Chapter1. Sets and Logic

1.1 Sets

1.2 Propositions

1.3 Conditional Propositions and Logical Equivalence

1.4 Arguments and Rules of Inference

1.5 Quantifiers

1.6 Nested Quantifier

1.1 Sets

- Sets : a collection of distinct unordered objects called
  element or member
  Regardless of the order of the elements, they may have
  duplicate elements

- Notations for sets
  By listing all of its elements
  $A = \{1,2,3,4\}$, $A = \{1,3,4,2\}$, $A = \{1,2,2,3,4\}$
  Set builder notation : by presenting conditions that can be
  members of a set
  $B = \{x \mid x$는 양수이고 짝수$\}$, '|' 는 such that로 읽는다.

- 수의 집합
  - **Z**는 정수 집합(정수를 뜻하는 독일어Zahlen에서 유래)
  - **Q**는 유리수 집합(몫을 뜻하는 Quotient)
  - **R**는 실수 집합(Real)
  - $\mathbf{Q}^{+}$는 양의 유리수 집합, $\mathbf{R}^{-}$는 음의 실수 집합
    $\mathbf{Z}^{nonneg}$는 음이 아닌nonnegative 정수 집합

1.1 Sets

- **기수**cardinality는 X가 유한집합일 때 원소의 개수. $|X|$로 표시
  - Ex1.1.1) 집합 $A = \{1,2,3,4\}$ 대해서 $|A| = 4$
- x가 집합 X의 원소이면 $x \in X$로 쓴다. 아니면, $x \notin X$로 쓴다.
- 원소를 갖지 않은 집합을 **공집합**empty(null, void)set이라 하고, $\emptyset$로 표시 즉, $\emptyset = \{\ \}$이다.
- 두 집합 X와 Y가 동일한 원소를 가질 때, X와 Y는 **동등**equivalent 하고, $X = Y$로 표시. 다음조건이 성립하면 $X = Y$이다.
  - 모든 x에 대해서 $x \in X$이면 $x \in Y$이다.
  - 모든 x에 대해서 $x \in Y$이면 $x \in X$이다.
- X와 Y가 집합이고, X의 모든 원소가 Y의 원소일 때, X를 Y의 **부분집합**subset이라 하고 $X \subseteq Y$로 표시한다.
- 만약 $X \subseteq Y$ 이고, $X \neq Y$면 **진부분집합**proper subset이라 하고, $X \subset Y$로 표시한다.

1.1 Sets

- 집합 X의 모든 부분집합의 집합은 X의 **멱집합**power set이라 하고, $\mathcal{P}(X)$로 표시한다
  - Ex 1.1.14) $A = \{a, b, c\}$라면

    $$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

    $|A| = 3$이고, $|\mathcal{P}(A)| = 2^3 = 8$이다
- **합집합**union

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

- **교집합**intersection

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

- **차집합**difference or 상대적 여집합 relative complement

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

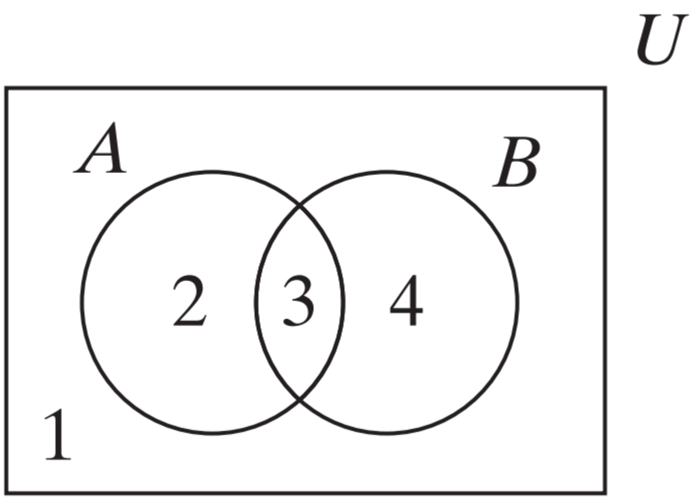
  - $A = \{1,3,5\},\ B = \{4,5,6\}$이면, $A - B = \{1,3\},\ B - A = \{4,6\}$

1.1 Sets

- 원소가 집합인 집합을 **집합들의 모임**collection of sets 또는 **집합족** family of sets이라고 부른다.
  - $\mathcal{S} = \{\{1,2\},\{1,3\},\{1,7,10\}\}$
- $X \cap Y = \emptyset$ 일 때 **서로소**disjoint라고 부른다.
  - {1,4,5}와 {2,6}은 서로소이다.
- 집합들의 모임인 $\mathcal{S}$에서 $\mathcal{S}$내의 모든 서로 다른 두 집합 X와 Y가 서로소이면 $\mathcal{S}$는 **쌍으로 소**pairwise disjoint라 한다.
  - Ex 1.1.7) $\mathcal{S} = \{\{1,4,5\},\{2,6\},\{3\},\{7,8\}\}$ 는 pairwise disjoint
- 때때로 집합들은 어떤 집합 U의 부분집합으로 다루어진다. 이 집합 U를 **전체집합**universal set이라 한다. U는 분명하게 제시되거나 문맥으로 추론될 수 있어야 한다.
- 전체집합 U와 U의 부분집합 X가 주어졌을 때, $U - X$를 X의 **여집합**complement이라 하고 $\overline{X}$로 표시한다.

