Today

- Comments on HW 7 (written)
- Gram-Schmidt orthonormalization
- Elementary matrices
- Matlab for HW 7

HW7

#1 (hint) linear $u(x)$ that maps $[1,2]$ onto $[-1,1]$

$\Rightarrow$ think equation of line

$$u(1) = -1$$
$$u(2) = 1$$

Gram-Schmidt

The G-S process can be summarized by: For $i = 1, 2, \ldots n$:

- Set $v_1 = x_1$
- For $i = 2, \cdots, n$ set ← orthogonalize

$$v_i = x_i - \sum_{j=1}^{i-1} \frac{\langle x_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j$$

$x_1 = (1,0,1)$

$x_2 = (-2,1,0)$

$x_3 = (0,-1,3)$

- For $i = 1, \ldots, n$, normalize:

$$q_i = \frac{v_i}{||v_i||}$$

where the norm $||v_i|| = \left(\langle v_i, v_i \rangle\right)^{1/2}$.

Ex 1 Use G-S to produce three orthonormal vectors in $\mathbb{R}^3$, given:

$$x_1 = (1,0,1)$$
$$x_2 = (-2,1,0)$$
$$x_3 = (0,-1,3)$$

Soln

$$v_1 = x_1 = \boxed{(1,0,1)}$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (-2,1,0) - \frac{(-2+0+0)}{(1+0+1)} (1,0,1)$$

$$= (-2,1,0) + (1,0,1)$$

$$= \boxed{(-1,1,1)}$$

$$v_3 = x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= (0,-1,3) - \frac{(0+0+3)}{(1+0+1)}(1,0,1)$$

$$- \frac{(0-1+3)}{(1+1+1)}(-1,1,1)$$

$$= \boxed{\left(\frac{-5}{6}, \frac{-5}{3}, \frac{5}{6}\right)}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1+1}}(1,0,1)$$

$$= \frac{1}{\sqrt{2}}(1,0,1)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{1+1+1}}(-1,1,1) = \frac{1}{\sqrt{3}}(-1,1,1)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{6}{\sqrt{150}}\left(\frac{-5}{6}, \frac{-5}{3}, \frac{5}{6}\right)$$

Ex/ The L^2 inner product for functions f, g is

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Consider the vectors in function space $\{1, x^2\}$. Use G-S to produce two orthonormal polynomials $\{q_1(x), q_2(x)\}$

Soln $v_1 = x_1 = \boxed{1}$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle x_2, v_1 \rangle = \langle x^2, 1 \rangle = \int_{-1}^{1} x^2 \cdot 1\, dx$$

$$= \left. \frac{x^3}{3} \right|_{x=-1}^{x=1} = \frac{2}{3}$$

$$\langle v_1, v_1 \rangle = \langle 1, 1 \rangle = \int_{-1}^{1} 1 \cdot 1\, dx = 2$$

$$v_2 = x^2 - \frac{2/3}{2} \cdot 1$$

$$= \boxed{x^2 - \frac{1}{3}}$$

Normalize:

$$\|v_1\| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{2}$$

$$q_1(x) = \frac{1}{\sqrt{2}}$$

$$\|v_2\| = \sqrt{\langle v_2, v_2 \rangle}$$

$$\langle v_2, v_2 \rangle = \langle x^2 - \tfrac{1}{3}, x^2 - \tfrac{1}{3} \rangle$$

$$= \int_{-1}^{1} (x^2 - \tfrac{1}{3})(x^2 - \tfrac{1}{3})\, dx$$

$$= \int_{-1}^{1} (x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9})\, dx$$

$$= \frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x \Big|_{x=-1}^{x=1}$$

$$= \left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) - \left(-\frac{1}{5} + \frac{2}{9} - \frac{1}{9}\right)$$

$$= \frac{8}{45} = \langle v_2, v_2 \rangle, \text{ so } \|v_2\| = \sqrt{\frac{8}{45}}$$

So $q_2(x) = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$

Elementary matrices

Elementary row operations

1) Row swap
2) Row multiplication
3) Row addition

1) Rowswap: [row i] $\longleftrightarrow$ [row j]
Matrix: I_n with row i & row j swapped

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{elementary matrix}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{pmatrix}$$

2) Row multiplication: [row i] $\rightarrow$ k [row i]
Matrix: I_n with row i mult. by k

$$\underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

3) Row addition: [row i] → [row i] + k [row j]

Matrix: I_n with k in the j^{th} entry of row i

$$\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}$$

Ex 1 Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -4 & 2 & 2 \\ 2 & -1 & 5 \end{pmatrix}$$

a) Find the E.R.O. to make A upper triangular (top to bottom)

Soln

$$\begin{pmatrix} 1 & -3 & 2 \\ -4 & 2 & 2 \\ 2 & -1 & 5 \end{pmatrix} \xrightarrow{4R_1 + R_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & -10 & 10 \\ 2 & -1 & 5 \end{pmatrix}$$

$$\xrightarrow{-2R_1+R_3} \begin{pmatrix} 1 & -3 & 2 \\ 0 & -10 & 10 \\ 0 & 5 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2+R_3} \underbrace{\begin{pmatrix} 1 & -3 & 2 \\ 0 & -10 & 10 \\ 0 & 0 & 6 \end{pmatrix}}_{U}$$

b) What are the 3 corresponding elementary matrices?

$$\underset{E_1}{\overset{4R_1+R_2}{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \quad \underset{E_2}{\overset{-2R_1+R_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}} \quad \underset{E_3}{\overset{\frac{1}{2}R_2+R_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}}}$$

$$E_3 E_2 E_1 A = U$$

It turns out

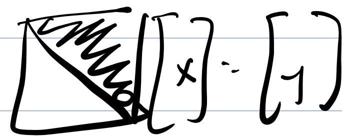
$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{\text{lower triangular}} U$$

HW7 Matlab: Back substitution

Keyline from lecture:

$Ux = y$ 

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}\, x_j\right)$$

i is going from $n \to 1$