제 3 장 Functions, Sequences, and Relations

3.1 Functions

3.2 Sequences and Strings

3.3 Relations

3.4 Equivalence Relations

3.5 Matrices of Relations

3.6 Relational Databases

3.1 Functions

- 4690358213754657 is a hypothetical credit card number

The first digit 4 shows that the card would be a Visa card. The last digit is a check digit that computed from the preceding digits

The check digit : Starting from the right and skipping the check digit, double every other number.

4	6	9	0	3	5	8	2	1	3	7	5	4	6	5	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Double every other digit.
8	6	18	0	6	5	16	2	2	3	14	5	8	6	10	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Add digits of two-digit numbers.
8	6	9	0	6	5	7	2	2	3	5	5	8	6	1	

Sum the resulting digits

$$8+6+9+0+6+5+7+2+2+3+5+5+8+6+1=73.$$

Luhn algorithm : the last digit of the sum is 0, the check digit is 0. Otherwise, subtract the last digit of the sum from 10 to get the check digit,10− 3 = 7

3.1 Functions

if 1 is changed to 7, the Luhn algorithm calculation becomes

$$\begin{array}{ccccccccccccccccl}
4 & 6 & 9 & 0 & 3 & 5 & 8 & 2 & 7 & 3 & 7 & 5 & 4 & 6 & 5 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{Double every other digit} \\
8 & 6 & 18 & 0 & 6 & 5 & 16 & 2 & 14 & 3 & 14 & 5 & 8 & 6 & 10 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{Add digits of two-digit numbers.} \\
8 & 6 & 9 & 0 & 6 & 5 & 7 & 2 & 5 & 3 & 5 & 5 & 8 & 6 & 1 &
\end{array}$$

and the sum becomes

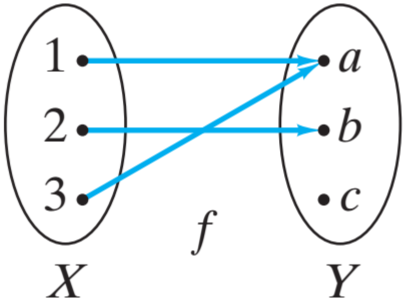
$$8 + 6 + 9 + 0 + 6 + 5 + 7 + 2 + 5 + 3 + 5 + 5 + 8 + 6 + 1 = 76.$$

Therefore the check digit changes to 4. Thus, if 1 is inadvertently transcribed as 7, the error will be detected. The Luhn algorithm gives an example of a function. A function assigns to each member of a set X exactly one member of a set Y.

The integer 469035821375465 is assigned the value 7, and the integer 469035827375465 is assigned the value 4.

3.1 Functions

- Definition 3.1.1 Let X and Y be sets.
  A **function** f from X to Y, denoted $f\colon X \to Y$, is a subset of the Cartesian product $X \times Y = \{(x, y) | x \in X, y \in Y\}$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$, denoted $f(x) = y$.
- The set X is called the **domain**정의역 of f, and
  the set Y is called the **codomain**공역 of f.
  The set $\{y \mid (x, y) \in f\}$ is called the **range**치역 of f.
- 예제 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.
  The domain of f is X,
  the codomain of f is Y, and
  the range of f is $\{a, b\}$.
  We may write $f(1) = a$, $f(2) = b$, and $f(3) = a$.



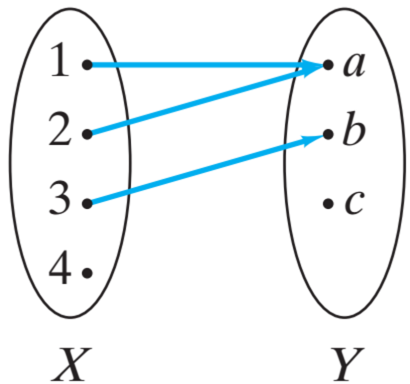
화살표 다이어그램

3.1 Functions

- 예제 3.1.4 The set

$$\{(1, a), (2, a), (3, b)\}$$

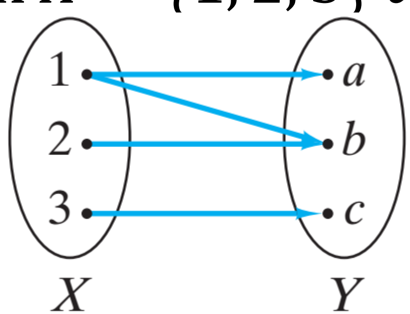
is not a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$.



- 예제 3.1.5 The set

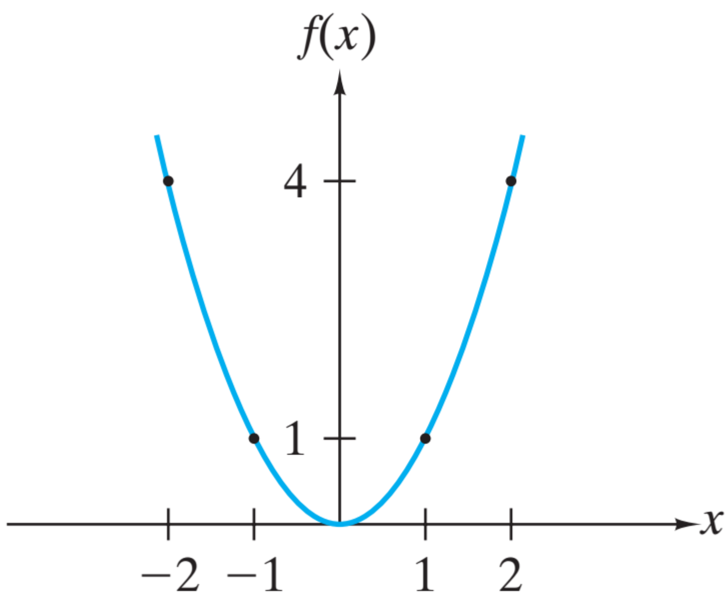
$$\{(1, a), (2, b), (3, c), (1, b)\}$$

is not a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.



