1

GRAPH THEORY

1.1. Introduction

Graph theory was born in 1736 with Euler's paper in which he solved the Konigsberg bridge problem. In 1947, G.R. Kirchoff developed the theory of trees for their application in electrical network. A Caylay also discovered trees while he was trying to enumerate the isomers of saturated hydrocarbon $C_n H_{2n+2}$. They also lay down four color conjective, which states that four colour are sufficient for colouring any atlas such that the countries with common boundaries have different colour.

Now a day Graph Theory is employed in many areas, such as Communications, Engineering, Physical Sciences, Social Sciences etc. On account of diversity of its application, it is useful to develop and study the subject in abstract form and then import its results. In general areas of computer science such as switching theory and logical design artificial intelligence, formal languages, computer graphics, operating system, graph theory is very useful.

In this chapter, we shall define the various components of the graph theory along with suitable examples. An attempt has been made to show that graphs can be useful to represent any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationships along them.

1.2.Terminology

Graph :

A graph (or undirected graph) is a diagram consisting of a collection of vertices together with edges joining certain pair of these vertices. Mathematically, we can write

A graph G = [V(G), E(G)]

where V(G) and E (G) are sets defined as

V(G) = Vertex set (points set or nodes set) of the graph G,

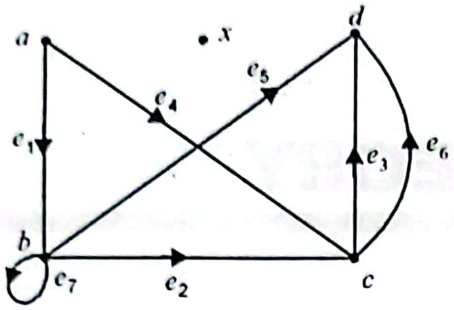
$E(G) \subseteq V(G) \times V(G)$, a relation on V(G), called edge set of G

Each element e of E(G) is assigned on **unordered pair** of vertices (a, b) called the end vertices of e.

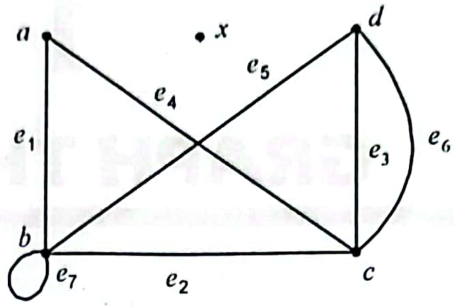
DIRECTED GRAPH.

A directed graph is a graph in which each element e of E (G) is assigned an **ordered pair of vertices** (a, b) along with arrow starting from a to b, where a is called the **initial vertex** and b is called the **terminal vertex** of the edge e.

The graphs directed and undirected are shown in the following figures :



(DIRECTED GRAPH)



(UNDIRECTED GRAPH)

REMARK : (*i*) A graph is represented by means of a diagram in which the vertices are denoted by points and edges are represented by line segments joining its end vertices.

(*ii*) It does not matter whether the joining of the two vertices in a graph is a straight line or a curve, longer or shorter.

Adjacent vertices

Two vertices u and v of a graph G = (V, E) are said to be adjacent if there is an edge $e = (u, v)$ connecting u and v. Also the edge e is said to be **incident** on each of its end points u and v.

For example :- In the above diagram a and b are adjacent vertices. Since there is an edge $e_1 = (a, b)$ joining a and b. Also the vertices a and d are not adjacent, as there is no edge joining the vertices a and d.

Loop (or self loop)

An edge that is incident from and into itself starts and ends at same vertex is called self loop or sting.

For Example :- In the above diagram the edge e_7 is a loop. Since the edge $e_7 = (b, b)$ starts and ends at b.

Isolated Vertex

A vertex of a graph G = (V, E), which is not joined to any vertex by an edge in G, is called an isolated vertex.

For example :- In the above diagram the vertex x is an isolated vertex.

Parallel edges

If two (or more) edges of a graph G have the same end vertices, then these edges are called parallel edges.

For example : In the above diagram the edges $e_3 = (c, d)$ and $e_6 = (c, d)$ are parallel edges.

Incidence

An edge e of a graph G = (V, E) is said to be **incident** with the vertex v if v is an end vertex of e (or u incident with e)

Adjacent edges

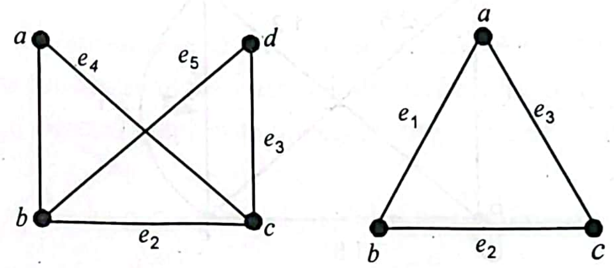
Two non-parallel edges of the graph are called adjacent if they have one common vertex.

For example :- In the above diagram the edges $e_1 = (a, b)$ and $e_4 = (a, c)$ are adjacent vertices, as they have a common end vertex a.

1.3. Types of Graphs

(i) Simple graph :

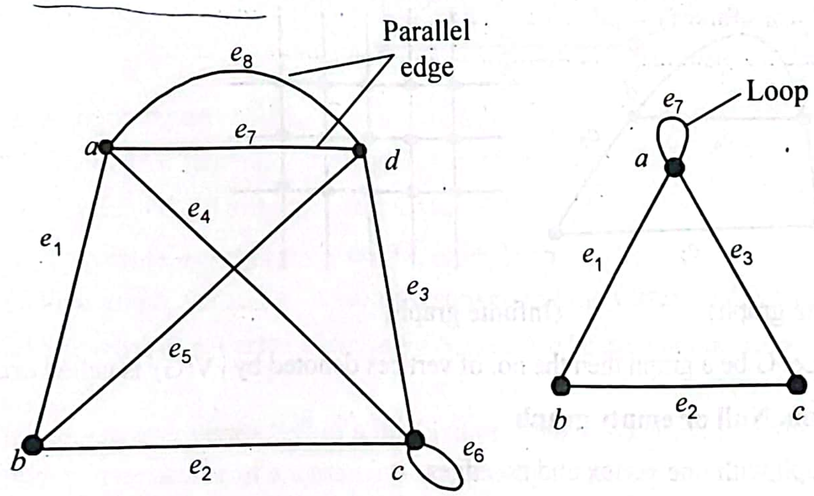
A graph which has neither loop nor parallel edge is called simple graph.



The above graphs are simple graphs.

(ii) General graph (or Multi graph)

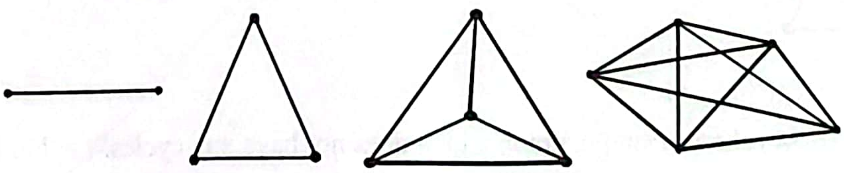
A graph which have either loop or parallel edge or both, is called a general graph or a multi graph.



The above graphs are general graphs.

(iii) Complete Graph : A simple graph in which there exists an edge between every pair of vertices is called a complete graph. It is also known as universal graph.

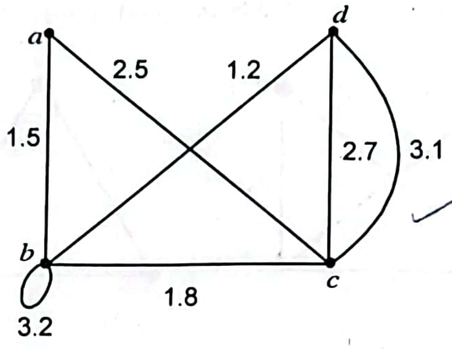
For example : Following graphs are complete graph.



Note I : A complete graph with n vertices is usually denoted by K_n.

II : A complete graph has $C(n, 2)$ edges.

