COSE436

Lecture 4: Transformation

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Outline

- Basic transformations in homogeneous coordinate
- Concatenate transformation

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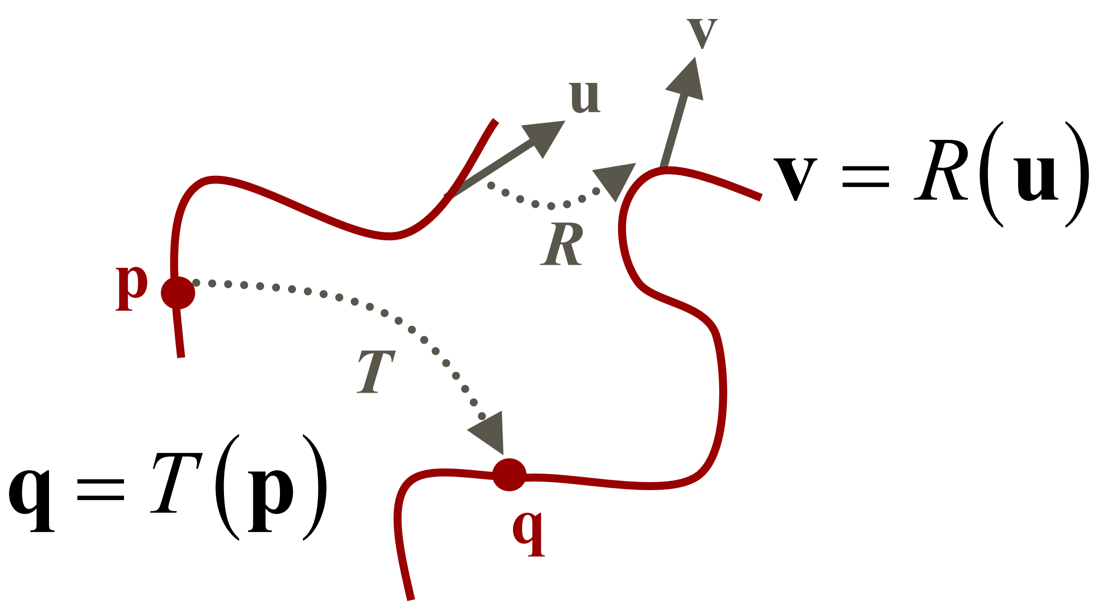
- Basic transformations in homogeneous coordinate
- Concatenate transformation

Transformation

- Mapping between two spaces
  - Point to point
  - Vector to vector



Transformations in Graphics

- Translation
- Rotation
- Scaling
- Shearing
- Reflection (Mirroring)
- ...

Linear Transformations

- Transformations that preserve
  - Vector addition
  - Scalar multiplication

$$f(\mathbf{x}) + f(\mathbf{y}) = f(\mathbf{x} + \mathbf{y})$$

$$\alpha f(\mathbf{x}) = f(\alpha \mathbf{x})$$

Linear Transformations

- Linear?

$$f(\mathbf{x}) = \mathbf{M}\mathbf{x},\ \ \mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},\ \ \mathbf{x} \in R^2$$

- Linear?

$$f(\mathbf{x}) = \mathbf{x} + \mathbf{d},\ \ \mathbf{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \ \mathbf{x} \in R^2$$

Linear Transformations

- Why linear transformations are preferred?
  - Can be expressed as a matrix multiplication
  - Multiple transformations can be concatenated

$$\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_n\mathbf{x} = \mathbf{M}'\mathbf{x}$$

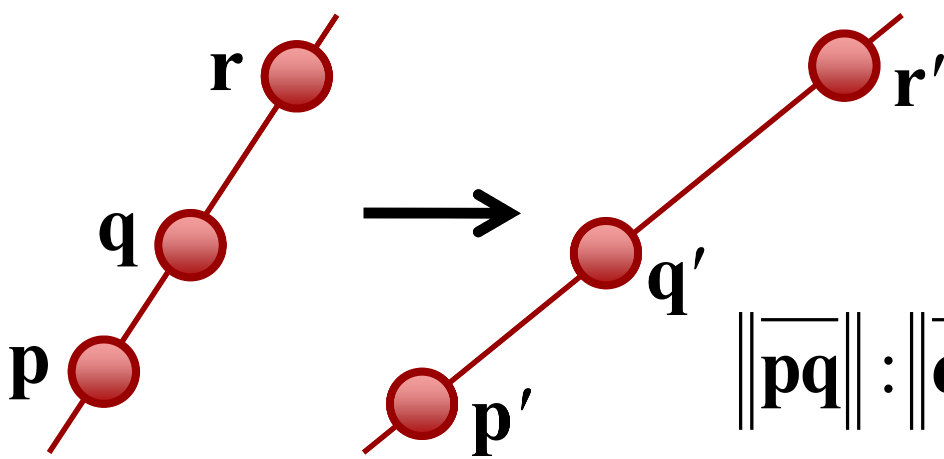
- Rotation, scaling, reflection, shearing are linear
- Translation is NOT linear

Affine Transformations

- General form

$$f(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{d}$$

  - Preserve co-linearity between points
  - Preserve ratios of distances along a line



$$\left\|\overline{\mathbf{pq}}\right\| : \left\|\overline{\mathbf{qr}}\right\| = \left\|\overline{\mathbf{p'q'}}\right\| : \left\|\overline{\mathbf{q'r'}}\right\|$$

Affine Transformations

- Rotation, scaling, reflection, shearing, and translation

- Is Affine Transformation Linear?

$$f(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{d}$$

- Can we make it linear?