3.1 Functions (Graph of Functions)

- The graph of a function f whose domain and codomain are subsets of the real numbers is obtained by plotting points in the plane that correspond to the elements in f.
- The domain is contained in the horizontal axis and the codomain is contained in the vertical axis.
- 예제 3.1.8 The graph of the function $f(x) = x^2$ is shown:

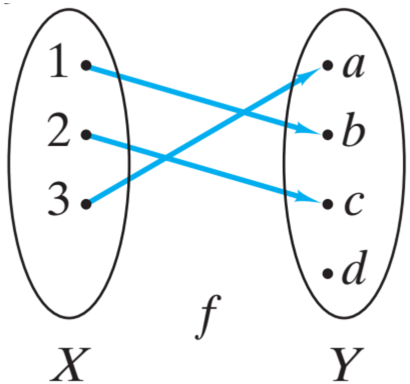


3.1 Functions (modulus operator, $\lfloor x \rfloor$, $\lceil x \rceil$)

- Definition 3.1.11 If x is an integer and y is a positive integer, we define x mod y to be the remainder when x is divided by y.
- 예제 3.1.12 $6 \bmod 2 = 0$, $5 \bmod 1 = 0$, $8 \bmod 12 = 8$, $199673 \bmod 2 = 1$.
- 예제 3.1.14 수요일에서 365일 후는 무슨 요일인가?
  수요일에서 7일 후는 수요일, 8일 후는 목요일…
  365 mod 7 = 1 이므로 목요일이다.
- Definition 3.1.17 The **floor** of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The **ceiling** of x, denoted $\lceil x \rceil$ is the least integer greater than or equal to x.
- 예제 3.1.18 $\lfloor 8.3 \rfloor = 8$, $\lfloor -8.7 \rfloor = -9$
  $\lceil 9.1 \rceil = 10$, $\lceil -11.3 \rceil = -11$, $\lceil 6 \rceil = 6$, $\lceil -8 \rceil = -8$

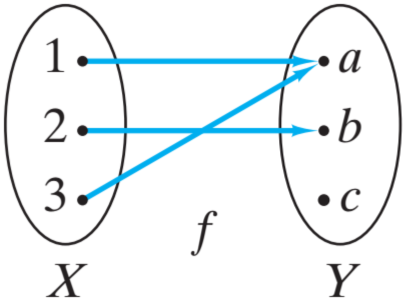
3.1 Functions (one-to-one)

- Definition 3.1.22 A function f from X to Y is said to be **one-to-one** (or **injective**) if
  for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$
- 예제 3.1.23 The function $f = \{(1, b), (3, a), (2, c)\}$ from
  $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is one-to-one.



- Each element in the codomain will have at most one arrow pointing to it.
- 예제 3.1.24

  The function $f = \{(1, a), (2, b), (3, a)\}$ is not one-to-one since $f(1) = a = f(3)$.



3.1 Functions (one-to-one)

- 예제 3.1.27 Prove that the function $f(n) = 2n + 1$ from the set of positive integers to the set of positive integers is one-to-one.
- Proof) We must show that for all positive integers n_1 and n_2,
  if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

  Suppose that $f(n_1) = f(n_2)$.
  Using the definition of f

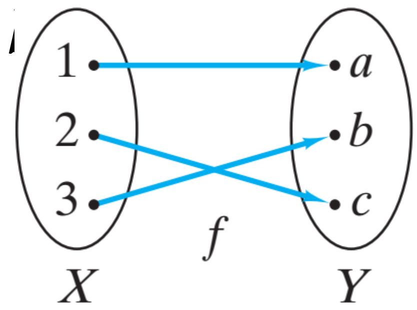
$$2n_1 + 1 = 2n_2 + 1$$

  Thus $n_1 = n_2$.
  Therefore, f is one-to-one.

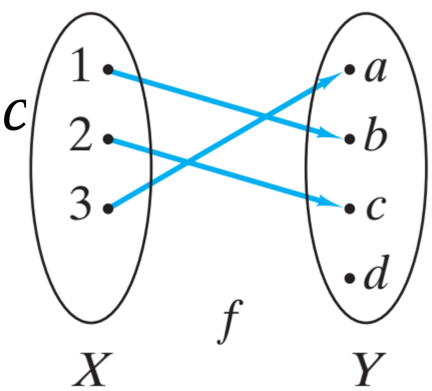
3.1 Functions (onto Y)

- **Definition 3.1.29** A function f from X to Y is said to be **onto** Y (or **surjective**) if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- 예제 3.1.30 The function $f = \{(1, a), (2, c), (3,$
  from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is onto Y

  and one-to-one .



- Each element in the codomain will have at least one arrow pointing to it.
- 예제 3.1.31 The function $f = \{(1, b), (3, a), (2, c$
  from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is not onto Y.



3.1 Functions (onto Y)

- 예제 3.1.33 Prove that the function $f(x) = 1/x^2$ from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.

- Proof) We must show that for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

  For every $y \in Y$, there exists $x = 1/\sqrt{y}$ such that

$$f(x) = f(1/\sqrt{y}) = \frac{1}{\left(1/\sqrt{y}\right)^2} = y$$

  Therefore, f is onto Y.

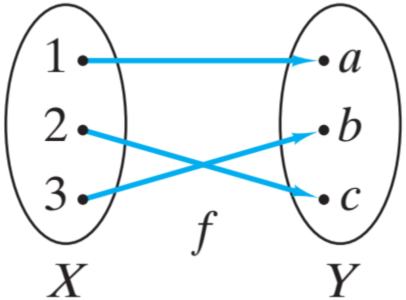
3.1 Functions (onto Y)

- 예제 3.1.34 Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is *not* onto Y.

- Proof) We must find an element $m \in Y$ such that for all $n \in X$, $f(n) \neq m$.
  Since $f(n)$ is an odd integer for all n, we may choose for y any positive, even integer, for example, $y = 2$. Then $y \in Y$ and $f(n) \neq y$ for all $n \in X$. Thus f is not onto Y.

3.1 Functions (bijection)

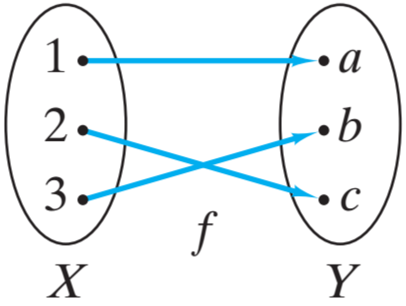
- Definition 3.1.35 A function that is both one-to-one and onto is called a **bijection**.
- 예제 3.1.36 The function $f = \{(1, a), (2, c), (3, b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is a bijection.

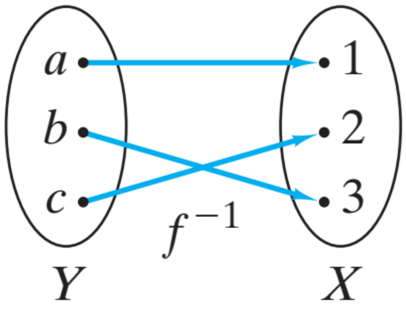


- If f is a bijection from a finite set X to a finite set Y, then $|X| = |Y|$, that is, the sets have the same cardinality.
- 예제 3.1.37 $f = \{(1, a), (2, b), (3, c), (4, d)\}$ is a bijection from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. Both sets have four elements.

