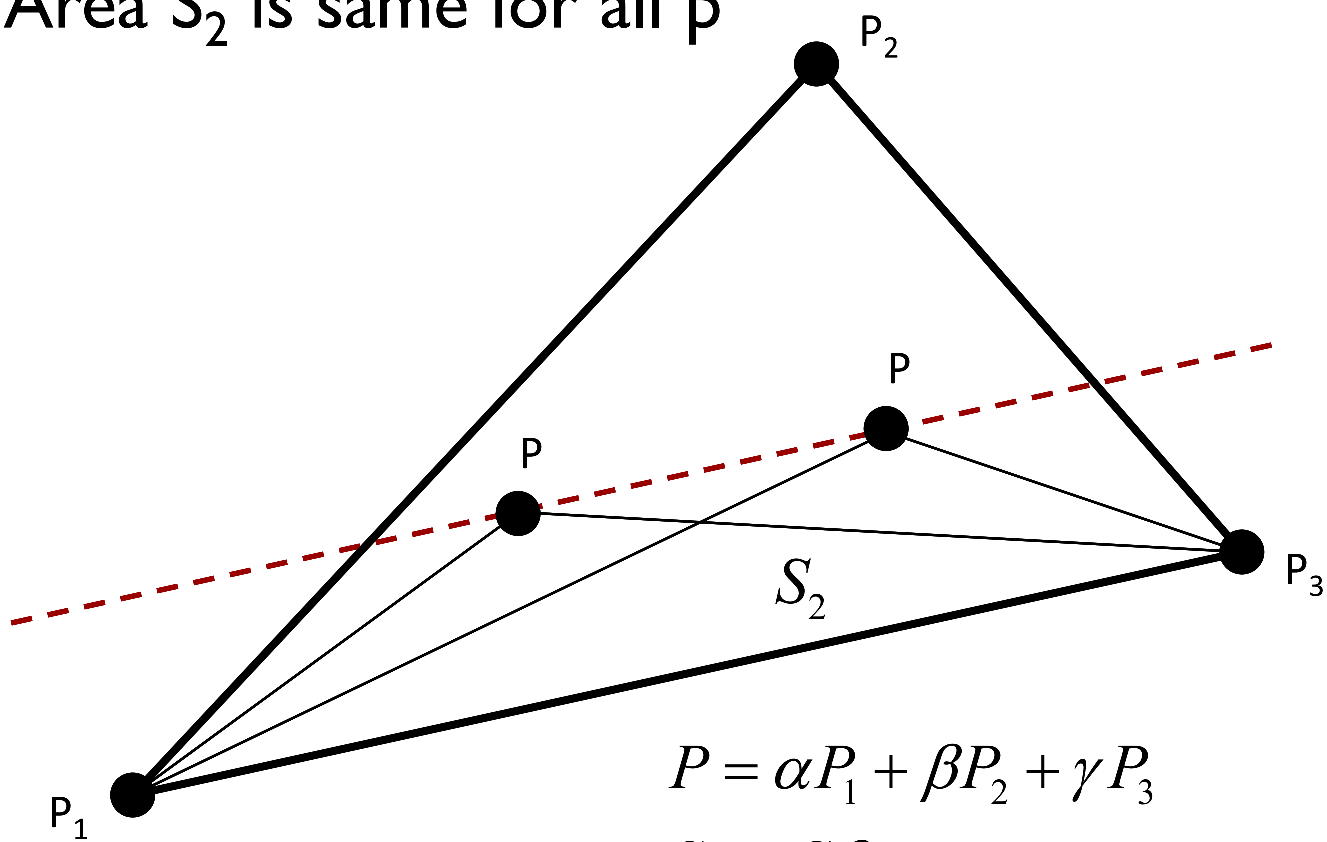
- Outside



$$p = \alpha A + \beta B + \gamma C$$
$$\alpha + \beta + \gamma = 1$$
$$\alpha = 1, \beta = 0.5, \gamma = -0.5$$

Barycentric Coordinates

- β is constant on lines parallel to an edge (P_1,P_3)
  - Area S_2 is same for all p