Affine Transformations in H.C

- 3D transformation

$$f(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{d}$$

$$\mathbf{p} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, \; f(\mathbf{p}) = \begin{pmatrix} & \mathbf{M} & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{p} + \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = \begin{pmatrix} & \mathbf{M} & & \mathbf{d} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{p}$$

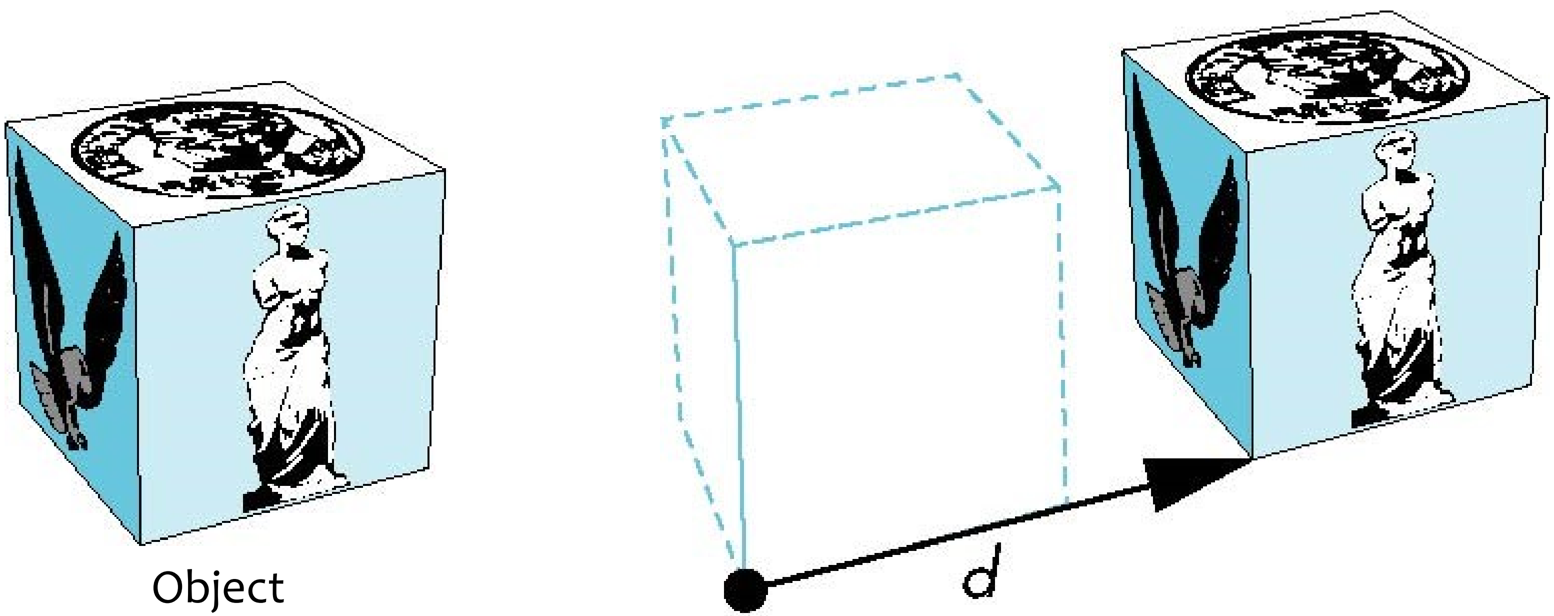
$$\mathbf{p}' = \mathbf{T}\mathbf{p}, \;\; \mathbf{T} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, T is linear transformation!

Translation

- Every point is displaced by a fixed distance along the same direction

$$\mathbf{p}' = \mathbf{p} + \mathbf{d}$$



Object

Object translated

Translation

- Homogeneous coordinate makes it linear

$$\mathbf{p}' = \mathbf{p} + \mathbf{d} \longrightarrow \mathbf{p}' = T\mathbf{p}$$

$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad \mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_x \\ d_y \\ d_z \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