1.1 Sets

- **벤다이어그램**Venn Diagram은 집합을 그림으로 보여준다.
  - 사각형은 전체집합을 나타낸다.
  - 부분집합은 원으로 표현한다.
  - 원의 내부는 원소를 나타낸다.
- 다음 그림에는 전체집합 U안에 2개의 집합 A와 B가 있다.
  - 영역 1은 $\overline{(A \cup B)}$, 즉, A 또는 B에 속하지 않는 원소
  - 영역 2은 $A - B$, 즉, A에 속하지만 B에 속하지 않는 원소
  - 영역 3은 $A \cap B$
  - 영역 4은 $B - A$



1.1 Sets

Theorem 1.1.21) U를 전체 집합이라 하고, A, B, C를 U의 부분 집합이라고 하면,

a) 결합법칙 Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

b) 교환법칙 Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

c) 분배법칙 Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

d) 항등법칙 Identity laws

$$A \cap U = A \qquad A \cup \varnothing = A$$

e) 여집합법칙 Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

1.1 Sets

f) 멱등법칙 Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

g) 경계법칙 Bound laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

h) 흡수법칙 Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

i) 대합법칙 Involution law

$$\overline{\overline{A}} = A$$

j) 0/1법칙 0/1 laws

$$\overline{\varnothing} = U \qquad \overline{U} = \varnothing$$

k) 드모르간 법칙 De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

1.1 Sets

- 임의의 집합족 $\mathcal{S}$의 합집합은 $\mathcal{S}$내의 적어도 하나의 집합 X에 속하는 원소 x들로 구성된 집합으로 정의한다.

$$\cup \mathcal{S} = \{x \mid \text{어떤 } X \in \mathcal{S}\text{에 대해서 } x \in X\}$$

- 임의의 집합족 $\mathcal{S}$의 교집합은 $\mathcal{S}$내의 모든 집합 X에 속하는 원소 x들로 구성된 집합으로 정의한다.

$$\cap \mathcal{S} = \{x \mid \text{모든 } X \in \mathcal{S}\text{에 대해서 } x \in X\}$$

- $\mathcal{S} = \{A_1, A_2, \dots, A_n\}$이라면

$$\cup \mathcal{S} = \bigcup_{i=1}^{n} A_i\,, \qquad \cap \mathcal{S} = \bigcap_{i=1}^{n} A_i$$

- $\mathcal{S} = \{A_1, A_2, \dots\}$이라면

$$\cup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i\,, \qquad \cap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i$$

1.1 Sets

- 집합 X의 **분할**은 X를 서로 겹치지 않는 부분집합으로 나눈다.
- 집합 X의 부분집합 중에서 공집합이 아닌 것들의 모임인 $\mathcal{S}$를 X의 모든 원소들이 정확히 $\mathcal{S}$의 하나의 원소에만 속할 때 집합 X의 **분할**partition이라고 부른다.
- A collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to *exactly one member* of $\mathcal{S}$.
- $\mathcal{S}$가 X의 분할이면 $\mathcal{S}$는 쌍으로 소이며 $\cup\,\mathcal{S} = X$이다.
- Ex 1.1.25 분할

$$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

의 각 원소들은

$$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$$

의 정확히 하나의 원소에만 속하므로 $\mathcal{S}$는 X의 분할이다

1.1 Sets

- An **ordered pair** of elements, written (a, b), is considered 좌표 distinct from the ordered pair (b, a) unless a=b . To put it another way, $(a, b) = (c, d)$ precisely when $a = c$ and $b = d$.
- If X and Y are sets, **Cartesian product** of X and Y is

$$X \times Y = \{(x, y) | \, x \in X, y \in Y\}$$

  - Ex 1.1.26)

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

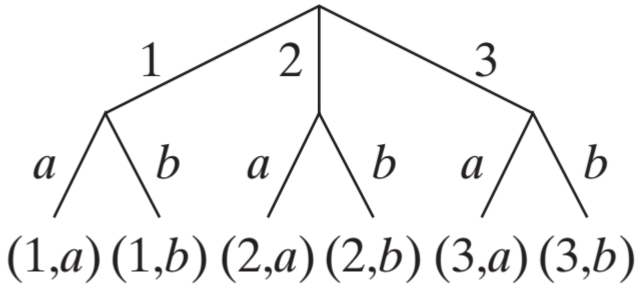
$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Y \times Y = \{(a, a), (a, b), (b, a), (b, b)\}.$$

1.1 Sets

- the previous 예제, $|X \times Y| = |X| \cdot |Y|$.
  The reason is that there are 3 ways to choose an element of X, there are 2 ways to choose an element of Y, and $3 \cdot 2 = 6$



- The preceding argument holds for arbitrary finite sets X and Y; it is always true that $|X \times Y| = |X| \cdot |Y|$.

