An Introduction to Bayes' Theorem

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1 Preliminary Remarks

A few preliminary definitions and theorems will make Bayes' Rule easy to derive.

1.1 Conditional Probability

First, we will define conditional probability. The following definition and theorem are reversed in some textbooks; that is, the theorem is given as the definition and vice versa. Since one implies the other, it really hardly matters.

Definition 1: ***Conditional Probability***
The conditional probability of an event A, given the event B, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) \neq 0$.

Theorem 1: ***Multiplication Theorem***
For any events A and B,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

Proof:
Since $P(A|B) = \frac{P(A \cap B)}{P(B)}$, it follows that $P(A \cap B) = P(A|B)P(B)$.

Notice that since $P(A \cap B) = P(B \cap A)$ and $P(B|A) = \frac{P(B \cap A)}{P(A)}$, we have $P(A \cap B) = P(B|A)P(A)$.

$\mathcal{Q.E.D.}$

Remember that, intuitively, the probability that an event A happened is simply the number of ways A can happen divided by the total number of scenarios.

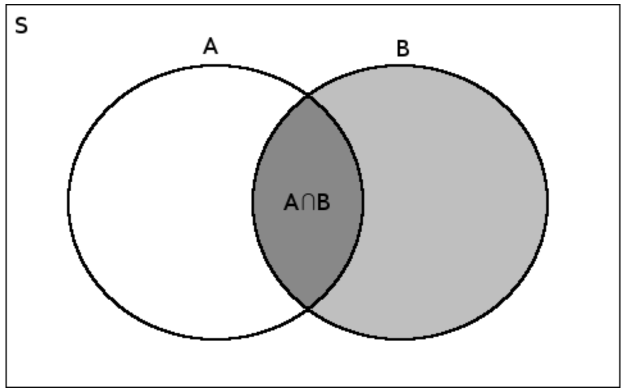

Figure 1: The Law of Conditional Probability

With this in mind, the definition of Conditional Probability should make sense upon closer inspection. When we say A given B, we are saying that B already happened. Thus, we restrict our sample space to B. Then, the probability that A happened is simply the probability of $A \cap B$ over the probability of B.

The Multiplication Theorem follows algebraically from the Law of Conditional Probability.

1.2 Total Probability

Theorem 2: ***Total Probability***
If $B_1, ..., B_k$ is a collection of mutually exclusive and exhaustive events, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

Proof:
The events $A \cap B_1, A \cap B_2, \ldots, A \cap B_k$ are mutually exclusive, so it follows that

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i).$$

The theorem results from applying the Theorem 1 to each term in this summation. $\mathcal{Q.E.D.}$

The Law of Total Probability, while nasty in notation, is really not so bad. It is best thought of as putting together a jigsaw puzzle. For example, in Figure 2, the total probability of A is the sum of the "pieces," the probabilities of A intersected with each B_i.

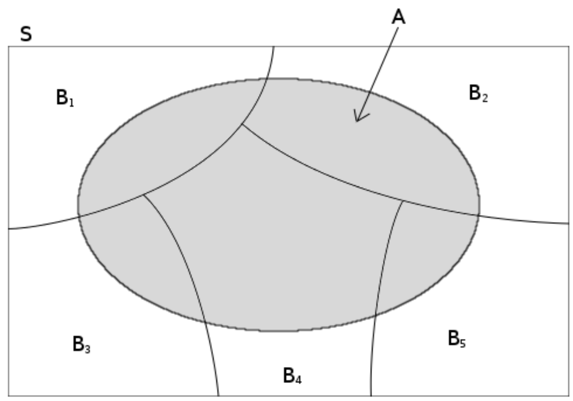

Figure 2: The Law of Total Probability

2 Bayes' Rule

Bayes' Rule is really a compendium of the previous concepts. The proof follows almost trivially from these results.

