Probability and Random Process

COSE382

Continuous Random Variables.

Definition 5.1.1 (Continuous r.v.).

- A *continuous random variable* is a random variable with a *continuous* distribution.
- A r.v. has a *continuous* distribution if its CDF, $F_X = P(X \leq x)$, is *differentiable* except finitely many points and continuous everywhere.

Definition 5.1.2 (Probability density function).

- The *probability density function* (PDF) of a continuous r.v. X with CDF F_X is the derivative of the CDF,

$$f_X(x) = \frac{d}{dx} F_X(x) = F'_X(x)$$

- The *support of* X (or the support of f_X), is the set of all x where $f_X(x) > 0$.

Proposition 5.1.3 (PDF to CDF). For a continuous r.v. X

$$F_X(x) = \int_{-\infty}^{x} f(t)dt \text{ and } P(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx$$

Theorem For a given $A \in \mathbb{R}$,

$$P(X \in A) = \int_A f_X(x)dx$$

Note that for continuous r.v. X,

$$P(X = x_0) = \int_{x_0} f(x)dx = 0, \text{ for all } x_0 \in \mathbb{R}$$

Thus, $P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b)$

Theorem 5.1.5 (Valid PDFs). The PDF f of a continuous r.v. must satisfy the following two criteria:

- Nonnegative: $f(x) \geq 0$;
- Integrates to 1: $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$.

Definition 5.1.9 (Expectation of a continuous r.v.). The *expected value* (also called the *expectation* or *mean*) of a continuous r.v. X with PDF f is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

• Note that not every distribution has a mean: a Cauchy distribution $f(x) = \frac{1}{\pi(1+x^2)}$,

$$E(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx = \text{ does not converge. } \left(\int_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} \log(1+x^2) \Big|_{0}^{\infty} = \infty \right)$$

Theorem 5.1.10 (LOTUS, continuous). If X is a continuous r.v. with PDF f and g is a function $g : \mathbb{R} \to \mathbb{R}$, then for $Y = g(X)$

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Uniform Distribution

Definition 5.2.1 (Uniform distribution). A continuous r.v. U is said to have the *Uniform distribution* on the interval (a, b) if its PDF is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{ortherwise.} \end{cases}$$

We denote this by $U \sim$Unif(a, b).

For the Uniform distributions, probability is proportional to length: Let $U \sim \text{Unif}(a, b)$ and let (c, d) be a subinterval of (a, b). Then

- **Proposition 5.2.2**

$$P(U \in (c, d)) = \frac{d-c}{b-a}$$

- **Proposition 5.2.4**

$$P(U \leq u | U \in (c, d)) = \frac{u-c}{d-c}$$

Uniform Distribution

For a $U \sim \mathrm{Unif}(a, b)$

- Mean

$$E(U) = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

- Variance

$$E(U^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a}$$

$$\mathrm{Var}(U) = E(U^2) - E(U)^2 = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$$

Universality of Uniform

Theorem 5.3.1 (Universality of the Uniform). Let X be a random variable with CDF F_X and $F_X : \mathbb{R} \to (0,1)$ be continuous and strictly increasing on its support, i.e. the inverse function $F_X^{-1} : (0,1) \to \mathbb{R}$ exists. Then,

1. $X = F_X^{-1}(U)$ for $U \sim \text{Unif}(0,1)$
2. $F_X(X) \sim \text{Unif}(0,1)$.

Proof.

1. Let $Y := F_X^{-1}(U)$. The range of Y is $\mathbb{R}$. For all real x,

   $$F_Y(x) = P(Y \le x) = P(F_X^{-1}(U) \le x) = P(U \le F_X(x)) = F_U(F_X(x)) = F_X(x),$$

   Since X and $Y = F^{-1}(U)$ have the same CDF F_X, $X = F^{-1}(U)$.

