THE SKEW-NORMAL APPROXIMATION OF THE BINOMIAL DISTRIBUTION

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INTRODUCTION

$X \sim Bin(n, p)$ where $0 < p < 1$ and $n = 1, 2, 3, \ldots$

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$$F_X(x) = P(X \leq x) = \sum_{k=0}^{x} f_X(k)$$

INTRODUCTION

The binomial cdf is easy to calculate for small n ...

But as n gets larger, it becomes increasingly difficult.

For example ...

INTRODUCTION

When $n = 3$,

$$F(1) = \binom{3}{1} p^1 q^2 + \binom{3}{0} p^0 q^3$$

When $n = 25$,

$$\begin{aligned}
F(12) = &\binom{25}{12} p^{12}q^{13} + \binom{25}{11} p^{11}q^{14} + \binom{25}{10} p^{10}q^{15} + \binom{25}{9} p^9 q^{16} \\
&+ \binom{25}{8} p^8 q^{17} + \binom{25}{7} p^7 q^{18} + \binom{25}{6} p^6 q^{19} + \binom{25}{5} p^5 q^{20} \\
&+ \binom{25}{4} p^4 q^{21} + \binom{25}{3} p^3 q^{22} + \binom{25}{2} p^2 q^{23} + \binom{25}{1} p^1 q^{24} \\
&+ \binom{25}{0} p^0 q^{25}
\end{aligned}$$

INTRODUCTION

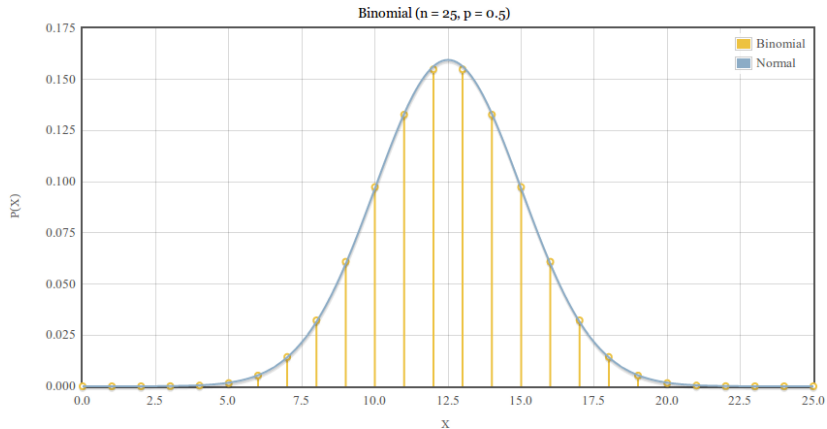Normal Approximation of the Binomial:

$$F_X(x) \approx \Phi\left(\frac{x + 0.5 - \mu}{\sigma}\right),$$

where $\mu = np$, $\sigma = \sqrt{np(1-p)}$, and Φ is the standard normal cdf.

When does this work well? ...

INTRODUCTION

Normal Approximation of the Binomial:

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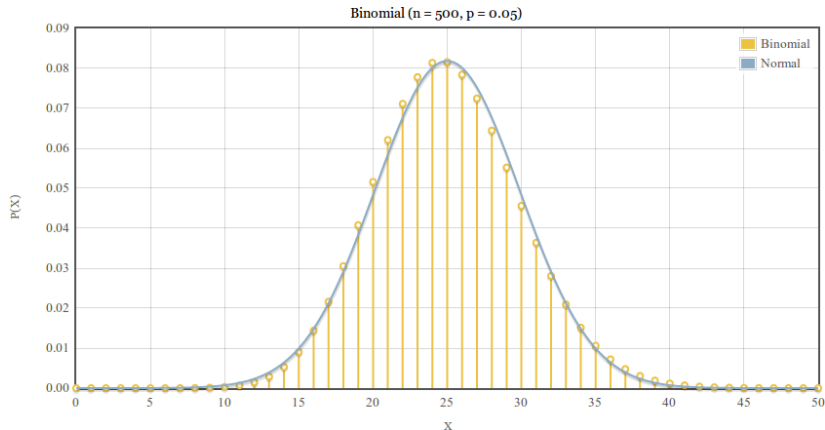When does this work well? ... In a nutshell, when the binomial is symmetric.

