Deriving λ with the Method of Moments

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1 Method of Moments

Let $B \sim Bin(n, p)$ and $Y \sim SN(\mu, \sigma^2, \lambda)$. We will find approximations for μ, σ, and λ by comparing the first, second, and third moments about the mean of B and Y.

1.1 The Moments of the Binomial

The first two moments, the mean and variance, are simply

$$E(B) = np, \quad Var(B) = np(1-p)$$

We can also easily find

$$E(B^2) = Var(B) + [E(B)]^2 = np(1-p) + n^2p^2 = np - np^2 + n^2p^2$$

which we will need for the third moment. We will also need $E(B^3)$, which we will get via the third factorial moment:

$$\begin{aligned}
E[B(B-1)(B-2)] &= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x} \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\, p^x q^{n-x} \\
&= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3} q^{n-x}
\end{aligned}$$

Let $y = x - 3$. Then $x = y + 3$, and $x = 3 \rightarrow y = 0$ and $x = n \rightarrow y = n - 3$.

$$
\begin{aligned}
&= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\; p^y q^{n-(y+3)} \\
&= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\; p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,p) \text{ summed from 0 to } n-3] = 1} \\
&= n(n-1)(n-2)p^3 \\
&= n^3p^3 - 3n^2p^3 + 2np^3
\end{aligned}
$$

Further expanding the left side and solving for $E(B^3)$,

$$
\begin{aligned}
E\left[B^3 - 3B^2 + 2B\right] &= n^3p^3 - 3n^2p^3 + 2np^3 \\
E(B^3) - 3E(B^2) + 2E(B) &= \\
E(B^3) - 3(np - np^2 + n^2p^2) + 2np &= \\
\Rightarrow \qquad E(B^3) &= n^3p^3 - 3n^2p^3 + 2np^3 + 3np - 3np^2 + 3n^2p^2 - 2np \\
&= n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np
\end{aligned}
$$

With these results (and a bit of elbow grease), we can obtain the third moment without too much trouble:

$$
\begin{aligned}
E\left([B - E(B)]^3\right) &= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) - 3np(np - np^2 + n^2p^2) + 2n^3p^3 \\
&= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3} \\
&= 2np^3 - 3np^2 + np \\
&= np(p-1)(2p-1)
\end{aligned}
$$

Our hard-earned results, restated for convenience:

$$
\begin{aligned}
E(B) &= np \\
E([B - E(B)]^2) &= np(1-p) \\
E([B - E(B)]^3) &= np(p-1)(2p-1)
\end{aligned} \tag{1}
$$

1.2 The Moments of the Skew Normal

Now we'll take a look at the moments of the skew normal. According to Equation 1 in Pewsey (2000)

$$
\begin{aligned}
E(Y) &= \mu + b\delta\sigma \\
E(Y^2) &= \mu^2 + 2b\delta\mu\sigma + \sigma^2 \\
Var(Y) &= \sigma^2(1 - b^2\delta^2) \\
E(Y^3) &= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3
\end{aligned}
$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. This takes care of our first two moments; again the third is a little more complicated:

$$
\begin{aligned}
E([Y - E(Y)]^3) &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) \\
&\quad + 2\,(\mu + b\delta\sigma)^3 \\
&= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{9b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} \\
&\quad - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\
&= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3 \\
&= b\delta^3\sigma^3(2b^2 - 1)
\end{aligned}
$$

We restate our results:

$$
\begin{aligned}
E(Y) = \mu + b\delta\sigma \qquad\qquad &= \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \\
E([Y - E(Y)]^2) = \sigma^2(1 - b^2\delta^2) \quad &= \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \\
E([Y - E(Y)]^3) = b\delta^3\sigma^3(2b^2 - 1) &= \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right)
\end{aligned}
\tag{2}
$$

1.3 Solving for μ, σ, λ

Now we set the two sets of moments equal to each other and, taking n and p as constants, solve for μ, σ and λ.

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{3a}$$

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{3b}$$

$$np(p-1)(2p-1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right) \tag{3c}$$

To get λ, we divide the cube of (3b) by the square of (3c):

$$\begin{aligned}
\frac{\sigma^6 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6 \cdot \frac{2}{\pi} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6 \left(\frac{4}{\pi} - 1\right)^2} &= \frac{n^3 p^3 (1-p)^3}{n^2 p^2 (p-1)^2 (2p-1)^2} \\
\Rightarrow \quad \frac{\left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi} \left(\frac{\lambda^2}{1+\lambda^2}\right)^3 \left(\frac{4}{\pi} - 1\right)^2} &= \frac{np(1-p)}{(1-2p)^2} \qquad (4)
\end{aligned}$$

The above equation (4) is a rational expression in λ^2 that can be solved with either a considerable amount of manual labor or, more efficiently, with a computer algebra system. Once we have λ^2, then λ is simply either the positive or negative square root, as determined by the sign of $(1-2p)$: When $p < 0.5$, the binomial skews left, so λ should be negative; when $p > 0.5$, the binomial skews right, so λ should be positive. Thus:

$$\lambda = \{\text{sign of } (1-2p)\} \sqrt{\lambda^2} \tag{5}$$

Note to Dr. Guffey: **I am trying to find an algebraic reason for the sign, but I'm not seeing it in the equations.**

Having secured λ, we can find σ using (3b):

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \quad \Rightarrow \quad \sigma = \sqrt{\frac{np(1-p)}{1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}}} \tag{6}$$

Similarly, with both λ and σ, equation (3a) yields μ:

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \quad \Rightarrow \quad \mu = np - \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{7}$$