3.1 Functions (inverse)

- Suppose that f is a one-to-one, onto function from X to Y. It can be shown that $\{(y,x) \mid (x,y) \in f\}$ is a one-to-one, onto function from Y to X. This new function, denoted f^{-1}, is called f **inverse**.
- 예제 3.1.38 For the function $f = \{(1,a),(2,c),(3,b)\}$, we have $f^{-1} = \{(a,1),(c,2),(b,3)\}$.
- 예제 3.1.39 Given the arrow diagram for a one-to-one, onto function f from X to Y, we can obtain the arrow diagram for f^{-1} simply by reversing the direction of each arrow.



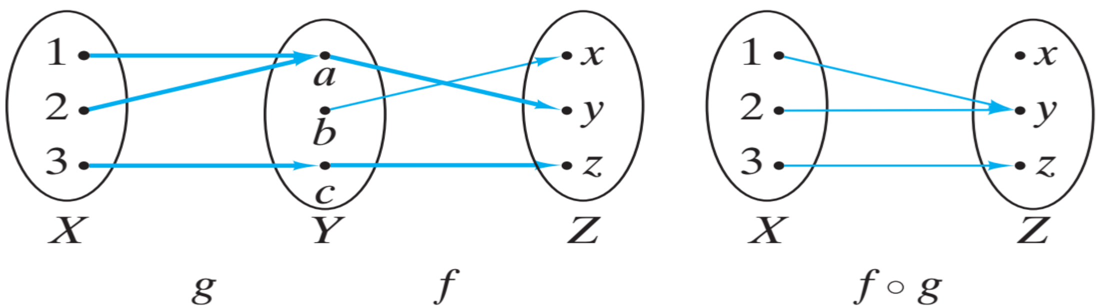


3.1 Functions (composition)

- Definition 3.1.41 Let g be a function from X to Y and let f be a function from Y to Z.
  The **composition** of f with g, denoted $f \circ g$, is the function
  $$(f \circ g)(x) = f\big(g(x)\big)$$
  from X to Z.

- 예제 3.1.42-43 Given $g = \{(1,a),(2,a),(3,c)\}$, a function from X to Y, and $f = \{(a,y),(b,x),(c,z)\}$, a function from Y to Z.
  $f \circ g = \{(1,y),(2,y),(3,z)\}$ is a function from X to Z.
  Here $X = \{1,2,3\}$, $Y = \{a,b,c\}$, $Z = \{x,y,z\}$.

- "following the arrows".



3.1 Functions (binary operator, unary operator)

- Definition 3.1.47 A function from $X \times X$ to X is called a **binary operator** on a set X.
- 예 Let $X = \{1, 2, \dots\}$. If we define $f(x, y) = x + y$, where $x, y \in X$, then f is a binary operator on X.
- Definition 3.1.50 A function from X to X is called a **unary operator** on a set X.
- 예제 3.1.51 Let U be a universal set. If we define

$$f(X) = \overline{X}$$

where $X \in \mathcal{P}(U)$, then f is a unary operator on $\mathcal{P}(U)$.

3.2 Sequences and Strings

- Definition 3.2.1 A **sequence** (수열) s is a function whose domain D is a subset of integers.
  The notation s_n is used instead of $s(n)$.
  The term n is called the **index** of the sequence.
  If D is a finite set, we call s a finite sequence;
  otherwise, s is an infinite sequence.

- A sequence s is denoted s or $\{s_n\}$ if n is the index of the sequence. We use the notation s_n to denote the *single* element of the sequence s at index n.

- If the domain is the set of positive integers $\mathbf{Z}^+$,
  s or $\{s_n\}$ denotes the entire sequence $s_1, s_2, s_3, \ldots$.

3.2 Sequences and Strings

- 예제 3.2.2 Consider the sequence s: $2, 4, 6, \ldots, 2n, \ldots$.
  The first element of the sequence is 2,
  the second element of the sequence is 4, and so on.
  The nth element of the sequence is $2n$.
  If the domain of s is $\mathbf{Z}^+$, we have
  $s_1 = 2, s_2 = 4, \ldots, s_n = 2n, \ldots$.
  The sequence s is an infinite sequence.

- 예제 3.2.3 Consider the sequence t: a, a, b, a, b.
  The first element of the sequence is a,
  the second element of the sequence is a, and so on.
  If the domain of t is $\{1, 2, 3, 4, 5\}$, we have
  $t_1 = a,\ t_2 = a,\ t_3 = b,\ t_4 = a$, and $t_5 = b$.
  The sequence t is a finite sequence.

3.2 Sequences and Strings ($\{s_n\}_{n=0}^{\infty}, \{s_n\}_{n=i}^{j}$)

- If the domain of a sequence s is the infinite set of consecutive integers $\{k,\ k+1,\ k+2, \dots\}$ and the index of s is n, we can denote the sequence s as $\{s_n\}_{n=k}^{\infty}$.
- A sequence s whose domain is $\mathbf{Z}^{nonneg}$ can be denoted $\{s_n\}_{n=0}^{\infty}$.
- If the domain of a sequence s is the finite set of consecutive integers $\{i, i+1, \dots, j\}$, we can denote the sequence s as $\{s_n\}_{n=i}^{j}$.
- 예제 3.2.4 The sequence $\{u_n\}$ defined by the rule $u_n = n^2 - 1$, for all $n \geq 0$, can be denoted $\{u_n\}_{n=0}^{\infty}$. The sequence u can also be denoted $\{u_m\}_{m=0}^{\infty}$. The formula for the term having index m is $u_m = m^2 - 1$, for all $m \geq 0$.

3.2 Sequences and Strings

- A sequence s is **increasing** if for all i and j in the domain of s, if $i < j$, then $s_i < s_j$.
- A sequence s is **decreasing** if for all i and j in the domain of s, if $i < j$, then $s_i > s_j$.
- A sequence s is **nondecreasing** if for all i and j in the domain of s, if $i < j$, then $s_i \leq s_j$.
- A sequence s is **nonincreasing** if for all i and j in the domain of s, if $i < j$, then $s_i \geq s_j$.
- 예제 3.2.8 The sequence 2, 5, 13, 104, 300 is increasing and nondecreasing.
- 예제 3.2.10 The sequence 100, 90, 90, 74, 74, 74, 30 is nonincreasing, but it is not decreasing.
- 예제 3.2.11 The sequence 100 (consisting of a single element) is increasing, decreasing, nonincreasing, and nondecreasing.

