The Skew-Normal Approximation of the Binomial Distribution

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Introduction

Definition (Binomial)

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and cdf

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{x} f_X(k).$$

INTRODUCTION

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But as n gets larger, it becomes increasingly difficult.

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For example ...

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When $n = 3$,

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When $n = 25$,

$$\begin{aligned} F(12) = &\binom{25}{12} p^{12} q^{13} + \binom{25}{11} p^{11} q^{14} + \binom{25}{10} p^{10} q^{15} + \binom{25}{9} p^{9} q^{16} \\ &+ \binom{25}{8} p^{8} q^{17} + \binom{25}{7} p^{7} q^{18} + \binom{25}{6} p^{6} q^{19} + \binom{25}{5} p^{5} q^{20} \\ &+ \binom{25}{4} p^{4} q^{21} + \binom{25}{3} p^{3} q^{22} + \binom{25}{2} p^{2} q^{23} + \binom{25}{1} p^{1} q^{24} \\ &+ \binom{25}{0} p^{0} q^{25} \end{aligned}$$

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A common technique is to use the normal distribution as an approximation:

$$F_X(x) \approx \Phi\left(\frac{x + 0.5 - \mu}{\sigma}\right),$$

where $\mu = np$, $\sigma = \sqrt{np(1-p)}$, and Φ is the standard normal cdf.

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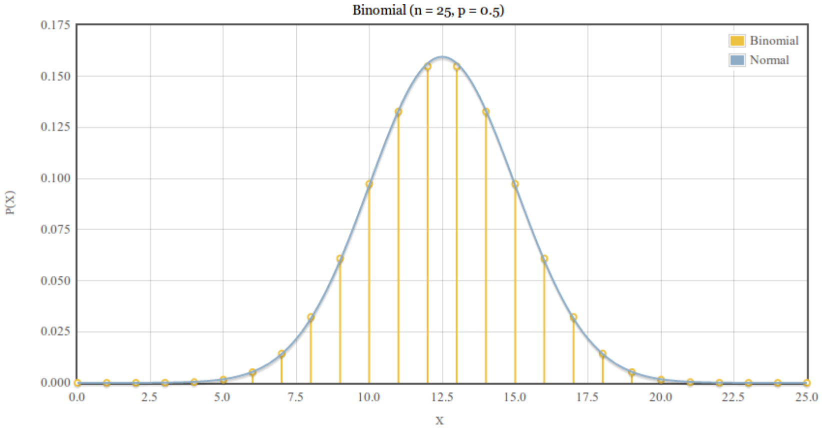
When does this work well? ... In a nutshell, when the binomial is symmetric.

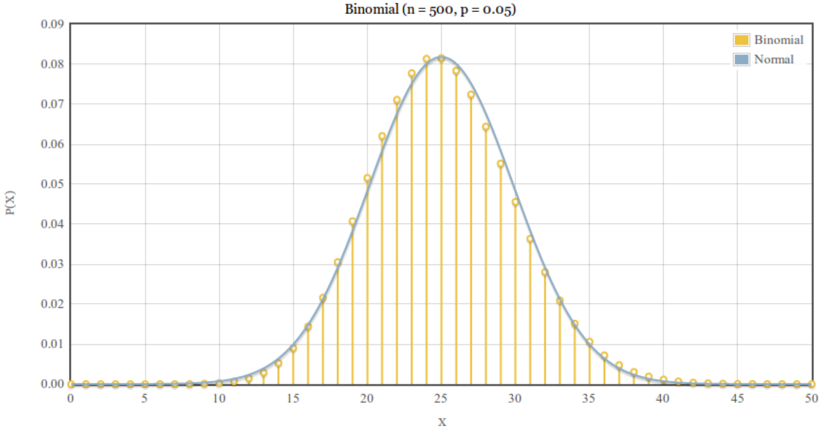
INTRODUCTION

The binomial is symmetric when $p = 0.5$



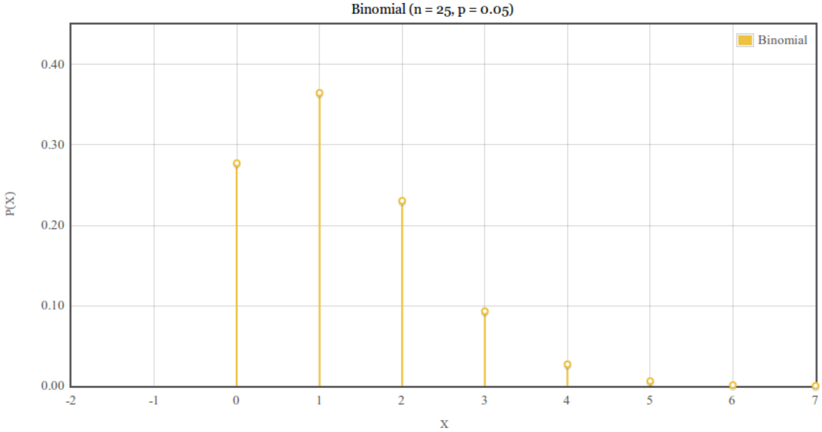
INTRODUCTION

The binomial is symmetric when $p = 0.5$ or n is very large.



INTRODUCTION

However, when n is medium and p is extreme ...



the binomial is very skewed ...

Introduction

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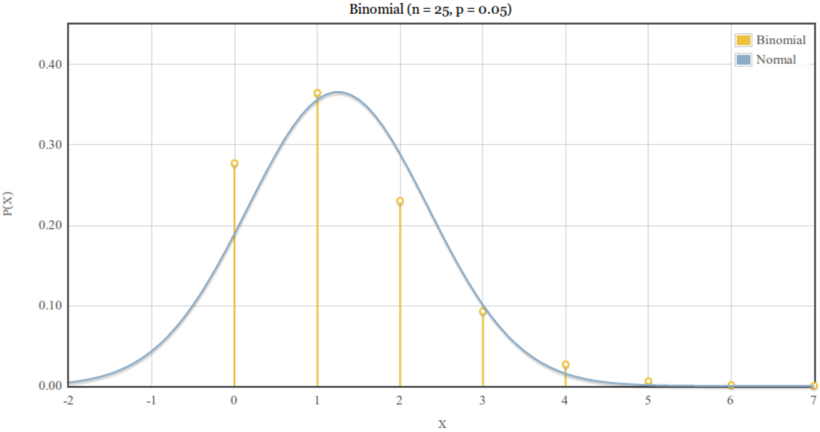

and the normal approximation doesn't work very well.

INTRODUCTION

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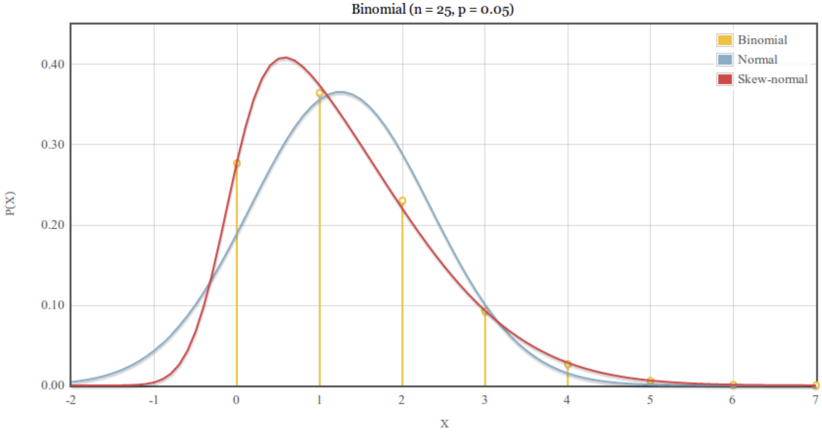

Introducing ... the skew-normal distribution.