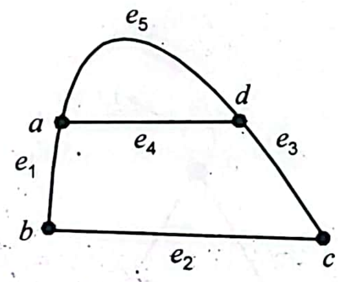
(iv) Weighted graph : Let G = (V, E) be any graph and $\omega : E \to R$ be a function from edge set E to set real numbers R. Then the graph G = (V, E, ω) in which each edge is assigned a number called the weight of the edge, is known as weighted graph.



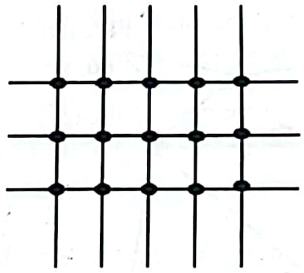
The above graph is a weighted graph, as each edge is assigned with a number.

(v) Finite graph : A graph G = (V, E) is called a finite graph if the vertex set V is a finite set.

(vi) Infinite graph : A graph G = (V, E) is called an infinite graph if the vertex set V is an infinite set.



(Finite graph)



(Infinite graph)

(vii) Order of a graph : Let G be a graph then the no. of vertices denoted by | V(G)| is called order of G.

(viii) Define Trivial graph, Null or empty graph

The trivial graph is the graph with one vertex and no edges.

The empty graph is the graph with No vertices and no edges.

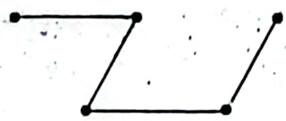
(ix) Edges in series : When two edges in a graph have exactly one vertex in common and this vertex is of degree two, then two edges are said to be in series.

Example.

e_1, e_2 are in series whereas e_2, e_3 are not in series.

(x) Acyclic : An Acyclic is a simple graph which does not have any cycles. *i.e.* No loop exists in such graphs.

Example :



1.4. Degree in a Graph

In-degree

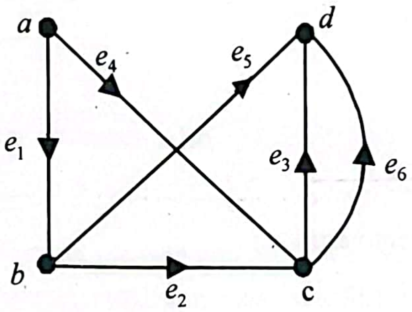
In a directed graph G, the **in-degree** of a vertex "*a*" is defined as the number of edges which have "*a*" as the terminal vertex. The in degree of the vertex a is denoted by deg $G^+(a)$ or $d^+(a)$.

Out-degree

In a directed graph G, the **out-degree** of a vertex "*a*" is defined as the number of edges which have "*a*" as the initial vertex. The out-degree of the vertex a is denoted by deg $G^-(a)$ or $d^-(a)$

Remark. (*i*) A vertex in a directed graph with in-degree zero is called a **source** and out-degree zero is called a **Sink**.

(*ii*) The direction of a loop in a directed graph has no significance.



In the above figure

deg $G^-(a) = 2$, deg $G^+(a) = 0$

deg $G^-(d) = 0$, deg $G^+(d) = 3$

deg $G^-(c) = 2$, deg $G^+(c) = 2$

In above graph, vertex "a" is called a **source** and the vertex "*d*" is called a **Sink**.

Even or odd (Parity) of a Vertex : The vertex V is said to be even or odd according as deg.(V) is even or odd.

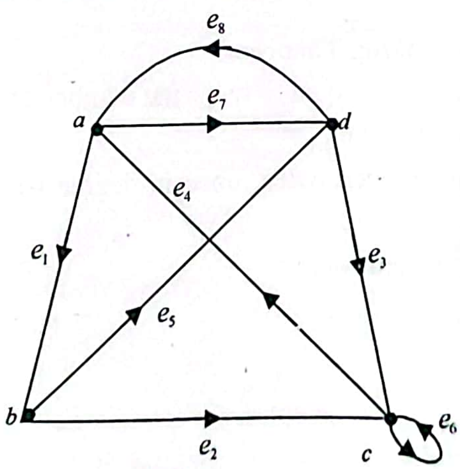
Degree

The degree of a vertex "*a*" in a directed or undirected graph is defined as the total number of edges incident with a. The degree of a vertex a is denoted by deg G (*a*) or *d* (*a*).

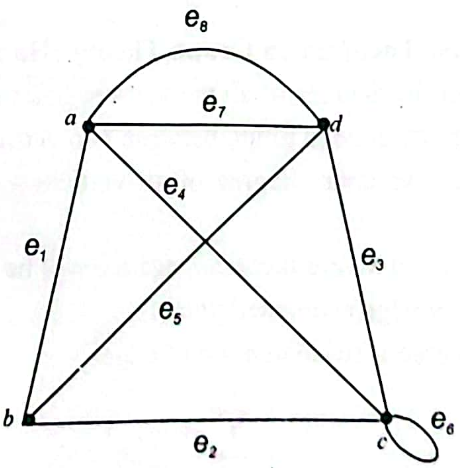
Thus in a direct graph deg $G(a) = \deg G^+(a) + \deg G^-(a)$.

Remark. For calculating degree of a *a* vertex in a general graph, *a* loop is counted twice.

For example : In the following graph directed or undirected we have



(Directed



(Undirected)

In directed graph

$\deg G(x) = \deg G^{+}(x) + \deg G^{-}(x)$

$\therefore\ \deg G(a) = 2 + 2 = 4$

$\deg G(b) = 1 + 2 = 3$

$\deg G(c) = 2 + 3 = 5$

$\deg G(d) = \cancel{3 + 1 = 4}$ $2+2=4$

In undirected graph

$\deg G(a) = 4$

$\deg G(b) = 3$

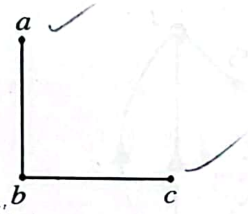
$\deg G(c) = 5$

$\deg G(d) = 4$

Pendent vertex (End vertex)

A vertex whose degree in a graph is one is called pendent vertex.

For example :- In the following graph



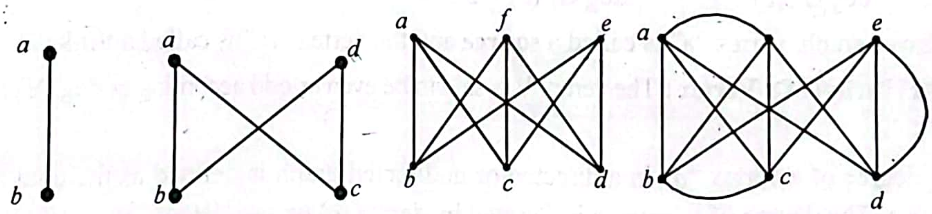
The vertices a and c are pendent vertices.

since $\deg G(a) = 1$ and $\deg G(c) = 1$

Definition Regular Graph : A graph in which all the vertices are of same degree is called a regular graph.

Definition k-Regular Graph : A graph in which all the vertices have the same degree equal to k, is called a k-Regular graph.

For example :



(1-Regular graph) (2-Regular graph) (3-Regular graph) (4-Regular graph)

Note : Complete Graph k_n is $n-1$ regular.

Next, we have a very important, but simple result on graph theory known as the first theorem on graph theory.

Theorem 1. First Theorem on Graph Theory (Handshaking Theorem)

The sum of the degrees of all the vertices in a graph G is equal to twice the number of edges in G.

Proof : Let e be any edge in graph between two vertices V_1 and V_2.

Now, when we count degree of all vertices e is counted twice, once in degree of V_1 and again in degree of V_2.

Also, if V_1 and V_2 are identical, again e will be counted twice. ($\because$ e is self-loop)

Hence every edge is counted twice.

So total degree is twice number of edges.

or $$\sum_{i=1}^{n} \deg(v_i) = 2e$$

Theorem 2. Prove that in a graph the number of vertices of odd degree is even.

Proof. Let $v_1, v_2, \ldots, v_n$ be n-vertices and $e_1, e_2, \ldots, e_e$ be e-edges in the graph G. Then by first theorem in graph theory

$$\sum_{i=1}^{n} d(v_i) = 2e \quad \ldots (1)$$

Now, divide the sum on the L.H.S of (1) in two parts

(*i*) One part contains the sum of degree of vertices which have even degree.