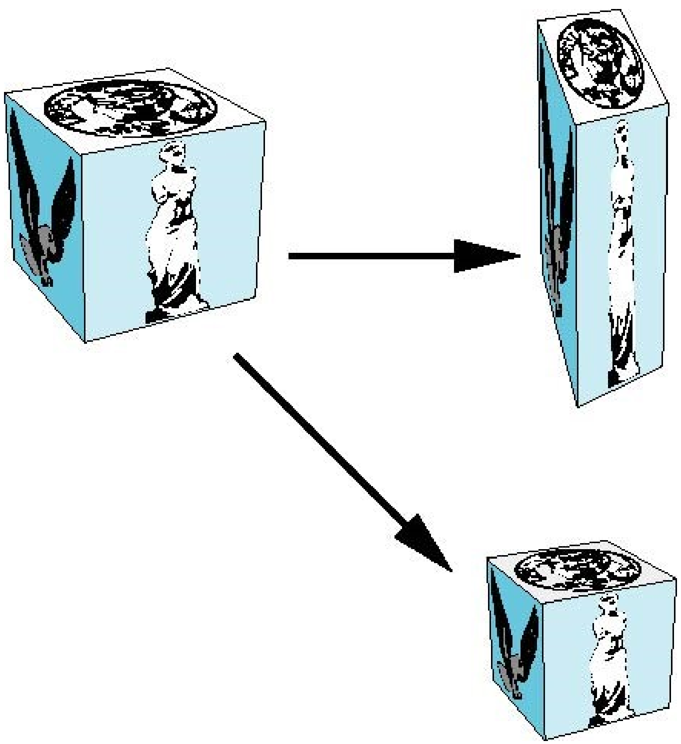
Translation

- Inverse?
  - Displace opposite direction

$$T^{-1}(\alpha_x, \alpha_y, \alpha_z) = T(-\alpha_x, -\alpha_y, -\alpha_z) = \begin{bmatrix} 1 & 0 & 0 & -\alpha_x \\ 0 & 1 & 0 & -\alpha_y \\ 0 & 0 & 1 & -\alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Expand or contract along each axis



Angel

Scaling

- Scaling matrix with a fixed point of the origin

$$x' = \beta_x x$$
$$y' = \beta_y y$$
$$z' = \beta_z z$$

$$\xrightarrow{S(\beta_x, \beta_y, \beta_z)}$$

$$\mathbf{p}' = S\mathbf{p}$$

?

Scaling

- Scaling matrix with a fixed point of the origin

$$x' = \beta_x x$$
$$y' = \beta_y y$$
$$z' = \beta_z z$$

$$\xrightarrow{S(\beta_x, \beta_y, \beta_z)}$$

$$\mathbf{p}' = S\mathbf{p}$$

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Scaling matrix with a fixed point of the origin

$$x' = \beta_x x$$
$$y' = \beta_y y$$
$$z' = \beta_z z$$

$$S(\beta_x, \beta_y, \beta_z)$$

$$\mathbf{p}' = S\mathbf{p}$$

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse of a scaling matrix

?

Scaling

- Scaling matrix with a fixed point of the origin

$$x' = \beta_x x$$
$$y' = \beta_y y$$
$$z' = \beta_z z$$

$$\xrightarrow{S(\beta_x, \beta_y, \beta_z)} \quad \mathbf{p}' = S\mathbf{p}$$

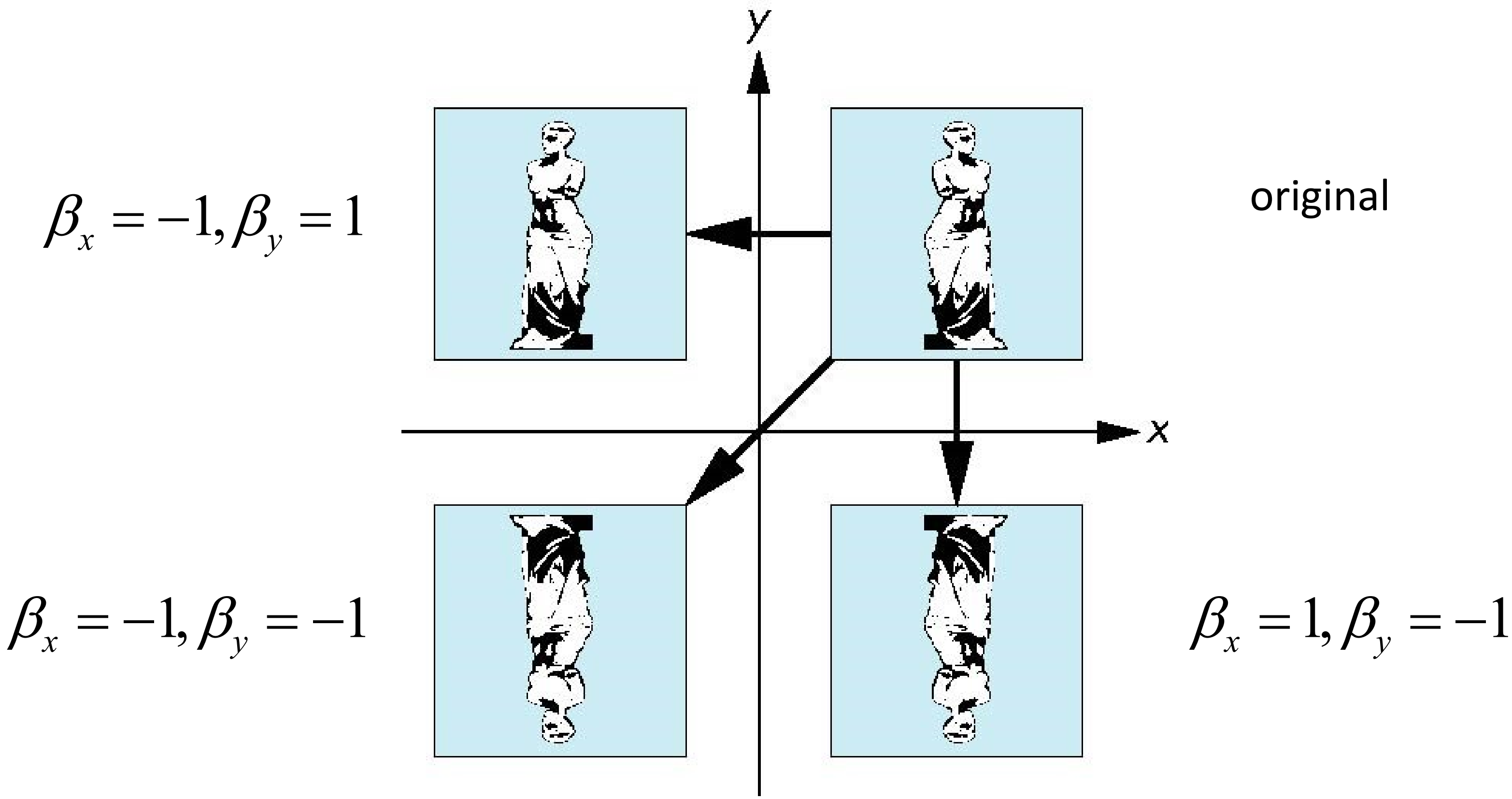
$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Inverse of a scaling matrix

