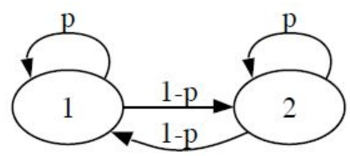
COSE 382 HW 8

Date: 2024. 12. 02 **Due:** 2024. 12. 09

1. Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities

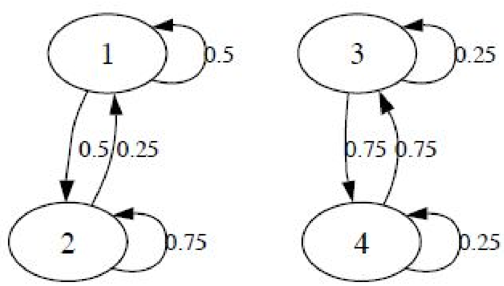


(a) Find the transition matrix Q

(b) Find the stationary distribution

(c) What happens to Q^n as $n \to \infty$

2. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicate transition probabilities

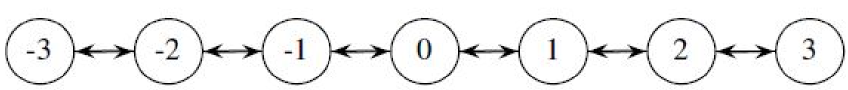


(a) Find the transition matrix Q

(b) Which states (if any) are recurrent? Which states (if any) are transient?

(c) Find two different stationary distributions for the chain

3. A Markov chain $X_0, X_1, \cdots$ with state space $\{-3, -2, -1, 0, 1, 2, 3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is X_{n-1} or X_{n+1}, each with probability 1/2. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.

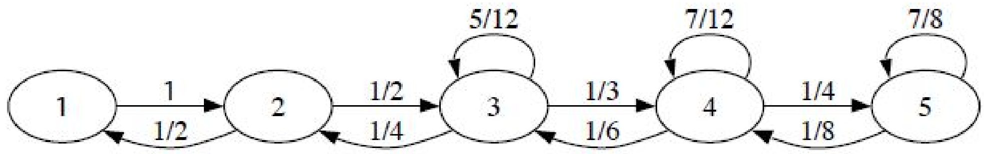


(a) Is $|X_0|, |X_1|, |X_2|, \cdots$ also a Markov chain?

(b) Let sgn be the sign function: $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = 0$. Is $\operatorname{sgn}(X_0), \operatorname{sgn}(X_1), \operatorname{sgn}(X_2), \cdots$ a Markov chain?

(c) Find the stationary distribution of the chain $X_0, X_1, X_2, \cdots$

4. Find the stationary distribution of the Markov chain shown below, without using matrices. The number above each arrow is the corresponding transition probability



5. Let $\{X_n\}$ be a Markov chain on states $\{0, 1, 2\}$ with transition matrix

$$\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

The chain starts at $X_0 = 0$. Let T be the time it takes to reach state 2 :

$$T = \min \{n : X_n = 2\}.$$

Find $E(T)$ and $\operatorname{Var}(T)$.

6. Let us consider random walk on a weighted undirected network. Suppose that an undirected network is given, where each edge (i, j) has a nonnegative weight w_{ij} assigned to it (we allow $i = j$ as a possibility). We assume that $w_{ij} = w_{ji}$ since the edge from i to j is considered the same as the edge from j to i. When (i, j) is not an edge, we set $w_{ij} = 0$

When at node i, the next step is determined by choosing an edge attached to i with probabilities proportional to the weights.

(a) Let $v_i = \sum_j w_{ij}$ for all nodes i. Show that the stationary distribution of node i is proportional to v_i.

(b) Show that every reversible Markov chain can be represented as a random walk on a weighted undirected network.

7. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

(a) Give the transition probabilities of the chain.

(b) Show that $(s_0, s_1, \ldots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

is the stationary distribution, by verifying the reversibility condition.