Outline

Today's itinerary:

1. Skew-Normal distribution – basic properties

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2. Method of Moments – derive an approximation

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1. Skew-Normal distribution – basic properties
2. Method of Moments – derive an approximation
3. Accuracy – examine the accuracy of our approximation

The Skew-Normal Distribution: Foundations

Definition (Skew-normal)

Let Y be a skew-normal distribution, with location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$, and shape parameter $\lambda \in \mathbb{R}$. Then Y has pdf

$$f(x|\mu,\sigma,\lambda) = \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot \Phi\left(\frac{\lambda(x-\mu)}{\sigma}\right), \quad x \in \mathbb{R},$$

where ϕ is the standard normal pdf and Φ is the standard normal cdf.

We write $Y \sim SN(\mu,\sigma,\lambda)$.

THE SKEW-NORMAL DISTRIBUTION: FOUNDATIONS

So where did this funny-looking pdf come from? ...

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Lemma

If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one-dimensional distribution function such that G' exists and is a density symmetric about 0, then

$$f(z) = 2 \cdot f_0(z) \cdot G\{w(z)\} \qquad (-\infty < z < \infty)$$

*is a density function for any odd function $w(\cdot)$. (Lemma 1, **?**)*

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- Kernel
- CDF

The Skew-Normal Distribution: Foundations

Basic properties:

$$\begin{aligned}
E(Y) &= \mu + b\delta\sigma \\
E(Y^2) &= \mu^2 + 2b\delta\mu\sigma + \sigma^2 \\
E(Y^3) &= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 \\
Var(Y) &= \sigma^2(1 - b^2\delta^2)
\end{aligned}$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. (**?**)

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The Skew-Normal Distribution: Foundations

What happens when $\lambda = 0$?

$$\begin{aligned} f(x|\mu, \sigma, \lambda = 0) &= \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot \Phi(0) \\ &= \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot 0.5 \end{aligned}$$

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&= \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot 0.5 \\
&= \frac{1}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right)
\end{aligned}$$

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&= \frac{1}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),
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$$

which is the pdf of the normal distribution (μ, σ).

The Skew-Normal Distribution: The Standard Skew-Normal

Definition (Standard skew-normal)

The $SN(0, 1, \lambda)$ distribution is called the standard skew-normal and has pdf

$$f_Z(x|\lambda) = 2 \cdot \phi(x) \cdot \Phi(\lambda x), \quad x \in \mathbb{R}.$$

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The $SN(0, 1, \lambda)$ distribution is called the standard skew-normal and has pdf

$$f_Z(x|\lambda) = 2 \cdot \phi(x) \cdot \Phi(\lambda x), \quad x \in \mathbb{R}.$$

Similar to the normal and standard normal, $Z = \dfrac{Y - \mu}{\sigma}$ and $Y = \sigma Z + \mu$.

The Skew-Normal Distribution: The Standard Skew-Normal

Property (1)

If $Z \sim SN(0,1,\lambda)$, *then* $(-Z) \sim SN(0,1,-\lambda)$.

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$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

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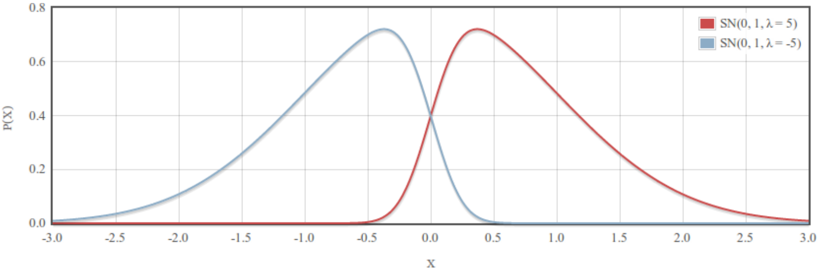
which is the pdf of $SN(0,1,-\lambda)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

The Skew-Normal Distribution: The Standard Skew-Normal

Property 1: $-SN(0, 1, \lambda) \sim SN(0, 1, -\lambda)$



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Property (2)

If $Z \sim SN(0, 1, \lambda)$, then $Z^2 \sim \chi^2_1$ (chi-square with 1 degree of freedom).

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Short proof:
Lemma 1 comes with a handy result, (**?**, page 161):

If $Y \sim f_0$ and $Z \sim f$, then $|Y| \stackrel{d}{=} |Z|$, where the notation $\stackrel{d}{=}$ denotes equality in distribution.

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If $Y \sim f_0$ and $Z \sim f$, then $|Y| \overset{d}{=} |Z|$, where the notation $\overset{d}{=}$ denotes equality in distribution.

Let $X \sim N(0, 1)$. Since $X^2 \sim \chi^2_1$ and $|X| \overset{d}{=} |Z|$, then $Z^2 \sim \chi^2_1$.
$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

The Skew-Normal Distribution: The Standard Skew-Normal

Property (3)

As $\lambda \to \pm\infty$, $SN(0, 1, \lambda)$ *tends to the half normal distribution,* $\pm|N(0, 1)|$.

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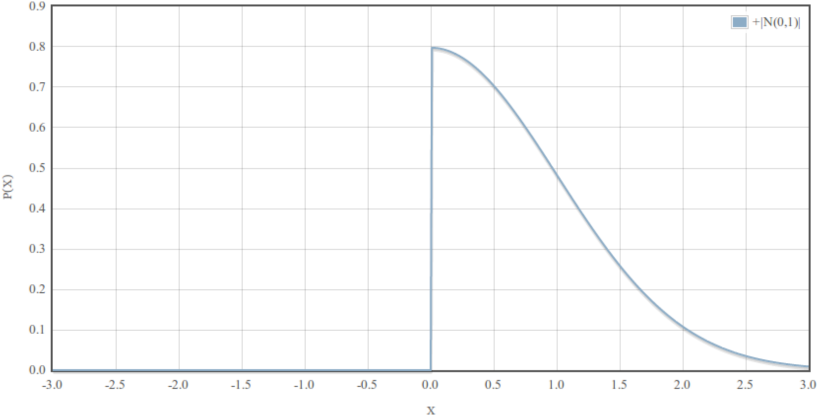
Let $X \sim |N(0, 1)|$. Then

$$f_X(x) = \begin{cases} 0 & \text{when } -\infty < x \leq 0 \\ 2\phi & \text{when } 0 < x < \infty \end{cases} .$$

The Skew-Normal Distribution: The Standard Skew-Normal

Property 3: $SN(0, 1, \lambda) \to +|N(0, 1)|$ as $\lambda \to \infty$:



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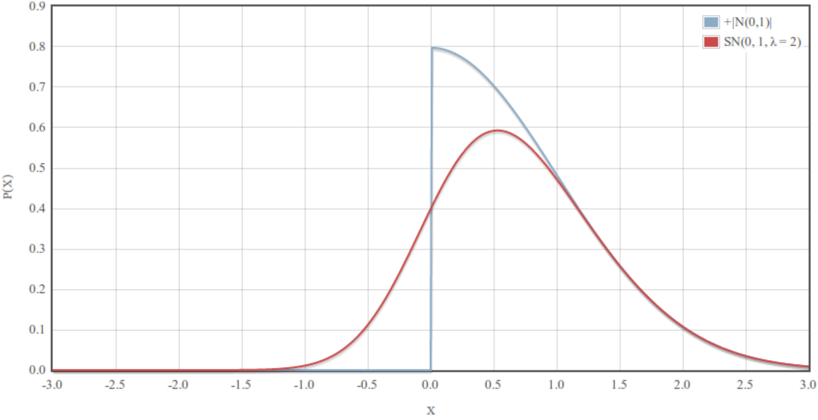

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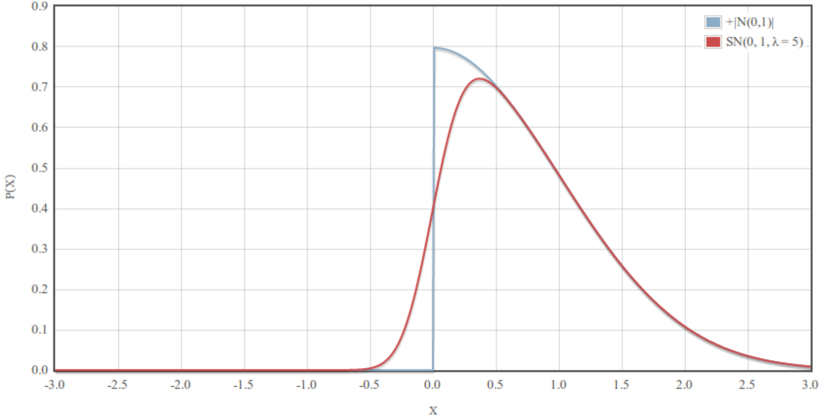

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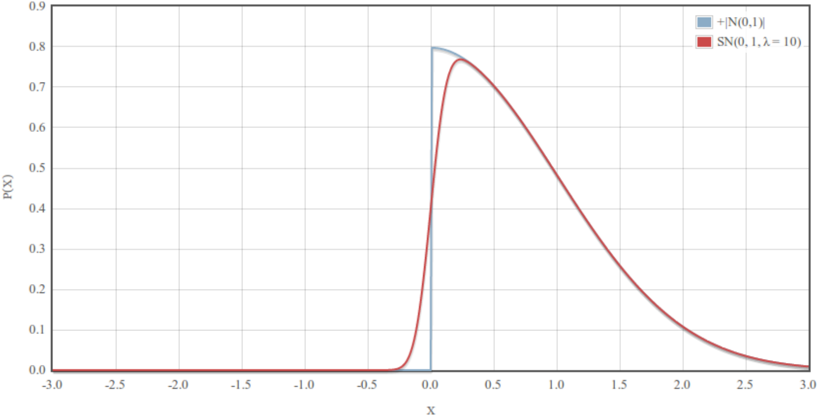

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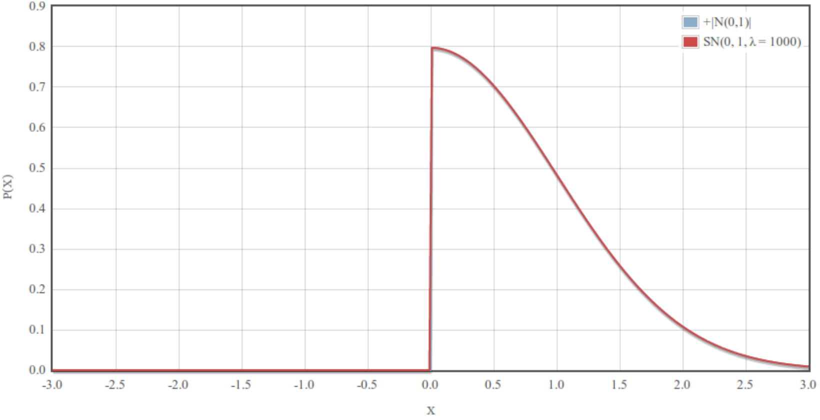

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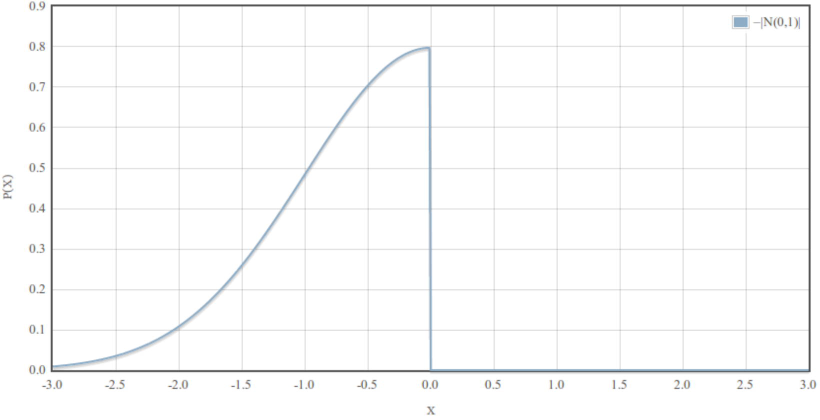

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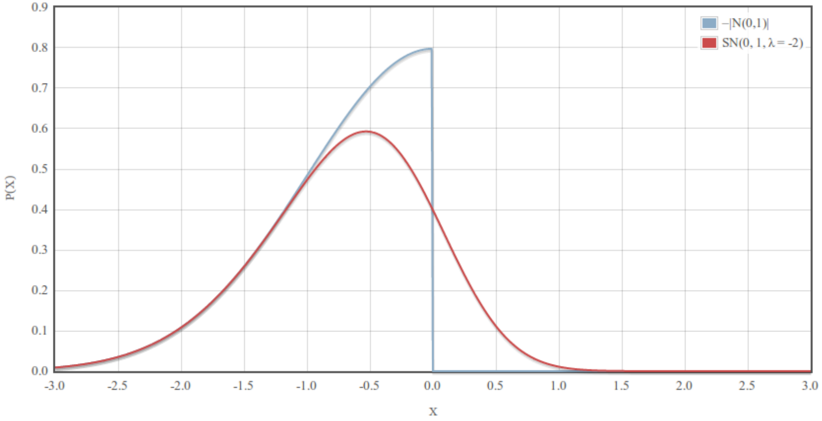

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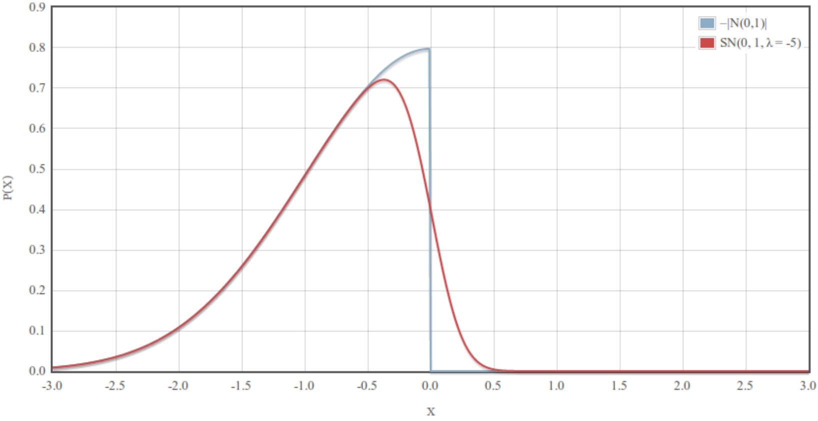