INTRODUCTION

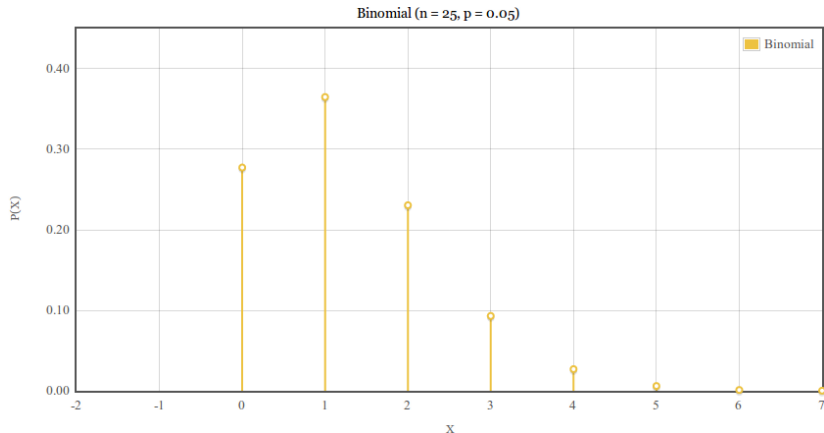The binomial is symmetric when $p = 0.5$

Binomial (n = 25, p = 0.5)

INTRODUCTION

The binomial is symmetric when $p = 0.5$ or n is very large.

Binomial (n = 500, p = 0.05)

INTRODUCTION

However, when *n* is medium and *p* is extreme ...

Binomial (n = 25, p = 0.05)
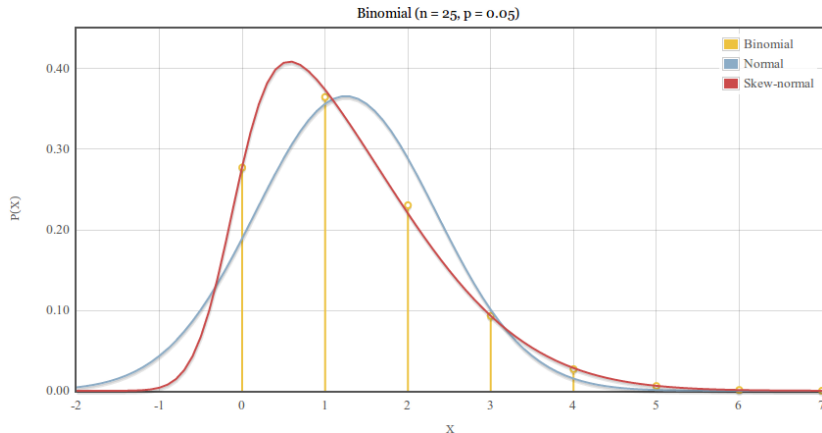
the binomial is very skewed ...

INTRODUCTION

However, when *n* is medium and *p* is extreme ...
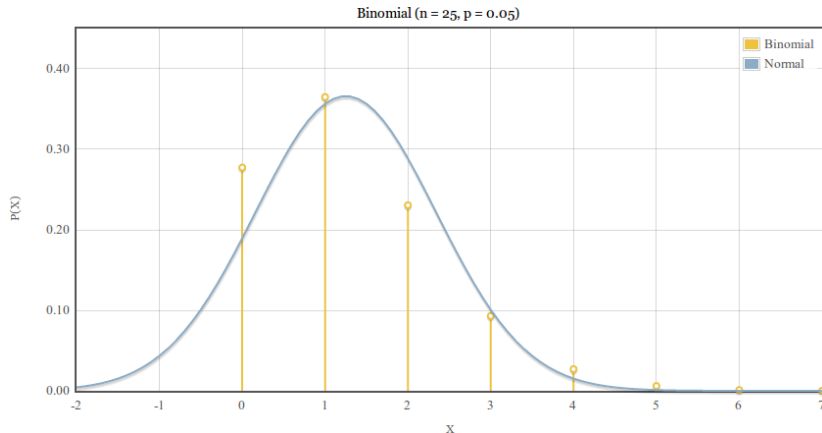
and the normal approximation doesn't work very well.

However, when *n* is medium and *p* is extreme ...

Introducing ... the skew-normal distribution.

What we're going to cover:

1. Skew-Normal distribution – basic properties

OUTLINE

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1. Skew-Normal distribution – basic properties
2. Method of Moments – derive an approximation

What we're going to cover:

1. Skew-Normal distribution – basic properties
2. Method of Moments – derive an approximation
3. Accuracy – compare to the normal approximation

DEFINITION (SKEW-NORMAL)

Let Y be a skew-normal distribution, with location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$, and shape parameter $\lambda \in \mathbb{R}$. Then Y has pdf

$$f(x|\mu, \sigma, \lambda) = \frac{2}{\sigma} \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \cdot \Phi\left(\frac{\lambda(x - \mu)}{\sigma}\right), \quad x \in \mathbb{R},$$

where ϕ is the standard normal pdf and Φ is the standard normal cdf.

We write $Y \sim SN(\mu, \sigma, \lambda)$.

So where did this funny-looking pdf come from? ...

THE SKEW-NORMAL DISTRIBUTION: FOUNDATIONS

So where did this funny-looking pdf come from? ...

