COSE 382 HW 6 Solutions

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1. For $Z \sim \mathcal{N}(0,1)$

   (a) Find the PDF of Z^3.

   (b) Find the PDF of Z^4.

Solution:

a)

$$\begin{aligned}
&Y = z^3, y = Z^3, z = y^{1/3}, \quad dy = 3z^2 dz \\
&\left|\frac{dz}{dy}\right| = \frac{1}{3z^2} = \frac{1}{3y^{\frac{2}{3}}} = \frac{1}{3}y^{-\frac{2}{3}} \\
&f_Y(y) = f_Z\left(y^{\frac{1}{3}}\right)\frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{3\sqrt{2\pi}}y^{-\frac{3}{3}}e^{-\frac{1}{2}y^{\frac{2}{3}}}
\end{aligned}$$

b) $Y = Z^4 \quad f(x) = x^4$ is not monotonically in/decreasing function. Hence, we use CDF approach,

$$\begin{aligned}
P(Y \leq y) &= P\left(z^4 \leq y\right) = P\left(-y^{1/4} \leq z \leq y^{1/4}\right) = \\
&= \Phi\left(y^{1/4}\right) - \Phi\left(-y^{1/4}\right) = 2\Phi\left(y^{1/4}\right) - 1 \\
f_Y(y) &= \frac{2}{4}y^{-3/4}\varphi\left(y^{1/4}\right) = \frac{1}{2\sqrt{2\pi}}y^{-\frac{3}{4}}e^{-\frac{y^{\frac{1}{2}}}{2}}
\end{aligned}$$

2. Let $U \sim \text{Unif}(0, \frac{\pi}{2})$. Find the PDF of $\sin(U)$.

   Solution:
   Let $Y = \sin(u)$, for $y = \sin(u)$

$$\begin{aligned}
&\frac{dy}{du} = \cos(u) \quad \left|\frac{du}{dy}\right| = \frac{1}{\cos u} = \frac{1}{\sqrt{1-y^2}} \\
&f_Y(y) = \frac{2}{\pi}\cdot\frac{1}{\sqrt{1-y^2}} = \frac{2}{\pi\sqrt{1-y^2}}, \text{ for } 0 < y < 1
\end{aligned}$$

3. Let X and Y have joint PDF $f_{X,Y}(x,y)$, and transform $(X,Y) \mapsto (T,W)$ linearly by letting

$$T = aX + bY \text{ and } W = cX + dY,$$

where a, b, c, d are constants such that $ad - bc \neq 0$.

   (a) Find the joint PDF $f_{T,W}(t,w)$ (in terms of $f_{X,Y}$ as a function of t and w).

   (b) For a special case where $T = X + Y,\ W = X - Y$, write down $f_{T,W}(t,w)$.

Solution:

a)

$$\begin{aligned} t = ax + by \\ w = cx + dy \end{aligned} \Rightarrow \begin{pmatrix} t \\ w \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} t \\ w \end{pmatrix}$$

$$\frac{\partial(t,w)}{\partial(x,y)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \left| \frac{\partial(t,w)}{\partial(x,y)} \right| = |ad - bc|$$

$$\Rightarrow \left| \frac{\partial(x,y)}{\partial(t,w)} \right| = \frac{1}{|ad - bc|}$$

$$f_{Tw}(t,w) = f_{XY}\left(\frac{dt - dw}{ad - bc}, \frac{-ct + aw}{ad - bc} \right) \frac{1}{|ad - bc|}$$

b) $a = b = c = 1, d = -1 \Rightarrow ad - bc = -2$

$$f_{Tw}(t,w) = \frac{1}{2} f_{XY}\left(\frac{t+w}{2}, \frac{t-w}{2} \right)$$

4. Let X and Y be independent positive r.v.s, with PDFs f_X and f_Y, respectively. Let T be the ratio X/Y and $W = X$.

(a) Find the joint PDF of T and W, using a Jacobian.

(b) Find the marginal PDF of T, as a single integral

Solution:

a) Let $t = x/y$ and $w = x$, so $x = w$ and $y = \frac{w}{t}$ $\quad (x, y > 0)$

$$\frac{\partial(x,y)}{\partial(t,w)} = \begin{pmatrix} 0 & 1 \\ \frac{-w}{t^2} & \frac{1}{t} \end{pmatrix}, \left| \det\left(\frac{\partial(x,y)}{\partial(t,w)} \right) \right| = \frac{w}{t^2}$$

$$f_{Tw}(t,w) = f_X(x) f_Y(y) \cdot \frac{w}{t^2} = f_X(w) f_Y\left(\frac{w}{t} \right) \cdot \frac{\omega}{t^2}$$

b)

$$f_T(t) = \int_0^\infty f_x(w) f_Y\left(\frac{w}{t} \right) \frac{w}{t^2} dw$$

5. Let X and Y be i.i.d. Expo(λ), and transform them to $T = X + Y, W = X/Y$.

(a) Find the joint PDF of T and W. Are they independent?

(b) Find the marginal PDFs of T and W.

