HW 4 Solution

1.

a) from $F(x) \geq 0$, $f(x) \geq 0$

$g(x) \geq 0$

$$\int g(x)\,dx = 2\int F(x)\,f(x)\,dx,$$ let $y = F(x)$

then $dy = f(x)\,dx$

$$= 2\int_0^1 y\,dy = 1$$

$F(-\infty) = 0$, $F(\infty) = 1$

$\therefore 0 \leq y \leq 1$

b) $h(x) \geq 0$,

$$\int_{-\infty}^{\infty} h(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx + \frac{1}{2}\int_{\infty}^{\infty} f(x)\,dx = 1$$

2. $0 \leq R \leq 1$.

a) Let $U \sim \mathrm{Unif}(0,1)$, and $X = \min(U, 1-U)$ $(0 \leq X \leq \frac{1}{2})$

$$P(R \leq r) = P(X \leq r(1-X)) = P\left(X \leq \frac{r}{1+r}\right)$$

$$P(X \leq x) = 1 - P(X > x) = 1 - P(\min(U, 1-U) > x)$$

$$= 1 - P(U > x \text{ and } 1-U > x)$$

$$= 1 - P(U > x \text{ and } 1-x > U)$$

$$= 1 - P(x < U < 1-x) = 2x \quad (0 \leq x \leq \tfrac{1}{2})$$

$$\therefore P(R \leq r) = \frac{2r}{1+r}$$

$$F_R(r) = \frac{2r}{1+r} \qquad 0 \leq r \leq 1$$

$$f_R(r) = \frac{2(1+r) - 2r}{(1+r^2)} = \frac{2}{(1+r)^2}$$

b)
$$E(R) = 2\int_0^1 \frac{r}{(1+r)^2}\,dr = 2\int_1^2 \frac{(t-1)}{t^2}\,dt$$

$$= 2\int_1^2 \frac{1}{t}\,dt - 2\int_1^2 \frac{1}{t^2}\,dt = 2\ln 2 - 1$$

3. $F(x) = 1 - e^{-x^3}$, $\quad y = 1 - e^{-x^3}$

$e^{-x^3} = 1-y$, $\quad -x^3 = \ln(1-y)$

$x = -(\lg(1-y))^{\frac{1}{3}}$, $\quad F^{-1}(u) = -(\lg(1-u))^{\frac{1}{3}}$

$\therefore \quad X = -(\lg(1-U))^{\frac{1}{3}}$

4. $E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I_{z>0}(z) \cdot z \cdot e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} z e^{-z^2/2} dz$

let $u = z^2/2$, then

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-u} du = \frac{1}{\sqrt{2\pi}} \left[-e^{-u}\right]_0^{\infty} = \frac{1}{\sqrt{2\pi}}$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2} = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$$

$$= \frac{1}{2}$$

$$Var(X) = E(X^2) - EX^2 = \frac{1}{2} - \frac{1}{2\pi} \quad (\geq 0)$$

5. Let t_h be the half-life of a particle

a) $P(T > t_h) = e^{-\lambda t_h} = \frac{1}{2}$, $\quad -\lambda t_h = \lg(\frac{1}{2})$