LEMMA

If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one-dimensional distribution function such that G' exists and is a density symmetric about 0, then

$$f(z) = 2 \cdot f_0(z) \cdot G\{w(z)\} \quad (-\infty < z < \infty)$$

*is a density function for any odd function $w(\cdot)$. (Lemma 1, **?**)*

Basic properties:

$$
\begin{aligned}
E(Y) &= \mu + b\delta\sigma \\
E(Y^2) &= \mu^2 + 2b\delta\mu\sigma + \sigma^2 \\
E(Y^3) &= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 \\
Var(Y) &= \sigma^2(1 - b^2\delta^2)
\end{aligned}
$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. (**?**)

THE SKEW-NORMAL DISTRIBUTION: FOUNDATIONS

What happens when $\lambda = 0$?

$$
\begin{aligned}
f(x|\mu, \sigma, \lambda = 0) &= \frac{2}{\sigma} \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \cdot \Phi(0) \\
&= \frac{2}{\sigma} \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \cdot 0.5 \\
&= \frac{1}{\sigma} \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right),
\end{aligned}
$$

which is the pdf of the normal distribution (μ, σ).

THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

DEFINITION (STANDARD SKEW-NORMAL)

The $SN(0, 1, \lambda)$ distribution is called the standard skew-normal and has pdf

$$f_Z(x|\lambda) = 2 \cdot \phi(x) \cdot \Phi(\lambda x), \quad x \in \mathbb{R}.$$

Similar to the normal and standard normal, $Z = \frac{Y - \mu}{\sigma}$ and $Y = \sigma Z + \mu$.

THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

PROPERTY (1)

If $Z \sim SN(0, 1, \lambda)$, then $(-Z) \sim SN(0, 1, -\lambda)$.

PROPERTY (2)

If $Z \sim SN(0, 1, \lambda)$, then $Z^2 \sim \chi_1^2$ (chi-square with 1 degree of freedom).

THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

PROPERTY (3)

As $\lambda \to \pm\infty$, $SN(0, 1, \lambda)$ *tends to the half normal distribution,* $\pm|N(0, 1)|$.

PROPERTY (4)

The moment generating function of $SN(0, 1, \lambda)$ *is*

$$M(t|\lambda) = 2 \cdot \Phi(\delta t) \cdot e^{t^2/2},$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$ *and* $t \in (-\infty, \infty)$.

Now we're going to derive the approximation!

Game plan:

1. Find the first three central moments of the binomial and the first three central moments of the skew-normal.

   What are central moments?: $E(X)$, $E([X - E(X)]^2)$, $E([X - E(X)]^3)$.

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2. Set them equal to each other.

METHOD OF MOMENTS: OVERVIEW

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2. Set them equal to each other.

3. Take *n* and *p* to be constants; solve for μ, σ, and λ.

The first two are easy:

$$E(B) = np$$
$$E([B - E(B)]^2) = Var(B) = np(1 - p)$$

The third one takes some elbow grease.

First we'll need to find $E(B^2)$ and $E(B^3)$.

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

$$E(B^2) = Var(B) + [E(B)]^2$$
$$= np(1-p) + n^2 p^2$$
$$= np - np^2 + n^2 p^2$$

We will get $E(B^3)$ via the third factorial moment, E[B(B-1)(B-2)].

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

$E[B(B-1)(B-2)]$

$= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$

$= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$

$= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x! \, (n-x)!} \, p^x q^{n-x}$

$= \sum_{x=3}^{n} \frac{n!}{(x-3)! \, (n-x)!} \, p^x q^{n-x}$

$= \sum_{x=3}^{n} n(n-1)(n-2) p^3 \cdot \frac{(n-3)!}{(x-3)! \, (n-x)!} \, p^{x-3} q^{n-x}$

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

Let $y = x - 3$; then $x = y + 3$, and $x = 3$, $x = n \Rightarrow y = 0$, $y = n - 3$:

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3}q^{n-x}$$

$$= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\, p^y q^{n-(y+3)}$$

$$= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\, p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,p) \text{ summed over its domain]} = 1}$$

$$= n(n-1)(n-2)p^3$$

$$= n^3 p^3 - 3n^2 p^3 + 2np^3$$

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

To get $E(B^3)$, we expand the left side of the previous equation:

$$E[B(B-1)(B-2)]$$
$$= E\left[B^3 - 3B^2 + 2B\right]$$
$$= E(B^3) - 3E(B^2) + 2E(B)$$
$$= E(B^3) - 3(np - np^2 + n^2p^2) + 2np$$
$$= E(B^3) - 3np + 3np^2 - 3n^2p^2 + 2np$$
$$= E(B^3) + 3np^2 - 3n^2p^2 - np$$

Left side: $E(B^3) + 3np^2 - 3n^2p^2 - np$

Right side: $n^3p^3 - 3n^2p^3 + 2np^3$

Set them equal and solve for $E(B^3)$:

$$E(B^3) + 3np^2 - 3n^2p^2 - np = n^3p^3 - 3n^2p^3 + 2np^3$$
$$\Rightarrow \quad E(B^3) = n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np$$