A Solving for λ

$$\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3 \Bigg/ \left[\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2\right] - \frac{np(1-p)}{(1-2p)^2}$$
$$\left(\frac{\pi(1+\lambda^2)-2\lambda^2}{\pi(1+\lambda^2)}\right)^3 \Bigg/ \left(\frac{2(4-\pi)^2\lambda^6}{\pi^3(1+\lambda^2)^3}\right) - \frac{np(1-p)}{(1-2p)^2}$$
$$\left(\frac{\pi^3(1+\lambda^2)^3-3\pi^2(1+\lambda^2)^2\cdot 2\lambda^2+3\pi(1+\lambda^2)\cdot 4\lambda^4-8\lambda^6}{\cancel{\pi^3(1+\lambda^2)^3}}\right)\cdot\left(\frac{\cancel{\pi^3(1+\lambda^2)^3}}{2(4-\pi)^2\lambda^6}\right) - \frac{np(1-p)}{(1-2p)^2}$$
$$\frac{\pi^3(1+3\lambda^2+3\lambda^4+\lambda^6)-6\pi^2\lambda^2(1+2\lambda^2+4\lambda^4)+12\pi\lambda^4(1+\lambda^2)-8\lambda^6}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$
$$\frac{\pi^3+3\pi^3\lambda^2+3\pi^3\lambda^4+\pi^3\lambda^6-6\pi^2\lambda^2-12\pi^2\lambda^4-24\pi^2\lambda^6+12\pi\lambda^4+12\pi\lambda^6-8\lambda^6}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$
$$\frac{\lambda^6(\pi^3-24\pi^2+12\pi-8)+\lambda^4(3\pi^3-12\pi^2+12\pi)+\lambda^2(3\pi^3-6\pi^2)+\pi^3}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$

Taking $c_1 = \pi^3 - 24\pi^2 + 12\pi - 8$, $c_2 = 3\pi^3 - 12\pi^2 + 12\pi$, $c_3 = 3\pi^3 - 6\pi^2$, $c_4 = \pi^3$, and $c_5 = 2(4-\pi)^2$, we can simplify this to

$$\frac{c_1\lambda^6+c_2\lambda^4+c_3\lambda^2+c_4}{c_5\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$
$$\frac{c_1(1-2p)^2\lambda^6+c_2(1-2p)^2\lambda^4+c_3(1-2p)^2\lambda^2+c_4(1-2p)^2-c_5\,np(1-p)\lambda^6}{c_5(1-2p)^2\lambda^6}$$
$$\frac{\left[c_1(1-2p)^2-c_5\,np(1-p)\right]\lambda^6+\left[c_2(1-2p)^2\right]\lambda^4+\left[c_3(1-2p)^2\right]\lambda^2+c_4(1-2p)^2}{c_5(1-2p)^2\,\lambda^6}$$

B Curiosity

As a curiosity, I was unable to get Pewsey and Azzalini to agree with each other on $E(Z^3)$. According to Pewsey (2000),

$$E(Y^3) = \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 \tag{8}$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}} \in (-1, 1)$. Since $Y = \mu + \sigma Z$, by the linearity of expected value, we also have

$$
\begin{aligned}
E(Y^3) &= E\left[(\mu+\sigma Z)^3\right] \\
&= E(\mu^3 + 3\mu^2\sigma Z + 3\mu\sigma^2 Z^2 + \sigma^3 Z^3) \\
&= \mu^3 + 3\mu^2\sigma \ E(Z) + 3\mu\sigma^2 \ E(Z^2) + \sigma^3 \ E(Z^3) \\
&= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + \sigma^3 \ E(Z^3) \qquad (9)
\end{aligned}
$$

By comparing equations (8) and (9) and eliminating terms, we arrive at

$$
\begin{aligned}
\sigma^3 \ E(Z^3) &= 3b\delta\sigma^3 - b\delta^3\sigma^3 \\
\Rightarrow \quad E(Z^3) &= 3b\delta - b\delta^3 \\
&= b\delta(3-\delta^2) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \cdot \left(3 - \frac{\lambda^2}{1+\lambda^2}\right) \qquad (10)
\end{aligned}
$$

However, according to equation (6.5?) in Azzalini (2005),

$$
E(Z^r) = \begin{cases} 1 \times 3 \times \cdots \times (r-1) & \text{if r is even} \\ \dfrac{\sqrt{2}\ (2k+1)!\ \lambda}{\sqrt{\pi}\ (1+\lambda^2)^{k+1/2}\ 2^k} \displaystyle\sum_{m=0}^{k} \dfrac{m!\ (2\lambda)^{2m}}{(2m+1)!\ (k-m)!} & \text{if } r = 2k+1 \text{ and } k = 0, 1, ... \end{cases} \qquad (11)
$$

So, for $E(Z^3)$, we have $r = 2k+1 = 3$ and $k = 1$:

$$
\begin{aligned}
E(Z^3) &= \frac{\sqrt{2} \cdot 3! \cdot \lambda}{\sqrt{\pi} \cdot (1+\lambda^2)^{3/2} \cdot 2} \sum_{m=0}^{1} \frac{m!\ (2\lambda)^{2m}}{(2m+1)!\ (1-m)!} \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\lambda}{(1+\lambda^2)^{3/2}} \cdot \left(\frac{0!(2\lambda)^0}{1!1!} + \frac{1!(2\lambda)^2}{3!0!}\right) \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\lambda}{(1+\lambda^2)^{3/2}} \cdot \left(1 + \frac{2}{3}\lambda^2\right) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{(\sqrt{1+\lambda^2})^3} \cdot (3+2\lambda^2) \qquad (12)
\end{aligned}
$$

Alas, equations (10) and (12) do not remotely line up.