TEACHER'S CORNER

In this department *The American Statistician* publishes articles, reviews, and notes of interest to teachers of the first mathematical statistics course and of applied statistics courses. The department includes the Accent on Teaching Materials section; suitable contents for the section are described under the section heading. Articles and notes for the department, but not intended specifically for the section, should be useful to a substantial number of teachers of the indicated types of courses or should have the potential for fundamentally affecting the way in which a course is taught.

Two Rules of Thumb for the Approximation of the Binomial Distribution by the Normal Distribution

MARTIN SCHADER and FRIEDRICH SCHMID*

Approximation of the binomial distribution by the Normal distribution is treated in most introductory books on statistical theory and in many applied textbooks. Though the problem has been investigated thoroughly and fairly precise—and rather difficult—approximations have been developed [see Peizer and Pratt (1968) and Molenaar (1970) for more recent surveys], often the following simple approximation is suggested:

$$F_{n,p}(k) \simeq \Phi\left((k + \tfrac{1}{2} - np)/\sqrt{np(1-p)}\right), \qquad k \in \{0, 1, \ldots, n\},$$

where $F_{n,p}(\cdot)$ denotes the distribution function of a binomial distribution with parameters $n \in N$ and $p \in (0, 1)$ and Φ denotes the distribution function of the standard Normal distribution. The term ½ is a continuity correction.

The accuracy of this approximation—considering absolute and relative error for various combinations of n and p—was computed by Raff (1956) and Peizer and Pratt (1968), respectively. In most introductory textbooks, however, so-called "rules of thumb" for the domain of application of the approximation are given. By far the most popular are

$$np(1-p) > 9 \tag{1}$$

and

$$np > 5 \text{ for } 0 < p \leq .5,$$
$$n(1-p) > 5 \text{ for } .5 < p < 1. \tag{2}$$

It may be of some interest to investigate the accuracy of the approximation if p and n obey these rules and n is of moderate size. Though from a purely mathematical point of view the relative error is more relevant than the absolute error, we decided to analyze the latter because a practitioner may be more interested in absolute errors. Let

$$\mathrm{MABS}(n, p) := \max_{k \in \{0,1,\ldots,n\}} \left| F_{n,p}(k) - \Phi\left((k + \tfrac{1}{2} - np)/\sqrt{np(1-p)}\right) \right|$$

be the maximal absolute error of the approximation. Let

$$n_1(p) := \min\{m \in N \mid mp(1-p) > 9\}$$

and

$$n_2(p) := \min\{m \in N \mid mp > 5\}$$

be the least values for n so that Rules (1) and (2) are satisfied, respectively. $n_1(p)$ and $n_2(p)$ were computed for $p = .01(.01).5$ and are displayed in Figure 1. $\mathrm{MABS}(n_1(p), p)$ and $\mathrm{MABS}(n_2(p), p)$ were computed for $p = .01(.01).5$ and are displayed in Figure 2. Rule (1) guarantees increased accuracy (see Fig. 2) at the cost of a larger minimum sample size n (see Fig. 1). Results for $p > .5$ are the same as in Figures 1 and 2 with p replaced by $1 - p$. It can be seen that $\mathrm{MABS}(n_1(p), p)$ and $\mathrm{MABS}(n_2(p), p)$ are very close to straight lines. Application of the method of least squares yields the regression lines

$$y = .022037 - .043778\,p$$

and

$$y = .029436 - .054565\,p$$

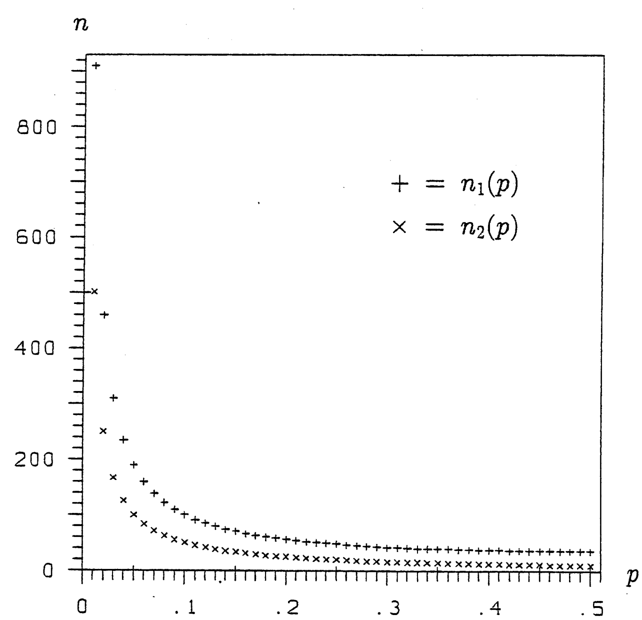

Figure 1. Minimum Required Sample Sizes $n_1(p)$ and $n_2(p)$, According to Rules (1) and (2), Respectively.