Theorem 3: ***Bayes' Rule***
If $B_1, ..., B_k$ is a collection of mutually exclusive and exhaustive events, then for each $j = 1, ..., k$,

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

Proof:
From Definition 1 and the Multiplication Theorem, we have

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(B_j)P(A|B_j)}{P(A)}$$

The result follows by applying Theorem 2 to the denominator. $\mathcal{Q.E.D.}$

3 Example

Scientists have developed a new medical test for a very rare disease. Only 0.1% of the population has this disease. When a patient has the disease, the test correctly returns a positive result 99% of the time. However, if a patient does not have the disease, the test returns a false positive 5% of the time.

Suppose a patient has tested positive. What is the probability that he has the disease?

Answer: Define the following events:

D = Patient has disease

D' = Patient does not have disease

T = Test returns a positive

T' = Test returns a negative

We know that a randomly selected patient has a 0.001 prior probability of having the disease: $P(D) = 0.001$.

We also are given that given a patient has the disease, the test returns positive 99% of the time: $P(T|D) = 0.99$.

Finally, if a patient does not have the disease, the test returns a false positive 5% of the time: $P(T|D') = 0.05$.

Our question is, if a patient tests positive, what is the probability he has the disease?: Find $P(D|T)$.

By the definition of conditional probability, we have

$$\begin{aligned}
P(D|T) &= \frac{P(D \cap T)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T \cap D) + P(T \cap D')} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D')P(D')} \\
&= \frac{0.05 \cdot 0.001}{0.05 \cdot 0.001 + 0.99 \cdot 0.999} \\
&= 0.019
\end{aligned}$$

Interpretation: When the test returns positive, the patient actually has the disease only 1.9% of the time. In other words, the test returns a false positive 98.1% of the time.

A tree diagram is helpful for understanding problems that are a series of steps, at each of which there are one or more choices. Figure 3 shows the tree diagram for this problem. Notice that the probabilities on the second layer are conditional probabilities, and that each set of branches emanating from a common point sum to 1.

Since we've been representing things with Venn diagrams so far, it might seem disorienting to go to a tree diagram all of the sudden. Figure 4 shows the same situation represented with a Venn diagram. As you can see, it is a good deal harder to understand.

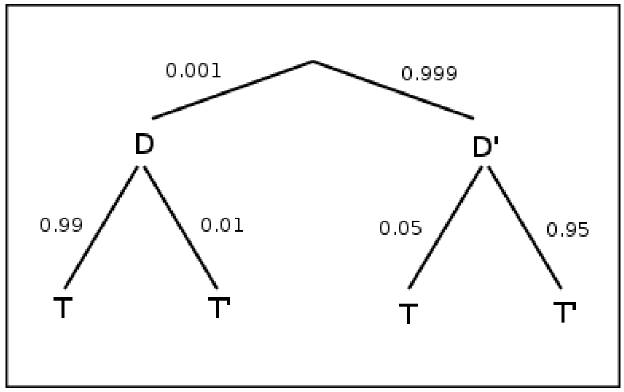

Figure 3: Tree Diagram

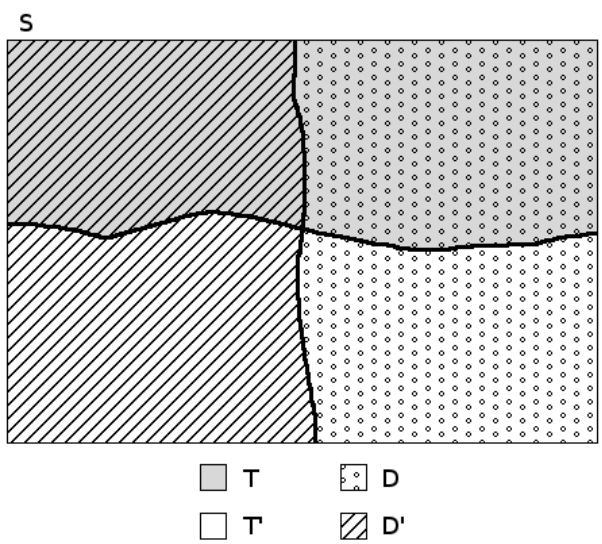

Figure 4: Venn Diagram