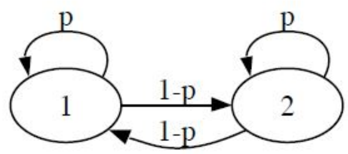
COSE 382 HW 8 Solution

---

1. Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities



a) Find the transition matrix Q

b) Find the stationary distribution

c) What happens to Q^n as $n \to \infty$

Solution:

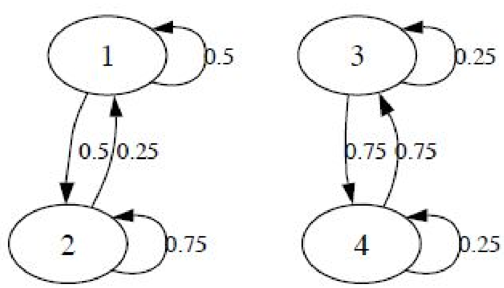
a) The transition matrix is

$$Q = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

b) Because Q is symmetric, the stationary distribution for the chain is the uniform distribution $(1/2, 1/2)$

c) The limit of Q^n as $n \to \infty$ is the matrix with the limit distribution $(1/2, 1/2)$ as each row, i.e., $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

2. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicate transition probabilities



a) Find the transition matrix Q

b) Which states (if any) are recurrent? Which states (if any) are transient?

c) Find two different stationary distributions for the chain

Solution:

a) The transition matrix is

$$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

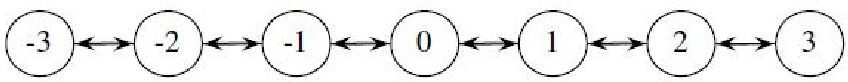
b) All of the states are recurrent. Starting at state 1, the chain will go back and forth between states 1 and 2 forever (sometimes lingering for a while). Similarly, for any starting state, the probability is 1 of returning to that state.

c) Solving

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix}$$
$$\begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{pmatrix} = \begin{pmatrix} c & d \end{pmatrix}$$

shows that $(a, b) = (1/3, 2/3)$, and $(c, d) = (1/2, 1/2)$ are stationary distributions on the 1,2 chain and on the 3,4 chain respectively, viewed as separate chains. It follows that $(1/3, 2/3, 0, 0)$ and $(0, 0, 1/2, 1/2)$ are both stationary for Q (as is any mixture $p(1/3, 2/3, 0, 0) + (1 - p)(0, 0, 1/2, 1/2)$ with $0 \le p \le 1$)

3. A Markov chain $X_0, X_1, \cdots$ with state space $\{-3, -2, -1, 0, 1, 2, 3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is X_{n-1} or X_{n+1}, each with probability $1/2$. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.



a) Is $|X_0|, |X_1|, |X_2|, \cdots$ also a Markov chain?

b) Let sgn be the sign function: $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = 0$. Is $\operatorname{sgn}(X_0), \operatorname{sgn}(X_1), \operatorname{sgn}(X_2), \cdots$ a Markov chain?

c) Find the stationary distribution of the chain $X_0, X_1, X_2, \cdots$

Solution:

a) Yes, $|X_0|, |X_1|, |X_2|, \ldots$ is also a Markov Chain. It can be viewed as the chain on state space 0,1,2,3 that moves left or right with equal probability, except that at 0 it bounces back to 1 and at 3 it bounces back to 2. Given that $|X_n| = k$, we know that $X_n = k$ or $X_n = -k$, and being given information about $X_{n-1}, X_{n-2}, \ldots$ does not affect the conditional distribution of $|X_{n+1}|$.

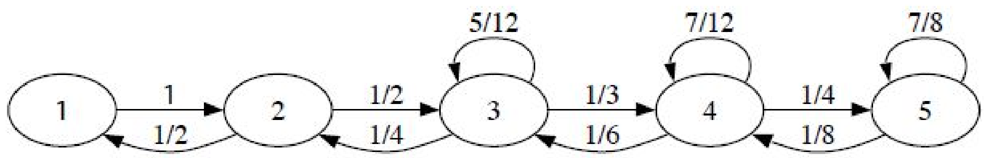
b) No, this is not a Markov chain because knowing that the chain was at 0 recently affects how far the chain can be from the origin. For example,

$$P(\operatorname{sgn}(X_2) = 1 \mid \operatorname{sgn}(X_1) = 1) > P(\operatorname{sgn}(X_2) = 1 \mid \operatorname{sgn}(X_1) = 1, \operatorname{sgn}(X_0) = 0)$$

since the conditioning information on the righthand side implies $X_1 = 1$, whereas the conditioning information on the lefthand side says exactly that X_1 is $1, 2,$ or 3 .

c) Using the result about the stationary distribution of a random walk on an undirected network, the stationary distribution is proportional to the degree sequence, $(1,2,2,2,2,2,1)$. Thus, the stationary distribution is $\frac{1}{12}(1,2,2,2,2,2,1)$

4. Find the stationary distribution of the Markov chain shown below, without using matrices. The number above each arrow is the corresponding transition probability



Solution:

We will show that this chain is reversible by solving for s (which will work out nicely since this is a birth-death chain). Let q_{ij} be the transition probability from i to j, and solve for $\mathbf{s}$ in terms of s_1. Noting that $q_{ij} = 2q_{ji}$ for $j = i+1($ when $1 \leq i \leq 4)$ we have that

$$\begin{aligned} s_1 q_{12} = s_2 q_{21} &\text{ gives } s_2 = 2s_1 \\ s_2 q_{23} = s_3 q_{32} &\text{ gives } s_3 = 2s_2 = 4s_1 \\ s_3 q_{34} = s_4 q_{43} &\text{ gives } s_4 = 2s_3 = 8s_1 \\ s_4 q_{45} = s_5 q_{54} &\text{ gives } s_5 = 2s_4 = 16s_1 \end{aligned}$$

The other reversibility equations are automatically satisfied since here $q_{ij} = 0$ unless $|i-j| \leq 1$. Normalizing, the stationary distribution is

$$\left(\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}\right)$$

5. Let $\{X_n\}$ be a Markov chain on states $\{0,1,2\}$ with transition matrix

$$\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

The chain starts at $X_0 = 0$. Let T be the time it takes to reach state 2 :

$$T = \min\{n : X_n = 2\}.$$

Find $E(T)$ and $\operatorname{Var}(T)$.