$$S^{-1}(\beta_x, \beta_y, \beta_z) = S\left(\frac{1}{\beta_x}, \frac{1}{\beta_y}, \frac{1}{\beta_z}\right) = \begin{bmatrix} 1/\beta_x & 0 & 0 & 0 \\ 0 & 1/\beta_y & 0 & 0 \\ 0 & 0 & 1/\beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection

- Negative scale factor



Rotation

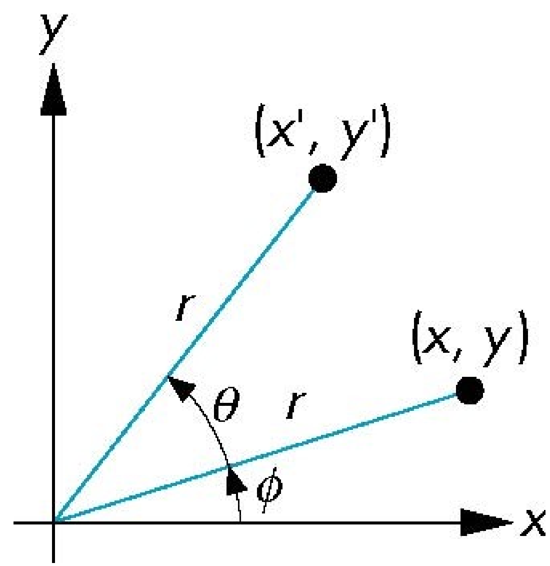
- 2D example

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$x' = r\cos(\theta+\varphi)$$

$$y' = r\sin(\theta+\varphi)$$



$$x' = r\cos\varphi\cos\theta - r\sin\varphi\sin\theta$$
$$= x\cos\theta - y\sin\theta$$
$$y' = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation in 3D

- Fixed point at origin

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$\xrightarrow{R_z(\theta)}$

$$\mathbf{p}' = R_z\mathbf{p}$$

?

Rotation matrix

Rotation in 3D

- Fixed point at origin

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\xrightarrow{R_z(\theta)}$$

$$\mathbf{p}' = R_z\mathbf{p}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x = R_x(\theta) = \quad ?$$

Rotation in 3D

- Fixed point at origin

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\xrightarrow{R_z(\theta)}$$

$$\mathbf{p}' = R_z\mathbf{p}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y = R_y(\theta) =$$

?

Rotation in 3D

- Fixed point at origin

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\xrightarrow{R_z(\theta)} \quad \mathbf{p}' = R_z\mathbf{p}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y = R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Inverse

$$R^{-1}(\theta) = R(-\theta)$$

$$\downarrow$$

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

$$\downarrow$$

$$R_z^{-1}(\theta) = R_z(-\theta) =$$

?