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

Now we can (finally!) compute the third central moment:

$$
E\left([B - E(B)]^3\right)
$$
$$
= E\left(B^3 - 3B^2 E(B) + 3B[E(B)]^2 - [E(B)]^3\right)
$$
$$
= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3
$$
$$
= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3
$$
$$
= (n^3 p^3 - 3n^2 p^3 + 2np^3 - 3np^2 + 3n^2 p^2 + np)
$$
$$
\quad - 3(np - np^2 + n^2 p^2)(np) + 2(np)^3
$$
$$
= \cancel{n^3 p^3} - \cancel{3n^2 p^3} + 2np^3 - 3np^2 + \cancel{3n^2 p^2} + np
$$
$$
\quad - \cancel{3n^2 p^2} + \cancel{3n^2 p^3} - \cancel{3n^3 p^3} + \cancel{2n^3 p^3}
$$
$$
= 2np^3 - 3np^2 + np
$$
$$
= np(p - 1)(2p - 1)
$$

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

Let's restate our results:

$$E(B) = np,$$
$$E([B - E(B)]^2) = np(1 - p),$$
$$E([B - E(B)]^3) = np(p - 1)(2p - 1)$$

The first and second central moments are the mean and variance.

$$E(Y) = \mu + b\delta\sigma$$
$$Var(Y) = \sigma^2(1 - b^2\delta^2)$$

METHOD OF MOMENTS: CENTRAL MOMENTS OF THE SKEW-NORMAL

Again, the third one is a little harder:

$$E([Y - E(Y)]^3)$$
$$= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3$$
$$= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3)$$
$$\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2(\mu + b\delta\sigma)^3$$
$$= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 - 3\mu^3 - 3b\delta\mu^2\sigma$$
$$\quad - 6b\delta\mu^2\sigma - 6b^2\delta^2\mu\sigma^2 - 3\mu\sigma^2 - 3b\delta\sigma^3 + 2\mu^3 + 6b\delta\mu^2\sigma$$
$$\quad + 6b^2\delta^2\mu\sigma^2 + 2b^3\delta^3\sigma^3$$
$$= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3$$
$$= b\delta^3\sigma^3(2b^2 - 1)$$

Our results, restated:

$$E(Y) = \mu + b\delta\sigma \qquad = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}}$$

$$E([Y - E(Y)]^2) = \sigma^2(1 - b^2\delta^2) \quad = \sigma^2\left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)$$

$$E([Y - E(Y)]^3) = b\delta^3\sigma^3(2b^2 - 1) = \sigma^3\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3\left(\frac{4}{\pi} - 1\right)$$

METHOD OF MOMENTS: DERIVING AN APPROXIMATION

Set the central moments of the binomial equal to the central moments of the skew-normal:

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{1a}$$

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{1b}$$

$$np(p-1)(2p-1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right) \tag{1c}$$

To get λ, divide the cube of (1b) by the square of (1c):

$$\frac{\sigma^6 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6 \cdot \frac{2}{\pi} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6 \left(\frac{4}{\pi} - 1\right)^2} = \frac{n^3 p^3 (1-p)^3}{n^2 p^2 (p-1)^2 (2p-1)^2}$$

$$\Rightarrow \quad \frac{\left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi} \left(\frac{\lambda^2}{1+\lambda^2}\right)^3 \left(\frac{4}{\pi} - 1\right)^2} = \frac{np(1-p)}{(1-2p)^2} \tag{2}$$

Equation (2) can be solved for λ^2. Then take λ to be

$$\lambda = \{\text{sign of } (1 - 2p)\}\sqrt{\lambda^2}.$$

METHOD OF MOMENTS: DERIVING AN APPROXIMATION

Once you have λ, solve for σ and then μ.

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1 + \lambda^2} \right) \quad \Rightarrow \quad \sigma = \sqrt{\frac{np(1-p)}{1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}}}$$

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}} \quad \Rightarrow \quad \mu = np - \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}}$$

METHOD OF MOMENTS: DERIVING AN APPROXIMATION

When $p = 0.5$, the binomial is symmetric, so $\lambda = 0$.

This takes us back to the usual normal approximation:

$$\sigma = \sqrt{\frac{np(1-p)}{1 - \frac{2}{\pi} \cdot \frac{0^2}{1+0^2}}} = \sqrt{\frac{np(1-p)}{1}} = \sqrt{np(1-p)}$$

$$\mu = np - \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{0}{\sqrt{1 + 0^2}} = np - 0 = np$$

METHOD OF MOMENTS: RESTRICTIONS

Let $u = \frac{\lambda^2}{1+\lambda^2}$ and $v = 1/u = \frac{1+\lambda^2}{\lambda^2}$.