$$t_h = \frac{\log 2}{\lambda} \simeq \frac{0.693}{\lambda}$$

b) $L = \min(T_1, \dots, T_n)$

$P(L>t) = P(T_1>t, \dots, T_n>t) = e^{-n\lambda t}$

$F_L(t) = 1 - e^{-n\lambda t}$

$E(L) = \frac{1}{n\lambda}$, $Var(L) = \frac{1}{(n\lambda)^2}$

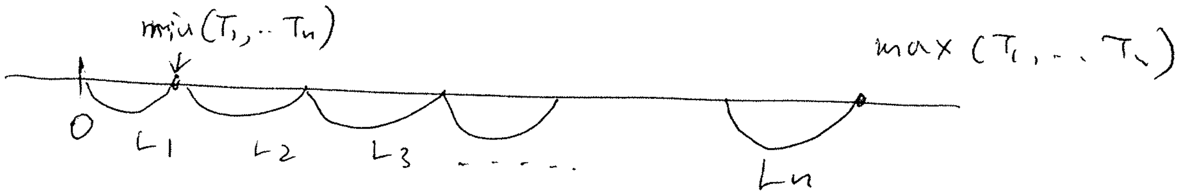
c) let $L_1 = \min(T_1, \dots T_n)$

$L_2 = \min(T_1 \dots T_n \text{ except } L_1) - L_1$

$L_3 = \min(T_1 \dots T_n \text{ except } L_1, L_2+L_1) - L_2$

$L_i = \min(T_1 \dots T_n \text{ except } \{L_1, L_1+L_2, \dots L_1+L_2+\dots+L_{i-1}\}) - L_{i-1}$

$L_n = \max(T_1, \dots T_n) - L_{n-1}$



Notice that $L_1 \sim Expo(\lambda n)$

$L_2 \sim Expo(\lambda(n-1))$ ← minimum among $(n-1)$ Expo.

$\vdots$

$L_i \sim Expo(\lambda(n-i+1))$

$\vdots$

$L_n \sim Expo(\lambda)$

Furthermore, L_i and L_j are independent $(i \neq j)$

Hence $L_1 + \dots L_n = M$

$$E(M) = \sum_{i=1}^{n} E(L_i) = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i}$$

$$Var(M) = \sum_{i=1}^{n} Var(L_i) = \frac{1}{\lambda^2} \sum_{i=1}^{n} \frac{1}{i^2}$$

6. The MGF of $U \sim \mathrm{Unif}(0,1)$

$$E(e^{tU}) = \int_0^1 e^{tu}\,du = \frac{1}{t}(e^t - 1)$$

let $X = U_1 + \cdots + U_{60}$,

$$E(e^{tX}) = (E(e^{tU}))^{60} = \frac{(e^t-1)^{60}}{t^{60}}$$

7. $$M_L(t) = E(e^{t(X-Y)}) = E(e^{tX})\,E(e^{-tY})$$

$$= \left(\frac{1}{1-t}\right)\left(\frac{1}{1+t}\right) = \frac{1}{1-t^2}$$

$$M_{Laplace}(t) = \frac{1}{2}\int_{-\infty}^{0} e^{tw} e^{+w}\,dw + \frac{1}{2}\int_0^{\infty} e^{tw} e^{-w}\,dw$$

$$= \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{1}{1-t^2}$$

Hence $M_L(t) = M_{Laplace}(t) \Rightarrow f_{(X)} = \frac{1}{2}e^{-|x|}$

8. $$M_W(t) = E(e^{t(X^2+Y^2)}) = E(e^{tX^2} e^{tY^2})$$

$$= E(e^{tX^2}) \cdot E(e^{tY^2})$$

$$= (1-2t)^{-1} = \frac{1}{1-2t}$$

let $X \sim \mathrm{Expo}(\frac{1}{2})$

$M_X(t) = \frac{1}{1-2t}$ $\quad\therefore\quad W \sim \mathrm{Expo}(\frac{1}{2})$.

9. a) CDF of $Y = X^3$,

$$P(Y \le y) = P(X^3 \le y) = P(X \le \sqrt[3]{y})$$

$$= 1 - e^{-y^{1/3}}$$

$$P(Y > s+t \mid Y > s) = \frac{P(Y > s+t)}{P(Y > s)} = \frac{e^{-(s+t)^{1/3}}}{e^{-s^{1/3}}}$$

$$\ne e^{-t^{1/3}} = P(Y > t)$$

b) From $E(X^n) = n!$,

$$E(Y^n) = E(X^{3n}) = (3n)!$$

c) $$E(e^{tY}) = E(e^{tX^3}) = \int_0^{\infty} e^{tx^3 - x}\, dx$$

for all $t > 0$ $\int_0^{\infty} e^{tx^3 - x}\, dx \to \infty$