(*ii*) Second part contains the sum of the vertices which have odd degree.

Then equation (1) can be written as

$$\sum_{\text{even}} d(v_i) + \sum_{\text{odd}} d(v_k) = 2e \quad \ldots (2)$$

Since the R.H.S of (2) is an even number. Also $\sum_{\text{even}} d(v_i)$ is also even. This implies that $\sum_{\text{odd}} d(v_k)$ is also even.

i.e. the sum of the degree of vertices having odd degrees is even

$\therefore$ The number of vertices having odd degree must be even.

Theorem 3. Prove that the maximum degree of any vertex in a simple graph having n vertices is $n-1$.

Proof. Since, in a simple graph, there are no parallel edges and no loops. Therefore a vertex can be connected to the remaining $n-1$ vertices at the most by $(n-1)$ edges.

Hence, the maximum degree of any vertex in a simple graph having n vertices is $n-1$.

Theorem 4. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be n-vertices and $e_1, e_2, \ldots, e_e$ be e-edges in a simple graph G. Then

By First theorem on graph theory $\sum_{i=1}^{n} d(v_i) = 2e$ ...(1)

Also, we know that

in a simple graph, the maximum degree of any vertex with n vertices is $n-1$.

$\therefore$ Sum of maximum degrees of n vertices $= \underbrace{(n-1) + (n-1) + \ldots\ldots + (n-1)}_{n \text{ terms}}$

$$= n(n-1)$$

$\therefore$ from (1), we have

$$2e = n(n-1)$$

$$e = \frac{n(n-1)}{2}$$

$\therefore$ The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Theorem 5. Prove that the number of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

Proof. Since every vertex in a complete graph is joined with every other vertex through one edge

$\therefore$ The degree of every vertex in a complete graph of n vertices is $n-1$.

$\therefore$ If e be the total number of edges in G. Then by first theorem on graph theory, we have

$$\sum_{i=1}^{n} d\,(v_i) = 2e$$

$$n\,(n-1) = 2e \qquad [\because d\,(v_i) = n-1 \text{ for } 1 \le i \le n]$$

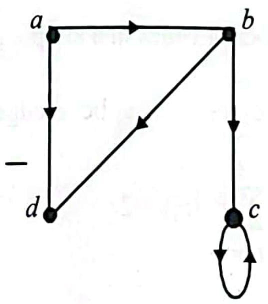
$$\Rightarrow \qquad e = \frac{n(n-1)}{2}$$

$\therefore$ Total number of edges in G $= \frac{n(n-1)}{2}$.

ILLUSTRATIVE EXAMPLES

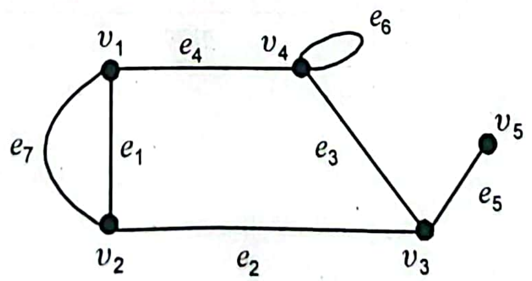
Example 1. If V = $\{a, b, c, d\}$ and E = $\{(a, b), (a, d), (b, c), (b, d), (c, c)\}$ be the vertex set and edge set of a graph G. Draw the directed graph G = (V, E). Is it a simple graph?

Sol. The directed graph G = (V, E) is as shown below :-



Since it contain a loop. Therefore it is not a simple graph.

Example 2. Find the degree of each vertices of the following graph.



Also verify first theorem on graph theory.

Sol. Here $d\,(v_1) = 3$, $d\,(v_2) = 3$, $d\,(v_3) = 3$, $d\,(v_4) = 4$, $d\,(v_5) = 1$

By first theorem of graph theory

$$\sum_{i=1}^{n} d\ (v_i) = 2e$$

where e is the number of edges and n is the number of vertices in the graph.

Here $n = 5$ and $e = 7$

Also $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3 + 3 + 3 + 4 + 1 = 14 = 2\ (7) = 2e$

Thus first theorem on graph theory is verified.

Example 3. A graph G has 21 edges, 3 vertices of degree 4 and all other vertices are of degree 3. Find the number of vertices in G.

Sol. Let n be the number of vertices in G.

According of first theorem on graph theory.

$$\sum_{i=1}^{n} d\ (v_i) = 2e, \text{ where } e \text{ is the no. of edges.}$$

Let v_1, v_2, v_3 be the vertices of degree 4 and $v_4, v_5, \ldots\ldots, v_n$ be the remaining vertices of degree 3

$$\therefore \quad \sum_{i=1}^{3} d\ (v_i) + \sum_{k=4}^{n} d\ (v_k) = 2\ (21)$$

$$3\times 4 + (n-3)\times 3 = 42$$

$$12 + 3n - 9 = 42$$

$$3n = 39$$

$$n = 13$$

$\therefore$ Number of vertices in G be 13. ✓

Example 4. Prove that there does not exist a graph with 5 vertices with degree equal to 1, 3, 4, 2, 3 respectively.

Sol. Here $n = 5$, Let e be the number of edges in the graph

By first Theorem on graph theory

$$\sum_{i=1}^{5} d\ (v_i) = 2\ e$$

$$\Rightarrow \quad 1+3+4+2+3 = 2\ e$$

$$\Rightarrow \quad 13 = 2\ e$$

$$\Rightarrow \quad e = \frac{13}{2}, \text{ which is not possible}$$ ✓

Hence there does not exist a graph with 5 vertices of given degrees.

Example 5. Is there a simple graph G with six vertices of degree 1, 1, 3, 4 , 6, 7 ?

Sol. Here number of vertices in the graph, $n = 6$

we know that

Maximum degree of any vertices in a simple graph $= n - 1 = 6 - 1 = 5$

But G has a vertices of degree 7, which is not possible in a simple graph.

Hence there is no simple graph G of six vertices having the given degrees.

Example 6. (*a*) Find k, if a k-regular graph with 8 vertices has 12 edges. Also draw k-regular graph.(*b*) Can there be a graph with 8 vertics and 29 edges ? Justify.

Sol. (*a*) We know that a graph G will be a k-regular graph if the degree of all the vertices in G are same and equal to k.

Also, number of vertices, $n = 8$

number of edges, $e = 12$

$\therefore$ By the first theorem on graph theory, we have

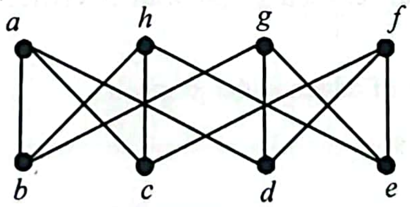
$$\sum_{i=1}^{n} d(v_i) = 2e.$$

$$\sum_{i=1}^{8} k = 2(12)$$

$\Rightarrow$ $8k = 24$

$\Rightarrow$ $k = 3$ so graph is 3-regular graph

and the 3-regular graph is



(3-Regular graph)

(*b*) Maximum no. of edges in graph with no multiple edges $= \dfrac{(n)(n-1)}{2}$

$n = 8$ $\therefore$ max. no. of edges. $= \dfrac{(8)(8-1)}{2} = 28$

but given no. of edges $= 29$

$\therefore$ It is not possible.

1.5. Isomorphic Graphs

Let G = (V, E) and G′ = (V′, E′) be two graphs. Then G is isomorphic to G′ written as $G \cong G'$ if there exists a bijection f, from V onto V′ such that $(v_i, v_j) \in E$, if and only if $(f(v_i), f(v_j)) \in E'$.

In other words, two graphs are isomorphic if there exists a one-one correspondence between their vertices and edges such that incidence relationship is preserved.