3.2 Sequences and Strings (subsequence)

- Definition 3.2.12 Let s be a sequence.
  A **subsequence**(부분 수열) of s is a sequence obtained from s by choosing certain terms of s in the same order in which they appear in s.

- 예제 3.2.13 The sequence b, c is a subsequence of the sequence a, a, b, c, q. (choosing the third and fourth terms).
  Notice that the sequence c, b is not a subsequence of the sequence since the order of terms is not maintained.

- 예제 3.2.14 The sequence

$$2, 4, 8, 16, \dots, 2^k, \dots$$

  is a subsequence of the sequence

$$2, 4, 6, 8, 10, 12, 14, 16, \dots, 2n, \dots.$$

3.2 Sequences and Strings (numerical sequence)

- Definition 3.2.17 If $\{a_i\}_{i=m}^{n}$ is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n, \qquad \prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot \cdots \cdot a_n$$

- i is called the **index**, m is called the lower limit하한, and n is called the upper limit상한.
- $\sum_{i=m}^{n} a_i$, $\prod_{i=m}^{n} a_i$
- If S is a finite set of integers and a is a sequence,

$$\sum_{i\in S} a_i \quad \text{and} \quad \prod_{i\in S} a_i$$

Where $\{a_i \mid i \in S\}$, respectively.

- 예제 3.2.22 If S is a set of prime numbers less than 20

$$\sum_{i\in S} \frac{1}{i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} = 1.455$$

3.2 Sequences and Strings

string: α, β
$a, b, c \in X$

- Definition 3.2.23 A **string**(문자열) over X, where X is a finite set, is a finite sequence of elements from X.
- 예제 3.2.24 Let $X = \{a, b, c\}$. $\beta_1 = b, \beta_2 = a, \beta_3 = a, \beta_4 = c$ is a string over X. This string is written $baac$.
- Since a string is a sequence, order is taken into account. e.g., the string $baac$ is different from the string $acab$.
- Repetitions in a string can be specified by superscripts. e.g., the string $bbaaac$ may be written b^2a^3c.
- The string with no elements is called the **null string** and is denoted λ.
- Let X^* denote the set of all strings over X, including the null string, and Let X^+ denote the set of all nonnull strings over X. e
  예제 3.2.25 Let $X = \{a, b\}$. Some elements in X^* are λ, $a, b, abab$, and $b^{20}a^5ba$.

3.2 Sequences and Strings

- The **length** of a string α, denoted $|\alpha|$, is the number of elements in α. 예제 3.2.26 $\alpha = aabab, |\alpha| = 5, \beta = a^3b^4a^{32}$ $|\beta| = 39$
- If α and β are two strings, the string consisting of α followed by β, written $\alpha\beta$, is called the **concatenation** of α and β. e.g.,

  예제 3.2.27 $\gamma = aab$ and $\theta = cabd$, then $\gamma\theta = aabcabd, \theta\gamma = cabdaab, \gamma\lambda = \gamma = aab, \ \ \lambda\gamma = \gamma = aab$.
- 예제 3.2.28 Let $X = \{a, b, c\}$. If we define $f(\alpha, \beta) = \alpha\beta$, where α and β are strings over X, then f is a binary operator on X^*.
- Definition 3.2.29 A string β is a **substring** of the string α if there are strings γ and δ with $\alpha = \gamma\beta\delta$.
- 예제 3.2.30 The string $\beta = add$ is a substring of the string $\alpha = aaaddad$ since, if we take $\gamma = aa$ and $\delta = ad$, we have $\alpha = \gamma\beta\delta$.

3.3 Relations

- A **relation** from one set to another can be thought of as a table that lists which elements of the first set relate to which elements of the second set.
  - The Table shows which students are taking which courses.

Relation of Students to Courses

Student	*Course*
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

3.3 Relations

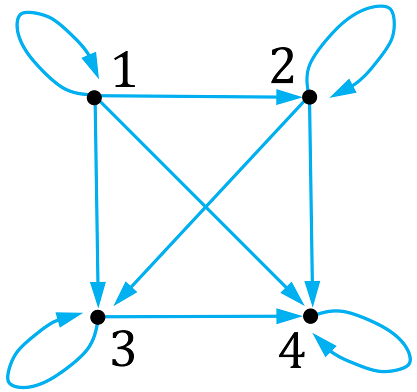
- Definition 3.3.1 A (binary) **relation** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$.
  If $(x, y) \in R$, we write $x\ R\ y$ and say that x is related to y.
  If $X = Y$, we call R a (binary) relation on X.

- 예제 3.3.2 If we let $X = \{Bill, Mary, Beth, Dave\}$ and $Y = \{CompSci, Math, Art, History\}$, our relation R of the previous Table can be written
  $$R = \{(Bill, CompSci), (Mary, Math), (Bill, Art), (Beth, History), (Beth, CompSci), (Dave, Math)\}.$$
  Since $(Beth, History) \in R$, we may write $Beth\ R\ History$.

- 예제 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$.
  If we define a relation R from X to Y by
  $$(x, y) \in R \text{ if } x \text{ divides } y,$$
  we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

3.3 Relations

- 예제 3.3.4 Let R be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y,\ \ x, y \in X$. Then
  $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
- To draw the **digraph** of a relation on a set X,
  we first draw vertices정점 to represent the elements of X.
  Next, if the element (x, y) is in the relation, we draw an arrow (called a **directed edge**) from x to y.
- A directed edge from x to x is called a **loop**.



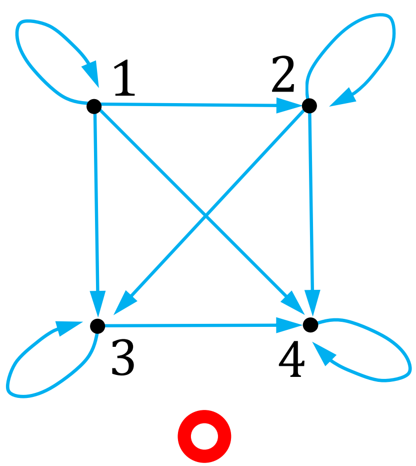
3.3 Relations (reflective)

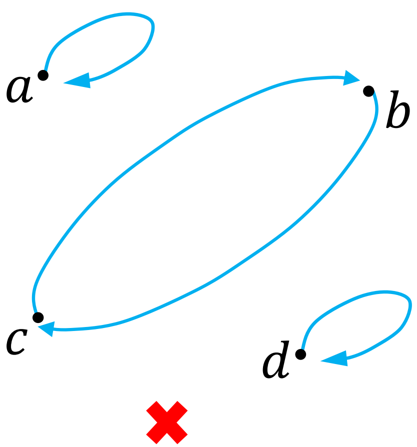
- **Definition 3.3.6** A relation R on a set X is **reflexive** if $(x, x) \in R$ for every $x \in X$.
- 예제 3.3.7 The relation R on $X = \{1, 2, 3, 4\}$ defined by

$$(x, y) \in R \text{ if } x \leq y \qquad x, y \in X$$

is reflexive; specifically, $(1, 1), (2, 2), (3, 3)$, and $(4, 4)$ are in R.