1.1 Sets

- An ordered **n-tuple**, written $(a_1, a_2, \ldots, a_n)$, takes order into account; that is,
$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$
precisely when
$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n.$$
- The Cartesian product of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, \ldots, n$; it is denoted $X_1 \times X_2 \times \cdots \times X_n$.
  - Ex 1.1.28)

If $X = \{1, 2\}$, $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}.$$

1.2 proposition 명제

- A 평서문(서술문)declarative sentence that is either true or false, but not both, is called a **proposition**.
- Propositions are the basic building blocks of any theory of logic.
- We will use *variables*, such as p, q, r, to represent propositions.
- We will also use the notation

$$p:\ 1+1=3$$

to define p to be the proposition $1+1=3$.

1.2 proposition명제 (논리연산)

- Let p and q be propositions.
  The **conjunction**논리곱 of p and q, denoted $p \wedge q$, is the proposition

$$p \text{ and } q$$

- The **disjunction**논리합 of p and q, denote $p \vee q$, is the proposition

$$p \text{ or } q$$

- The **negation**부정 of p, denoted $\neg p$, is the proposition

$$\text{not } p$$

- The **exclusive-or**배타적 합, denoted $p\ exor\ q$, is true if p or q, but not both, is true, and false otherwise.

1.2 proposition 명제 (진리표 truth table)

- The truth values of propositions can be described by **truth tables**.
- T denoting true and F denoting false, and for each such combination lists the truth value of P.
- A truth table for a proposition P made up of n propositions has $r = 2^n$ rows.
- Truth tables for *and, or, exclusive or, not*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p exor q
T	T	F
T	F	T
F	T	T
F	F	F

p	$\neg p$
T	F
F	T

1.2 proposition 명제 (연산자 우선 순위 Operator precedence)

- ¬, ∧, ∨, →
- Assuming that p is true, q is false, and r is true, find the truth value of each proposition.
  - $p \wedge q \rightarrow r$
    We first evaluate $p \wedge q$ because → is evaluated last. Since p is true and q is false, $p \wedge q$ is false. Therefore, $p \wedge q \rightarrow r$ is true.
  - $p \vee q \rightarrow \neg r$
    We first evaluate $\neg r$. Since r is true, $\neg r$ is false.
    We next evaluate $p \vee q$. Since p is true and q is false, $p \vee q$ is true.
    Therefore, $p \vee q \rightarrow \neg r$ is false.
  - $p \wedge (q \rightarrow r)$
    Since q is false, $q \rightarrow r$ is true. Since p is true, $p \wedge (q \rightarrow r)$ is true.
  - $\neg p \vee q \wedge r$
    We first evaluate $\neg p$, which is false. We next evaluate $q \wedge r$, which is false. Finally, we evaluate $\neg p \vee q \wedge r$, which is false.

1.3 조건 명제와 논리적 동치

- If p and q are propositions, the proposition

  $$\text{if } p \text{ then } q$$

  is called a **conditional proposition** 조건 명제 and is denoted

  $$p \to q$$

- The p is called the **hypothesis** 가설 or **antecedent** 전제,
  The q is called the **conclusion** 결론 or **consequent** 결과.

- **Definition 1.3.3**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

  - p: 오늘 비가 온다, q: 오늘 우산을 가지고 간다.
    오늘 비가 온다면 우산을 가지고 간다

1.3 조건 명제와 논리적 동치

Motivation for defining $p \to q$ to be true when p is false

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Most people would agree that the proposition
  For all real numbers x, if $x > 0$, then $x^2 > 0$.
  is true.
- In case $x = -2$
  $x^2 = 4$
  In order for the proposition to be true in this case, we must define $p \to q$ to be true when p is false and q is true.
- In case $x = 0$
  $x^2 = 0$
  In order for the proposition to be true in this case, we must define $p \to q$ to be true when both p and q are false.

1.3 조건 명제와 논리적 동치(필요조건Necessary condition)

- 필요조건은 특정 결과가 이루어지기 위한 필요조건으로,
  그 조건이 이루어 지지 않으면 결과는 생기지 않는다.
  조건이 이루어 지면 결과를 보장하지는 않는다.
  즉 결과가 이루어 졌다는 것으로 필요조건이 달성되었다는 것이다
  - 시카고 컵스가 우승하기위한 필요조건은 우완 구원투수와 계약하는 것이다

동치 구성은
'시카고 컵스가 우승을 한다면 우완구원투수와 계약을 한 것이다'
결론이 필요조건을 표현한다

하지만 또 다른 명제
'우완구원투수와 계약을 한다면 시카고 컵스가 우승을 할 것이다'
필요조건이 이루어지면 결과를 보장하지 않으므로
우완 구원투수와 계약은 우승을 보장 할 수 없다.
하지만 필요조건이 이루어지지 않으면 결과를 보장하지 않으므로
우완 구원투수와 계약을 안한다면 우승하지 못한다는 것을 보장한다.