Rotation

- Inverse

$$R^{-1}(\theta) = R(-\theta)$$

$$\downarrow$$

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

$$\downarrow$$

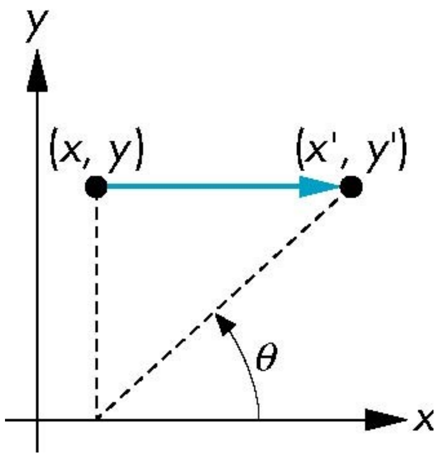
$$R_z^{-1}(\theta) = R_z(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

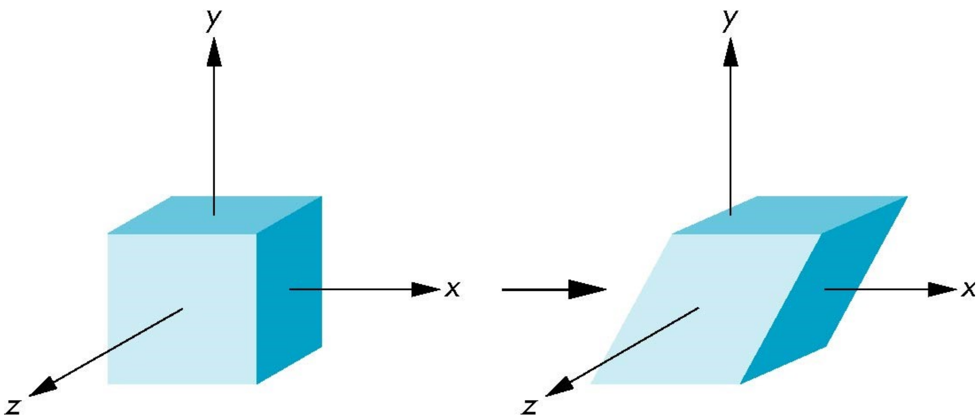
$$\downarrow$$

$$\boxed{R^{-1} = R^T}$$: Orthogonal matrix

Shearing

- Pulling faces in opposite direction





$$x' = x + y \cot \Theta$$
$$y' = y$$
$$z' = z$$

$$\mathbf{T} = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outline

- Basic transformations in homogeneous coordinate
- Concatenate transformation

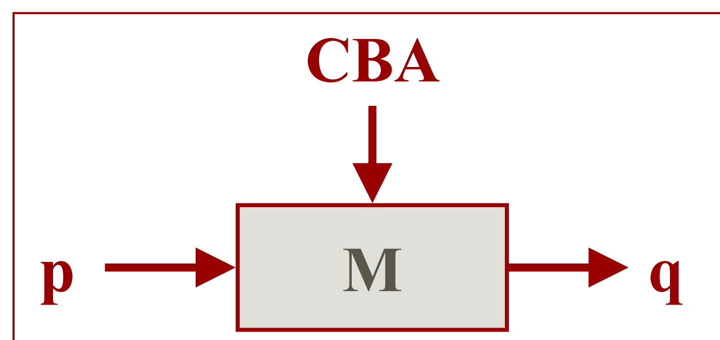
Concatenation

- Arbitrary many affine transforms can be represented by a single matrix
  - Thanks to Homogeneous coordinate
- Order is important (right to left)