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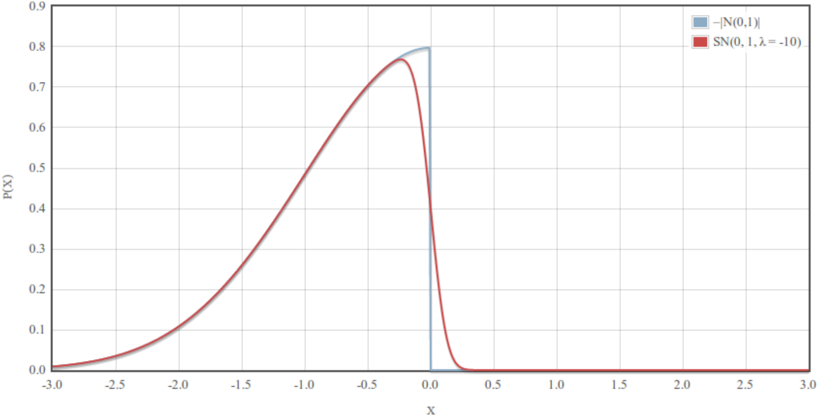

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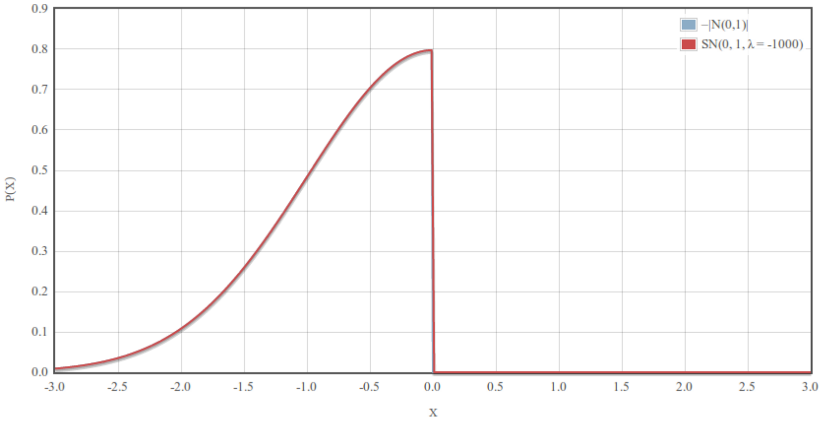

The Skew-Normal Distribution: The Standard Skew-Normal

Property (4)

The moment generating function of $SN(0, 1, \lambda)$ is

$$M(t|\lambda) = 2 \cdot \Phi(\delta t) \cdot e^{t^2/2},$$

where $\delta = \dfrac{\lambda}{\sqrt{1+\lambda^2}}$ *and* $t \in (-\infty, \infty)$.

THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

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The moment generating function of $SN(0, 1, \lambda)$ *is*

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where $\delta = \dfrac{\lambda}{\sqrt{1+\lambda^2}}$ *and* $t \in (-\infty, \infty)$.

According to Equation 5 in **?**, the mgf of $SN(\mu, \sigma, \lambda)$ is

$$M(t) \;=\; E\{e^{tY}\} \;=\; 2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot \Phi(\delta \sigma t).$$

Our result follows.

Method of Moments: Overview

Now we're going to derive the approximation!

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Game plan:

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   What are central moments?: $E(X)$, $E([X - E(X)]^2)$, $E([X - E(X)]^3)$.

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2. Set them equal to each other.
3. Take n and p to be constants; solve for μ, σ, and λ.

Method of Moments: Central Moments of the Binomial

Let's start with the binomial ...

Method of Moments: Central Moments of the Binomial

The first two central moments are just the mean and variance:

Method of Moments: Central Moments of the Binomial

The first two central moments are just the mean and variance:

$$E(B) = np$$
$$E([B - E(B)]^2) = Var(B) = np(1-p)$$

Method of Moments: Central Moments of the Binomial

The third one takes some elbow grease.

Method of Moments: Central Moments of the Binomial

The third one takes some elbow grease.

First we'll need to find $E(B^2)$ and $E(B^3)$.

Method of Moments: Central Moments of the Binomial

Recall that $Var(B) = E(B^2) - [E(B)]^2$.

Method of Moments: Central Moments of the Binomial

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$$E(B^2) = Var(B) + [E(B)]^2$$

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Recall that $Var(B) = E(B^2) - [E(B)]^2$. Thus

$$\begin{aligned} E(B^2) &= Var(B) + [E(B)]^2 \\ &= np(1-p) + n^2p^2 \end{aligned}$$

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$$\begin{aligned} E(B^2) &= Var(B) + [E(B)]^2 \\ &= np(1-p) + n^2p^2 \\ &= np - np^2 + n^2p^2. \end{aligned}$$

Method of Moments: Central Moments of the Binomial

We will get $E(B^3)$ via the third factorial moment, E[B(B-1)(B-2)].

Method of Moments: Central Moments of the Binomial

$$E[B(B-1)(B-2)]$$

Method of Moments: Central Moments of the Binomial

$$E[B(B-1)(B-2)]$$
$$= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$$

Method of Moments: Central Moments of the Binomial

$$E[B(B-1)(B-2)]$$
$$= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$$
$$= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$$

Method of Moments: Central Moments of the Binomial

$$
\begin{aligned}
&E[B(B-1)(B-2)] \\
&= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}
\end{aligned}
$$

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&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x} \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\, p^x q^{n-x}
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

$$
\begin{aligned}
&E[B(B-1)(B-2)] \\
&= \sum_{x=0}^{n} x(x-1)(x-2)\cdot\left\{\binom{n}{x}p^x q^{n-x}\right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2)\cdot\left\{\binom{n}{x}p^x q^{n-x}\right\} \\
&= \sum_{x=3}^{n} x(x-1)(x-2)\cdot\frac{n!}{x!\,(n-x)!}\,p^x q^{n-x} \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\,p^x q^{n-x} \\
&= \sum_{x=3}^{n} n(n-1)(n-2)p^3\cdot\frac{(n-3)!}{(x-3)!\,(n-x)!}\,p^{x-3} q^{n-x}
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3} q^{n-x}$$

Method of Moments: Central Moments of the Binomial

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3}q^{n-x}$$

Let $y = x - 3$; then $x = y + 3$, and $x = 3$, $x = n \;\Rightarrow\; y = 0$, $y = n - 3$:

Method of Moments: Central Moments of the Binomial

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3} q^{n-x}$$

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$$= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\, p^{y} q^{n-(y+3)}$$

Method of Moments: Central Moments of the Binomial

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3}q^{n-x}$$

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$$= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\, p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,\,p) \text{ summed over its domain] = 1}}$$