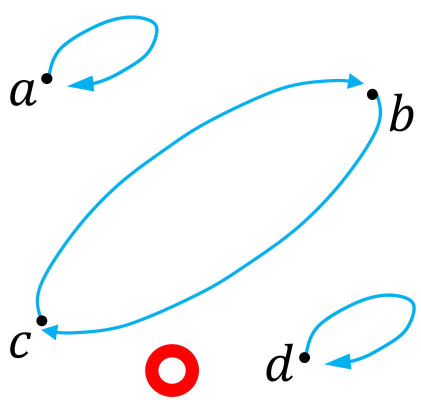
- The digraph: a loop at every vertex.

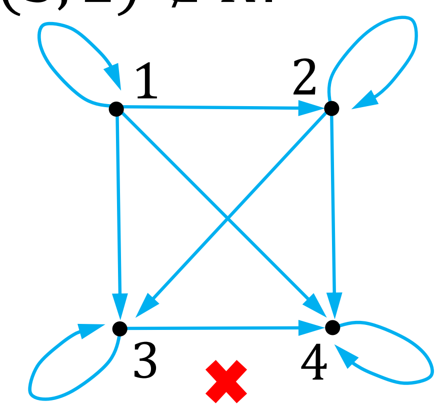




3.3 Relations (symmetric)

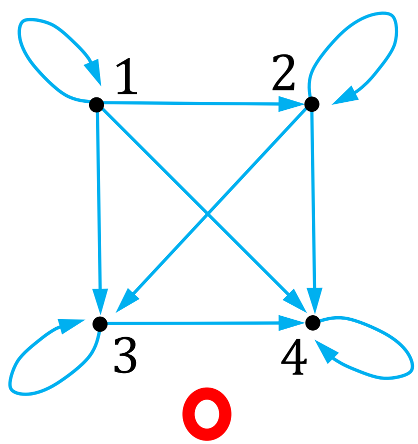
- Definition 3.3.9 A relation R on a set X is **symmetric** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.
- 예제 3.3.10 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is symmetric. e.g., $(b, c) \in R$, $(c, b) \in R$.
- The digraph on symmetric : whenever there is a directed edge from v to w,there is also a directed edge from w to v.
- 예제 3.3.11 The relation R on $X = \{1, 2, 3, 4\}$ defined by
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
is not symmetric. e.g., $(2, 3) \in R$, but $(3, 2) \notin R$.

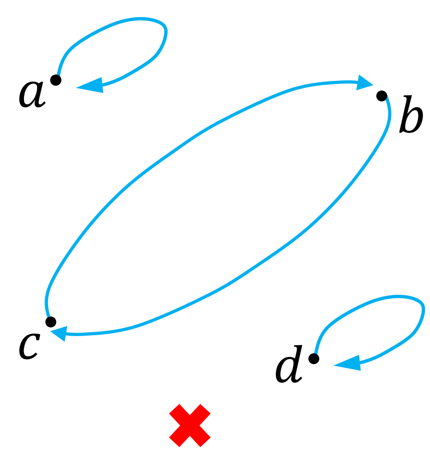




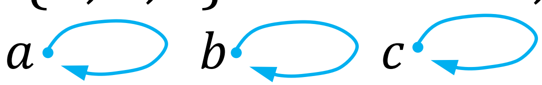
3.3 Relations (antisymmetric)

- **Definition 3.3.12** A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$
- The contrapositive: if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$
- 예제 3.3.13 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is antisymmetric because for all x,y if $(x, y) \in R$ (i.e., $x \leq y$) and $(y, x) \in R$ (i.e., $y \leq x$), then $x = y$.
- The digraph: between any two distinct vertices there is at most one directed edge.





3.3 Relations

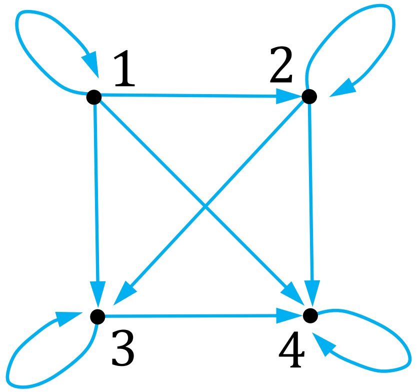
- Let R be a relation on a set X
- R is **reflexive:** $\forall x \in X, (x, x) \in R$
  - a loop at every vertex
- R is **symmetric:** $\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R$.
  - whenever there is a directed edge from v to w, there is also a directed edge from w to v
- R is **antisymmetric:** $\forall x, y \in X, (x, y) \in R \land (y, x) \in R \rightarrow x = y$
  $\forall x, y \in X, x \neq y \rightarrow (x, y) \notin R \lor (y, x) \notin R$
  - between any two distinct vertices there is at most one directed edge
- The relation $R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$ is reflexive, symmetric, antisymmetric.
  

  "antisymmetric" is not the same as "not symmetric"
- The relation $R = \{(a, a), (a, b), (b, a)\}$ on $X = \{a, b\}$ is symmetric, but is not antisymmetric, reflexive.

3.3 Relations (transitive)

- Definition 3.3.17 A relation R on a set X is **transitive** if for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
- 예제 3.3.18 The relation
$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
on $X = \{1,2,3,4\}$ is transitive. Note that if $x = y$ or $y = z$, we need not explicitly verify that the condition.

(x, y)	(y, z)	(x, z)
(1, 2)	(2, 3)	(1, 3)
(1, 2)	(2, 4)	(1, 4)
(1, 3)	(3, 4)	(1, 4)
(2, 3)	(3, 4)	(2, 4)



- The digraph:
whenever there are directed edges from x to y and from y to z, there is also a directed edge from x to z.