Then we can rewrite (2):

$$\frac{\left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3 \left(\frac{4}{\pi} - 1\right)^2}$$

$$\vdots$$

$$(magic)$$

$$\vdots$$

$$\left(v - \frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) = g(v)$$

METHOD OF MOMENTS: RESTRICTIONS

$g(v)$ is increasing in $v = \frac{1+\lambda^2}{\lambda^2} \geq 1$. Therefore:

$$\min_v g(v) = g(1) = \left(1 - \frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) = 1.009524 \approx 1$$

To be able to solve (2) for λ, we must have

{right hand side of (2)} $\geq$ {min of left hand side of (2)}

$$\frac{np(1-p)}{(1-2p)^2} \geq 1$$

$$np(1-p) \geq (1-2p)^2. \tag{3}$$

From (3), we can answer two questions:

METHOD OF MOMENTS: RESTRICTIONS

One:

Given p, what is the least n necessary?

$$n \geq \frac{(1 - 2p)^2}{p(1 - p)}$$

METHOD OF MOMENTS: RESTRICTIONS

Least possible *n*, given a fixed *p*:
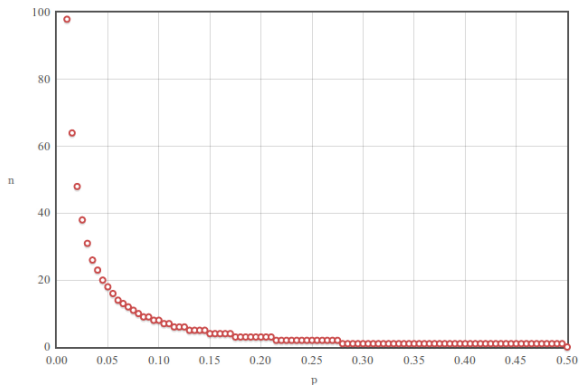
METHOD OF MOMENTS: RESTRICTIONS

Two:

Given n, what is the range of possible p's?

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}} \leq p \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}}$$

Range of possible *p*, given a fixed *n*:
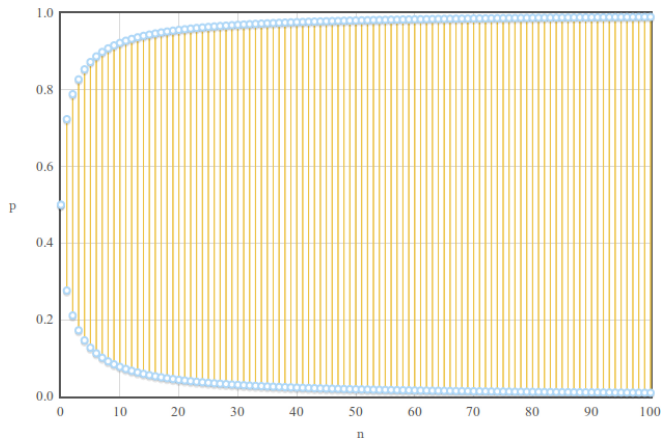
DEMONSTRATING IMPROVED ACCURACY

We have an approximation!!

But is it more accurate? ...

We have an approximation!!

But is it more accurate? ... Answer: Yes!