Remark. (*a*) Two graphs which are isomorphic will have

(*i*) same number of vertices

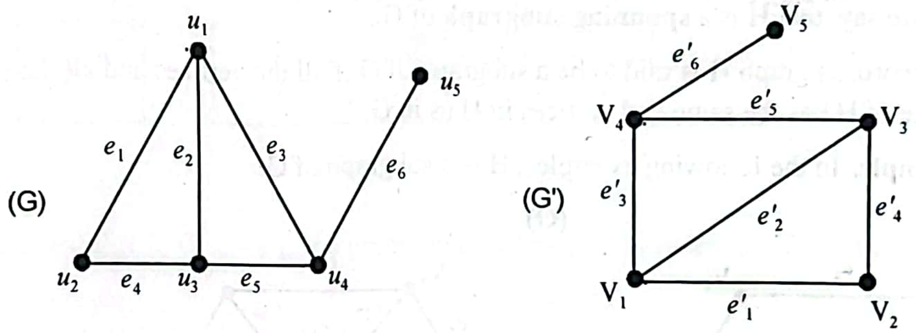
(*ii*) same number of edges

(*iii*) an equal number vertices with given degrees

(*b*) The converse of (*a*) need not be true.

For example.

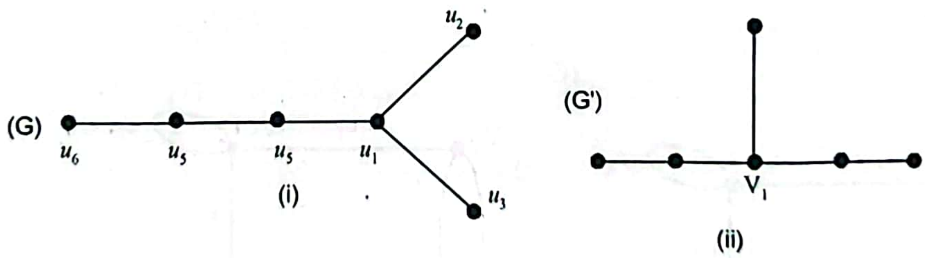
(*a*) The following two graphs are isomorphic to each other



Because ∃ a mapping $u_i \xrightarrow{f} v_i$ for $i = 1, 2, 3, 4, 5$

and $e_i \xrightarrow{f} e'_i$ for $i = 1, 2, 3, 4, 5$

(*b*) The two graphs may be non-isomorphic even though they have the same number of vertices and edges and an equal number of vertices of given degrees.



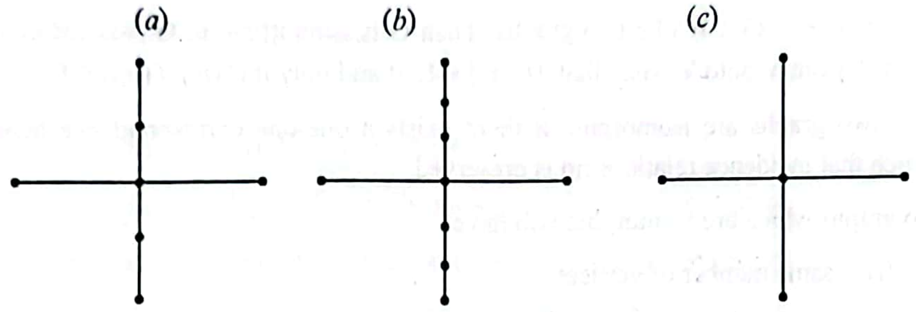
If the graph G are the to be isomorphic to graph G′, then the vertex u_1 must corresponds to v_1, because there is no other vertex of degree 3 in G′. Also in G′ there is only one pendent vertex v_2 adjacent to v_1, while in G there are two pendent vertices u_2 and u_3 adjacent to u_1.

Hence $G \not\cong G'$. (of degree 1).

Homeomorphic Graphs

Given any graph G, obtain a new graph by dividing an edge of G with additional vertices.

e.g.



(a) and (b) Homeomorphic obtained from (c).

Sub-Graphs

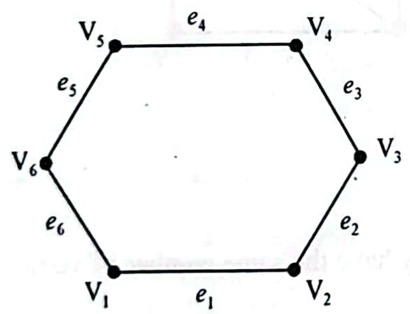
Let G and H be two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we call H as a **Subgraph** of G (or G as a supergraph of H).

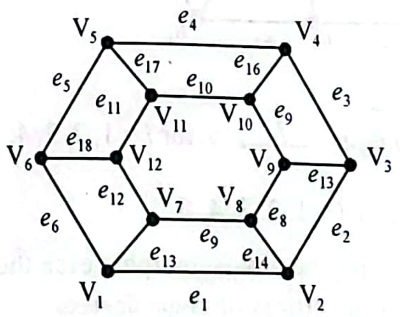
If $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then H is a **Proper subgraph** of G and if $V(H) = V(G)$ and $E(H) \subset (G)$ then we say that H is a **spanning subgraph** of G.

In other words, a graph H is said to be a subgraph of G if all the vertices and all the edges of H are in G, and each edge of H has the same end vertices in H as in G.

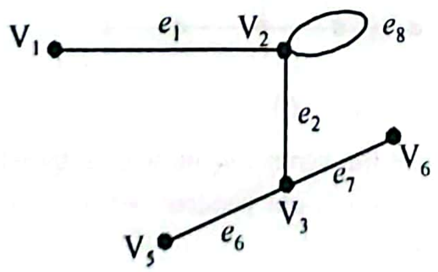
For example. In the following examples, H is a subgraph of G.

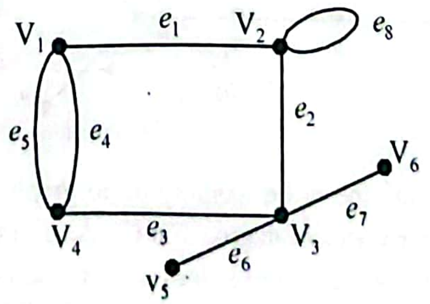
(i) (H) (G)





(ii) (H) (G)





Full Subgraph

Suppose $H(V', E')$ be a subgraph of $G(V, E)$. H is called full subgraph of G if E' contains all the edges of E whose end points lie in V'. It is called subgraph of G generated by V'.

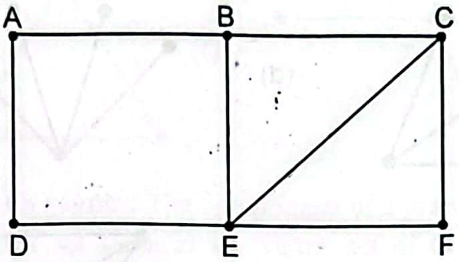
Define G-V

G-V is a subgraph of G obtained by deleting the vertex V from vertex set V(G) and deleting all the edges in E(G) which are incident on V.

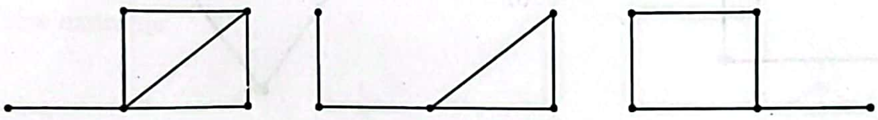
Cut Vertex

A vertex V is called a cut vertex for G if G-V is disconnected.

Example. Let G be the graph find G-A, G-B, G-C



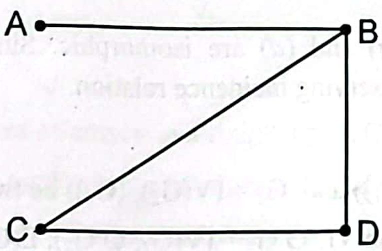
Sol.



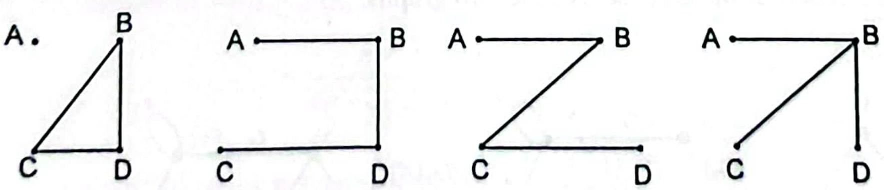
(*a*) (*b*) (*c*)

Define : G – *e*, *e* is an edge in G. G– *e* is the graph obtained by simply deleting *e* from the edge set of G.