1.3 조건 명제와 논리적 동치(충분조건sufficient condition)

- 충분조건은 특정 결과를 보장하기에 충분한 조건으로,
  그 조건이 이루어 지지 않더라도 결과는 이루어 질 수도, 이루어 지지 않을 수도 있다.
  조건이 이루어 지면 결과를 보장한다.
  - 마리아가 프랑스를 방문하기위한 충분조건은 에펠탑에 가는 것이다
  에펠탑에 가지 않고 리옹에 가더라도 결과 프랑스를 방문 할 수 다

  가설은 충분조건을 표현하여
  '마리아가 에펠탑에 가면, 프랑스를 방문한 것이다'
  동치 구성이 된다

  하지만 또 다른 명제
  '마리아가 프랑스를 방문하면, 에펠탑에 간다 ' 는
  충분조건을 만족하지 않더라도 결과가 이루어 질 수 있기 때문에
  에펠탑을 가지 않더라도 프랑스를 방문 할 수 있어 동치가 아니다

1.3 조건 명제와 논리적 동치(역명제 converse)

- We call the proposition $q \rightarrow p$ the **converse** of the proposition $p \rightarrow q$.
- Ex1.3.7) 다음 명제와 그 역명제를 기호와 말로 표현하라.
  제리가 장학금을 받는다면, 대학에 진학할 것이다.
  또한 제리가 장학금을 받지 못하고 복권에 당첨되어 대학에 진학한다고 가정하고, 본래 명제와 그 역명제의 진리값을 구하라.
- sol) p: 제리는 장학금을 받는다.
  q: 제리는 대학에 진학한다.
  원래 명제는 $p \rightarrow q$의 기호로 로 나타낼 수 있고,
  가설 p가 거짓이므로 조건 명제는 참이다.

  역명제는 제리가 대학에 진학하면, 장학금을 받는다.
  역명제는 $q \rightarrow p$의 기호로 나타낼 수 있다.
  가설 q가 참이고 결론 p는 거짓이므로, 역명제는 거짓이 된다.

1.3 조건 명제와 논리적 동치(Biconditional Proposition)

- If p and q are propositions, the proposition

$$p \text{ if and only if } q$$

  is called a **biconditional proposition**,

$$p \leftrightarrow q$$

- The true table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

if and if only
= iff
필요충분조건

- In mathematical definitions, "if" means "if and only if." e.g., the definition of set equality: If sets X and Y have the same elements, then X and Y are equal.
  The meaning of this definition is that sets X and Y have the same elements if and only if X and Y are equal.

1.3 조건 명제와 논리적 동치(logically equivalent)

- Definition 1.3.10) Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$.
  We say that P and Q are *logically equivalent* and write $P \equiv Q$, provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

  Ex 1.3.11) Verify De Morgan's Law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  By writing the truth tables for $P = \neg(p \vee q)$ and $Q = \neg p \wedge \neg q$, we can verify that, given any truth values of p and q, either P and Q are both true or P and Q are both false:

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

De Morgan's Law for logic
$\neg(p \vee q) \equiv \neg p \wedge \neg q$
$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Thus P and Q are logically equivalent.

1.3 조건 명제와 논리적 동치(logically equivalent)

$$\boxed{\neg(p \to q) \equiv (p \wedge \neg q)}$$

- Ex 1.3.13) Show that the negation of $p \to q$ is logically equivalent to $p \wedge \neg q$.

sol) By writing the truth tables for $P = \neg(p \to q)$ and $Q = p \wedge \neg q$, we can verify that, given any truth values of p and q, either P and Q are both true or P and Q are both false:

p	q	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Thus P and Q are logically equivalent.

1.3 조건 명제와 논리적 동치(logically equivalent)

$$\boxed{p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)}$$

- In words,

  p if and only if q

  is logically equivalent to

  if p then q and if q then p.

- Ex 1.3.15) The truth table shows that

  $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

1.3 조건 명제와 논리적 동치(대우contrapositive)

- Definition 1.3.16 The **contrapositive** of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- 정리 1.3.18

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

The truth table shows that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

1.4 논법argument과 추론규칙rule of inference

- Definition 1.4.1) An **argument** is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ \therefore\ q \end{array}$$

or

$$p_1, p_2, \dots, p_n / \therefore q$$

The argument is **valid**유효 provided that if p_1 and p_2 and $\cdots$ and p_n are all true, then q must also be true; otherwise, the argument is **invalid**무효 (or fallacy오류),.

- The symbol $\therefore$ is read "therefore." The propositions $p_1, p_2, \dots, p_n$ are called the hypotheses가설 (or premises전제), and the proposition q is called the conclusion결론.

1.4 논법과 추론규칙(유효한 논법valid argument)

- In a valid argument, we sometimes say that the conclusion *follow from* the hypotheses.
- We are not saying that the conclusion is true;
  we are only saying that if you grant인정하다 the hypotheses, you must also grant the conclusion.
  An argument is valid because of its form, not because of its content.
- 추론 규칙은 논리학에서 논리식으로부터 다른 논리식을 이끄는 규칙을 말한다

1.4 논법과 추론규칙(긍정식 논법 Modus ponens rule)

- Ex 1.4.2 Is the argument valid?