$$\mathbf{q} = (\mathbf{C}(\mathbf{B}(\mathbf{A}\mathbf{p}))) = \mathbf{CBAp}$$

$$\mathbf{M} = \mathbf{CBA}$$

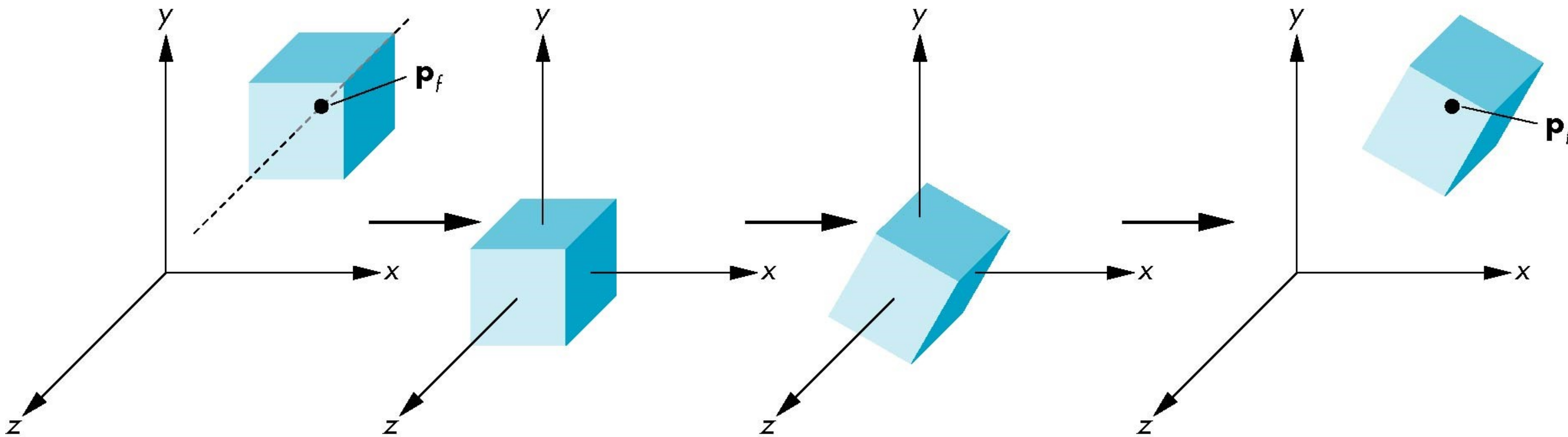
$$\mathbf{q} = \mathbf{Mp}$$



Rotation about a Fixed Point

- Move fixed point to origin
- Rotate
- Move fixed point back

$$\mathbf{M} = \mathbf{T}(\mathbf{p}_f)\mathbf{R}(\mathbf{q})\mathbf{T}(-\mathbf{p}_f)$$

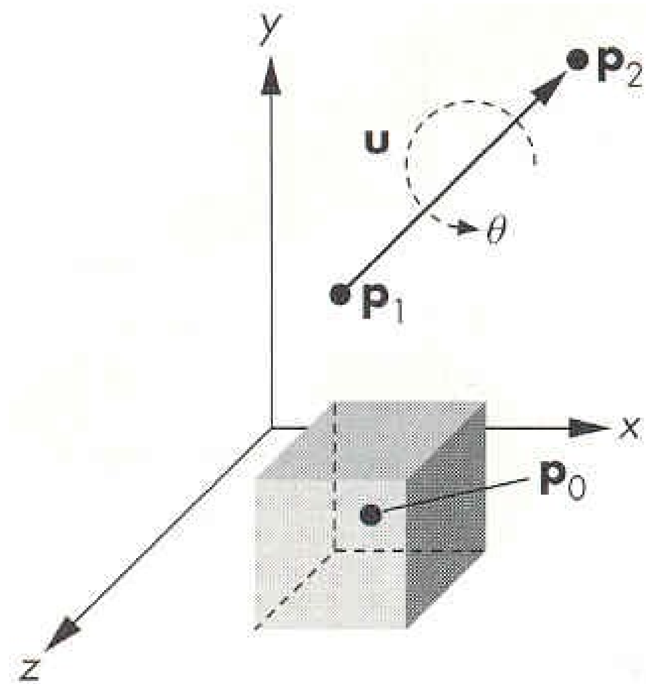


Rotation about Arbitrary Axis

- Fixed point: $\mathbf{p}_1$
- Rotation angle: θ
- Rotation axis: vector $\mathbf{p}_2 - \mathbf{p}_1$

$$\mathbf{u} = \mathbf{p}_2 - \mathbf{p}_1$$

$$\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$



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Strategy

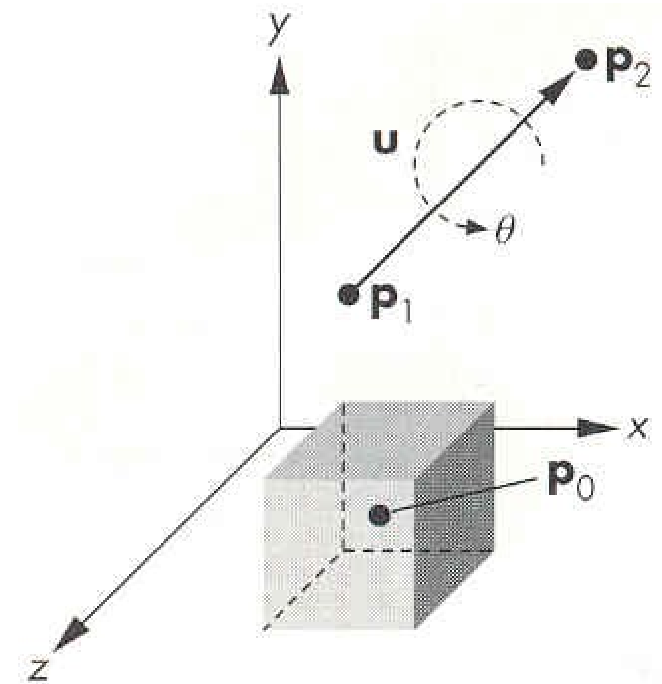
1. Translate fixed point to origin
2. Align rotation axis to z-axis
3. Rotate θ about z-axis
4. Revert rotation axis from z-axis
5. Translate fixed point from origin

1. Translate fixed point to origin

- Translate $-\mathbf{p}_1$ and normalize $\mathbf{u}$

$$T(-\mathrm{P}_1) = \begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{P}_1 = (p_x, p_y, p_z)$$