The easiest way of gauging accuracy is by visual inspection.
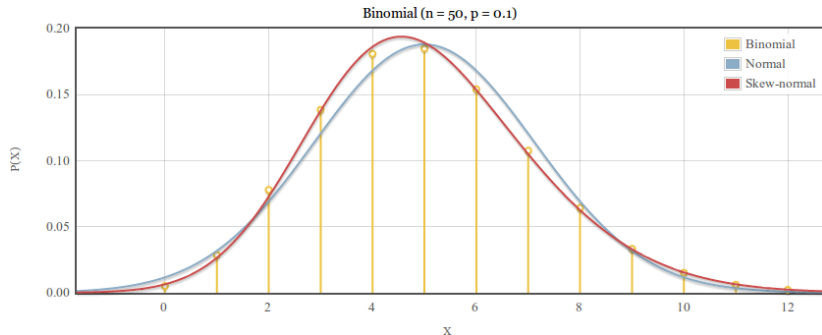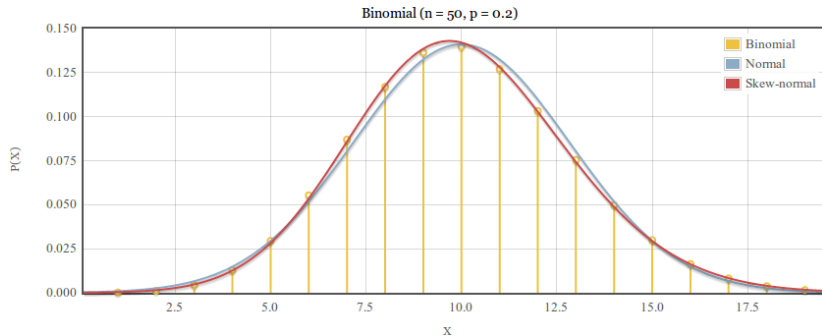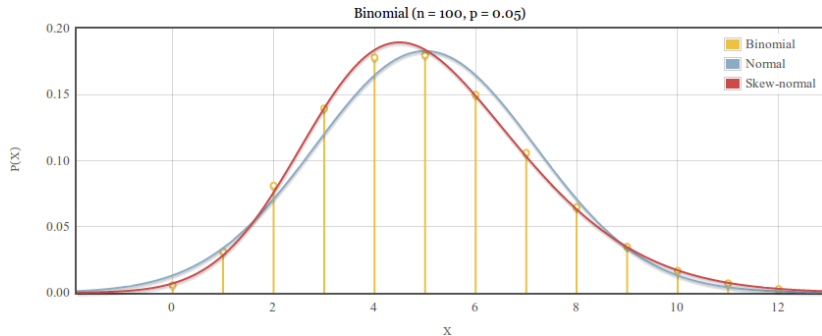
DEMONSTRATING IMPROVED ACCURACY: VISUAL
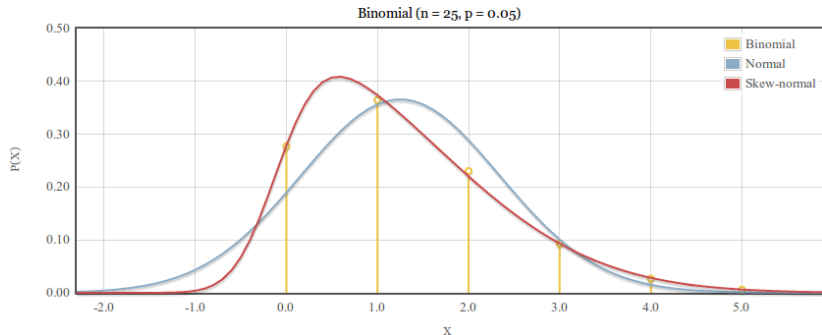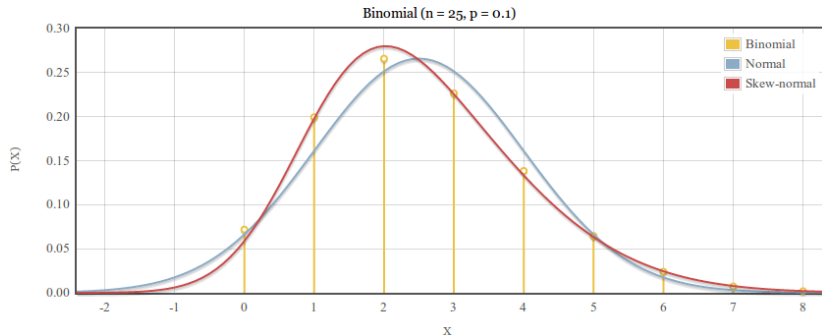
DEMONSTRATING IMPROVED ACCURACY: VISUAL

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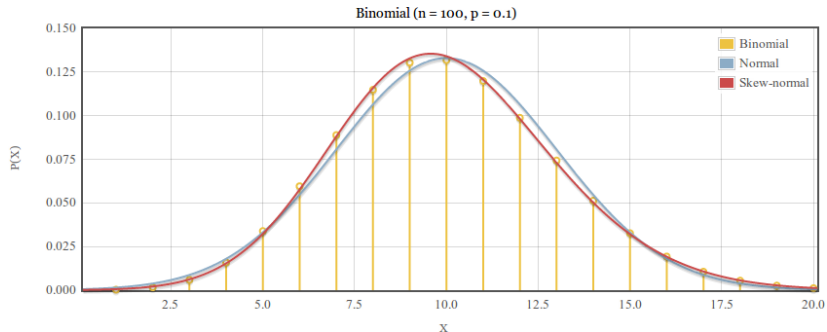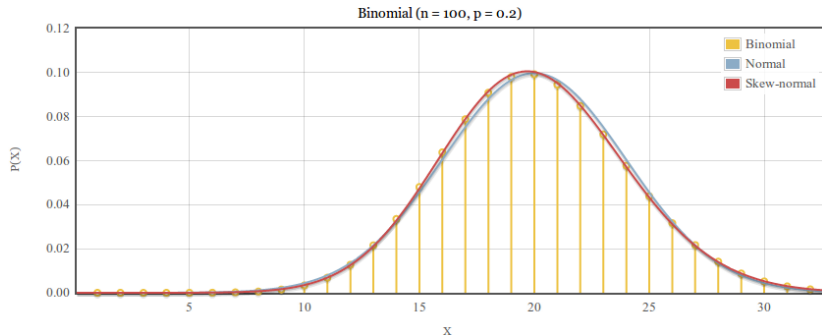Binomial (n = 50, p = 0.05)

