Probability and Random Process

COSE382

Moments

Definition 6.2.1 (Moments). Let X be a r.v. with mean μ and variance σ^2. For any positive integer n;

- The nth *moment* of X is
$$E(X^n)$$
- The nth *central moment* is
$$E((X-\mu)^n)$$
- The nth *standardized moment* is
$$E\left(\left(\frac{X-\mu}{\sigma}\right)^n\right)$$

Throughout the previous sentence, if it exists is left implicit.

x

Moment Generating Functions

Definition 6.4.1 (Moment generating function). The *moment generating function* (MGF) of a r.v. X is

$$M(t) = E(e^{tX})$$

as a function of t, if this is finite valued on some open interval $(-a, a)$ containing 0. Otherwise we say the MGF of X does not exist.

Note that $M(0) = 1$ for any valid MGF M;

Example 6.4.2 (Bernoulli MGF). For $X \sim \text{Bern}(p)$,

$$M(t) = E(e^{tX}) = e^{t\cdot 1}p + e^{t\cdot 0}(1-p) = e^t p + 1 - p, \quad \text{for all } t \in \mathbb{R}$$

Example 6.4.3 (Geometric MGF). For $X \sim \text{Geom}(p)$,

$$\begin{aligned} M(t) &= E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}(1-p)^k p = p\sum_{k=0}^{\infty}((1-p)e^t)^k \\ &= \frac{p}{1-(1-p)e^t}, \text{ for } (1-p)e^t < 1 \end{aligned}$$

i.e., for $t \in (-\infty, \log(1/(1-p)))$, which is an open interval containing 0.

Example 6.4.4 (Uniform MGF). Let $U \sim \text{Unif}(a, b)$.

$$M(t) = E(e^{tU}) = \frac{1}{b-a}\int_a^b e^{tu}du = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

Theorem 6.4.5 (Moments via derivatives of the MGF). Given the MGF of X, we can get the nth moment of X by evaluating the nth derivative of the MGF at 0

$$E(X^n) = \frac{d^n}{dt^n} M(t)\bigg|_{t=0} = M^{(n)}(0)$$

Proof. This can be seen by noting that the Taylor expansion of $M(t)$ at 0 is

$$M(t) = \sum_{n=0}^{\infty} M^{(n)}(0) \frac{t^n}{n!},$$

while on the other hand, we also have

$$M(t) = E(e^{tX}) = E\left(\sum_{n=0}^{\infty} X^n \frac{t^n}{n!} \right) = M(t) = \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!}.$$

Matching the coefficients of the two expansions, we get $E(X^n) = M^{(n)}(0)$.

Theorem 6.4.6 (MGF determines the distribution).

If two r.v.s have the same MGF even in a tiny interval $(-a, a)$ containing 0, they must have the same distribution.

This is a difficult result in analysis, so we will not prove it here.

Theorem 6.4.7 (MGF of a sum of independent r.v.s). If X and Y are independent, then the MGF of $X+Y$ is the product of the individual MGFs:

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t).$$

If X and Y are independent, we have $E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY})$

Example 6.4.8 (Binomial MGF). The MGF of a Bern(p) r.v. is $pe^t + q$, so the MGF of a Bin(n, p) r.v. is

$$M(t) = (pe^t + q)^n.$$

Proposition 6.4.11 If X has MGF $M(t)$ then the MFG of $a + bX$ is

$$E(e^{t(a+bX)}) = e^{at}E(e^{btX}) = e^{at}M(bt)$$

Example 6.4.12 (Normal MGF). The MGF of a standard Normal r.v. Z is

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2}dz = e^{t^2/2},$$

The MGF of $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$ is

$$M_X(t) = e^{\mu t}M_Z(\sigma t) = e^{\mu}e^{(\sigma t)^2/2} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Example 6.4.13 (Exponential MGF). The MGF of $X \sim \text{Expo}(1)$ is

$$M(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}e^{-x}dx = \int_0^{\infty} e^{-x(1-t)}dx = \frac{1}{1-t} \quad \text{for} \quad t < 1.$$

So the MGF of $Y = X/\lambda \sim \text{Expo}(\lambda)$ is $M_Y(t) = M_X(\frac{t}{\lambda}) = \frac{\lambda}{\lambda - t} \quad \text{for} \quad t < \lambda.$

Generating moments with MGFs

Example 6.5.1 (Exponential moments).

Let $X \sim \text{Expo}(1)$. The MGF of X is $M(t) = 1/(1-t)$ for $t < 1$. For $|t| < 1$,

$$M(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} n! \frac{t^n}{n!}.$$

On the other hand,

$$M(t) = \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!}.$$

Hence, $E(X^n) = n!$ for all n.

For $Y \sim \text{Expo}(\lambda)$, we have $Y = X/\lambda$, $Y^n = X^n/\lambda^n$, and

$$E(Y^n) = \frac{n!}{\lambda^n}.$$

Example 6.5.2 (Standard Normal moments). Let $Z \sim \mathcal{N}(0,1)$.

$$M(t) = e^{t^2/2} = \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!} \frac{t^{2n}}{(2n)!}.$$

Therefore

$$E(Z^n) \quad = \quad \begin{cases} \dfrac{n!}{2^{n/2}(n/2)!} & \text{for even } n \\ 0 & \text{for odd} n \end{cases}$$

Or,

$$E(Z^{2n}) \quad = \quad (2n-1)!!$$

$E(Z) = 0, E(Z^2) = 1, E(Z^3) = 0, E(Z^4) = 1 \cdot 3, E(Z^5) = 0, E(Z^6) = 1 \cdot 3 \cdot 5, \cdots$

Sum of independent r.v.s via MGFs

Example 6.6.1 (Sum of independent Poissons). Sum of indepdent Poissons is Poisson: First let's find the MGF of X ~Pois(λ):

$$E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Now let $Y \sim$ Pois(μ) be independent of X. The MGF of $X+Y$ is

$$E(e^{tX})E(e^{tY}) = e^{\lambda(e^t-1)} e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)},$$

which is the Pois($\lambda + \mu$) MGF.

Example 6.6.3 (Sum of independent Normals). If we have $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ independently, then the MGF of $X_1 + X_2$ is

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

which is the $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ MGF. Thus the sum of independent Normals is Normal, and the means and variances simply add.

Example 6.6.4 (Sum is Normal). A converse to the previous example also holds: If X_1 and X_2 are independent and $X_1 + X_2$ is Normal, then X_1 and X_2 must be Normal.

Proof. I.i.d. case only (general version is known as *Cramer's theorem*). Let X_1 and X_2 be i.i.d. with MGF $M(t)$. Without loss of generality, we can assume $X_1 + X_2 \sim \mathcal{N}(0, 1)$, and then its MGF is

$$e^{t^2/2} = E(e^{t(X_1+X_2)}) = E(e^{tX_1})E(e^{tX_2}) = (M(t))^2,$$

so $M(t) = e^{t^2/4}$, which is the $\mathcal{N}(0, 1/2)$ MGF. Thus, $X_1, X_2 \sim \mathcal{N}(0, 1/2)$.