Method of Moments: Central Moments of the Binomial

$$= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\, p^{x-3} q^{n-x}$$

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Method of Moments: Central Moments of the Binomial

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Let $y = x - 3$; then $x = y + 3$, and $x = 3,\ x = n \ \Rightarrow\ y = 0$, $y = n - 3$:

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$$= n(n-1)(n-2)p^3$$

$$= n^3p^3 - 3n^2p^3 + 2np^3$$

Method of Moments: Central Moments of the Binomial

To get $E(B^3)$, we expand the left side of the previous equation:

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To get $E(B^3)$, we expand the left side of the previous equation:

$$
\begin{aligned}
& E[B(B-1)(B-2)] \\
& = E\left[B^3 - 3B^2 + 2B\right]
\end{aligned}
$$

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& = E(B^3) - 3E(B^2) + 2E(B)
\end{aligned}
$$

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$$
\begin{aligned}
& E[B(B-1)(B-2)] \\
& = E\left[B^3 - 3B^2 + 2B\right] \\
& = E(B^3) - 3E(B^2) + 2E(B) \\
& = E(B^3) - 3(np - np^2 + n^2p^2) + 2np
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

To get $E(B^3)$, we expand the left side of the previous equation:

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\begin{aligned}
&E[B(B-1)(B-2)] \\
&= E\left[B^3 - 3B^2 + 2B\right] \\
&= E(B^3) - 3E(B^2) + 2E(B) \\
&= E(B^3) - 3(np - np^2 + n^2p^2) + 2np \\
&= E(B^3) - 3np + 3np^2 - 3n^2p^2 + 2np
\end{aligned}
$$

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$$
\begin{aligned}
&E[B(B-1)(B-2)] \\
&= E\left[B^3 - 3B^2 + 2B\right] \\
&= E(B^3) - 3E(B^2) + 2E(B) \\
&= E(B^3) - 3(np - np^2 + n^2p^2) + 2np \\
&= E(B^3) - 3np + 3np^2 - 3n^2p^2 + 2np \\
&= E(B^3) + 3np^2 - 3n^2p^2 - np
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

Left side: $E(B^3) + 3np^2 - 3n^2p^2 - np$

Right side: $n^3p^3 - 3n^2p^3 + 2np^3$

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Left side: $E(B^3) + 3np^2 - 3n^2p^2 - np$

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Set them equal and solve for $E(B^3)$:

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Left side: $E(B^3) + 3np^2 - 3n^2p^2 - np$

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Set them equal and solve for $E(B^3)$:

$$E(B^3) + 3np^2 - 3n^2p^2 - np = n^3p^3 - 3n^2p^3 + 2np^3$$

Method of Moments: Central Moments of the Binomial

Left side: $E(B^3) + 3np^2 - 3n^2p^2 - np$

Right side: $n^3p^3 - 3n^2p^3 + 2np^3$

Set them equal and solve for $E(B^3)$:

$$E(B^3) + 3np^2 - 3n^2p^2 - np = n^3p^3 - 3n^2p^3 + 2np^3$$
$$\Rightarrow \qquad E(B^3) = n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np$$

Method of Moments: Central Moments of the Binomial

Now we can (finally!) compute the third central moment:

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$$E\left([B - E(B)]^3\right)$$
$$= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right)$$

Method of Moments: Central Moments of the Binomial

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$$
\begin{aligned}
& E\left([B - E(B)]^3\right) \\
&= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3
\end{aligned}
$$

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&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3
\end{aligned}
$$

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&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) \\
&\quad - 3(np - np^2 + n^2p^2)(np) + 2(np)^3
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

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\begin{aligned}
& E\left([B - E(B)]^3\right) \\
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& = E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
& = E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
& = (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) \\
& \quad - 3(np - np^2 + n^2p^2)(np) + 2(np)^3 \\
& = \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np \\
& \quad - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3}
\end{aligned}
$$

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$$
\begin{aligned}
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&= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) \\
&\quad - 3(np - np^2 + n^2p^2)(np) + 2(np)^3 \\
&= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np \\
&\quad - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3} \\
&= 2np^3 - 3np^2 + np
\end{aligned}
$$

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Now we can (finally!) compute the third central moment:

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\begin{aligned}
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&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) \\
&\quad - 3(np - np^2 + n^2p^2)(np) + 2(np)^3 \\
&= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np \\
&\quad - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3} \\
&= 2np^3 - 3np^2 + np \\
&= np(p-1)(2p-1)
\end{aligned}
$$

Method of Moments: Central Moments of the Binomial

Let's restate our results:

$$\begin{aligned} E(B) &= np, \\ E([B - E(B)]^2) &= np(1-p), \\ E([B - E(B)]^3) &= np(p-1)(2p-1) \end{aligned}$$

Method of Moments: Central Moments of the Skew-Normal

Now lets move on to skew-normal ...

Method of Moments: Central Moments of the Skew-Normal

Again, the first and second central moments are the mean and variance.

Method of Moments: Central Moments of the Skew-Normal

Again, the first and second central moments are the mean and variance.

$$E(Y) = \mu + b\delta\sigma$$
$$Var(Y) = \sigma^2(1 - b^2\delta^2)$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$E([Y - E(Y)]^3)$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$
\begin{aligned}
&E([Y - E(Y)]^3) \\
&= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3
\end{aligned}
$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$
\begin{aligned}
&E([Y - E(Y)]^3) \\
&= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\
&\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2\,(\mu + b\delta\sigma)^3
\end{aligned}
$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$
\begin{aligned}
& E([Y - E(Y)]^3) \\
& = E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
& = (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\
& \quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2\,(\mu + b\delta\sigma)^3 \\
& = \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} \\
& \quad - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} \\
& \quad + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3
\end{aligned}
$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$
\begin{aligned}
&E([Y-E(Y)]^3) \\
&= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\
&\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2\,(\mu + b\delta\sigma)^3 \\
&= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} \\
&\quad - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} \\
&\quad + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\
&= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3
\end{aligned}
$$

Method of Moments: Central Moments of the Skew-Normal

Again, the third one is a little harder:

$$
\begin{aligned}
&E([Y - E(Y)]^3) \\
&= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\
&\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2\,(\mu + b\delta\sigma)^3 \\
&= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} \\
&\quad - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} \\
&\quad + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\
&= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3 \\
&= b\delta^3\sigma^3(2b^2 - 1)
\end{aligned}
$$

Method of Moments: Central Moments of the Skew-Normal

Our results, restated:

$$
\begin{aligned}
E(Y) &= \mu + b\delta\sigma &&= \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \\
E([Y - E(Y)]^2) &= \sigma^2(1 - b^2\delta^2) &&= \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \\
E([Y - E(Y)]^3) &= b\delta^3\sigma^3(2b^2 - 1) &&= \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right)
\end{aligned}
$$

Method of Moments: Deriving an Approximation

We're finally ready to derive our approximation!

Method of Moments: Deriving an Approximation

We start by setting the central moments of the binomial equal to the central moments of the skew-normal.

Method of Moments: Deriving an Approximation

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$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{1a}$$

Method of Moments: Deriving an Approximation

We start by setting the central moments of the binomial equal to the central moments of the skew-normal.