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Sol 1) We construct a truth table for all the propositions involved:

p	q	$p \to q$	p	q
T	T	$\boxed{\text{T}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

We observe that whenever the hypotheses $p \to q$ and p are true, the conclusion q is also true; therefore, the argument is valid.

- Sol 2) Suppose that $p \to q$ and p are true. Then q must be true, for otherwise $p \to q$ would be false. Therefore, the argument is valid.

1.4 논법과 추론규칙(명제에 대한 추론 규칙)

Rule of Inference	*Name*	*Rule of Inference*	*Name*
$p \rightarrow q$ $\underline{p\qquad}$ $\therefore q$	긍정식 논법 Modus ponens	p $\underline{q\qquad}$ $\therefore p \wedge q$	논리곱 Conjunction
$p \rightarrow q$ $\underline{\neg q\qquad}$ $\therefore \neg p$	부정식 논법 Modus tollens	$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	가설적 삼단논법 Hypothetical syllogism
$\underline{p\qquad}$ $\therefore p \vee q$	추가법 Addition	$p \vee q$ $\underline{\neg p\qquad}$ $\therefore q$	논리합 삼단논법 Disjunctive syllogism
$\underline{p \wedge q}$ $\therefore p$	단순화 Simplification		

1.4 논법과 추론규칙(결론 확정의 오류)

- **Ex 1.4.4.** Represent the argument

  If 2 = 3, then I ate my hat.
  I ate my hat.
  ∴ 2 = 3

  symbolically and determine whether the argument is valid.

Sol) If we let

p: 2 = 3, q: I ate my hat,

the argument may be written

$$\begin{array}{l} p \to q \\ \underline{q \qquad} \\ \therefore p \end{array}$$

Suppose that $p \to q$ and q are true. This is possible if p is false and q is true. In this case, p is not true; thus the argument is invalid.

1.4 논법과 추론규칙

- Ex 1.4.5. The bug is either in module 17 or in module 81.
  The bug is a numerical error.
  Module 81 has no numerical error.
  ___
  ∴ The bug is in module 17.

$$\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \underline{\quad \neg p} \\ \therefore q \end{array}$$

- Sol) p: The bug is in module 17.
  q: The bug is in module 81.
  r: The bug is a numerical error.
  the argument may be written

1. $p \vee q$
2. r
3. $r \to \neg q$
4. $\neg q$ m.p. : 3, 2
5. p d.s. : 1, 4

-

$$\begin{array}{l} p \vee q \\ r \\ \underline{r \to \neg q} \\ \therefore\ p \end{array}$$

From $r \to \neg q$ and r, we may use modus ponens긍정식 논법

to conclude $\neg q$. From $p \vee q$ and $\neg q$, we may use the disjunctive syllogism논리합 삼단논법 to conclude p. Thus the conclusion p follows from the hypotheses and the argument is valid.

1.5 한정기호Quantifiers

- p: n is an odd integer홀수.
  The statement p is not a proposition, because whether p is true or false depends on the value of n.
  - p is true if $n = 103$ and false if $n = 8$.
- Definition 1.5.1 Let $P(x)$ be a statement involving the variable x and let D be a set.
  We call P **a propositional function**명제함수 or **predicate**술어 (with respect to D) if for each $x \in D$, $P(x)$ is a proposition.
  We call D the domain of discourse논의영역 of P.
  - Ex 1.5.2 Let $P(n)$ be the statement
    n is an odd integer.
    Then P is a propositional function with domain of discourse $\boldsymbol{Z}^+$ since for each $n \in \boldsymbol{Z}^+$, $P(n)$ is a proposition

1.5 한정기호(전칭 한정된 문장universally quantified statement)

- The symbol ∀ means "for every" and is called a **universal quantifier**전칭 한정기호.

- Definition 1.5.4

  Let P be a propositional function with domain of discourse D. The statement

  $$\text{for every } x, P(x)$$

  is said to be a **universally quantified statement** 전칭 한정문장. It may be written

  $$\forall x\ P(x)$$

  It is true if $P(x)$ is true for every x in D,
  it is false if $P(x)$ is false for at least one x in D.

1.5 한정기호(전칭 한정된 문장 universally quantified statement)

- Ex 1.5.5 Consider the universally quantified statement
$$\forall x(x^2 \geq 0).$$
The domain of discourse is $\boldsymbol{R}$.
The statement is true because, for every real number x, it is true that the square of x is positive or zero.
- A value x in D that makes $P(x)$ false is called a **counterexample**반례 to the statement.
- Ex 1.5.6 Determine whether the universally quantified statement
$$\forall x(x^2 - 1 > 0)$$
is true or false. The domain of discourse is $\boldsymbol{R}$.
- Sol) The statement is false since, if $x = 1$, the proposition
$$1^2 - 1 > 0$$
is false. The value 1 is a *counterexample* to the statement

1.5 한정기호(전칭 한정된 문장 universally quantified statement)

- We call the variable x in the propositional function $P(x)$ a **free variable**. (The idea is that x is "free" to roam배회하다 over the domain of discourse.)
- We call the variable x in the universally quantified statement $\forall x\ P(x)$ a **bound variable**. (The idea is that x is "bound" by the quantifier $\forall$.)
- A propositional function does not have a truth value.
- Definition 1.5.4 assigns a truth value to the quantified statement $\forall x\ P(x)$.
- A statement with free (unquantified) variables is not a proposition, and a statement with no free variables (no unquantified variables) is a proposition.

