Probability and Random Process

COSE 382

Probability

- Why study probability ?
  - Because it's here... everywhere.
- What is probability ? Two views on probability:
  - Frequentist: Probability represents a long run-frequency over a large number of repetitions of an **event**.
    A coin has probability 1/2 of Heads
  - Bayesian: Probability represents a degree of belief about an **event** in question even if it isn't possible to repeat.
    Probability that the defendant is guilty
- In modern mathematics, probability is defined as a function that assigns a number to a set (event) - set theory is a powerful language for describing an event.

Sample Space and Event

- Sample space
  - Probability basically assigns a number to an event in the universe.
  - This does not mean that we deal with all the events in the universe simultaneously .
  - Instead, we focus on specific events of interest.
  - The sample space is a set that contains all the events we are interested in.
  - To formally define a sample space, we assume an experiment and the set of all possible outcomes is called the sample space.
  - Set of all possible outcomes of an experiment $S = \{\omega | \omega \text{ is an outcome of an experiment}\}$
- Event A
  - Subset of S, $A \subset S$
  - We say an event A occurs if the actual outcome is in A

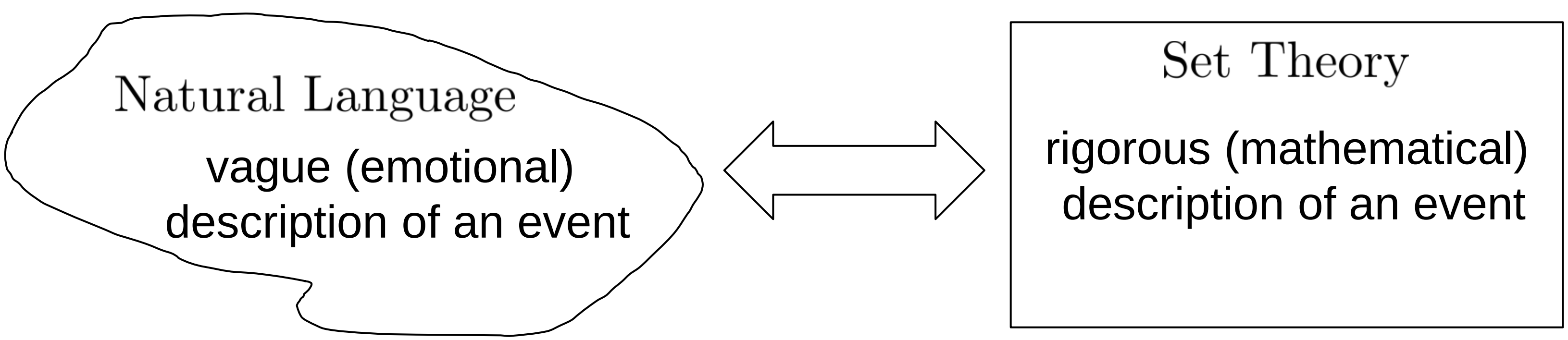
Example

Experiment	Sample Space
Exp 1: Tossing a coin	S1={H, T}
Exp 2: Tossing a coin two times	S2={(H,H), (H,T), (T,H), (T,T)}
Exp 3: Tossing two coins	S3={(H,H), (H,T), (T,H), (T,T)}
Exp 4: Number of Heads in E2	S4={2, 1, 0}
Exp 5: Measuring a room temperature	S5=[-50, 90]

Event	Subset of S
A 1: Consecutive Heads in Exp 2	$\{(H,H)\} \subset S2$
A 2: Same faces in Exp 3	$\{(H,H), (T,T)\} \subset S3$

Events and Set theory

- An event is often complicated composition of several other events.
- Any composition of events can be represented by set operations (union, joint, complement)
- Set theory is a powerful tool for working with events;
  - build new events from already-defined ones
  - provide new interpretation or expression of an event
  - For example: $A \cup B$, $A \cap B$, A^c
  - *The logical aspect of our language can be fully described by the set theory*