$$P = \alpha P_1 + \beta P_2 + \gamma P_3$$
$$S_2 = C\beta$$

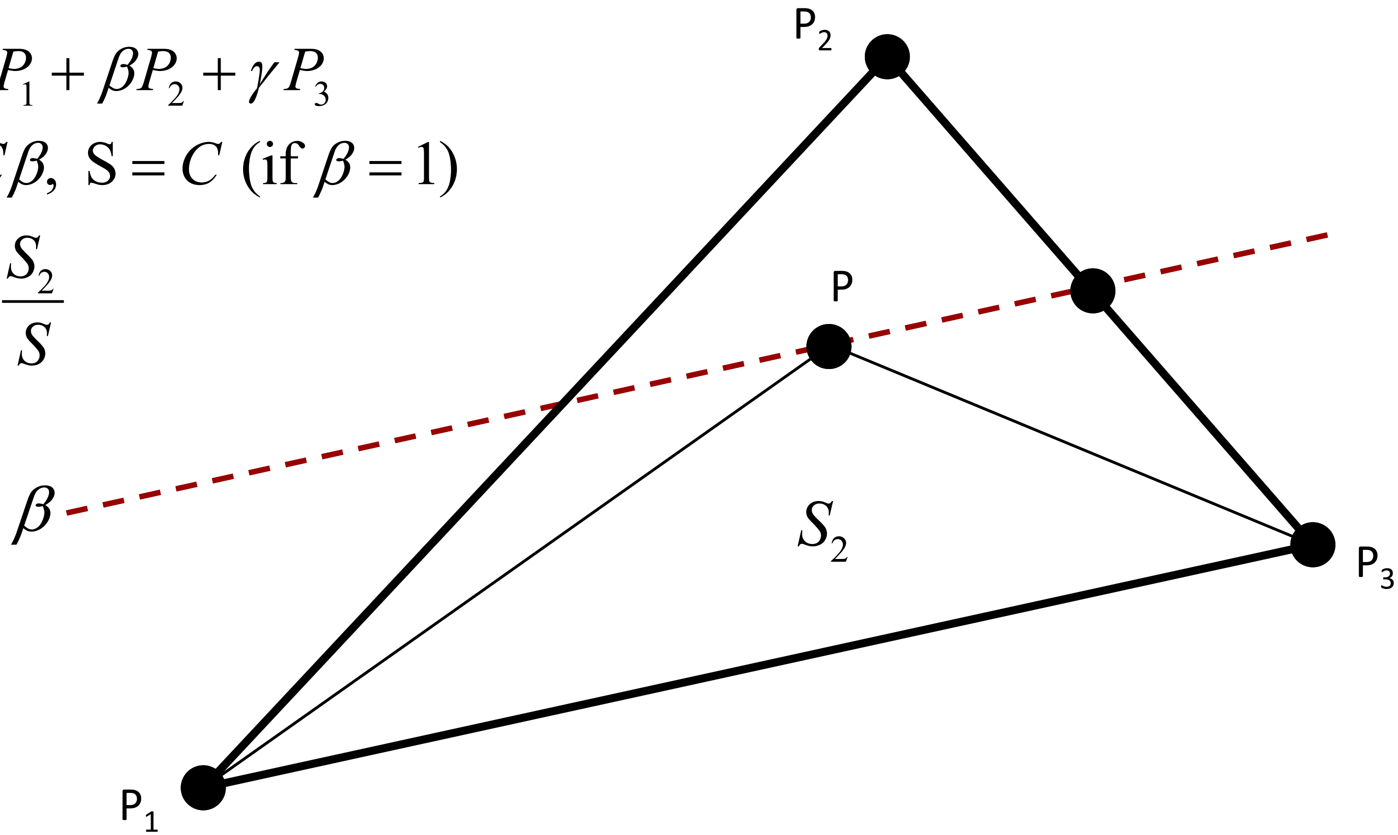
Barycentric Coordinates

- What is β ?