Example. Let G be graph.



Find (*a*) G – {A, B} (*b*) G – {B, C} (*c*) G – {B, D} (*d*) G – {C, D}

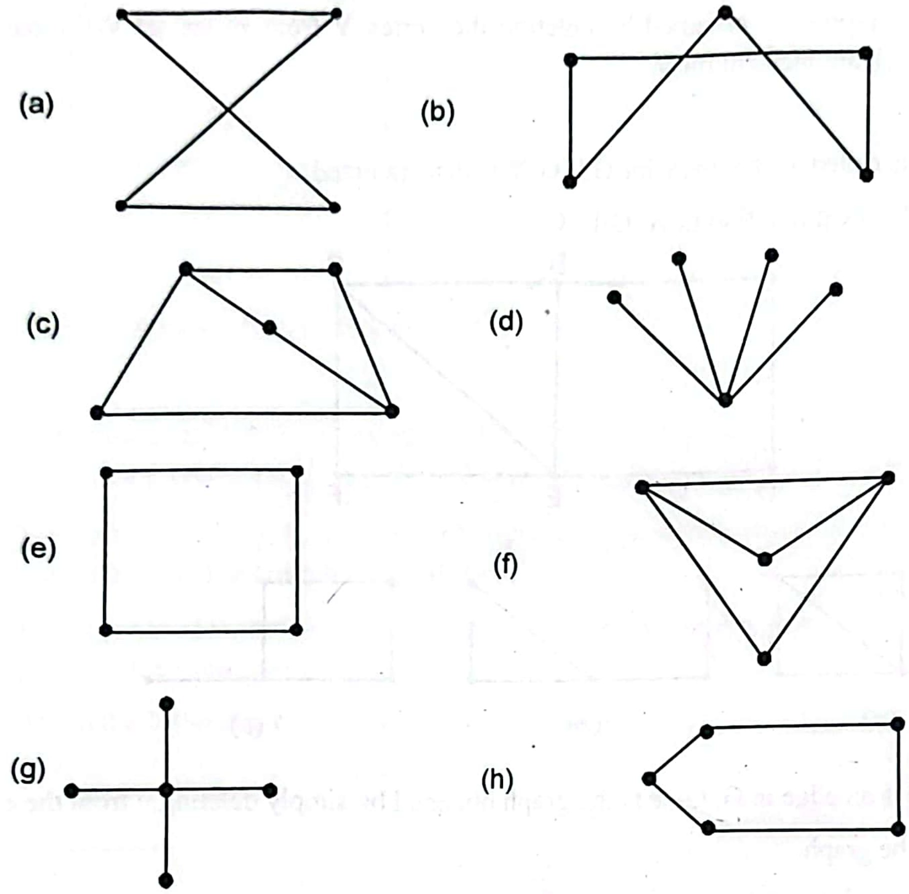


G – {A, B} G – {B, C} G – {B, D} G – {C, D}

Remark. (*i*) Every graph is its own subgraph

(*ii*) The null graph obtained from G by deleting all the edges of G is a subgraph of G.

Example 7. Which of the following pair of graphs are isomorphic ?



Sol. The graphs (*a*) and (*e*) ; (*b*) and (*h*) ; and (*g*) and (*d*) are isomorphic. Since there a one-one correspondence between the vertex and the edge set preserving incidence relation.

Operations of Graph

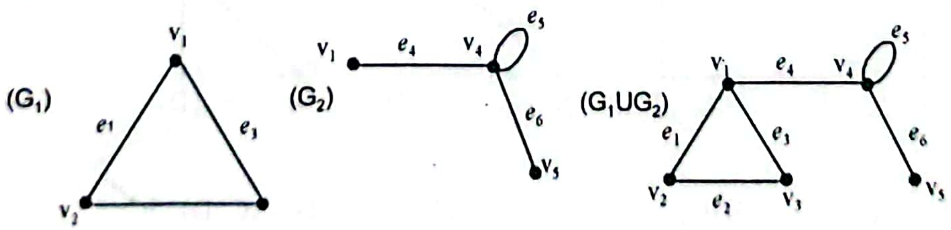
***(i)* Union of two graphs :** Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), (G_2))$ be two graphs.

Then their union is denoted by $G_1 \cup G_2$, is a graph $G_1 \cup G_2 = (V(G_1, \cup G_2), E(G_1, \cup G_2))$

such that $V(G_1, \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

In other words, union of two graphs is a graph whose vertex set is the union of the vertex sets of the two graphs and edge set is the union of the edge sets of the two graphs.

For example.



***(ii)* Intersection of two graphs :** Let $G_1 = (V(G_1), E(G_2))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. Then their intersection is denoted by $G \cap G_2$, is a graph

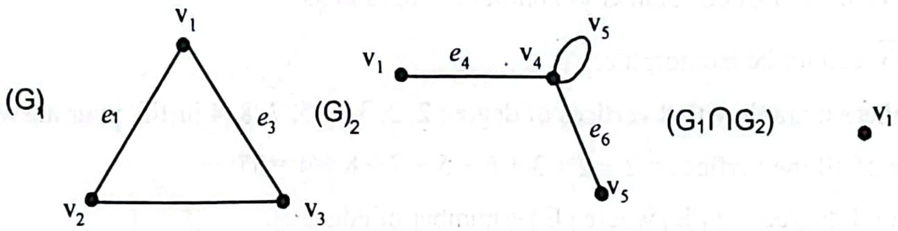
$$G_1 \cap G_2 = (V(G_1 \cap G_2), E(G_1 \cap G_2))$$

such that $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$

$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

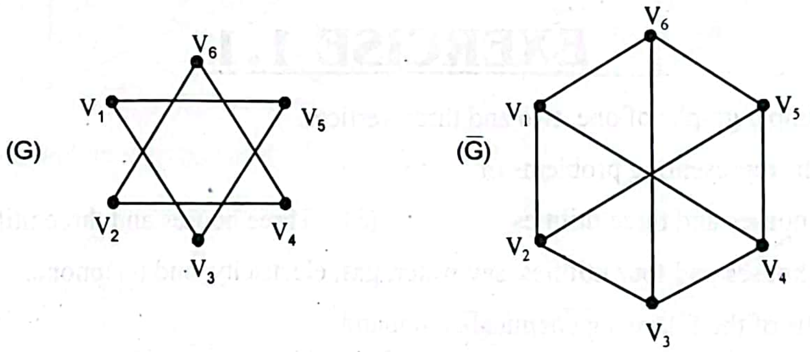
In other words, intersection of two graphs is a graph whose vertex set is the intersection of the vertex sets of the two graphs and edge set is the intersection of the edge sets of the two graphs.

For example.



***(iii)* Complement of a graph :** The compliment of a graph G is denoted by $\overline{G}$ and is defined as the simple graph with the vertex set same as the vertex set of G together with the edge set satisfying the property that there is an edge between two vertices in $\overline{G}$, when there is no edge between these vertices in G.

For example



Note :- If the degree of a vertex v in a simple graph G having n vertices is k. Then degree of v in $\overline{G}$ is n-k-1.

Example 8. What is total number of edges in k_n, the complete graph on n vertices ? Justify your answer ?

Sol. We know number of vertices in $k_n = n$

also in complete graph there is an edge between every two vertices.

So we have to make pairs of n vertices

for this number of ways = $c\,(n, 2)$

$$= \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

$\frac{n(n-1)(n-2)!}{(n-2)!\,2!}$

$\therefore$ number of edges in complete graph = $\dfrac{n(n-1)}{2}$

Example 9. Can a graph with seven vertices be isomorphic to its complement ? Justify.

Sol. Let G be the given graph and $\overline{G}$ is complement of G. We know, if an edge $e \in G$ then $e \notin \overline{G}$.

So total number of edges in G and $\overline{G}$ = Maximum number of possible edges in complete graph.