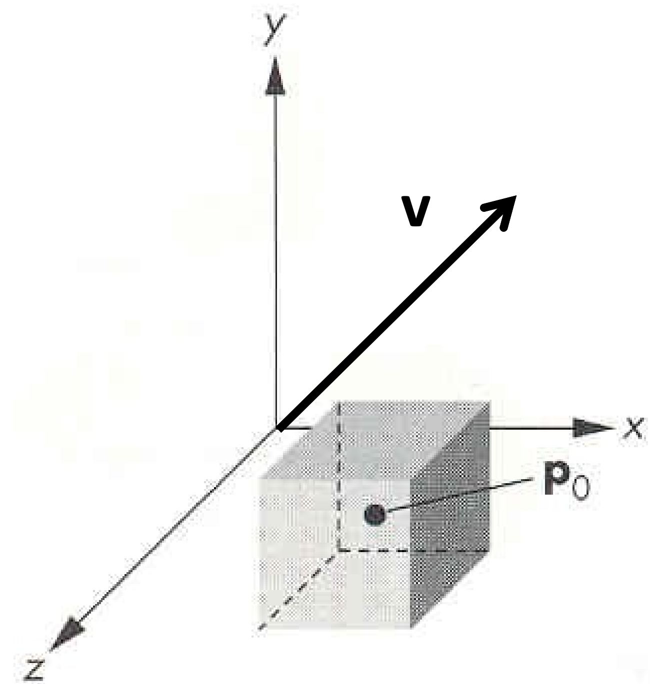
1. Translate fixed point to origin

- Translate $-\mathbf{p}_1$ and normalize $\mathbf{u}$

$$T(-\mathrm{P}_1)=\begin{bmatrix} 1 & 0 & 0 & -p_x \\ 0 & 1 & 0 & -p_y \\ 0 & 0 & 1 & -p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{P}_1=(p_x,p_y,p_z)$$

$$v=\frac{u}{|u|}=(\alpha_1,\alpha_2,\alpha_3)$$



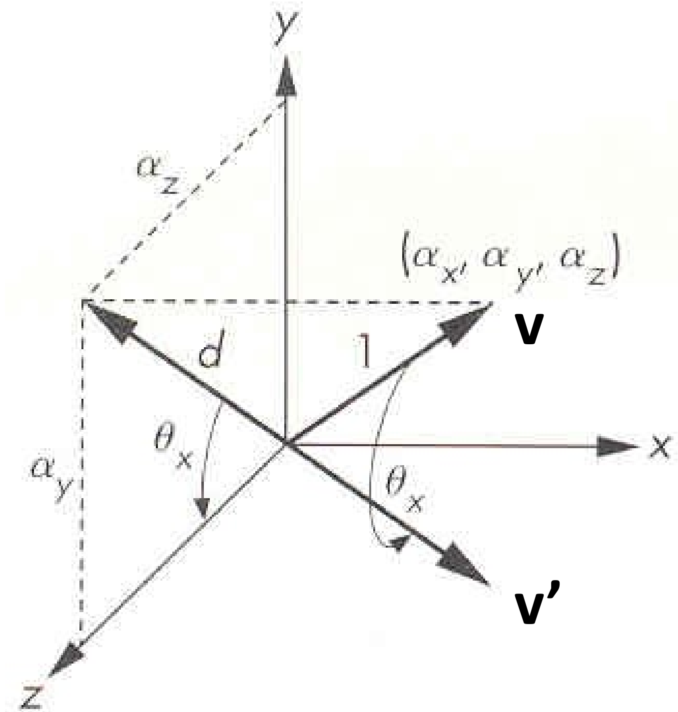
2.1. Align rotation axis to z-axis

- Rotate about x-axis
  - Project **v** onto x-z plane

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & ? & ? & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_x = ?,\ \ \sin\theta_x = ?$$

$$d = ?$$



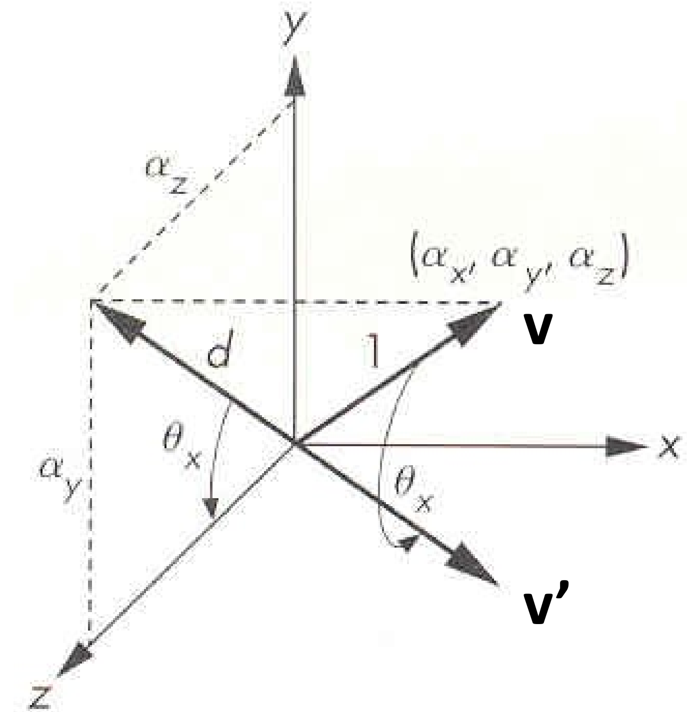
2.1. Align rotation axis to z-axis

- Rotate about x-axis
  - Project **v** onto x-z plane

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_z / d & -\alpha_y / d & 0 \\ 0 & \alpha_y / d & \alpha_z / d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_x = \alpha_z / d, \ \sin\theta_x = \alpha_y / d$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