$$P = \alpha P_1 + \beta P_2 + \gamma P_3$$

$$S_2 = C\beta,\ \mathrm{S} = C\ (\text{if } \beta = 1)$$

$$\therefore \beta = \frac{S_2}{S}$$



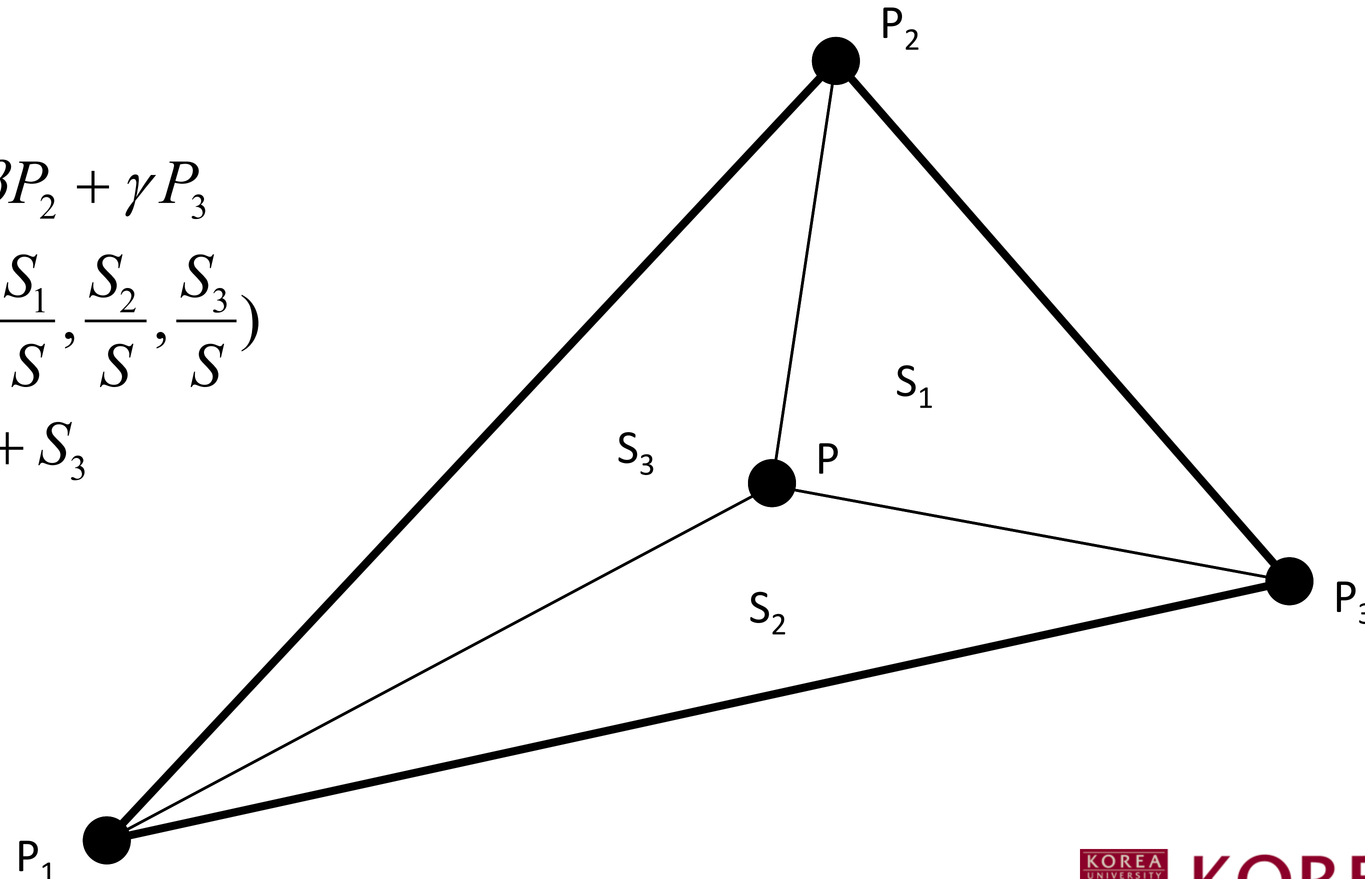
Barycentric Coordinates

- Proportional to the signed areas of subtriangles

$$P = \alpha P_1 + \beta P_2 + \gamma P_3$$

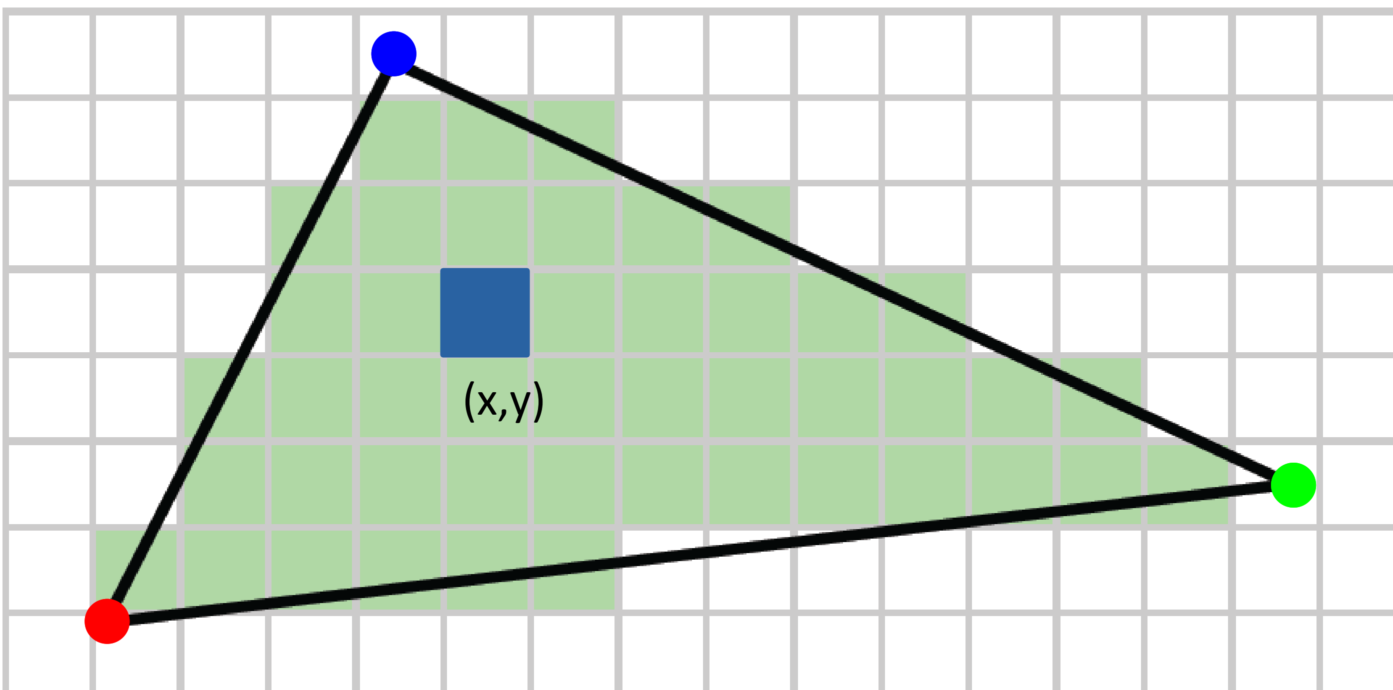
$$(\alpha, \beta, \gamma) = (\frac{S_1}{S}, \frac{S_2}{S}, \frac{S_3}{S})$$