Here we have number of vertices = 7

Using 7 vertices max. number of edge = $\frac{7(7-1)}{2}$

$= 21$

So number of edges in G and $\overline{G}$ = 21

which means number of edges in G ≠ number of edges in $\overline{G}$ [∵ 21 is odd]

So G and $\overline{G}$ cannot be isomorphic.

Example 10. Is there a graph with 8 vertices of degree 2, 2, 3, 6, 5, 7, 8, 4 justify your answer.

Sol. Total degree of all the vertices = 2 + 2 + 3 + 6 + 5 + 7 + 8 + 4 = 37

We know total degree = 2 | E | where | E | = number of edges

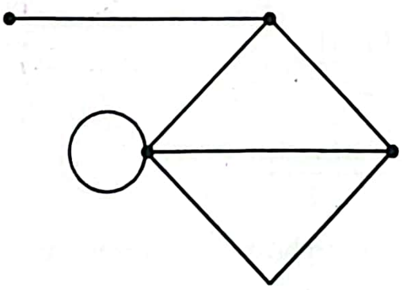
$\therefore \quad 37 = 2|E| \Rightarrow |E| = \frac{37}{2}$

which is not possible.

Hence given graph does not exists.

EXERCISE 1.1

1. Draw all simple graphs of one, two and three vertices.
2. Draw graphs representing problems of
   (*a*) Two houses and three utilities (*b*) Three houses and three utilities
   (*c*) Four houses and four utilities, say water, gas, electricity, and telephone.
3. Draw graphs of the following chemical compounds
   (*a*) CH_4 (*b*) C_2H_6 (*c*) C_6H_6
4. Differentiate between directed graph and undirected graphs.
5. How many nodes are necessary to construct a 2-regular graph with exactly 6 edges ?
6. Is it possible to construct a graph with 12 edges such that two of its vertices have degree 3 and remaining vertices have degree 4 ?
7. Find the degree of each vertices in the following graph :



8. Does there exist a graph with 6 vertices with degree equal to 3, 2, 4, 1, 3, 2 respectively.
9. Find *k*, if a *k*-regular graph with 7 vertices has 14 edges. Also draw the k-regular graph.
10. Find *n*, if a complete graph having *n* vertices has 15 edges.

1.6. Walks, Paths and Circuits

Walk : A walk in a graph G is finite sequence

$$W = V_0, e_1, V_1, e_2, \ldots\ldots, V_{k-1}, e_{k-1}, V_k$$

whose terms are alternatively vertices and edges such that for $1 \le i \le k-1$, the edge e_i has end vertices v_{i-1} and v_i. The vertex V_0 is called the initial and the vertex V_k is called terminal of the walk W. Vertices $V_1, V_2, \ldots, V_{k-1}$ are called internal vertices. A walk is also refereed as an edge train or chain.

Remark. (*i*) Each edge can appear only once in a walk, however vertices may appear more than once.

Open Walk : If a walk begin and end with the different vertices, it is called an open walk.

Closed Walk : If the initial and terminal vertices of a walk are same, it is called a closed walk.

Remark. A walk containing no edge and has length zero is called a **Trivial walk.**

PATH : An open walk in which no vertex appear more than once is called a path or simple path.

Note : A path do not intersect it self.

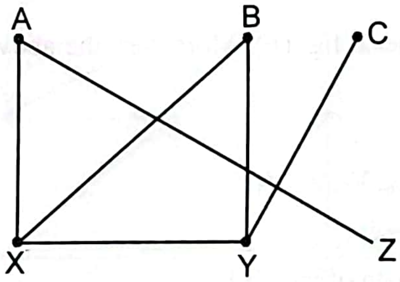
Simple Path and Trail

A path is **simple** if all vertices are distinct. The path is a **trail** if all the edges are distinct.

Example. Let G be the graph. Determine whether or not each of the following sequences of edges forms a path

(*a*) {(A, X), (X, B), (C, Y), (Y, X)} (*b*) {(A, X), (X, Y), (Y, Z), (Z, A)}

(*c*) {(X, B), (B, Y), (Y, C)} (*d*) {(B, Y), (X, Y), (A, X)}



Sol. (*a*) No, the edge {X, B} is not followed by edge (C, Y).

(*b*) No, Graph has not edge (Y, Z)

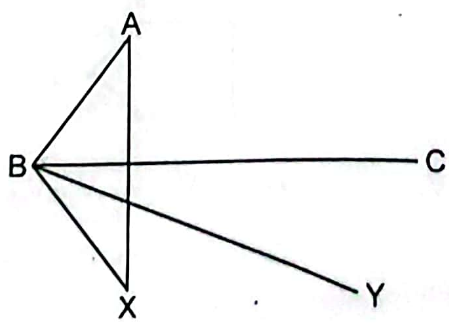
(*c*) Yes

(*d*) Yes, sequence can be written as {(B, Y), (Y, X), (X, A)}

[In undirected graph (Y, X) and (X, Y) are same]

Example. Let G be the graph. Determine whether each of the following is a closed path, trail, simple path or cycle.

(*a*) (B, A, X, B) (*b*) (X, A, B, Y) (*c*) (B, X, Y, B)



(*a*) This path is a cycle since it is closed and has distinct vertices

(*b*) This path is simple since it vertices are distinct. It is not cycle since it is not closed.

(*c*) This not even a path since {X, Y} is not an edge.

Length of path : The number of edges appearing in the sequence of the path is called the length of the path.

For example. Consider the following graph

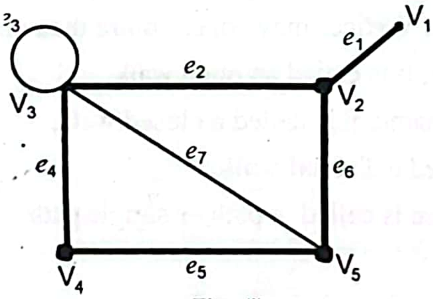


Fig. (i)

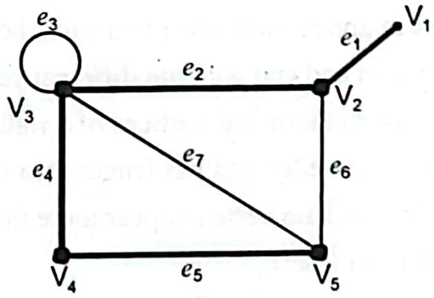


Fig. (ii)

$W = V_1, e_1, V_2, e_2, V_3, e_3, V_3, e_4, V_4, e_5, V_5, e_6, V_2.$

Then W is a walk of length 6 as shown by the bold line in fig. (*i*). The above walk is not a path as the vertices V_3 and V_2 appear twice in the walk W. However the walk

$W' = V_1, e_1, V_2, e_2, V_3, e_4, V_4, e_5, V_5$

is a path of length 4 as shown by the bold line in fig. (*ii*). Moreover, the above walk W and W′ are open walk as their terminus vertices are different.

But the walk

$W'' = V_1, e_1, V_2, e_3, V_3, e_3, V_3, e_4, V_4, e_5, V_5, e_6, V_2, e_1, V_1$

is a closed walk as the terminus vertices are same.

Remark. (*i*) An edge which is not a self loop is a path of length 1.

(*ii*) A self loop can be included in a walk but not in a path.

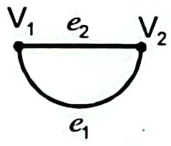
(*iii*) The terminus vertices of a path are of degree 1 and the internal vertices of the walk are of degree 2.

CIRCUIT : A circuit is a closed walk in which no vertex (except the initial and terminal vertex) appears more than once.

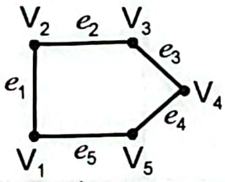
In other words, a circuit is a closed, non-intersecting walk. A circuit is also called the **cycle** or **elementary cycle** or **circular path** or **polygon.**

For example.

$W = V_1 e V_1$



$W = V_1 e_1 V_2 e_2 V_1$



$W = V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5 e_5 V_1$

are all circuits.

***k*-cycle :** A cycle with *k*-edges is called a k-cycle and it is denoted by C_k.

Remark (*i*) A self loop is also a circuit, but converse is not true.

(*ii*) The degree of every vertex in a circuit is two.