3.3 Relations (partial order)

- Relations can be used to order elements of a set. e.g., the relation R defined on the set of integers by
$$(x, y) \in R \text{ if } x \leq y$$
orders the integers. Notice that the relation R is reflexive, antisymmetric, and transitive.

- Definition 3.3.20 A relation R on a set X is a **partial order** if R is reflexive, antisymmetric, and transitive.

- If R is a partial order on a set X, the notation $x \preceq y$ is sometimes used to indicate that $x \; R \; y$, i.e., $(x, y) \in R$.

- Suppose that R is a partial order on a set X.
If $x, y \in X$ and either $x \preceq y$ or $y \preceq x$, we say that x and y are comparable비교 가능.
If every pair of elements in X is comparable, we call R a total order전순서.

3.3 Relations (inverse)

- Definition 3.3.23 Let R be a relation from X to Y.
  The **inverse** of R, denoted R^{-1}, is the relation from Y to X defined by
  $$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$
- 예제 3.3.24 If we define a relation R from $X = \{2, 3, 4\}$ to $Y = \{3, 4, 5, 6, 7\}$ by
  $$(x, y) \in R \text{ if } x \text{ divides } y$$
  we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.
  The inverse of this relation is
  $$R^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}$$

3.3 Relations (combination of relations)

- Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

  $$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

- 예제 3.3.26 The composition of the relations

  $$R_1 = \{(1,2), (1,6), (2,4), (3,4), (3,6), (3,8)\}$$

  and

  $$R_2 = \{(2,u), (4,s), (4,t), (6,t), (8,u)\}$$

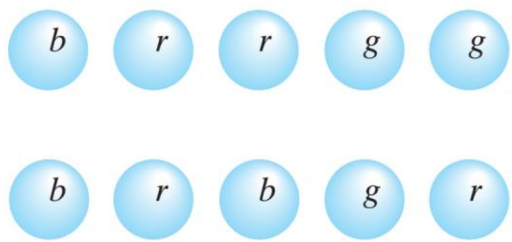
  is

  $$R_2 \circ R_1 = \{(1,u), (1,t), (2,s), (2,t), (3,s), (3,t), (3,u)\}.$$

  e.g., $(1,u) \in R_2 \circ R_1$ because $(1,2) \in R_1$ and $(2,u) \in R_2$.

3.4 Equivalence relations

- Suppose that we have a set X of 10 balls, each of which is either red, blue, or green. If we divide the balls into sets R, B, and G according to color, the family $\{R, B, G\}$ is a partition of X.
- A collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to *exactly one member* of $\mathcal{S}$.



A set of colored balls

- A partition can be used to define a relation.
  If $\mathcal{S}$ is a partition of X, we may define $x\ R\ y$ to mean that for some set $S \in \mathcal{S}$, both x and y belong to S.
- For the example of above Figure, the relation obtained could be described as "is the same color as."

3.4 Equivalence relations

- Theorem 3.4.1 Let $\mathcal{S}$ be a partition of a set X.
  Define $x\ R\ y$ to mean that for some set S in $\mathcal{S}$, $x \in S$ and $y \in S$.
  $$R = \{(x, y) \mid S \in \mathcal{S}, x \in S, y \in S\}$$
  Then R is reflexive, symmetric, and transitive.

- Proof) Let $x \in X$. By the definition of partition, x belongs to some member S of $\mathcal{S}$. Thus $x\ R\ x$ and R is reflexive.

  Suppose that $x\ R\ y$. Then both x and y belong to some set $S \in \mathcal{S}$. Since both y and x belong to S, $y\ R\ x$ and R is symmetric.

  Finally, suppose that $x\ R\ y$ and $y\ R\ z$. Then both x and y belong to some set $S \in \mathcal{S}$ and both y and z belong to some set $T \in \mathcal{S}$.
  Since y belongs to exactly one member of $\mathcal{S}$, $S = T$.
  Therefore, both x and z belong to S and $x\ R\ z$.
  We have shown that R is transitive.

3.4 Equivalence relations

- 예제 3.4.2 Consider the partition $\mathcal{S} = \{\{1,3,5\},\{2,6\},\{4\}\}$ of $X = \{1,2,3,4,5,6\}$.
  Define $x \; R \; y$ to mean that for some set S in $\mathcal{S}$, both x and y belong to S.

- $R = \{(x,y) \mid S \in \mathcal{S}, x \in S, y \in S\}$
  $R = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),$
  $(2,2),(2,6),(6,2),(6,6),(4,4)\}.$

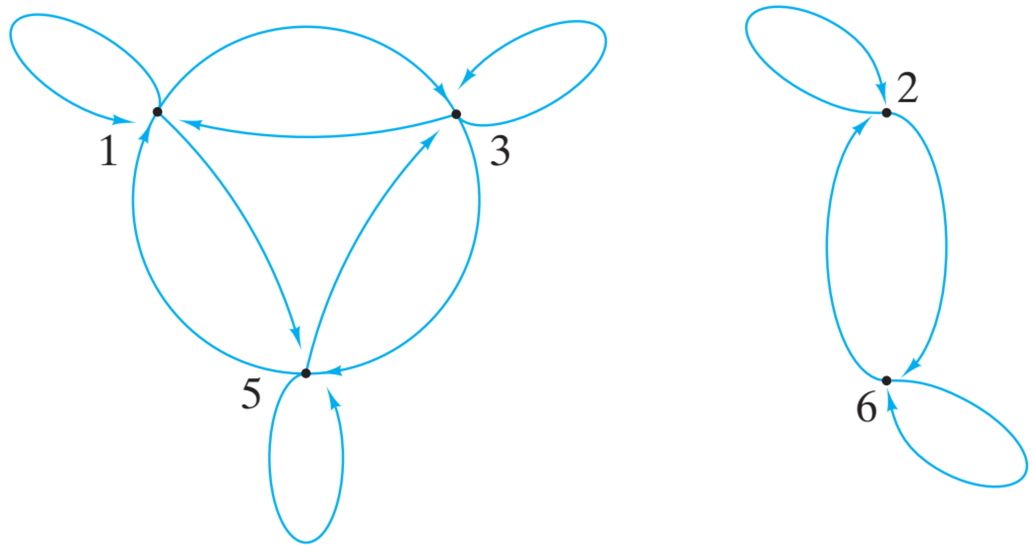
- R is reflexive, symmetric, and transitive.