$$S = S_1 + S_2 + S_3$$



Derive Barycentric Coordinate

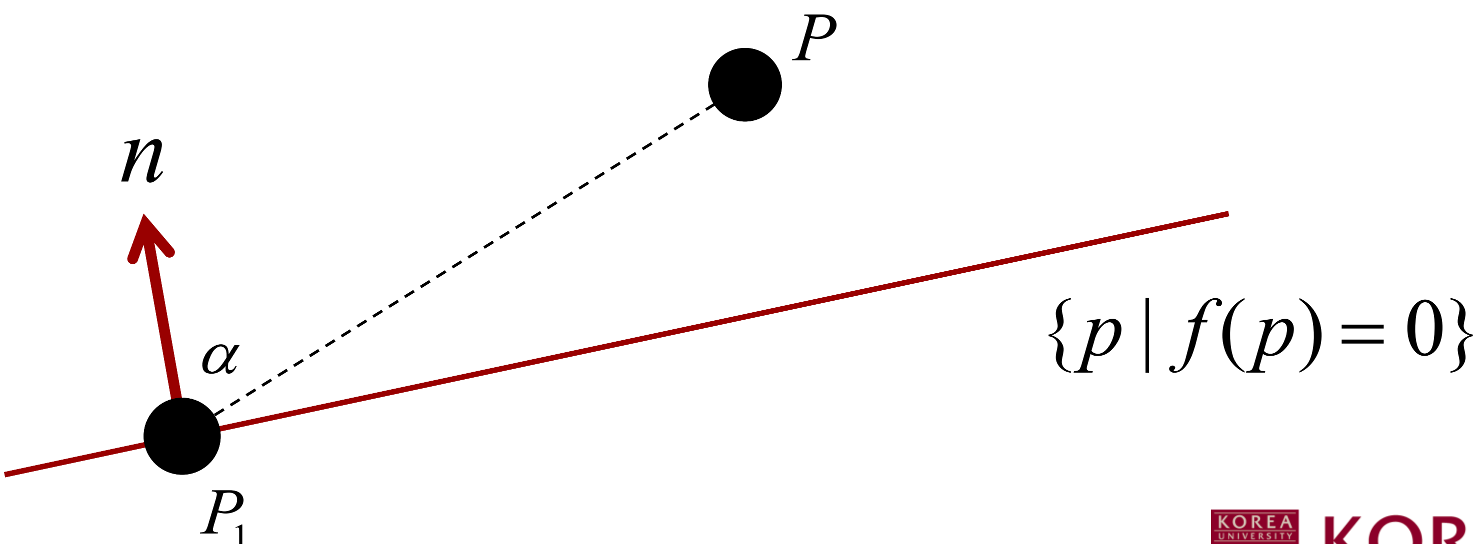
- Goal
  - Compute α, β, γ from arbitrary P(x,y)



Edge Equation

- Implicit form
  - if |n|=1 then edge equation is the distance from point to line orthogonal to n