1.5 한정기호 (전칭 한정된 문장의 증명)

- The symbol ∀ may be read "for every," "for all," or "for any."
- To prove that $\forall x\ P(x)$ is true, we must, in effect, examine *every* value of x in the domain of discourse and show that for *every* x, $P(x)$ is true.
- One technique for proving that $\forall x\ P(x)$ is true is to let x denote an *arbitrary* element of the domain of discourse D. The argument then proceeds using the symbol x. Whatever is claimed about x must be true *no matter what value* x might have in D. The argument must conclude by proving that $P(x)$ is true.
- Sometimes to specify the domain of discourse D, we write a universally quantified statement as

$$\text{for every } x \text{ in } D,\ P(x).$$

1.5 한정기호 (전칭 한정된 문장의 증명)

- Ex 1.5.8) Verify that the universally quantified statement
  for every real number x, if $x > 1$, then $x + 1 > 1$ is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Sol) Let x be any real number.
  Then $x \leq 1$ or $x > 1$ is true.
  If $x \leq 1$, the hypothesis $x > 1$ is false
  and the conditional proposition is true
  Now suppose that $x > 1$. Regardless of the specific value of x, $x + 1 > x$. Since $x + 1 > x$ and $x > 1$, $x + 1 > 1$. So the conclusion is true. If $x > 1$, the hypothesis and conclusion are both true; hence the conditional proposition is true.

  We have shown that for every real number x, the proposition is true. Therefore, the universally quantified statement for every real number x, if $x > 1$, then $x + 1 > 1$ is true.

1.5 한정기호(존재 한정된 문장existentially quantified statement)

- The symbol ∃ means "there exists." and is called an existential quantifier존재 한정기호.
- Definition 1.5.9

  Let P be a propositional function with domain of discourse D. The statement

  there exists x, $P(x)$

  is said to be an **existentially quantified statement**. It may be written

$$\exists x\, P(x).$$

  It is true if $P(x)$ is true for at least one x in D,
  it is false if $P(x)$ is false for every x in D.

1.5 한정기호 (존재 한정된 문장의 증명)

- Ex 1.5.10 Consider the existentially quantified statement

$$\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right).$$

The domain of discourse is $\boldsymbol{R}$. The statement is true because it is possible to find at least one real number x for which the proposition

$$\frac{x}{x^2+1} = \frac{2}{5}$$

is true. e.g., if $x = 2$, we obtain the true proposition

$$\frac{2}{2^2+1} = \frac{2}{5}$$

1.5 한정기호 (존재 한정된 문장의 증명)

- Ex 1.5.11 Verify that $\exists x \in R\left(\frac{1}{x^2+1} > 1\right)$ is false.

- Sol) We show that $\frac{1}{x^2+1} > 1$ is false for every real number x.
  Now $\frac{1}{x^2+1} > 1$ is false precisely when $\frac{1}{x^2+1} \leq 1 \qquad (eq)$ is true.
  Thus, we must show that (eq) is true for every real number x.

  Let x be any real number. Since $0 \leq x^2$, $1 \leq x^2+1$. If we divide both sides of this last inequality by x^2+1, we obtain (eq).

  Therefore, the statement (eq) is true for every real number x.
  Thus the statement $\frac{1}{x^2+1} > 1$ is false for every real number x.

  We have shown that the existentially quantified statement
  $\exists x \in R\left(\frac{1}{x^2+1} > 1\right)$ is false.

1.5 한정기호 (Pseudocode의사코드 for quantified statements)

- Ex 1.5.7) Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \ldots, d_n\}$. The following pseudocode determines whether $\forall x\, P(x)$ is true or false:

```
for i = 1 to n
    if (¬P(d_i))
        return false
return true
```

- Ex 1.5.12) Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \ldots, d_n\}$. The following pseudocode determines whether $\exists x\, P(x)$ is true or false:

```
for i = 1 to n
    if (P(d_i))
        return true
return false
```

1.5 한정기호 (Generalized De Morgan's Laws for Logic)

- Theorem 1.5.14) Generalized De Morgan's Laws for Logic
  If P is a propositional function with domain of discourse D,
  (a) $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$
  (b) $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$
- Proof a) Suppose that the proposition $\neg(\forall x\ P(x))$ is true.
  Then the proposition $\forall x\ P(x)$ is false.
  By Definition 1.5.4, when $P(x)$ is false for at least one $x \in D$, $\forall x\ P(x)$ is false. But $P(x)$ is false for at least one $x \in D$, $\neg P(x)$ is true for at least one $x \in D$.
  By Definition 1.5.9, when $\neg P(x)$ is true for at least one $x \in D$, the proposition $\exists x\ \neg P(x)$ is true.
  Thus, if $\neg(\forall x\ P(x))$ is true, $\exists x\ \neg P(x)$ is true.
- Similarly, if the proposition $\neg(\forall x\ P(x))$ is false, the proposition $\exists x\ \neg P(x)$ is false.
- Therefore, the pair of propositions in part (a) always has the same truth values.

