COSE 382 HW 7

Date: 2024. 11. 18 **Due:** 2024. 11.25

1. A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

2. Let X_1, X_2 be i.i.d., and let $\bar{X} = \frac{1}{2}(X_1 + X_2)$ be the sample mean. In many statistics problems, it is useful or important to obtain a conditional expectation given $\bar{X}$. As an example of this, find $E(w_1 X_1 + w_2 X_2|\bar{X})$, where w_1, w_2 are constants with $w_1 + w_2 = 1$.

3. Let X be the height of a randomly chosen adult man, and Y be his father's height, where X and Y have been standardized to have mean 0 and standard deviation 1. Suppose that (X, Y) is Bivariate Normal, with $X, Y \sim N(0, 1)$ and $\mathrm{Corr}(X, Y) = \rho$.

(a) Find a constant c (in terms of ρ) and an r.v. V such that $Y = cX + V$, with V independent of X.

(b) Find a constant d (in terms of ρ) and an r.v. W such that $X = dY + W$, with W independent of Y.

(c) Find $E(Y|X)$ and $E(X|Y)$.

4. Let X and Y be random variables with finite variances, and let $W = Y - E(Y|X)$. This is a residual: the difference between the true value of Y and the predicted value of Y based on X.

(a) Compute $E(W)$ and $E(W|X)$.

(b) Compute $Var(W)$, for the case that $W|X \sim N(0, X^2)$ with $X \sim N(0, 1)$.

5. Show that if $E(Y|X) = c$ is a constant, then X and Y are uncorrelated.

6. In a national survey, a random sample of people are chosen and asked whether they support a certain policy. Assume that everyone in the population is equally likely to be surveyed at each step, and that the sampling is with replacement. Let n be the sample size, and let $\hat{p}$ and p be the proportion of people who support the policy in the sample and in the entire population, respectively. Show that for every $c > 0$,

$$P(|\hat{p} - p| > c) \leq \frac{1}{4nc^2}$$

7. For i.i.d. r.v.s $X_1, \cdots, X_n$ with mean μ and variance σ^2, find a value of n which will ensure that there is at least a 99% chance that the sample mean will be within 2 standard deviations of the true mean μ.

8. Let X and Y be i.i.d. positive r.v.s, and let $c > 0$. For each part below, fill in the appropriate equality or inequality symbol. If no relation holds in general, write "?".

(a) $E(X)$ $\qquad \sqrt{E(X^2)}$

(b) $P(X > c)$ $\qquad E(X^3)/c^3$

(c) $E(X^3)$ $\qquad \sqrt{E(X^2)E(X^4)}$

(d) $P(|X+Y| > 2)$ $\qquad \frac{1}{10}E((X+Y)^4)$

(e) $E(Y|X)$ $\qquad E(Y|X+3)$

(f) $P(X+Y > 2)$ $\qquad (EX+EY)/2$

9. Consider i.i.d. Pois(λ) r.v.s $X_1, X_2, \cdots$. The MGF of X_j is $M(t) = e^{\lambda(e^t-1)}$.

(a) Find the MGF $M_n(t)$ of the sample mean $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$

(b) Find the limit of $M_n(t)$ as $n \to \infty$

10. Let $Y = e^X$, with $X \sim$ Expo(3).

(a) Find the mean and variance of Y .

(b) For $Y_1, \cdots, Y_n$ i.i.d. with the same distribution as Y, what is the approximate distribution of the sample mean $\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$ when n is large?