$$f(p) = n \cdot (p - p_1) = \|n\| \|p - p_1\| \cos\alpha$$

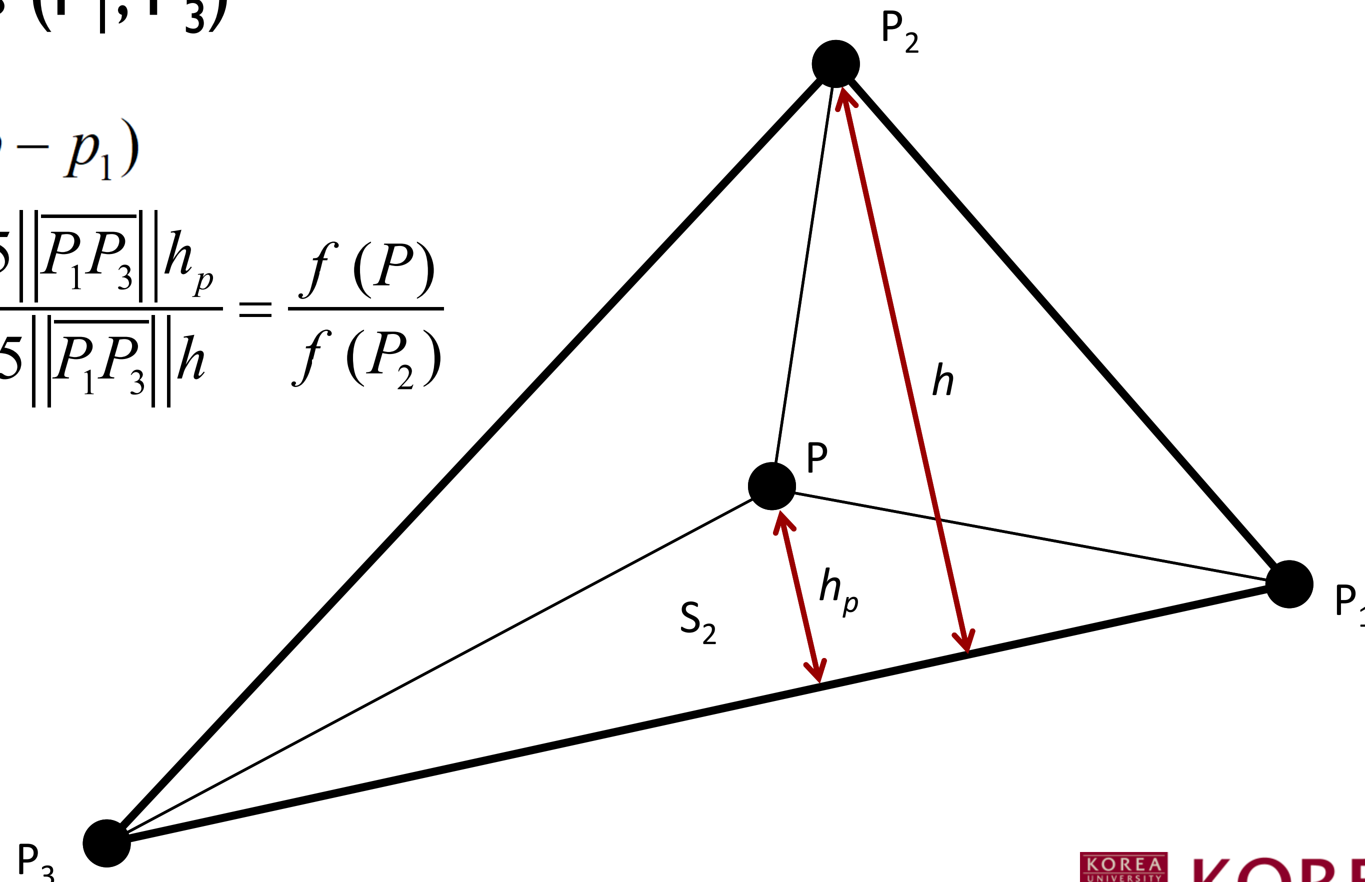


B.C from Edge Equation

- *f(p)* is the height of the triangle
  - Base is (P_1, P_3)

$$f(p) = n \cdot (p - p_1)$$

$$\beta = \frac{S_2}{S} = \frac{0.5\left\|\overline{P_1P_3}\right\|h_p}{0.5\left\|\overline{P_1P_3}\right\|h} = \frac{f(P)}{f(P_2)}$$



Outline

- Basic geometry
- Coordinate systems

Linear Combination

- For a set of vectors $v_1, v_2, \ldots, v_n$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, linear combination of vectors is defined as $\alpha_1 v_1 + \alpha_2 v_2 + .. \alpha_n v_n$
- A set of vectors is *linearly independent* if its linear combination is 0 iff $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$
- Meaning?
  – One <u>cannot</u> be represented in terms of the others

Example

- (1,0) and (0,1)
- a(1,0) + b(0,1) = (a,b) = (0,0) iff a=b=0
  - Linearly independent

- (1,0), (2,0), and (0,1)
- a(1,0) + b(2,0) +c(0,1) = (a+2b,c) = (0,0) if a=2, b=-1, and c=0
  - Linearly dependent
  - (1,0)=0.5(2,0)

Dimension of a Vector Space

- Maximum number of linearly independent vectors
- Any set of linearly independent vectors form a *basis*
- For a given basis, any vector in that space can be uniquely represented by a linear combination of basis

Example: 2D Vector Space

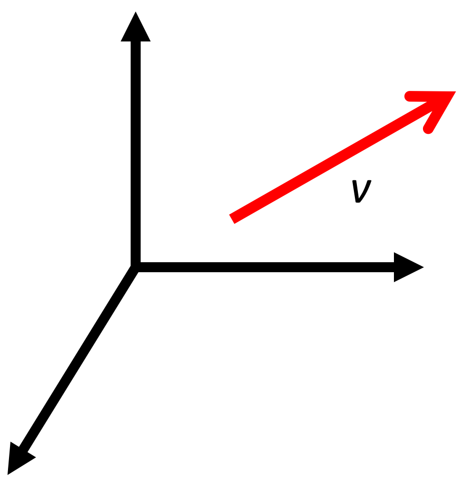
- Dimension?
- Basis?
  - Any set of 2D vectors that are linearly independent
  - (1,0), (0,1)
  - (0.5,0.5), (0.2, 1.0)
  - (0.2, 0.4), (0.4, 0.8)
  - (1,0), (0.5,0.5), (0,1)

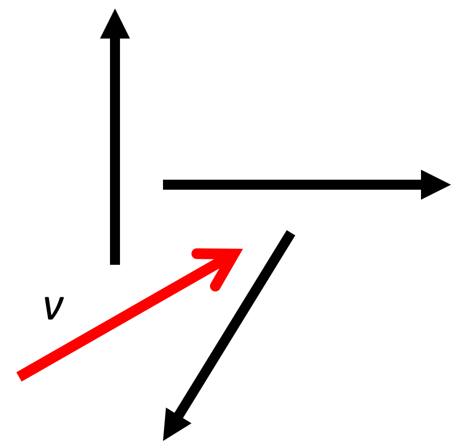
Coordinate Systems

- Consider a basis $v_1, v_2, \ldots, v_n$
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the *representation (coordinate)* of v with respect to the given basis
  - Coordinate system

Coordinate Systems

- Which one is correct?