(*iii*) 1 cycle is loop, 2 cycle is a pair of parallel edges, 3 cycle is a triangle,......, n cyclic is a polygon of n sides.

CONNECTED GRAPHS, DISCONNECTED GRAPHS, AND COMPONENTS

Connectivity : An undirected graph is said to be connected, if for any pair of vertices of the graph the two vertices are reachable from one another.

Strongly Connected : If any pair of vertices of the digraph both the vertices of the pair are reachable from another, then graph is strongly connected.

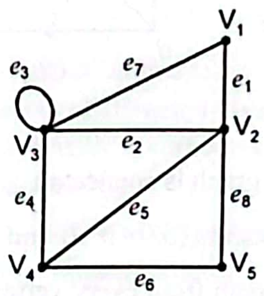
Unilaterally Connected : A simple directed graph is said to unilaterally connected if for any pair of vertices of the graph, at least one of the vertices of the pair is reachable from other vertex.

Weakly Connected Digraph : A directed graph is called weakly connected if its undirected graph is connected.

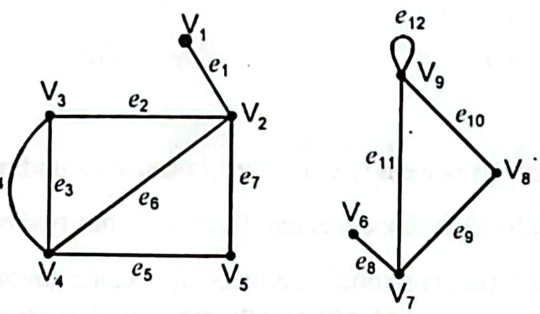
Connected graph : A graph G is said to be connected graph if there is atleast one path between every pair of vertices in G.

Disconnected graph : A graph which is not a connected graph is called disconnected graph.

For example.



Connected Graph



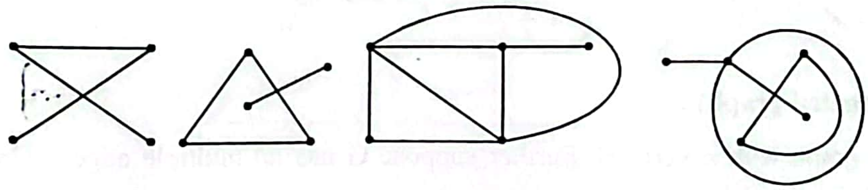
Disconnected Graph
(with two components)

Component : Each connected subgraph of a disconnected graph are called component.

Example. Consider the multigraph which of them are

(*a*) connected (*b*) loop-free (*c*) simple graphs

X



I II III IV

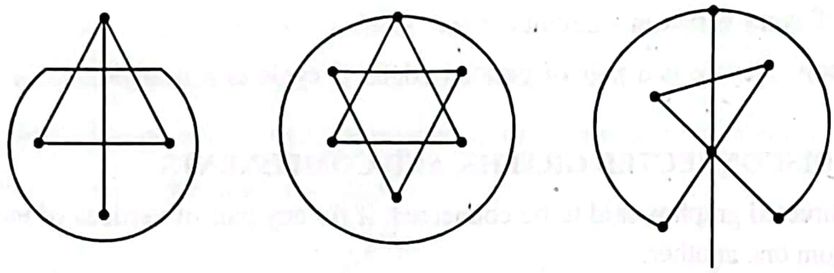
(*a*) I and III are connected.

(*b*) only IV has a loop.

(*c*) only (I) and (II) are simple graphs

(III) has multiple edges and IV has multiple edges and a loop.

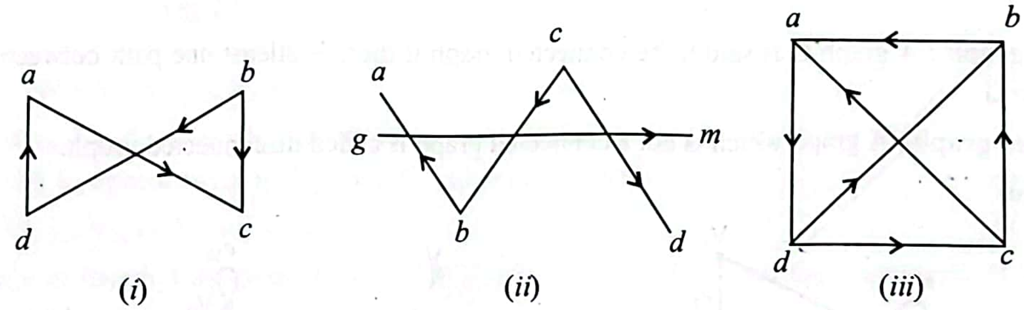
Example 1. Which of following are connected, graphs.



(*a*) Only III is connected.

(*b*) All are graphs.

Example 2. Classify the following graphs as : Strongly connected graph, unilaterally connected graph, weakly connected graph and disconnected graph.



Sol.

(*a*) The graph (*i*) is weakly connected because its undirected graph is connected.

(*b*) The graph (*ii*) is disconnected. Because it has two components $\{a, b, c, d\}$ and $\{g, m\}$

(*c*) The graph (*iii*) is strongly connected, because there is a path from every vertex u to v and v to u. It is also weekly and unilaterally connected as strongly connected graph obey properties of both these.

1.7. Matrix Representation of Graphs

A graph can be represented by a matrix in two ways :

(*i*) Adjacency matrix

(*ii*) Incidence matrix.

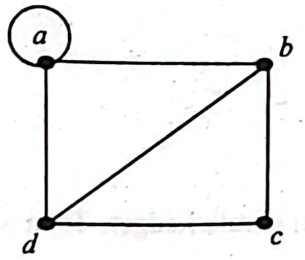
Adjacency Matrix (for undirected graph) :

Let G be an undirected graph with n vertices. Further suppose G has no multiple edges. Then G is represented by $n \times n$ matrix defined as $M = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{if } a_i \text{ and } a_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

i.e. an entry is 1 if there is an edge between a_i and a_j.

xample : Consider the graph



Find Adjacency matrix.

Sol.

	a	b	c	d
a	1	1	0	1
b	1	0	1	1
c	0	1	0	1
d	1	1	1	0

So $$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

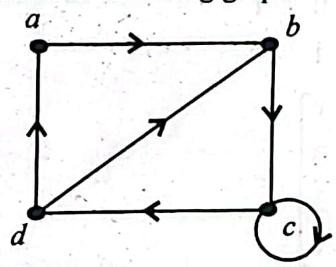
Note : Adjacency matrix of undirected graph is always symmetric.

Adjacency matrix of Directed Graph.

Let G be digraph with n vertices having no multiple edges. Then G can be represented by $n \times n$ adjacency matrix m defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is edge from } a_i \text{ to } a_j \\ 0 & \text{otherwise} \end{cases}$$

Example : Write adjacency matrix of following graph



Sol.

	a	b	c	d
a	0	1	0	0
b	0	0	1	0
c	0	0	1	1
d	1	1	0	0

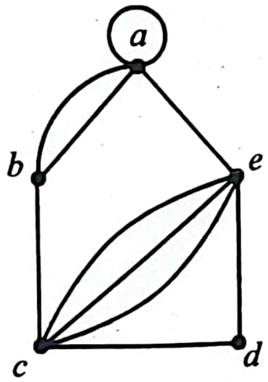
So $$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Adjacency matrix of multi-graph (undirected)

Let G be undirected graph of n vertices that may contain parallel edges. Then adjacency matrix M is $n \times n$ matrix defined by $M = [a_{ij}]_{n \times n}$

where $a_{ij} = \begin{cases} n, & n \text{ is number of edges between } a_i \text{ and } a_j \\ 0 & \text{otherwise} \end{cases}$

Example 1. Find adjacency matrix of Multi-graph.



Sol.

	a	b	c	d	e
a	1	2	0	0	1
b	2	0	1	0	0
c	0	1	0	1	3
d	0	0	1	0	1
e	1	0	3	1	0

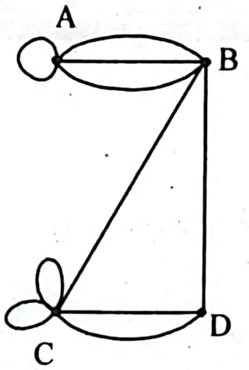
$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 0 \end{bmatrix}$$

Note : In similar way we can find adjacency matrix of directed multi-graph.