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{1a}$$

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{1b}$$

Method of Moments: Deriving an Approximation

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$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{1a}$$

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{1b}$$

$$np(p-1)(2p-1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right) \tag{1c}$$

Method of Moments: Deriving an Approximation

To get λ, divide the cube of (1b) by the square of (1c):

Method of Moments: Deriving an Approximation

To get λ, divide the cube of (1b) by the square of (1c):

$$\frac{\sigma^6\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6\cdot\frac{2}{\pi}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6\left(\frac{4}{\pi}-1\right)^2}=\frac{n^3p^3(1-p)^3}{n^2p^2(p-1)^2(2p-1)^2} \tag{2}$$

Method of Moments: Deriving an Approximation

To get λ, divide the cube of (1b) by the square of (1c):

$$\frac{\sigma^6\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6\cdot\frac{2}{\pi}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6\left(\frac{4}{\pi}-1\right)^2}=\frac{n^3p^3(1-p)^3}{n^2p^2(p-1)^2(2p-1)^2}$$

$$\Rightarrow\quad\frac{\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2}=\frac{np(1-p)}{(1-2p)^2}\tag{2}$$

Method of Moments: Deriving an Approximation

Equation (2) can be solved for λ^2.

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Then take λ to be

$$\lambda = \{\text{sign of } (1 - 2p)\}\sqrt{\lambda^2}.$$

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Why?

Method of Moments: Deriving an Approximation

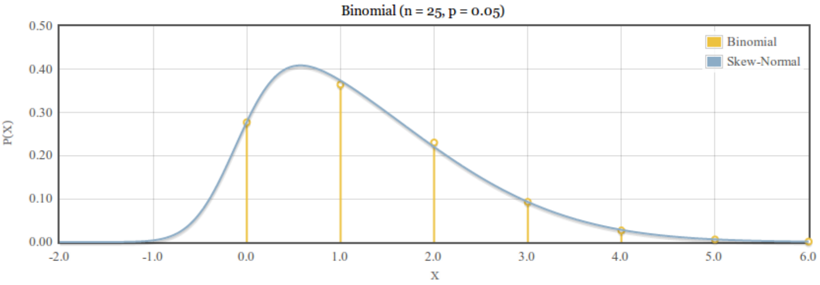
Equation (2) can be solved for λ^2.

Then take λ to be

$$\lambda = \{\text{sign of } (1 - 2p)\}\sqrt{\lambda^2}.$$

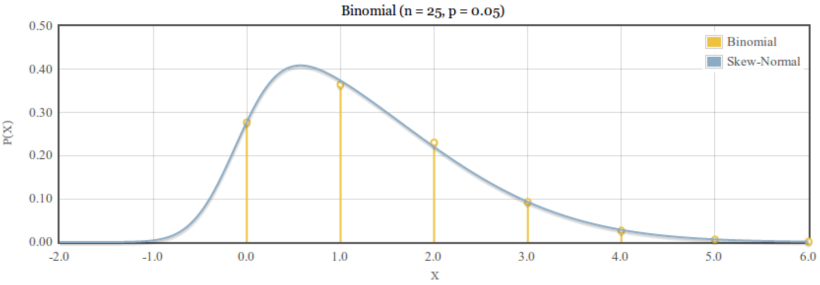
Why? Recall Property 3 ...

Method of Moments: Deriving an Approximation



When $p < 0.5$:

Method of Moments: Deriving an Approximation



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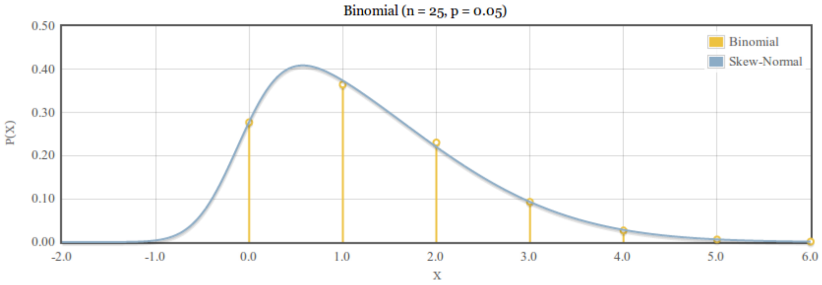
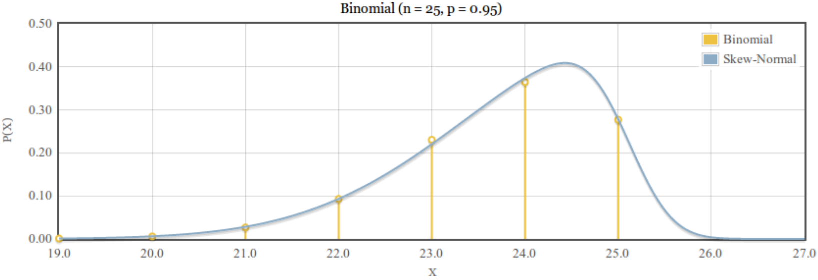


When $p < 0.5$:

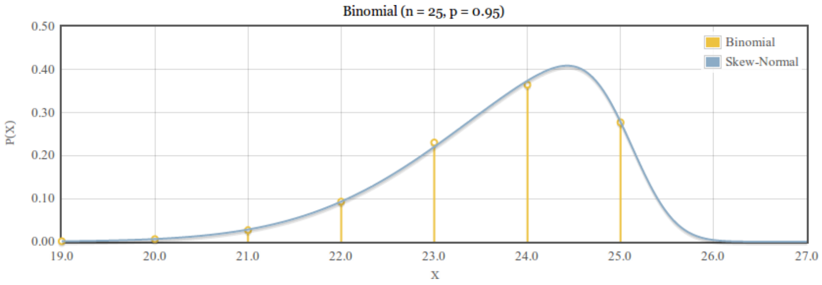
- The binomial skews right (weight shifts left) and approaches $+|N(0,1)| \longrightarrow \lambda$ is positive
- $(1 - 2p)$ is positive

Method of Moments: Deriving an Approximation



When $p > 0.5$:

Method of Moments: Deriving an Approximation



When $p > 0.5$:

- The binomial skews left (weight shifts right) and approaches $-|N(0,1)| \longrightarrow \lambda$ is negative

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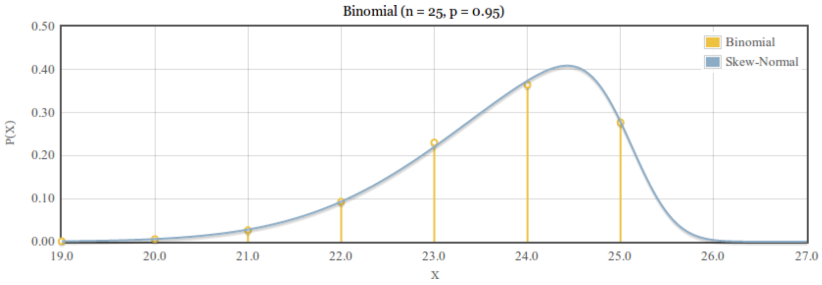

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Once you have λ, solve for σ and then μ.

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$$np = \mu + \sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}} \quad \Rightarrow \quad \mu = np - \sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}}$$

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Method of Moments: Restrictions

Unfortunately, though better than the normal approximation, our skew-normal method isn't universal.