2. Let $Y := F_X(X)$. The range of Y is $(0,1)$. For $u \in (0,1)$,

   $$F_Y(u) = P(Y \le u) = P(F_X(X) \le u) = P(X \le F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u = F_U(u).$$

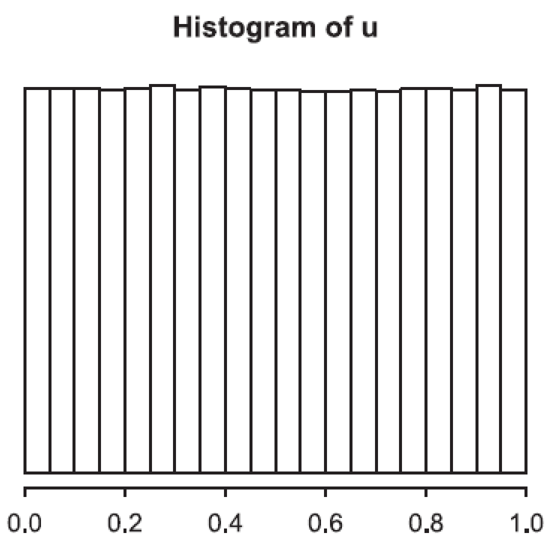
   Since $F_Y = F_U$, $Y = F(X) = U$.

Example 5.3.4 (Universality with Logistic). The Logistic CDF is

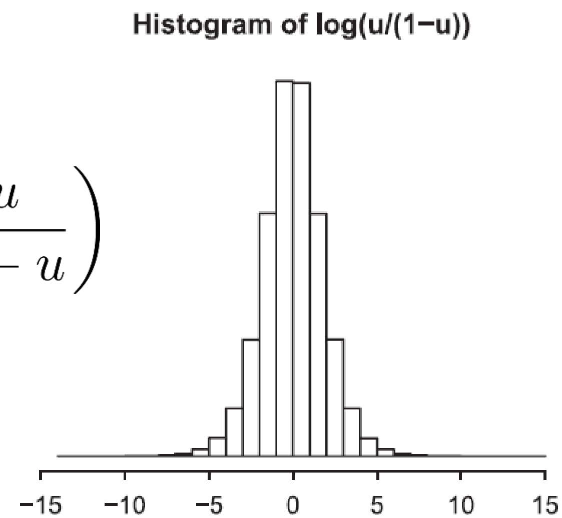
$$F(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

For $U \sim \mathrm{Unif}(0,1)$, $F^{-1}(U) = \log\left(\dfrac{U}{1-U}\right)$.

Therefore, $\log\left(\frac{U}{1-U}\right) \sim$ Logistic. Logistic PDF is $f(x) = \dfrac{e^x}{(1+e^x)^2}$



$$u \mapsto \log\left(\frac{u}{1-u}\right)$$



Normal (Gaussian) distribution

Definition 5.4.1 (Standard Normal distribution). A continuous r.v. Z is said to have the *standard Normal distribution* if its PDF φ is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

We write this as $Z \sim \mathcal{N}(0, 1)$ since, as we will show, Z has mean 0 and variance 1.

(Standard Normal CDF). The standard Normal CDF is given as

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t)dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Definition 5.4.3 (Normal distribution). If $Z \sim \mathcal{N}(0, 1)$, then

$$X = \mu + \sigma Z$$

is said to have the *Normal distribution* with mean μ and variance σ^2. We denote this by $X \sim \mathcal{N}(\mu, \sigma^2)$.

- Validity of standard Normal CDF

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-z^2/2} dz\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = \int_0^{2\pi}\left(\int_0^{\infty} e^{-v} dv\right) d\theta \\
&= \int_0^{2\pi} d\theta = 2\pi
\end{aligned}
$$

- Mean: $E(Z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z e^{-z^2/2} dz = 0$

- Variance:

$$
\begin{aligned}
\mathrm{Var}(z) &= E(Z^2) - (EZ)^2 = E(Z^2) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} z^2 e^{-z^2/2} dz \\
&= \frac{2}{\sqrt{2\pi}}\left(-z e^{z^2/2}\Big|_0^{\infty} + \int_0^{\infty} e^{-z^2/2} dz\right) = \frac{2}{\sqrt{2\pi}}\left(0 + \frac{\sqrt{2\pi}}{2}\right) \\
&= 1
\end{aligned}
$$

