HW 3 Solution

1.

$$E(X) = \sum_{k=1}^{\infty} k\,P(X=k) = c\sum_{k=1}^{\infty} p^k = \frac{cp}{1-p}$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2\,P(X=k) = c\sum_{k=1}^{\infty} k\,p^k$$

$$= cp\sum_{k=1}^{\infty} k\,p^{k-1} = \frac{cp}{(1-p)^2}$$

$$VAR(X) = \frac{cp}{(1-p)^2} - \frac{c^2p^2}{(1-p)^2} = \frac{cp(1-cp)}{(1-p)^2}$$

2. Let X be the number of children needed, from the 2nd child, to obtain different gender from the 1st baby. Then ~~is~~ $X-1$ is the Geom$(\frac{1}{2})$, hence $E(X) = 2$, hence the Expected total number of children $= 3$.

3. 20 drops per square inch per minute

\# of drops ~~per~~ 5 square inch per 3-second

$= 100/20 = 5$, $\lambda = 5$

P(no rain drops per 5 inch^2 in 3 secs)

$$= p_{\lambda=5}(k=0) = e^{-5}\frac{\lambda^{k=0}}{(k=0)!} = e^{-5} = 0.0067$$

4.

a) Let I_i be the indicator r.v. for the ith person being a mutual friend.

$$E\left(\sum_{i=1}^{1000} I_i\right) = 1000\, E(I_i) = 1000\, P(I_i = 1)$$
$$= 1000 \left(\frac{5}{100}\right)^2 = 2.5$$

b) $$P(X=k) = \frac{\binom{50}{k}\binom{950}{50-k}}{\binom{1000}{50}} \qquad X \sim \text{HGeom}(50, 950, 1000)$$

5. Let I_i be the indicator r.v. for the ith box being empty

The # of empty boxes $= I_1 + \cdots + I_N$

$$E\left(\sum_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} E(I_i) = n\left(1-\frac{1}{n}\right)^k$$

6. a) Let N be the number of trials to see both success

$$\underbrace{(X = S,\ Y = S)}_{P_1 P_2},\ (X=S, Y=F),\ (X=F, Y=S),\ (X=F, Y=F)$$

Hence, $N \sim FS(P_1 P_2)$ and $P(N=n) = P_1P_2(1-P_1P_2)^{n-1}$

$$E(N) = \frac{1}{P_1 P_2}$$

b) Let T be the number of trials to see the first any success

$$\underbrace{(X=S, Y=S),\ (X=S, Y=F),\ (X=F, Y=S)}_{1-(1-P_1)(1-P_2)},\ \underset{(1-P_1)(1-P_2)}{(X=F, Y=F)}$$

$T \sim FS(1-(1-P_1)(1-P_2))$
$$E(T) = \frac{1}{1-(1-P_1)(1-P_2)}$$

c) $$P(T_X = T_Y) = \sum_{n=1}^{\infty} P(T_X = n \mid T_Y = n) P(T_Y = n)$$
$$= \sum_{n=1}^{\infty} P(T_X = n) P(T_Y = n) = \sum_{n=1}^{\infty} P_1^2 (1-P_1)^{2(n-1)}$$
$$= \frac{P}{2-P}$$

$$1 = P(T_X < T_Y) + P(T_Y < T_X) + P(T_X = T_Y) = 2P(T_X < T_Y) + P(T_X = T_Y)$$

$$\therefore\ P(T_X < T_Y) = \frac{1}{2}\cdot\left(1 - \frac{P}{2-P}\right) = \frac{1-P}{2-P}$$

7.

a) $$E(Xg(X)) = e^{-\lambda}\sum_{k=0}^{\infty} k\, g(k)\frac{\lambda^k}{k!}$$
$$= \lambda e^{-\lambda}\sum_{k=1}^{\infty} g(k)\frac{\lambda^{k-1}}{(k-1)!}$$
$$= \lambda e^{-\lambda}\sum_{k=0}^{\infty} g(k+1)\frac{\lambda^k}{k!}$$
$$= \lambda E(g(X+1))$$

b) $$E(X^3) = E(X\cdot X^2) = \lambda E(\cancel{X}+1)^2$$
$$= \lambda E(X^2) + 2\lambda E(X) + \lambda$$
$$E(X) = \lambda E(1) = \lambda$$
$$E(X^2) = \lambda E(X+1) = \lambda^2+\lambda,$$
$$\therefore E(X^3) = \lambda(\lambda^2+\lambda+2\lambda+1) = \lambda^3+3\lambda^2+\lambda$$

8.

a) $$E(e^{tX}) = e^{-\lambda}\sum_{k=0}^{\infty} e^{tk}\frac{\lambda^k}{k!} = e^{-\lambda}\sum\frac{(e^t\lambda)^k}{k!}$$
$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

b) $$E(e^{tX}) = p\sum_{k=0}^{\infty} e^{tk}(1-p)^k = p\sum_{k=0}^{\infty}((1-p)e^t)^k$$
$$= \frac{p}{1-(1-p)e^t} \quad ((1-p)e^t < 1)$$