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To be able to solve for λ, we must put a few restrictions on n and p.

Method of Moments: Restrictions

Let $u = \dfrac{\lambda^2}{1+\lambda^2}$ and $v = 1/u = \dfrac{1+\lambda^2}{\lambda^2}$.

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Then we can rewrite (2):

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$$\left(v - \frac{2}{\pi}\right)^3\left(\frac{\pi^3}{2(4-\pi)^2}\right) = g(v)$$

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$g(v)$ is increasing in $v = \dfrac{1+\lambda^2}{\lambda^2} \geq 1$.

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$g(v)$ is increasing in $v = \dfrac{1+\lambda^2}{\lambda^2} \geq 1$. Therefore:

$$\min_v g(v) = g(1) = \left(1 - \frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) = 1.009524 \approx 1$$

Method of Moments: Restrictions

To be able to solve (2) for λ, we must have

$$\{\text{right hand side of (2)}\} \geq \{\text{min of left hand side of (2)}\} \tag{3}$$

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To be able to solve (2) for λ, we must have

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np(1-p) &\geq \frac{1}{(1-2p)^2}. \qquad (3)
\end{aligned}$$

From (3), we can answer two questions:

Method of Moments: Restrictions

One:

Given p, what is the least n necessary?

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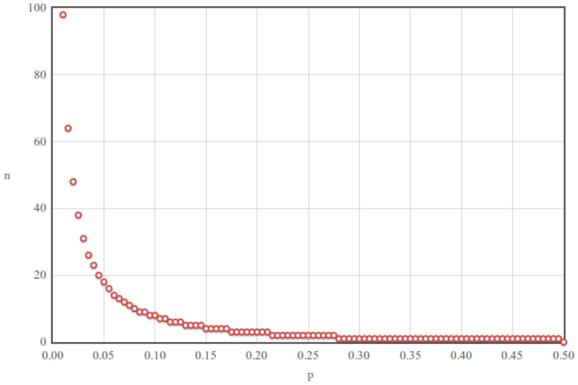
$$n \geq \frac{(1-2p)^2}{p(1-p)}$$

Method of Moments: Restrictions

Least possible n, given a fixed p:



Method of Moments: Restrictions

Two:

Given n, what is the range of possible p's?

Method of Moments: Restrictions

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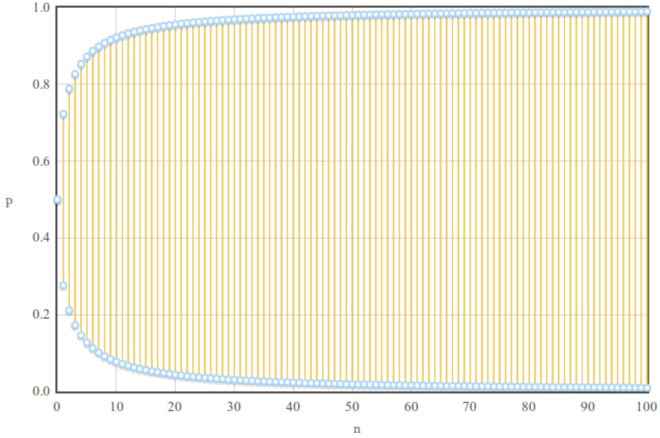
Given n, what is the range of possible p's?

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}} \;\leq\; p \;\leq\; \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}}$$

Method of Moments: Restrictions

Range of possible p, given a fixed n:



Demonstrating Improved Accuracy

We have an approximation!!

But is it more accurate? ...

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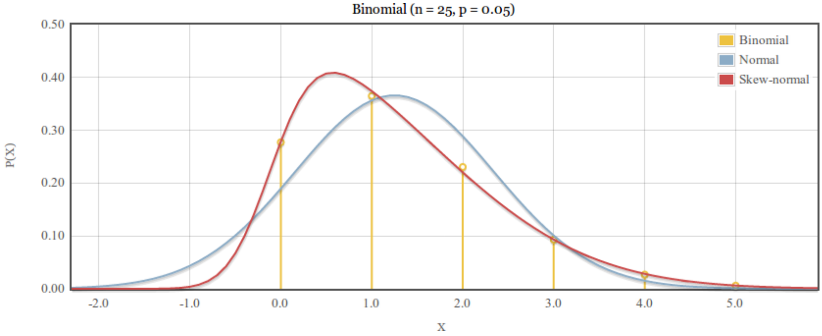
But is it more accurate? ... Answer: Yes!

Demonstrating Improved Accuracy: Visual

The easiest way of gauging accuracy is by visual inspection.

Demonstrating Improved Accuracy: Visual



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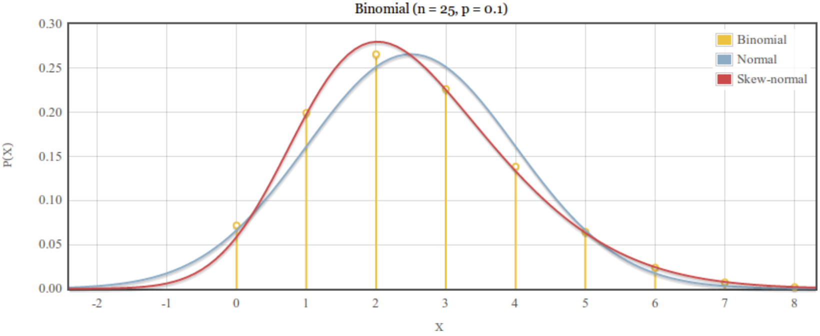

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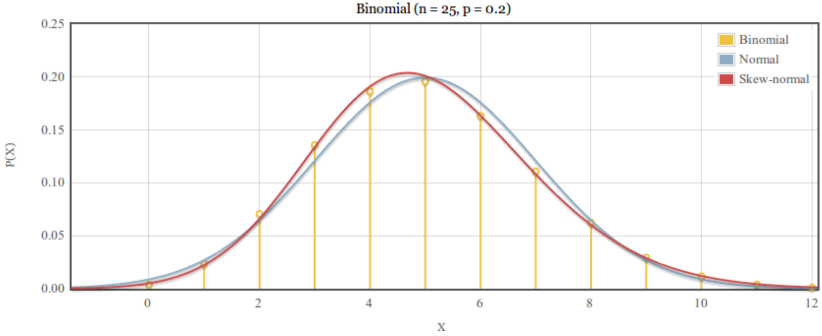
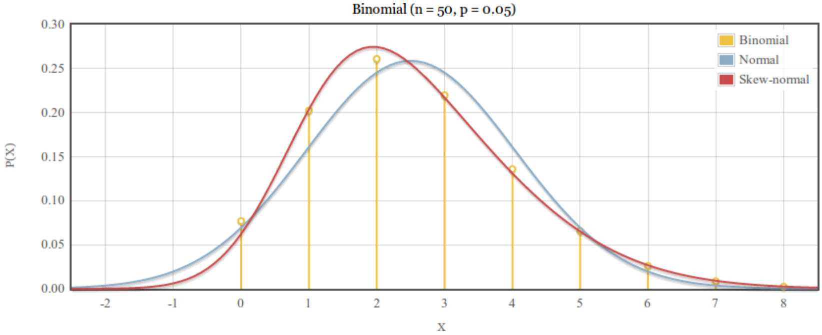
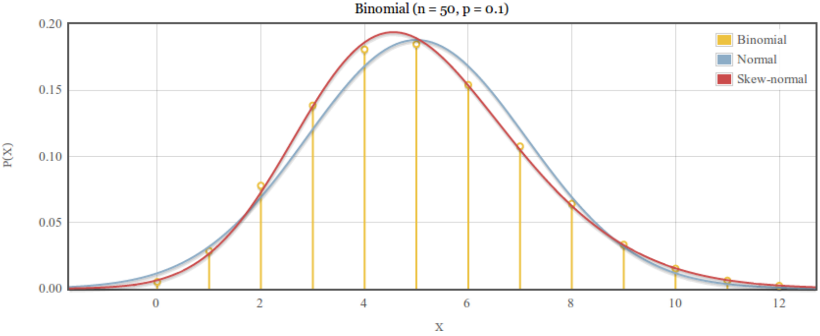