Example 1.2.2

- Experiments: A coin is flipped 10 times (Head: 1, Tail: 0)
  - Outcome s is an element of $S = \{0,1\}^{10}$

$$S = \{(0,\cdots,0),\cdots,(1,\cdots,1)\}$$

  - Let A_i be the event that the i-th flip is H. As a set,

$$A_i = \{(s_1,\cdots,s_{i-1},1,s_{i+1},\cdots,s_{10})|s_j \in \{0,1\} \text{ for } j \neq i\}$$

  - Let B be the event that at least one flip was H. As a set,

$$B = \bigcup_{j=1}^{10} A_j$$

  - Let C be the event that all the flips were H. As a set,

$$C = \{(1,1,\cdots,1)\} = \bigcap_{j=1}^{10} A_j$$

  - Let D be the event that there were at least two consecutive H. As a set,

$$D = \bigcup_{j=1}^{9} (A_j \cap A_{j+1})$$

More on Events and Set theory

- Any set operation is a composition of following two elementary operations:

Event	Set Operation	Name	Definition
A or B	$A \cup B$	union	$x \in A \cup B \iff x \in A$ or $x \in B$
not A	A^C	complement	$x \in A^C \iff x \in \Omega$ and $x \notin A$

  - Other compositions can be expressed by the above two

Event	Set Operation	Name	Expression
A and B	$A \cap B$	Intersection	$(A^c \cup B^c)^c$
A alone	$A - B$	Difference	$A \cap B^c$

- (Logical) relation between events can be expressed by set operations

Event	Set operation
Events A and B are mutually exclusive	Disjoint, $A \cap B = \phi$
The event A implies the event B $(A \to B)$	Subset, $A \subset B$
A is equivalent to B $(A = B)$	$A = B$, $A \subset B$ and $B \subset A$

Definition: $A \subset B \iff \forall x \in A, x \in B$

- Set operations with Venn diagram provide a geometrical intuition

Properties of set operations

Operations

- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Assoicate: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- DeMorgan's Rules: $(A \cap B)^C = A^C \cup B^C$, $(A \cup B)^C = A^C \cap B^C$

You can prove all above using only previous definitions...

Notations

- Difference: $A - B := A \cap B^C$
- Disjoint union: $A \dot{\cup} B := A \cup B$ with $A \cap B = \phi$
- Unions: $\displaystyle\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots A_n$
- Intersections: $\displaystyle\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots A_n$
- Disjoint Unions: $\dot{\bigcup}_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots A_n$ with $A_i \cap A_j = \phi, \forall i \neq j$
- Infinite Unions and Intersections: $\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k$
- Unions of mutually disjoint sets: $\dot{\bigcup}_{k=1}^{\infty} A_k$

Naïve Definition of Probability

- Let A be an event for an experiment with a finite sample space S. The naive probability of A is defined as

$$P(A) = \frac{|A|}{|S|}$$

  - $|A|$: number of elements (outcomes) in A
  - $|S|$: number of elements (outcomes) in S

- This naive definition works only for
  - finite sample space S
  - equal likelihood for each outcome

Example: The probability of life on Mars is 1/2 (*either there is or isn't*)?

How to Count

Theorem 1.4.1(Multiplication rule)

The number outcomes of a *compound* experiment of two (indpendent) experiment A (with a outcomes) and experiment B (with b outcomes) is ab

Theorem 1.4.5(Sampling with replacement)

There are n^k possible outcomes for making k choices from n objects, *once at a time with replacement*

Theorem 1.4.6(Sampling without replacement)

There are $n(n-1)\cdots(n-k+1) = n!/(n-k)!$ possible outcomes for making k choices from n objects, *once at a time without replacement*

Example 1.4.8 (Birthday Problem)

There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year. What is the probability that two or more people have the same birthday ?