1.5 한정기호 (한정된 문장의 부정)

- Ex 1.5.17 Write the statement

  Some birds cannot fly,

  symbolically. Write its negation symbolically and in words.

- Sol) Let $P(x)$ be the propositional function "x flies."

  The given statement can be written symbolically as

  $$\exists x \neg P(x)$$

  The domain of discourse is the set of birds.

- By Theorem 1.5.14, part (b), the negation $\neg(\exists x \neg P(x))$ of the preceding proposition is equivalent to

  $$\forall x \neg\neg P(x)$$

  or, equivalently,

  $$\forall x\ P(x)$$

  In words, this last proposition can be stated as:

  Every bird can fly.

1.5 한정기호 (한정된 문장에 대한 추론법칙)

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore P(d) \text{ if } d \in D}$	전칭 예시화 Universal instantiation
$\dfrac{P(d) \text{ for every } d \in D}{\therefore \forall x\, P(x)}$	전칭 일반화 Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore P(d) \text{ for some } d \in D}$	존재 예시화 Existential instantiation
$\dfrac{P(d) \text{ for some } d \in D}{\therefore \exists x\, P(x)}$	존재 일반화 Existential generalization

1.5 한정기호 (한정된 문장에 대한 추론법칙)

- Ex 1.5.21 Given that for every positive integer n, $n^2 \geq n$ is true, we may use universal instantiation to conclude that $54^2 \geq 54$ since 54 is a positive integer.
- Ex 1.5.22 Let $P(x)$ denote the propositional function "x owns a laptop computer," where the domain of discourse is the set of students taking the discrete mathematics.
  Suppose that Taylor, who is taking the discrete mathematics, owns a laptop computer; in symbols, $P(Taylor)$ is true.
  We may then use existential generalization to conclude that $\exists x\ P(x)$ is true.

1.5 한정기호 (한정된 문장에 대한 추론법칙)

- Ex 1.5.23 Show that the argument is valid.

  For every real number x,
  if x is an integer, then x is a rational number유리수.

  $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

- Sol) Let $P(x)$ denote "x is an integer",
  $Q(x)$ denote "x is rational."

$$\begin{array}{l} \forall x \in \boldsymbol{R}\ (P(x) \to Q(x)) \\ \neg Q(\sqrt{2}) \\ \hline \therefore \neg P(\sqrt{2}) \end{array}$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Since $\sqrt{2} \in \boldsymbol{R}$, we may use universal instantiation전칭 예시화 to conclude $P(\sqrt{2}) \to Q(\sqrt{2})$.
Combining $P(\sqrt{2}) \to Q(\sqrt{2})$ and $\neg Q(\sqrt{2})$, we may use modus tollens to conclude $\neg P(\sqrt{2})$. Thus the argument is valid.

1.5 한정기호 (한정된 문장에 대한 추론법칙)

- Ex 1.5.24 We are given these hypotheses:
  Everyone loves either Microsoft or Apple.
  Lynn does not love Microsoft.
  Show that the conclusion, Lynn loves Apple, follows from the hypotheses.

- Sol) Let $P(x)$ denote "x loves Microsoft," and
  let $Q(x)$ denote "x loves Apple."
  The first hypothesis is $\forall x(P(x) \vee Q(x))$.
  By universal instantiation, we have $P(Lynn) \vee Q(Lynn)$.
  The second hypothesis is $\neg P(Lynn)$.
  The disjunctive syllogism rule of inference now gives $Q(Lynn)$, which represents the proposition "Lynn loves Apple."
  We conclude that the conclusion follows from the hypotheses.

1.6 다중 한정 기호nested quantifiers

- Multiple quantifiers such as $\forall x \forall y$ are said to be *nested quantifiers*.
  - Ex 1.6.1 Restate $\forall m \exists n (m < n)$ in words. The domain of discourse is the set $\boldsymbol{Z} \times \boldsymbol{Z}$.
  - Sol) For every m, there exists n such that $m < n$.
    Less formally, this means that if you take any integer m whatsoever, there is an integer n greater than m.
    Another restatement is then: There is no greatest integer.
  - Ex 1.6.2 Write the assertion
    Everybody loves somebody,
    symbolically, letting $L(x, y)$ be the statement "x loves y."
  - Sol) $\forall x \exists y\ L(x, y)$. In words, for every person x, there exists a person y such that x loves y.

1.6 다중 한정 기호nested quantifiers ($\forall x \, \forall y \, P(x,y)$ 진리값)

- $\forall x \forall y \, P(x,y)$ with domain of discourse $X \times Y$, is true if, for every $x \in X$ and for every $y \in Y$, $P(x,y)$ is true. $\forall x \forall y \, P(x,y)$ is false if there is at least one $x \in X$ and at least one $y \in Y$ such that $P(x,y)$ is false.