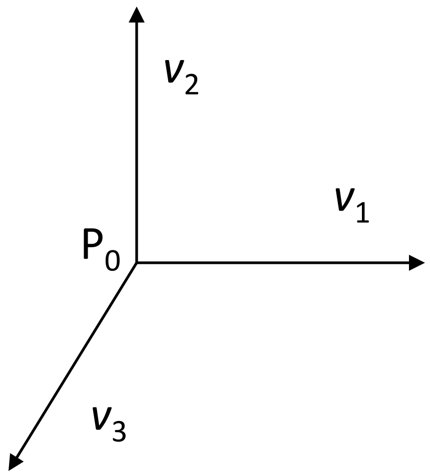
Frame

- Basis + origin
  - $(v_1, v_2, v_3, \ldots, v_n, P_0)$
- Every vector can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots. + \alpha_n v_n$$

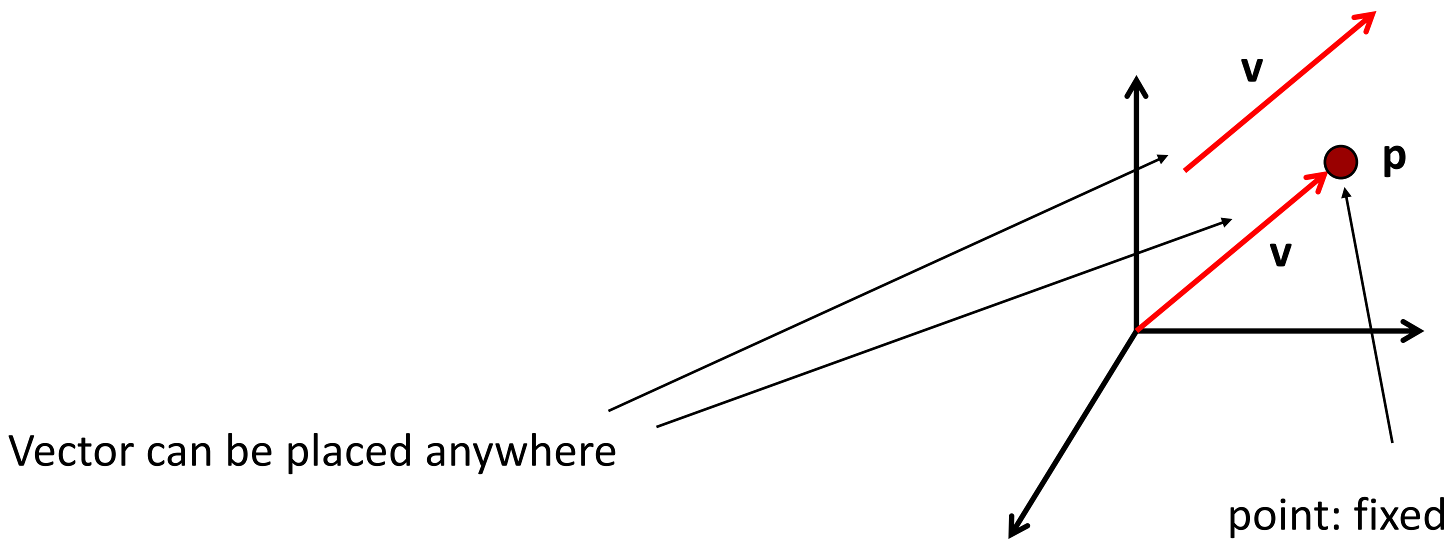
- Every point can be written as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots. + \beta_n v_n$$



Points and Vectors

- Same n-tuple representation
- $\mathbf{p}=[\beta_1\ \beta_2\ \beta_3]$, $\mathbf{v}=[\alpha_1\ \alpha_2\ \alpha_3]$
- How can we distinguish?



Homogeneous Coordinate

- *n+1* dimension to represent *n* dimension
  - For 3D, (*x,y,z,w*)
- Points

$$(wx, wy, wz, w) \rightarrow (x, y, z),\ w \neq 0 \text{ (commonly 1)}$$

- Vectors

$$(x, y, z, 0) \text{ is the vector in the direction of } (x, y, z)$$

- Easy to distinguish points and vectors!