* Martin Schader is Professor of Statistics, Institut für Informatik, Universität der Bundeswehr, Holstenhofweg 85, 2000 Hamburg 70, West Germany. Friedrich Schmid is Professor of Statistics, Institut für Statistik und Ökonometrie, Universität Hamburg, Von-Melle-Park 5, 2000 Hamburg 13, West Germany.

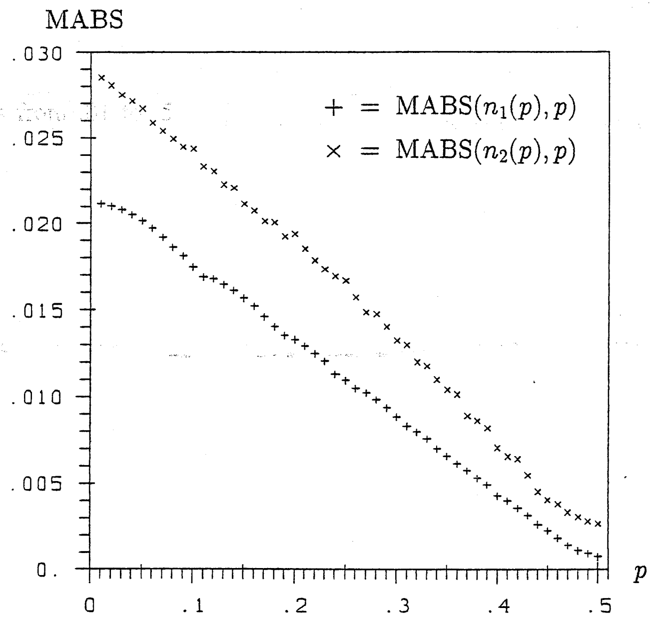

Figure 2. Maximal Absolute Error of the Approximation of the Binomial by the Normal Distribution: MABS($n_1(p), p$) and MABS($n_2(p), p$), According to Rules (1) and (2), Respectively.

for Rules (1) and (2), respectively.

Figure 2 clearly demonstrates that the value of p strongly influences the size of the absolute error for both rules. Indeed, $n_1(p)p(1 - p)$ is roughly equal to 9 for all p, but MABS($n_1(p)$, p) decreases from .0211611 to .0007444 as p increases from .01 to .5. $n_2(p)p$ is roughly equal to 5 for all p, but MABS($n_2(p)$, p) decreases from .0284163 to .0025054 as p increases from .01 to .5.

Our computations show that the bound

$$\text{MABS}(n, p) < .140/\sqrt{np(1 - p)}$$

[given by Johnson and Kotz (1970, p. 64)], which is valid for all $n \in N$ and $p \in (0, 1)$, is not very useful for moderate values of n. According to this bound, one can conclude that

$$\text{MABS}(n_1(p), p) < .140/\sqrt{9} = .0466,$$

whereas for $p = .5$ we have MABS($n_1(.5)$,.5) = .0007444—and even for $p = .01$ we have MABS($n_1(.01)$,.01) = .0211611.

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Pocket-Calculator Approximation for Areas Under the Standard Normal Curve

ROBERT M. NORTON*

1. WHY YET ANOTHER APPROXIMATION?

There are numerous well-known and not-so-well-known methods for approximating areas under the standard normal curve. Let $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-x^2/2)\, dx$. For any t, $\Phi(t)$ may be approximated to any degree of accuracy by the partial sum of a series. However, there are times when expediency must be considered. Shah (1985) considered occasions when, acting as an expert witness in statistics, he had a need to approximate such areas without the aid of a calculator or tables. He suggested the following choice-function approximation for $\Phi(t) - .5$, $t \geq 0$:

$$\begin{array}{ll} t(4.4-t)/10, & 0 \leq t \leq 2.2 \\ .49, & 2.2 < t < 2.6 \\ .50, & t \geq 2.6. \end{array} \qquad (1.1)$$

Shah pointed out a number of sources, including Feller (1968) and Johnson and Kotz (1970), in which approximations appear, and he argued that his approximation was simple and extremely accurate.

Shah's estimate is not designed to approximate right tail areas for $t \geq 2.6$. It would be desirable to approximate such small probabilities. My experience as an expert witness has been that a pocket calculator may be used when one is available.

2. GUIDELINES AND BASIS FOR THE APPROXIMATION

The preceding remarks suggest considering the following criteria for approximation of $\overline{\Phi}(t) = 1 - \Phi(t)$, $t \geq 0$:

1. The error in approximation is small.
2. Overestimates are preferable to underestimates. (From the judicial perspective, p-values are regarded as related to guilt.)

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*Robert M. Norton is Professor, Department of Mathematics, College of Charleston, Charleston, SC 29424. The author is grateful for helpful referee and editorial suggestions.