3.4 Equivalence relations

- Let $\mathcal{S}$ and R be as in Theorem 3.4.1. If $S \in \mathcal{S}$, we can regard the members of S as equivalent in the sense of the relation R, which motivates calling relations that are reflexive, symmetric, and transitive equivalence relations.
- In the example of Figure of a set of colored balls,
  the relation is “is the same color as”;
  hence equivalent means “is the same color as.”
  Each set in the partition consists of all the balls of a particular color.
- Definition 3.4.3
  A relation that is reflexive, symmetric, and transitive on a set X is called an **equivalence relation**(동치 관계) on X.
- Any partition $\mathcal{S}$ has a corresponding equivalence relation R.

$$R = \{(x, y) \mid S \in \mathcal{S}, x \in S, y \in S\}$$

3.4 Equivalence relations

- 예제 3.4.4 Show that the relation of Ex 3.4.2 $R = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,2),(2,6),(6,2),(6,6),(4,4)\}$ is an equivalence relation on $\{1, 2, 3, 4, 5, 6\}$.
- 

- We see that R is reflexive (there is a loop at every vertex), symmetric (for every directed edge from v to w, there is also a directed edge from w to v), and
transitive (if there is a directed edge from x to y and a directed edge from y to z, there is a directed edge from x to z).

3.4 Equivalence relations

- 예제 3.4.5 Consider the relation $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$ on $\{1, 2, 3, 4, 5\}$.
  The relation is reflexive because $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \in R$.
  The relation is symmetric because whenever (x, y) is in R, (y, x) is also in R.
  The relation is transitive because whenever (x, y) and (y, z) are in R, (x, z) is also in R.
  Since R is reflexive, symmetric, and transitive, R is an equivalence relation on $\{1, 2, 3, 4, 5\}$.

3.4 Equivalence relations

- Given an equivalence relation on a set X, we can partition X by grouping related members of X. Elements related to one another may be thought of as equivalent.

- Theorem 3.4.8 Let R be an equivalence relation on a set X. For each $a \in X$, let $[a] = \{x \in X \mid x\ R\ a\}$. Then

$$\mathcal{S} = \{[a] \mid a \in X\}$$

is a partition of X. $[a]$ is the set of all elements in X that are related to a.

- Proof) We must show that every element in X belongs to exactly one member of $\mathcal{S}$.

Let $a \in X$. Since $a\ R\ a$, $a \in [a]$. Thus every element in X belongs to at least one member of $\mathcal{S}$.

3.4 Equivalence relations

- Lemma: For all $c, d \in X$, if $c\ R\ d$, then $[c] = [d]$.
  Suppose that $c\ R\ d$. Let $x \in [c]$. Then $x\ R\ c$.
  Since $c\ R\ d$ and R is transitive, $x\ R\ d$.
  Therefore, $x \in [d]$ and $[c] \subseteq [d]$.
  Similarly, $[d] \subseteq [c]$. Thus $[c] = [d]$.
- "every element in X belongs to *exactly one* member of $\mathcal{S}$"
  Show that if $x \in X$ and $x \in [a]$ and $x \in [b]$, then $[a] = [b]$.
  Assume $x \in X$ and $x \in [a]$ and $x \in [b]$. Then $x\ R\ a$ and $x\ R\ b$.
  By above Lemma, $[x] = [a]$ and $[x] = [b]$. Thus $[a] = [b]$.

- Definition 3.4.9 Let R be an equivalence relation on a set X.
  The sets $[a]$ defined in Theorem 3.4.8 are called the **equivalence classes** of X given by the relation R

3.4 Equivalence relations

- 예제 3.4.10 The relation $R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,6), (6,2), (6,6), (4,4)\}$ on $X = \{1, 2, 3, 4, 5, 6\}$ is an equivalence relation.
- $[a] = \{x \in X \mid x\ R\ a\}$
- The equivalence class $[1]$ containing 1 consists of all x such that $(x, 1) \in R$. Therefore, $[1] = \{1, 3, 5\}$. Similarly, $[3] = [5] = \{1, 3, 5\}$, $[2] = [6] = \{2, 6\}$, $[4] = \{4\}$.
- 예제 3.4.13 The relation $R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$ is reflexive, symmetric, and transitive.
  Thus R is an equivalence relation.
  The equivalence classes are $[a] = \{a\}$, $[b] = \{b\}$, $[c] = \{c\}$.

3.4 Equivalence relations

- 예 3.3.14 Let $X = \{1, 2, \dots, 10\}$. Define $x\ R\ y$ to mean that 3 divides $x\ -\ y$. Then R is an equivalence relation on X.

$R = \{(1,1), (1,4), (1,7), (1,10), (2,2), (2,5), (2,8), (3,3), (3,6), \dots\}$

Determine the members of the equivalence classes.

- Sol) The equivalence class $[1]$ consists of all x with $x\ R\ 1$.
  Thus $[1] = \{x \in X \mid 3 \text{ divides } x - 1\} = \{1, 4, 7, 10\}$.
  Similarly, $[2] = \{2, 5, 8\}$ and $[3] = \{3, 6, 9\}$.
  These three sets partition X. Note that $[1] = [4] = [7] = [10]$,
  $[2] = [5] = [8]$, $[3] = [6] = [9]$.

For this relation, equivalence is "has the same remainder when divided by 3."

3.4 Equivalence relations

- 예 3.4.15 Show that if a relation R on a set X is symmetric and transitive but *not* reflexive, the collection of sets $[a]$, $a \in X$, defined in Theorem 3.4.8 does not partition X.

- Proof) We define "pseudo equivalence classes"

$$[a] = \{x \in X \mid x\ R\ a\}$$

Since R is not reflexive, there exists $b \in X$ such that $(b, b) \notin R$.

We show that b is not in pseudo equivalence classes.
Suppose, by way of contradiction, that $b \in [a]$ for some $a \in X$. Then $(b, a) \in R$. Since R is symmetric, $(a, b) \in R$.
Since R is transitive, $(b, b) \in R$. But we assumed that $(b, b) \notin R$. This contradiction shows that b is not in any pseudo equivalence class.
Thus the collection of pseudo equivalence classes does not partition X.

3.4 Equivalence relations

- Theorem 3.4.16
  Let R be an equivalence relation on a finite set X.
  If each equivalence class has r elements, there are $|X|/r$ equivalence classes.

- Proof) Let $X_1, X_2, \ldots, X_k$ denote the distinct equivalence classes.
  Since these sets partition X,
  $|X| = |X_1| + |X_2| + \cdots + |X_k| = r + r + \cdots + r = kr$
  and the conclusion follows.