Using Frame in H.C

- Points

$$\mathbf{P} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{P}_0 \end{bmatrix}$$

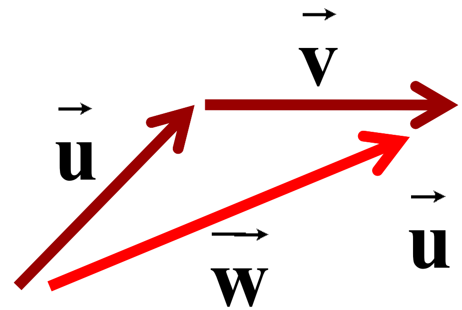
- Vectors

$$\mathbf{v} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{P}_0 \end{bmatrix}$$

Point & Vector Relationship

- Vector + Vector = Vector

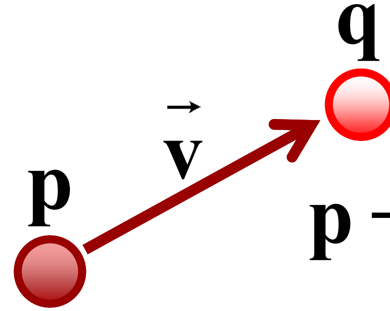
$$\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{u}+\mathbf{v} \\ 0 \end{pmatrix}$$



$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{w}}$$

- Point + Vector = Point

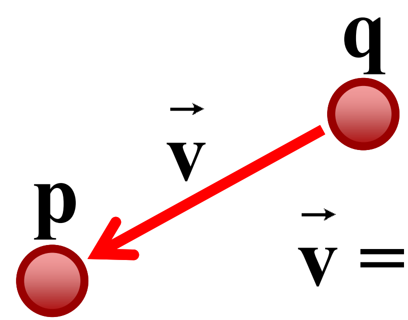
$$\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{p}+\mathbf{v} \\ 1 \end{pmatrix}$$



$$\mathbf{p} + \vec{\mathbf{v}} = \mathbf{q}$$

- Point – Point = Vector

$$\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} - \begin{pmatrix} \mathbf{q} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{p}-\mathbf{q} \\ 0 \end{pmatrix}$$



$$\vec{\mathbf{v}} = \mathbf{p} - \mathbf{q}$$

Changes of Coordinate Systems

- Two basis: $\{ v_1, v_2, v_3 \}$, $\{ u_1, u_2, u_3 \}$

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$
$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$
$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

$\Downarrow$ ***matrix***

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\mathbf{u} = M\ \mathbf{v}$$

Changes of Coordinate Systems

- Vector: w

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$w = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \text{ where } \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \Rightarrow w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \text{ where } \mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

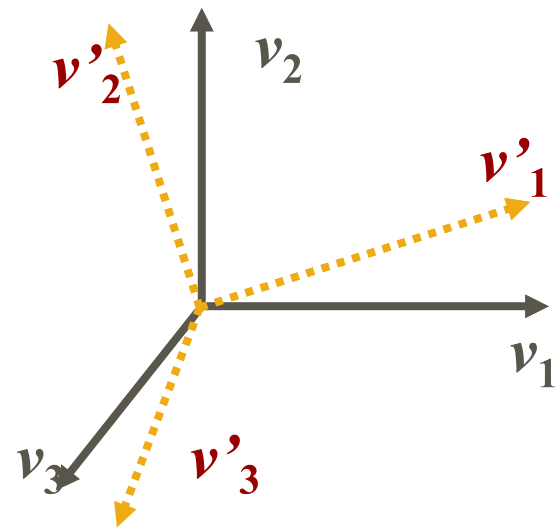
$$w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{a} = M^T \mathbf{b}$$

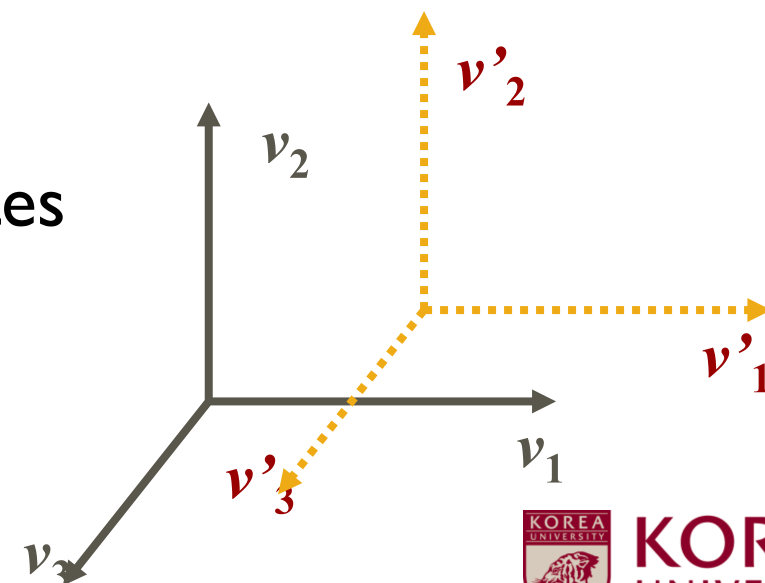
$$\mathbf{b} = \left(M^T\right)^{-1} \mathbf{a}$$

Change of Basis

- Origin unchanges
  - Rotations and scaling of basis



- Origin changes
  - Translation of origin
  - Homogeneous coordinates



Change of Frames

- Frames A(v_1, v_2, v_3, P_0), B(u_1, u_2, u_3, Q_0)