Example 2. Example 2. Draw multigraph G whose adjacency matrix is given by

$$M = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Sol. The corresponding multigraph is given as :

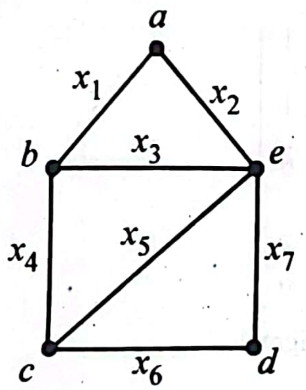


Incidence matrix :

Let G be a graph have m vertices and n edges. Then incidence matrix of graph is $m \times n$ matrix written as $A(G) = [a_{ij}]_{m \times n}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Example : Write incident matrix of graph.



Sol. Number of vertices = 5

Number of edges = 7

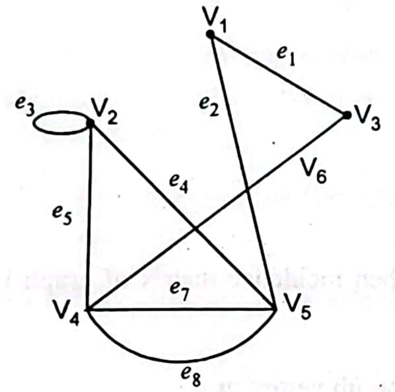
So **incidence** matrix is 5×7 matrix.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	1	1	0	0	0	0	0
b	1	0	1	1	0	0	0
c	0	0	0	1	1	1	0
d	0	0	0	0	0	1	1
e	0	1	1	0	1	0	1

so
$$A(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

ILLUSTRATIVE EXAMPLES

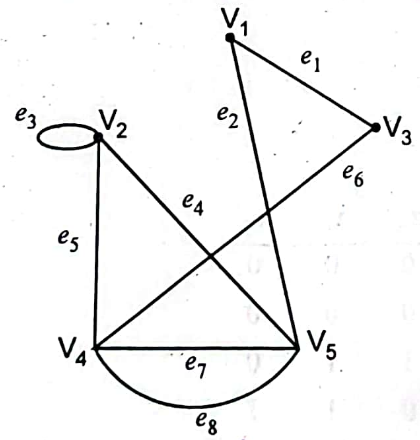
Example 1. Find the adjacency matrix A of the multigraph.



Set $a_{ij} = n$, where n is the no. of edges between V_i and V_j and set $a_{ij} = 0$ otherwise.

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

Example 2. Find the incidence matrix M of the multigraph.

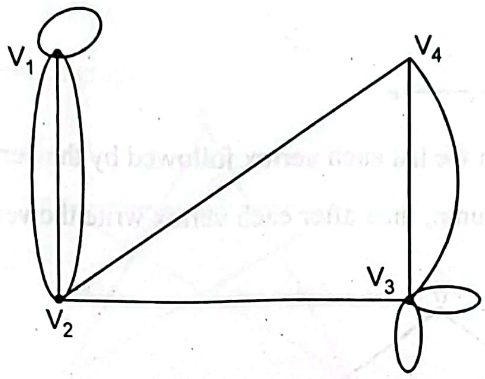


$$M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 3. Draw the multigraph G whose adjacency matrix A = follows :

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Sol.

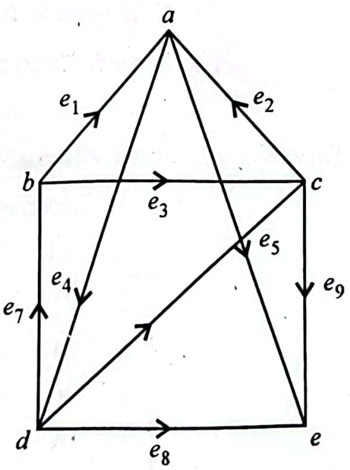


Example 4. Draw the Directed graph G whose incidence matrix M_1 is

(I)

$$M_1 = \begin{array}{c|ccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ \hline a & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ c & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{array}$$

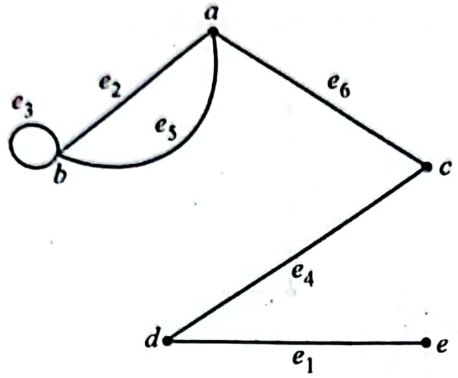
Directed graph is



(II)

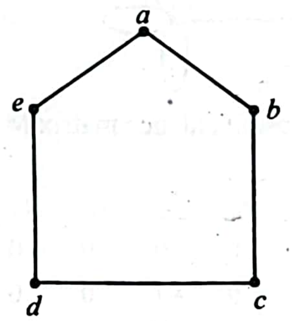
$$M_1 = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline a & 0 & 1 & 0 & 0 & 1 & 1 \\ b & 0 & 1 & 1 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 0 & 0 & 0 & 0 \\ e & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Undirected graph is



Adjacency list : In adjacency list of a graph we list each vertex followed by the vertices adjacent to it. First write vertices of graph in a vertical column, then after each vertex write the vertices adjacent to it.

Example 5. For a graph



I. Write the adjacency list,

II. Find the adjacency matrix

III. Find the incidence matrix

IV. Draw complement graph.

Sol. I. adjacency list is

a ; e, b

b ; a, c

c ; b, d

d ; c, e

e ; a, d

II $V = \{a, b, c, d, e\}, |V| = 5$

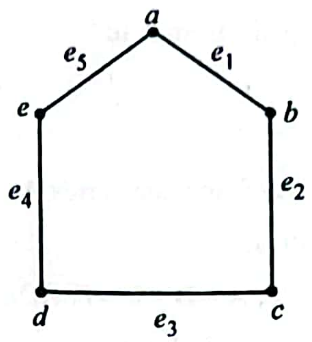
adjacency matrix will be a square matrix ;

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

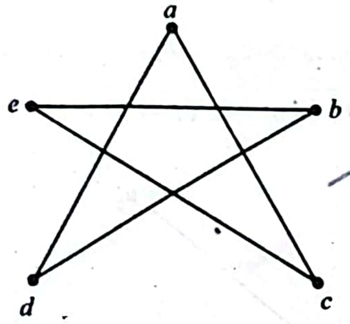
III. $|V| = 5, \; |E| = 5$

$\therefore$ Incidence matrix is

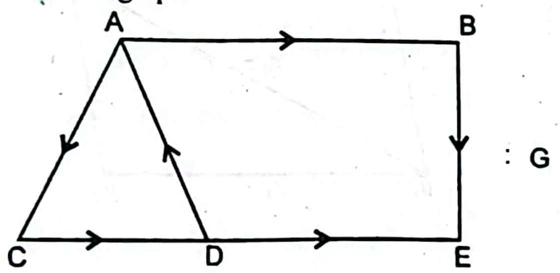
$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline a & 1 & 0 & 0 & 0 & 1 \\ b & 1 & 1 & 0 & 0 & 0 \\ c & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 1 & 1 & 0 \\ e & 0 & 0 & 0 & 1 & 1 \end{array}$$



IV. Complement of graph



Example 6. Consider the graph



(*i*) Express G by its adjacency table.

(*ii*) Find all the simple paths from A to E.

(*iii*) Find all cycles in G.

(*iv*) Show that G is unilaterally connected by exhibiting a spanning path of G.

(*v*) Is G strongly connected ?

I.
$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 1 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 1 \\ C & 0 & 0 & 0 & 1 & 0 \\ D & 1 & 0 & 0 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 \end{array}$$

II. A – B – E, A – C – D – E

III. A – C – D – A

IV. G is unilaterally connected.

V. G is not strongly connected.

[$\because$ adjacency matrix of strongly connected digraphs has all entries = 1]