Solution:

a) Let $U = \frac{X}{X+Y}$, then T and U are independent with $T \sim \text{Gama}(2, \lambda)$ and $U \sim \text{Unif}(0, 1)$

Note that T and W are also independent from

$$W = \frac{X(X+Y)}{Y(X+Y)} = \frac{U}{1-U}$$

$$P(W \le w) = P(u \le w/(w+1)) = \frac{w}{w+1}$$
$$f_W(w) = \frac{w+1-w}{(w+1)^2} = \frac{1}{(w+1)^2}$$
$$f_{TW}(t, w) = \lambda^2 t e^{-\lambda t} \cdot \frac{1}{(w+1)^2}$$

b)

$$f_T(t) = \lambda^2 t e^{-\lambda t}$$
$$f_W(w) = \frac{1}{(w+1)^2}$$

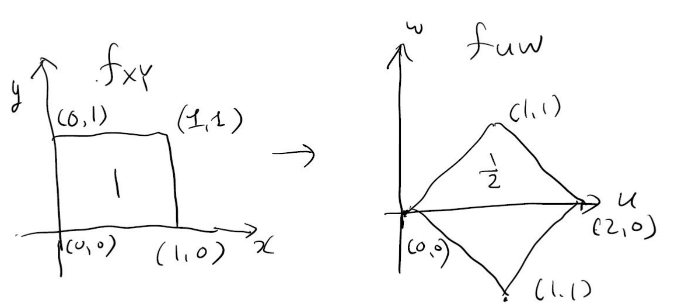
6. Let X and Y be i.i.d. Unif(0,1). Find the joint distribution of $U = X + Y$ and $V = X - Y$.

Solution:

$$\begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} X+Y \\ X-Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} U+W \\ U-W \end{pmatrix}$$
$$\frac{\partial(u,w)}{\partial(x,y)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \det\left(\frac{\partial(u,w)}{\partial(x,y}\right) = -2$$

Note that $0 \le u \le 2, -1 \le w \le 1$

$$f_{UW}(u, w) = \begin{cases} \frac{1}{2} & \text{for } 0 \le u+w \le 2, 0 \le u-w \le 2 \\ 0 & \text{else} \end{cases}$$



7. Let X and Y be i.i.d. Gaussian Normal $\mathcal{N}(0,1)$. Let

$$R = \sqrt{X^2 + Y^2} \text{ and}$$

$$U = \begin{cases} \tan^{-1}(Y/X) & x > 0 \\ \tan^{-1}(Y/X) + \pi & x < 0, y \geq 0 \\ \tan^{-1}(Y/X) - \pi & x < 0, y < 0 \end{cases}$$

Find the pdf of R.

Solution:
Note that (X,Y) is the Cartesian Coordinate and (R,U) is the polar Coordinate.

$$\begin{aligned}
&X = R\cos U, \quad 0 \leq R < \infty, -\pi < U < \pi \\
&Y = R\sin U \\
&\frac{\partial(x,y)}{\partial(r,u)} = \begin{pmatrix} \cos u & -r\sin u \\ \sin u & r\cos u \end{pmatrix} \\
&\det\left(\frac{\partial(x,y)}{\partial(r,u)}\right) = r\cos^2 u + r\sin^2 u = r \\
&f_{RU}(r,u) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot r = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \cdot r = \frac{1}{2\pi} r e^{-\frac{r^2}{2}}. \\
&f_R(r) = \int_{-\pi}^{\pi} \frac{1}{2\pi} r e^{-\frac{r^2}{2}} du = r e^{-\frac{r^2}{2}}
\end{aligned}$$

8. Let T and V be random variables with the joint pdf

$$f_{T,V}(t,v) = \frac{1}{\sqrt{\pi}\Gamma(n/2)} \frac{1}{2^{(n+1)/2}} \frac{1}{\sqrt{n}} v^{(n+1)/2-1} e^{-(v/2)\left(1+t^2/n\right)} \quad \text{(for } v > 0\text{)}.$$

Compute the marginal pdf of T.

Solution:
Let

$$C := \frac{1}{\sqrt{\pi}\Gamma(n/2)} \frac{1}{2^{(n+1)/2}} \frac{1}{\sqrt{n}}$$

Then,

$$f_{T,V}(t,v) = Cv^{(n+1)/2-1} e^{-(v/2)\left(1+t^2/n\right)}$$

$$\begin{aligned}
f_T(t) &= \int Cv^{(n+1)/2-1} e^{-\frac{1}{2}\left(1+t^2/n\right)v} dv \\
&= C\left(\frac{1}{2}(1+t^2/n)\right)^{-(n+1)/2} \int \left(\frac{1}{2}(1+t^2/n)\right)^{(n+1)/2} v^{(n+1)/2-1} e^{-\frac{1}{2}\left(1+t^2/n\right)v} dv \\
&= C\left(\frac{1}{2}(1+t^2/n)\right)^{-(n+1)/2} \Gamma(\frac{n+1}{2}) \\
&= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(n/2)} \left(1+\frac{t^2}{n}\right)^{-(n+1)/2} \frac{1}{\sqrt{n}}
\end{aligned}$$