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$
$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$
$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$
$$Q_0 = \gamma_{41} v_1 + \gamma_{42} v_2 + \gamma_{43} v_3 + P_0$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

$$\mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\mathbf{b} = \left(\mathbf{M}^T\right)^{-1} \mathbf{a}$$

Example I

- Change of frames $(v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)$

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$
$$Q_0 = P_0$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Point $P = [1 \;\; 2 \;\; 3 \;\; 1]^T \rightarrow P' = [-1 \; -1 \;\; 3 \;\; 1]^T$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{(\boldsymbol{M}^T)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}}_{\boldsymbol{a}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}}_{\boldsymbol{b}}$$

Example 2

- Change of frames $(v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)$

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_1 + v_2 \\
u_3 &= v_1 + v_2 + v_3 \\
Q_0 &= P_0 + v_1 + 2v_2 + 3v_3
\end{aligned}
\quad \Rightarrow \quad
M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \boxed{1 \quad 2 \quad 3} & & & 1 \end{bmatrix}
$$

- Point $P = [1 \;\; 2 \;\; 3 \;\; 1]^T \rightarrow P' = [0 \;\; 0 \;\; 0 \;\; 1]^T$

$$
\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{(M^T)^{-1}}
\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}}_{a}
=
\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}
=
\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{b}
$$

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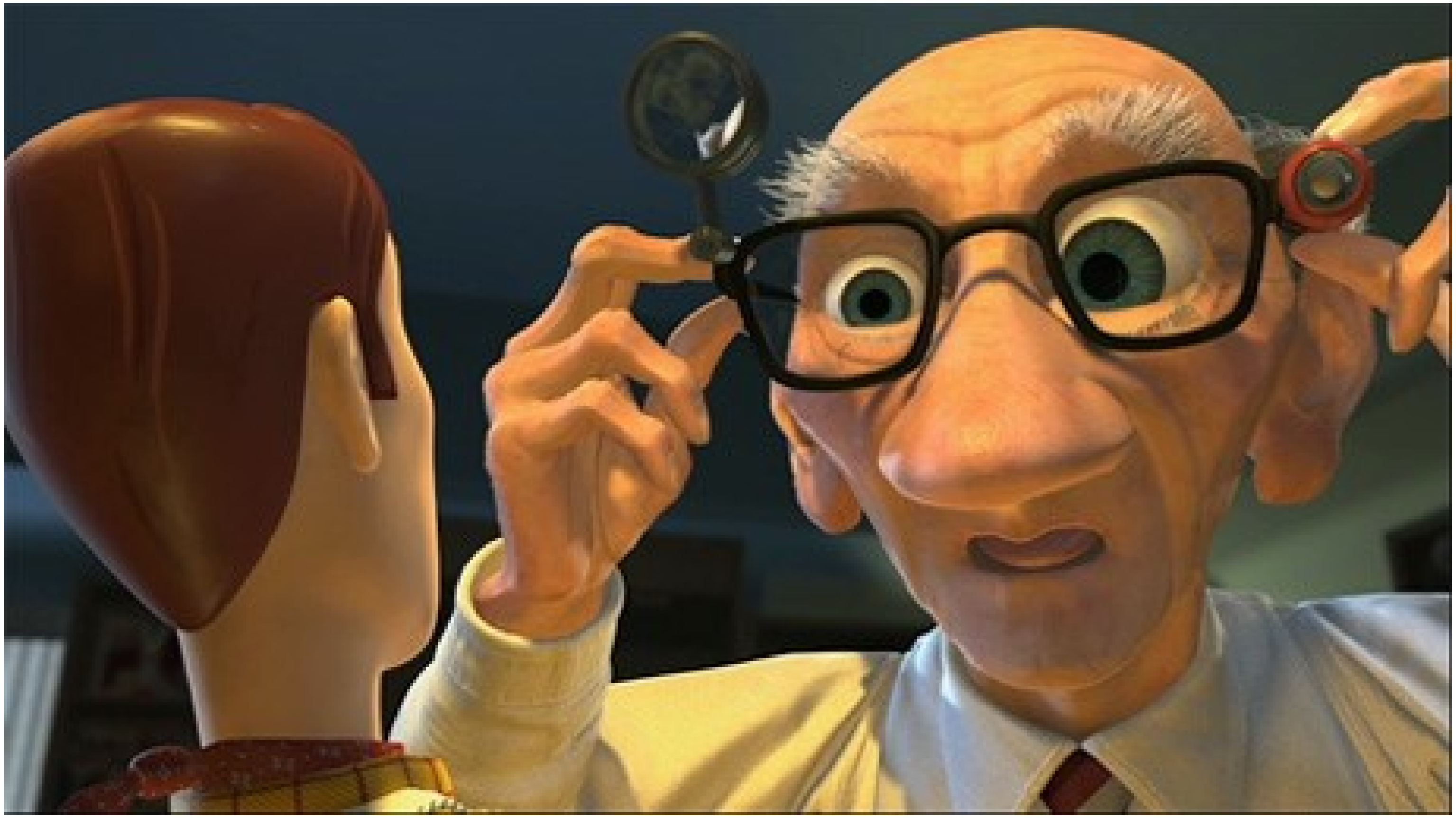
Questions?



Toy Story 2