3.5 Matrices of Relations

- **Matrix of the relation R**: can be used by a computer.
  - Label the rows with the elements of X in some arbitrary order, label the columns with the elements of Y in some arbitrary order.
  - Set the entry in row x and column y to 1 if $x\ R\ y$ and to 0 otherwise

- 예제 3.5.1-2 The matrices of the relation

$$R = \{(1,b),(1,d),(2,c),(3,c),(3,b),(4,a)\}$$

from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$:

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{c c} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

3.5 Matrices of Relations

- 예제 3.5.3 The matrix of the relation R from $\{2, 3, 4\}$ to $\{5, 6, 7, 8\}$, relative to the orderings $2, 3, 4$ and $5, 6, 7, 8$, defined by $x\ R\ y$ if x divides y
  is

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3.5 Matrices of Relations

- When we write the matrix of a relation R on a set X, we use the same ordering for the rows as we do for the columns.
  - The matrix of a relation on a set X is always a square matrix.
- 예제 3.5.4 The matrix of the relation

$$R = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$$

on $\{a,b,c,d\}$, relative to the ordering a,c,b,d, is

$$\begin{array}{cc} & \begin{array}{cccc} a & c & b & d \end{array} \\ \begin{array}{c} a \\ c \\ b \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3.5 Matrices of Relations

- Consider a relation R on a set X, and the matrix A of R (relative to some ordering).
- The relation R is reflexive if and only if A has 1's on the main diagonal. (The main diagonal of a square matrix consists of the entries on a line from the upper left to the lower right.)
  The R is reflexive if and only if $(x, x) \in R$ for all $x \in X$. But this last condition holds precisely when the main diagonal consists of 1's
- The relation R is symmetric if and only if matrix A is symmetric
  The reason is that R is symmetric if and only if whenever (x, y) is in R, (y, x) is also in R. But this last condition holds precisely when A is symmetric about the main diagonal.

3.5 Matrices of Relations

- 예제 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$, and let R_2 be the relation from Y to $Z = \{x, y, z\}$ defined by $R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.
- The matrix of R_1 and the matrix of R_2 are

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

- The product of these matrices is

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

3.5 Matrices of Relations

- The ikth entry in A_1A_2 is computed as

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ i & \begin{pmatrix} s & t \end{pmatrix} \end{array} \quad \begin{array}{c} k \\ \begin{pmatrix} u \\ v \end{pmatrix} \end{array} = su + tv$$

- If this value is nonzero, then either su or tv is nonzero. Suppose that $su \neq 0$. (The argument is similar if $tv \neq 0$.) Then $s \neq 0$ and $u \neq 0$. This means that $(i, a) \in R_1$ and $(a, k) \in R_2$. This implies that $(i, k) \in R_2 \circ R_1$.
  We have shown that if the ikth entry in A_1A_2 is nonzero, then $(i, k) \in R_2 \circ R_1$.

- Assume that $(i, k) \in R_2 \circ R_1$. Then, either 1. $(i, a) \in R_1$ and $(a, k) \in R_2$ or 2. $(i, b) \in R_1$ and $(b, k) \in R_2$.
  If 1 holds, then $s = 1$ and $u = 1$, so $su = 1$ and $su + tv$ is nonzero. Similarly, if 2 holds, $tv = 1$ and $su + tv$ is nonzero. We have shown that if $(i, k) \in R_2 \circ R_1$, then the ikth entry in A_1A_2 is nonzero.

3.5 Matrices of Relations

- We have shown that $(i, k) \in R_2 \circ R_1$ if and only if the ikth entry in A_1A_2 is nonzero; thus A_1A_2 is "almost" the matrix of the relation $R_2 \circ R_1$. To obtain the matrix of the relation $R_2 \circ R_1$, we need only change all nonzero entries in A_1A_2 to 1. Thus the matrix of the relation $R_2 \circ R_1$, relative to the previously chosen orderings 1, 2, 3 and x, y, z, is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & x \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

- Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_2 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product A_1A_2 by 1.

3.5 Matrices of Relations

- Let A be the matrix of a relation R (relative to some ordering).
- The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

Justification)
R is transitive if and only if
whenever (i, k) and (k, j) are in R, (i, j) is in R.
Entry i, j in A^2 is nonzero if and only if there are elements (i, k) and (k, j) in R.
(i, j) is in R if and only if entry i, j in A is nonzero.

Therefore, R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations

- 예제 3.5.7 The matrix of the relation $R = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$ on $\{a,b,c,d\}$, relative to the ordering a, b, c, d, is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Since whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero, R is transitive.

- 예제 3.5.8: R is *not* transitive

- $A = \begin{pmatrix} 1 & \mathbf{0} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & \mathbf{1} & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3.6 Relational databases

- *A binary* relation R is a relation among two sets X and Y, already defined as $R \subseteq X \times Y$.
- An *n-ary* relation R is a relation among n sets $X_1, X_2, \ldots, X_n$, i.e. a subset of the Cartesian product, $R \subseteq X_1 \times X_2 \times \ldots \times X_n$
- Thus, R is a set of n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_k \in X_k$, $1 \leq k \leq n$.
- A *database* is a collection of records that are manipulated by a computer. They can be considered as n sets X_1 through X_n, each of which contains a list of items with information.
- *Database management systems* are programs that help access and manipulate information stored in databases.

3.6 Relational databases

- Relational database model

  Columns of an *n*-ary relation are called *attributes*

  An attribute is a *key* if no two entries have the same value

  - e.g. social security number

  A *query* is a request for information from the database

- 예 3.6.3 The selection operator chooses certain n-tuples from a relation. The relation PLAYER given in Table 3.6.1, PLAYER [Position=c] will select the tuples

  (23012, Johnsonbrough, c, 22) (84341, Cage, c, 30)

- 예 3.6.4 The projection operator choose column. PLAYER [Name, Position] will select

  (Johnsonbrough, c), (Glover, of),…,,(Singleton, 2b)

- The selection and projection operators manipulate a single relation. Join manipulates two operators. If the join condition is satisfied, the tuples are combined to form a new tuple.

3.6 Relational databases

- 예 3.6.5 Perform a join operator on Table 3.6.1 and Table 3.6.2 by the condition ID number = PID

- Sol) Take a row form T 3.6.1 and a row form 3.6.2

  if ID number = PID, for example ID in 5th row of T 3.6.1 matches PID in 4th row of T 3.6.2. These tuples are combined

  (01180, Homer, 1b, 37, Mutts)

  This operator is expressed as

  PLAYER [ID number = PID] ASSIGNMENT

  The relation obtained by this join is shown in T 3.6.3