DEMONSTRATING IMPROVED ACCURACY: VISUAL

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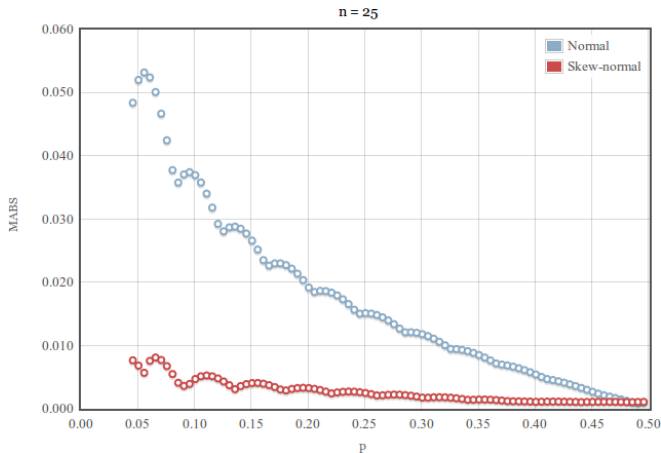
Binomial (n = 100, p = 0.2)
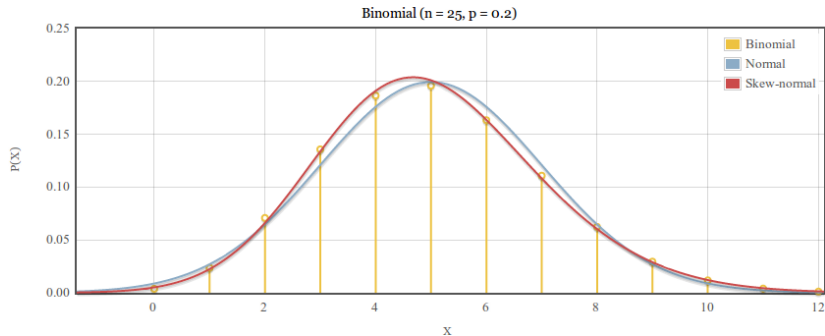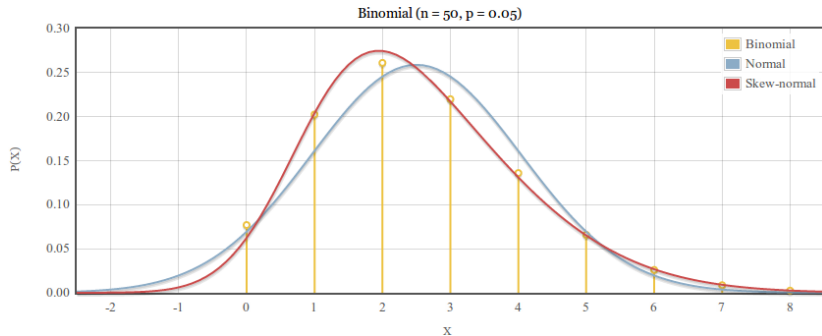
A more numerical way of gauging accuracy is the *MABS*, defined by **?** as

$$\text{MABS}(n, p) = \max_{k \in \{0, 1, \ldots, n\}} \left| F_{B(n,p)}(k) - F_{\text{appr}(n,p)}(k + 0.5) \right|.$$
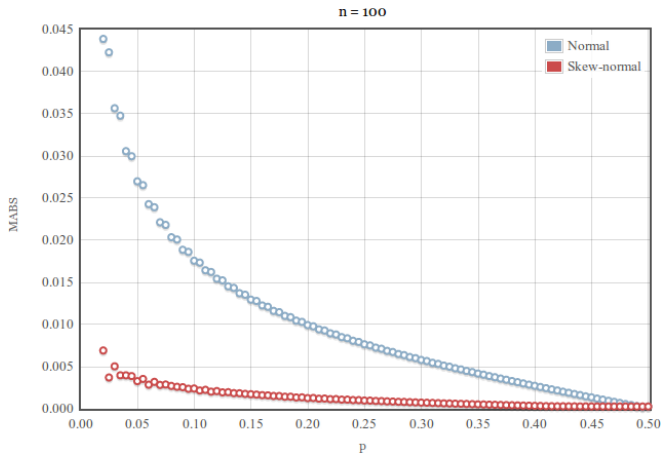
MABS as a function of *p*, with fixed $n = 25$:

DEMONSTRATING IMPROVED ACCURACY: MABS
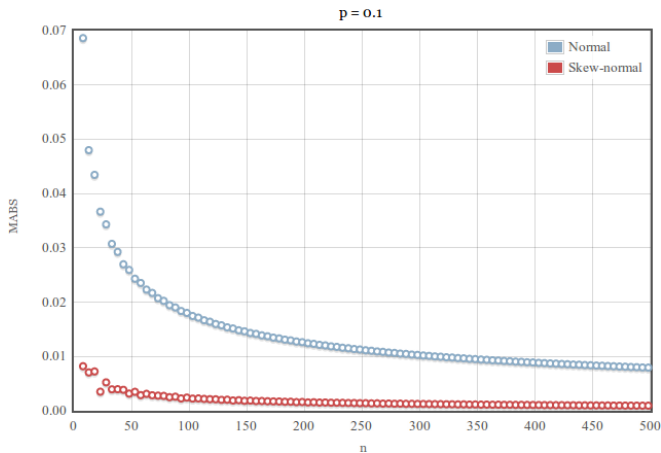
MABS as a function of *p*, with fixed $n = 100$:

MABS as a function of n, with fixed $p = 0.05$:

MABS as a function of n, with fixed $p = 0.1$:

RESOURCES

Estimations of $SN(\mu, \sigma, \lambda)$ for $Bin(n, p)$

		n			
		25	50	100	250	500
	0.05	(-0.11, 1.74, 4.56)	(0.79, 2.30, 2.54)	(2.85, 3.06, 1.86)	(9.58, 4.52, 1.38)	(21.32, 6.11, 1.15)
	0.10	(0.89, 2.20, 2.31)	(2.97, 2.94, 1.74)	(7.44, 3.94, 1.40)	(21.53, 5.88, 1.10)	(45.62, 8.01, 0.94)
	0.15	(2.02, 2.49, 1.79)	(5.32, 3.34, 1.43)	(12.25, 4.51, 1.19)	(33.77, 6.77, 0.96)	(70.30, 9.27, 0.82)
	0.20	(3.23, 2.67, 1.50)	(7.76, 3.61, 1.24)	(17.18, 4.89, 1.04)	(46.18, 7.39, 0.85)	(95.18, 10.16, 0.74)
	0.25	(4.49, 2.79, 1.29)	(10.28, 3.78, 1.09)	(22.20, 5.15, 0.93)	(58.71, 7.83, 0.76)	(120.22, 10.80, 0.67)
	0.30	(5.80, 2.85, 1.12)	(12.86, 3.88, 0.95)	(27.31, 5.32, 0.82)	(71.34, 8.12, 0.68)	(145.39, 11.24, 0.60)
	0.35	(7.17, 2.86, 0.96)	(15.50, 3.92, 0.83)	(32.49, 5.39, 0.72)	(84.09, 8.28, 0.60)	(170.70, 11.50, 0.53)
	0.40	(8.59, 2.83, 0.80)	(18.23, 3.89, 0.70)	(37.76, 5.39, 0.61)	(96.96, 8.32, 0.51)	(196.18, 11.60, 0.45)
p	0.45	(10.12, 2.73, 0.61)	(21.08, 3.79, 0.53)	(43.21, 5.29, 0.47)	(110.07, 8.23, 0.40)	(221.93, 11.54, 0.35)
	0.50	(12.50, 2.50, 0.00)	(25.00, 3.54, 0.00)	(50.00, 5.00, 0.00)	(125.00, 7.91, 0.00)	(250.00, 11.18, 0.00)
	0.55	(14.88, 2.73, -0.61)	(28.92, 3.79, -0.53)	(56.79, 5.29, -0.47)	(139.93, 8.23, -0.40)	(278.07, 11.54, -0.35)
	0.60	(16.41, 2.83, -0.80)	(31.77, 3.89, -0.70)	(62.24, 5.39, -0.61)	(153.04, 8.32, -0.51)	(303.82, 11.60, -0.45)
	0.65	(17.83, 2.86, -0.96)	(34.50, 3.92, -0.83)	(67.51, 5.39, -0.72)	(165.91, 8.28, -0.60)	(329.30, 11.50, -0.53)
	0.70	(19.20, 2.85, -1.12)	(37.14, 3.88, -0.95)	(72.69, 5.32, -0.82)	(178.66, 8.12, -0.68)	(354.61, 11.24, -0.60)
	0.75	(20.51, 2.79, -1.29)	(39.72, 3.78, -1.09)	(77.80, 5.15, -0.93)	(191.29, 7.83, -0.76)	(379.78, 10.80, -0.67)
	0.80	(21.77, 2.67, -1.50)	(42.24, 3.61, -1.24)	(82.82, 4.89, -1.04)	(203.82, 7.39, -0.85)	(404.82, 10.16, -0.74)
	0.85	(22.98, 2.49, -1.79)	(44.68, 3.34, -1.43)	(87.75, 4.51, -1.19)	(216.23, 6.77, -0.96)	(429.70, 9.27, -0.82)
	0.90	(24.11, 2.20, -2.31)	(47.03, 2.94, -1.74)	(92.56, 3.94, -1.40)	(228.47, 5.88, -1.10)	(454.38, 8.01, -0.94)
	0.95	(25.11, 1.74, -4.56)	(49.21, 2.30, -2.54)	(97.15, 3.06, -1.86)	(240.42, 4.52, -1.38)	(478.68, 6.11, -1.15)

RESOURCES

All values in my project were computed using a Python library,
which is freely available online:

http://github.com/joycetipping/skew-normal-capstone/

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