- The total number of possible birthdays for k people: $|S| = 365^k$ (Sampling with replacement)
- The number of possible birthdays that no one has the same birthday (Sampling without replacement)

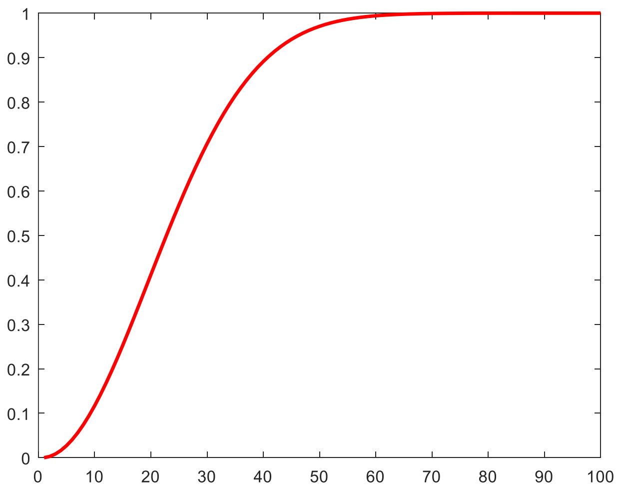
$$|A| = \frac{365!}{(365-k)!}$$

- The probability that no one has the same birthday

$$\frac{|A|}{|S|} = \frac{365!}{365^k(365-k)!}$$

- The probability we want to know:

$$P = 1 - \frac{365!}{365^k(365-k)!}$$



- As $k > 23$, $P > \frac{1}{2}$

Example 1.4.10 (Leibniz's Mistake)

If we roll two fair dice, which is more likely: a sum of 11 or a sum of 12 ?

- Leibniz's answer : they are equally likely since 12=6+6 and 11=5+6 can be done only one manner (in contrast to 10=5+5=4+6).
- Our answer:
  - Total number of possible outcomes: 6^2
  - Number of possible outcomes that make 11: $(6, 5)$ and $(5, 6)$
  - Number of possible outcomes that make 12: $(6, 6)$
  - Probability for a sum of 11 : 1/18, Probability for a sum of 11 : 1/36,
- What was Leibniz's mistake?
  - He made mistake of treating two dices indistinguishable
  - Consequently, violated "equal likelihood for each outcome"

Formal Definition of Probability

Definition 1.6.1 (General definition of probability)

Probability space consisted of a sample space S and a probability function P which takes an event $A \subset S$ as input and returns a non-negative real number. The probability function P must satisfy the following axioms:

A1) (Unit measure)

$$P(S) = 1$$

A2) (Sigma-additivity) For any (finite or countable) collection of mutually disjoint events $A_1, A_2, \cdots$ ($A_i \cap A_j = \phi$ for all $i \neq j$)

$$P\left(\dot{\bigcup}_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Note that $P(\phi) = 0$ and $P(A) \in [0, 1]$

Properties of Probability

Theorem 1.6.2 (Properties of probability)

Probability has the following properties

1) $P(A^c) = 1 - P(A)$
2) If $A \subset B$, then $P(A) \leq P(B)$
3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Theorem 1.6.3 (Incursion-exclusion)

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

For example $n = 3$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Example 1.6.4 (de Montmort's matching problem)

Consider a well-shuffled deck of n cards, labeled 1 through n. You flip over the cards one by one. You win the game if, at some point, the ith card you flipped is labeled i. What is the probability of winning ?

Let A_i be the event that the ith card is labeled i, then

$$P(A_i) = \frac{\text{number of possible sequences having the fixed } i\text{-th}}{\text{number of all possible sequences}} = \frac{(n-1)!}{n!},$$

$$P\left(\bigcap_{i=1}^{k} A_i\right) = \frac{\text{number of possible sequences having the fixed } 1, \cdots, k\text{-th}}{\text{number of all possible sequences}} = \frac{(n-k)!}{n!}$$

The probability of winning is $P(\bigcup_{i=1}^{n} A_i)$ with $P(A_i \cap A_j) = P(A_1 \cap A_2)$ and $P(A_i \cap A_j \cap A_k) = P(A_1 \cap A_2 \cap A_3)$ and etc., we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P\left(\bigcap_{i=1}^{k} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

As $n \to \infty$, since $e^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}$ (from $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$),

$$\lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - e^{-1} \approx 0.6321$$