Solution:

To get from state 0 to state 2 , the chain needs to get from state 0 to state 1 and then get from state 1 to state 2 . So $T = T_1 + T_2$, where T_1 is the time it takes to reach state 1 and T_2 is the additional time it takes to reach state 2 . Then T_1 and T_2 are independent (by the Markov property), with $T_1 \sim \mathrm{FS}(0.2)$ and also $T_2 \sim \mathrm{FS}(0.2)$. So

$$E(T) = 5 + 5 = 10 \text{ and } \operatorname{Var}(T) = 20 + 20 = 40$$

6. Let us consider random walk on a weighted undirected network. Suppose that an undirected network is given, where each edge (i, j) has a nonnegative weight w_{ij} assigned to it (we allow $i = j$ as a possibility). We assume that $w_{ij} = w_{ji}$ since the edge from i to j is considered the same as the edge from j to i. When (i, j) is not an edge, we set $w_{ij} = 0$. When at node i, the next step is determined by choosing an edge attached to i with probabilities proportional to the weights.

a) Let $v_i = \sum_j w_{ij}$ for all nodes i. Show that the stationary distribution of node i is proportional to v_i.

b) Show that every reversible Markov chain can be represented as a random walk on a weighted undirected network.

Solution:

a) Let p_{ij} be the transition probability from i to j. Let $c = \sum_i v_i$. It suffices to show that $v_i p_{ij} = v_j p_{ji}$ for all nodes i and j, since then the reversibility condition $s_i p_{ij} = s_j p_{ji}$ holds for $s_i = v_i / c$. If $w_{ij} = 0$, then both sides are 0 . So assume $w_{ij} > 0$. Then

$$v_i p_{ij} = v_i \frac{w_{ij}}{\sum_k w_{ik}} = v_i \frac{w_{ij}}{v_i} = w_{ij} = w_{ji} = v_j \frac{w_{ji}}{v_j} = v_j p_{ji}.$$

b) Consider a reversible Markov chain with state space $\{1, 2, \ldots, M\}$, transition probability q_{ij} from i to j, and stationary distribution s. Let $w_{ij} = s_i q_{ij}$.

Now consider a weighted undirected network with nodes $1, 2, \ldots, M$ and an edge (i, j) with weight w_{ij} for each (i, j) with $w_{i,j} > 0$ (and no edge (i, j) if $w_{ij} = 0$). Then random walk on this weighted undirected network has transition probability q_{ij} from i to j, since the probability of going from i to j in one step is

$$\frac{w_{ij}}{\sum_k w_{ik}} = \frac{s_i q_{ij}}{\sum_k s_i q_{ik}} = \frac{q_{ij}}{\sum_k q_{ik}} = q_{ij}.$$

7. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

a) Give the transition probabilities of the chain.

b) Show that $(s_0, s_1, \ldots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

is the stationary distribution, by verifying the reversibility condition.

Solution:

a) Let p_{ij} be the transition probability from i to j. The number of black balls changes by at most 1 at each step, so $p_{ij} = 0$ for $|i-j| > 1$. Note that if urn 1 has i black balls and $N-i$ white balls, then urn 2 has i white balls and $N-i$ black balls. So

$$p_{i,i+1} = \left(\frac{N-i}{N}\right)^2,$$

since to get from state i to state $i+1$ we need to swap a white ball from urn 1 with a black ball from urn 2. Similarly,

$$p_{i,i-1} = \left(\frac{i}{N}\right)^2.$$

For the number of black balls to stay the same, we need to choose two black balls or two white balls. So

$$p_{i,i} = \frac{2i(N-i)}{N^2}$$

b) Case 1: $j = i+1$:

$$s_i p_{ij} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}\left(\frac{N-i}{N}\right)^2 = \frac{\binom{N}{i}^2 (N-i)^2}{\binom{2N}{N} N^2}$$

and

$$s_j p_{ji} = \frac{\binom{N}{i+1}\binom{N}{N-(i+1)}}{\binom{2N}{N}}\left(\frac{i+1}{N}\right)^2 = \frac{\binom{N}{i+1}^2 (i+1)^2}{\binom{2N}{N} N^2}.$$

Thus, $s_i p_{ij} = s_j p_{ji}$ from

$$\binom{N}{i}(N-i) = \binom{N}{i+1}(i+1)$$

Case 2: $j = i-1$:

$$s_i p_{ij} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}\left(\frac{i}{N}\right)^2 = \frac{\binom{N}{i}^2 i^2}{\binom{2N}{N} N^2}$$

and

$$s_j p_{ji} = \frac{\binom{N}{i-1}\binom{N}{N-(i-1)}}{\binom{2N}{N}}\left(\frac{N-(i-1)}{N}\right)^2 = \frac{\binom{N}{i-1}^2 (N-(i-1))^2}{\binom{2N}{N} N^2}.$$

So again we have $s_i p_{ij} = s_j p_{ji}$ from

$$\binom{N}{i} i = \binom{N}{i-1} (N - (i-1))$$

Hence, the chain is reversible, with stationary distribution $(s_0, s_1, \ldots, s_N)$.