Exponential distribution

Definition 5.5.1 (Exponential distribution). Exponential r.v. X with parameter $\lambda > 0$, denoted by $X \sim \text{Expo}(\lambda)$, has PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$$

The corresponding CDF is

$$F(x) = 1 - e^{-\lambda x}, x > 0.$$

– Note that if $X \sim \text{Expo}(\lambda)$, then $Y = \lambda_0 X \sim \text{Expo}(\lambda/\lambda_0)$

– Mean and Variance: For $X \sim \text{Expo}(1)$

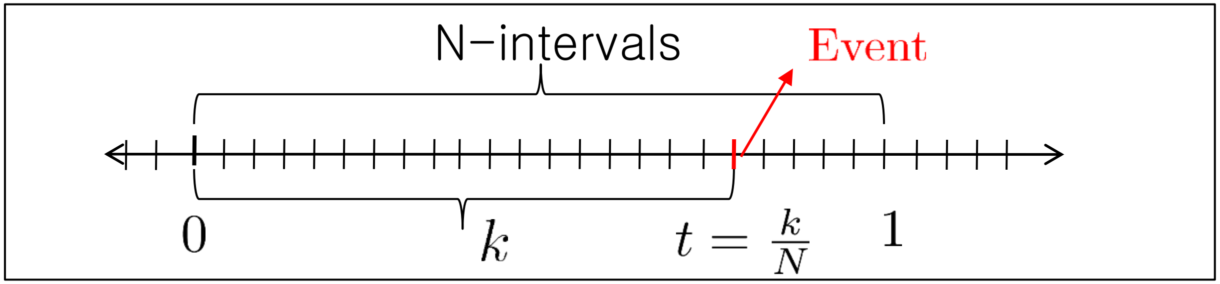
$$\begin{aligned} E(X) &= \int_0^\infty x e^{-x} dx = 1, \quad E(X^2) = \int_0^\infty x^2 e^{-x} dx = 2, \\ \text{Var}(X) &= E(X^2) - (EX)^2 = 1. \end{aligned}$$

For $Y \sim \text{Expo}(\lambda)$ we then have $Y = \frac{1}{\lambda} X$ and

$$E(Y) = \frac{1}{\lambda} E(X) = \frac{1}{\lambda}, \quad \text{Var}(Y) = \frac{1}{\lambda^2} \text{Var}(X) = \frac{1}{\lambda^2},$$

Exponential RV and Geometric RV

Divide each unit time interval into N subintervals
Assume the event occurrence in each subinterval is i.i.d. Bern(p)
Let λ be the averaged number of events occurring in a unit time interval, then $\lambda = pN$



- The number of subintervals until the occurrence of an event is Geom(p).

$$P(G \geq k) = \sum_{n=k}^{\infty} (1-p)^n p = (1-p)^k$$

- The t unit time corresponds to the Nt-th interval.

- As $N \to \infty$ keeping $Np = \lambda$ constant, we have continuous time and

$$P(X > t) = \lim_{N\to\infty} P(G \geq Nt) = \lim_{N\to\infty} (1-p)^{Nt} = \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^{Nt} = e^{-\lambda t}$$

- Geometric in discrete (number of trials to see an event), Exponential in continuous (waiting time to see an event).

Properties of Exponential RVs

Definition 5.5.2 (Memoryless property).

A random variable X is said to have the *memoryless property* if for all $s, t > 0$

$$P(X \geq s + t | X \geq s) = P(X \geq t)$$

Properties of Exponential R.V.

- Exponential distribution has the memoryless property: For $X \sim \text{Expo}(\lambda)$

$$P(X \geq s + t | X \geq s) = \frac{P(X \geq s + t)}{P(X \geq s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t).$$

- Let X and Y be i.i.d. $\text{Exp}(\lambda)$
  - $\min(X, Y) \sim \text{Exp}(2\lambda)$
  - $\max(X, Y) - \min(X, Y)$ is independent to $\min(X, Y)$ and $\sim \text{Exp}(\lambda)$