- Ex 1.6.3 Consider the statement

$$\forall x \forall y \big((x > 0) \wedge (y > 0) \rightarrow (x + y > 0) \big).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x and for every real number y, the conditional proposition

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

is true. In words, for every real number x and for every real number y, if x and y are positive, their sum is positive.

1.6 다중 한정 기호nested quantifiers ($\forall x\, \forall y\, P(x, y)$ 진리값)

- Ex 1.6.4 Consider the statement

$$\forall x \forall y \big((x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)\big).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is false because if $x = 1$ and $y = -1$, the conditional proposition $(x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)$ is false.
We say that the pair $x = 1$ and $y = -1$ is a counterexample.

- Ex 1.6.5 Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. The following pseudocode determines whether $\forall x \forall y\, P(x, y)$ is true or false

```
for i = 1 to n
    for j = 1 to n
        if (¬P(d_i, d_j))
            return false
return true
```

1.6 다중 한정 기호 ($\forall x \exists y\, P(x, y)$ 진리값)

- $\forall x \exists y\, P(x, y)$ with domain of discourse $X \times Y$, is true if, for every $x \in X$, there is at least one $y \in Y$ for which $P(x, y)$ is true. $\forall x \exists y\, P(x, y)$ is false if there is at least one $x \in X$ such that $P(x, y)$ is false for every $y \in Y$.
- Ex 1.6.6 Consider the statement $\forall x \exists y (x + y = 0)$. The domain of discourse is $\mathbf{R} \times \mathbf{R}$.
  This statement is true because, for every real number x, there is at least one y (namely $y = -x$) for which $x + y = 0$ is true.
- Ex 1.6.7 Consider the statement $\forall x \exists y (x > y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$.
  This statement is false because there is at least one x, namely $x = 1$, such that $x > y$ is false for every positive integer y.

1.6 다중 한정 기호 ($\forall x \exists y\, P(x, y)$ 진리값)

- Ex 1.6.8 Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$.
- The following pseudocode determines whether $\forall x \exists y\, P(x, y)$ is true or false

```
for i = 1 to n
   if (¬exists_dj(i))
      return false
return true
exist_dj(i) {
   for j = 1 to n
      if (P(d_i, d_j))
         return true
   return false
}
```

1.6 다중 한정 기호 ($\exists x \forall y\, P(x,y)$ 진리값)

- $\exists x \forall y\, P(x,y)$, with domain of discourse $X \times Y$, is true if there is at least one $x \in X$ such that $P(x,y)$ is true for every $y \in Y$.
  $\exists x \forall y\, P(x,y)$ is false if, for every $x \in X$, there is at least one $y \in Y$ such that $P(x,y)$ is false.
- Ex 1.6.9 Consider the statement $\exists x \forall y (x \leq y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$.
  This statement is true because there is at least one positive integer x (namely $x = 1$) for which $x \leq y$ is true for every positive integer y.
  In words, there is a smallest positive integer (namely 1).
- Ex 1.6.10 Consider the statement $\exists x \forall y (x \geq y)$ with $\mathbf{Z}^+ \times \mathbf{Z}^+$.
  This statement is false because, for every positive integer x, there is at least one positive integer y, namely $y = x + 1$, such that $x \geq y$ is false.
  In words, there is no greatest positive integer.

1.6 다중 한정 기호 ($\exists x \exists y\, P(x, y)$ 진리값)

- $\exists x \exists y\, P(x, y)$, with domain of discourse $X \times Y$, is true if there is at least one $x \in X$ and at least one $y \in Y$ such that $P(x, y)$ is true.
  $\exists x \exists y\, P(x, y)$ is false if, for every $x \in X$ and for every $y \in Y$, $P(x, y)$ is false.
- Ex 1.6.11 Consider $\exists x \exists y((x > 1) \wedge (y > 1) \wedge (xy = 6))$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$.
  This statement is true because there is at least one integer $x > 1$ (namely $x = 2$) and at least one integer $y > 1$ (namely $y = 3$) such that $xy = 6$.
- Ex 1.6.12 Consider $\exists x \exists y((x > 1) \wedge (y > 1) \wedge (xy = 7))$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$.
  This statement is false because
  for every positive integer x and for every positive integer y, $(x > 1) \wedge (y > 1) \wedge (xy = 7)$ is false.

1.6 다중 한정 기호 (드모르간 법칙)

- Ex 1.6.13
  $\neg(\forall x \exists y\, P(x,y)) \equiv \exists x \neg(\exists y\, P(x,y)) \equiv \exists x \forall y\, \neg P(x,y)$.
- Ex 1.6.14
  Write the negation of $\exists x \forall y(xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

- Sol) Using the generalized De Morgan's laws for logic,
  $\neg(\exists x \forall y(xy < 1)) \equiv \forall x \neg(\forall y(xy < 1)) \equiv \forall x \exists y \neg(xy < 1) \equiv \forall x \exists y(xy \geq 1)$.
  The given statement $\exists x \forall y(xy < 1)$ is true because there is at least one x (namely $x = 0$) such that $xy < 1$ for every y.
  Since the given statement is true, its negation is false.