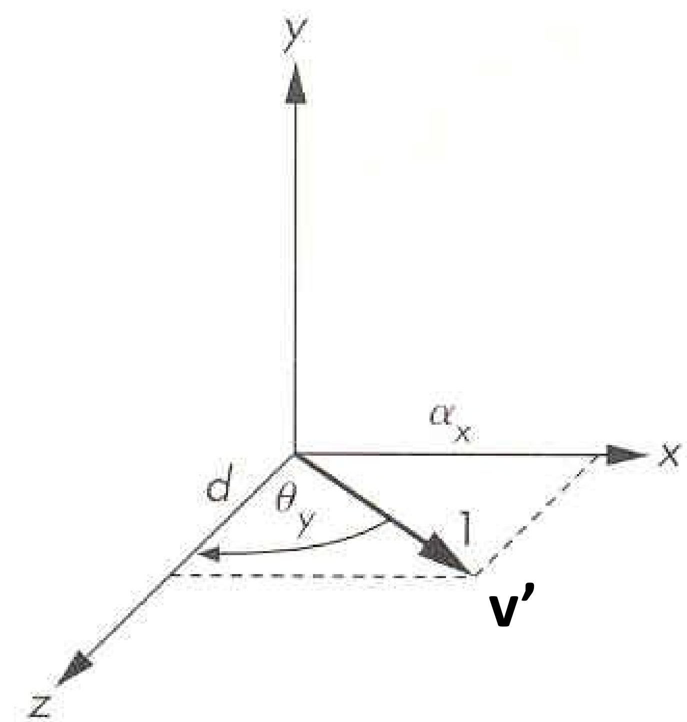
2.1. Align rotation axis to z-axis

- Rotation about y-axis
  - Align **v'** onto z-axis

$$R_y(\theta_y) = \begin{bmatrix} ? & 0 & ? & 0 \\ 0 & 1 & 0 & 0 \\ ? & 0 & ? & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_y = ?,\ \sin\theta_y = ?$$

$$d = ?$$



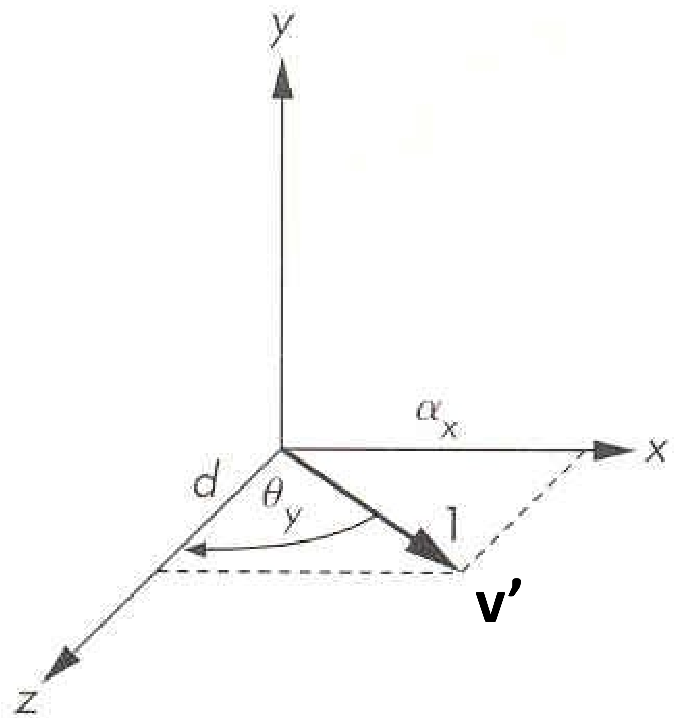
2.1. Align rotation axis to z-axis

- Rotation about y-axis
  - Align **v'** onto z-axis

$$R_y(\theta_y) = \begin{bmatrix} d & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_y = d,\ \sin\theta_y = \alpha_x$$

$$d = \sqrt{\alpha_y^2 + \alpha_z^2}$$



Strategy

1. Translate fixed point to origin
2. Align rotation axis to z-axis
3. Rotate θ about z-axis
4. Revert rotation axis from z-axis
5. Translate fixed point from origin

$$\mathbf{M} = T(\mathbf{p}_0)R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)T(-\mathbf{p}_0)$$

$$\mathbf{R} = R_x(-\theta_x)R_y(-\theta_y)R_z(\theta)R_y(\theta_y)R_x(\theta_x)$$

$$\mathbf{M} = T(\mathbf{p}_0)\mathbf{R}T(-\mathbf{p}_0)$$

Questions?

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