DEMONSTRATING IMPROVED ACCURACY: VISUAL



Demonstrating Improved Accuracy: Visual



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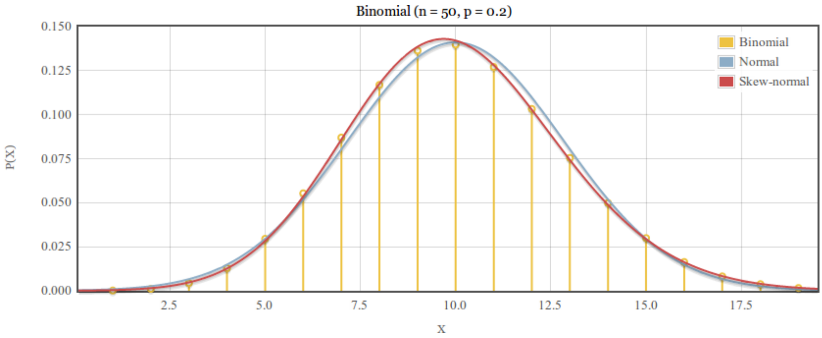

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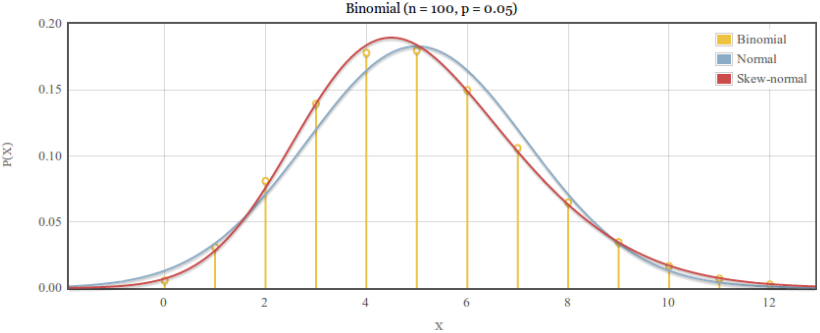

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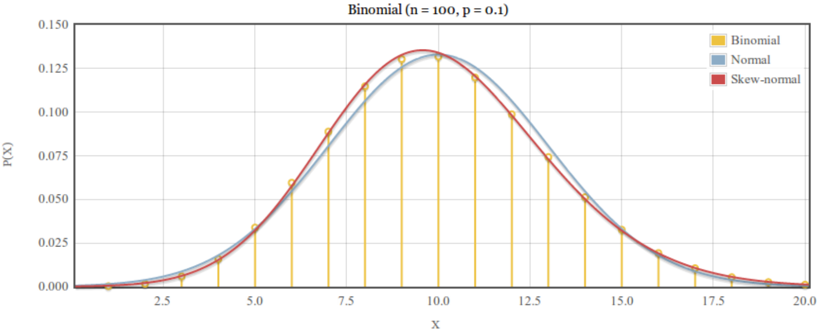

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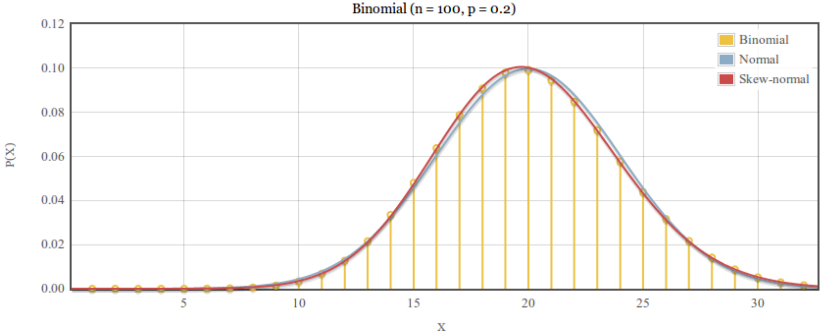


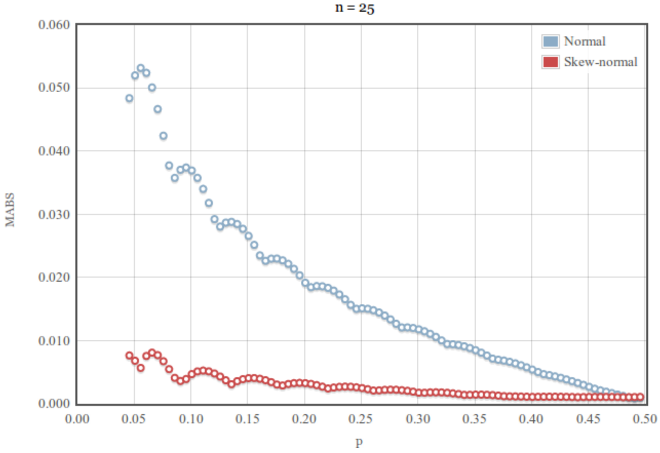
Demonstrating Improved Accuracy: MABS

A more numerical way of gauging accuracy is the *MABS*, defined by **?** as

$$\mathrm{MABS}(n,p) \;=\; \max_{k\in\{0,1,\ldots,n\}} \left| F_{B(n,p)}(k) - F_{\mathrm{appr}(n,p)}(k+0.5) \right| .$$

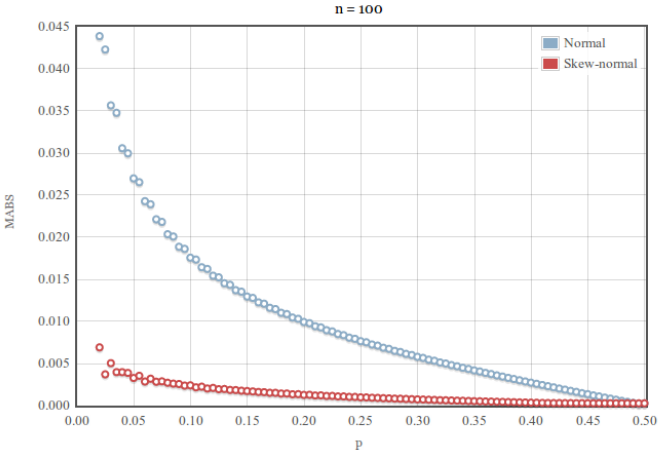
Demonstrating Improved Accuracy: MABS

MABS as a function of p, with fixed $n = 25$:



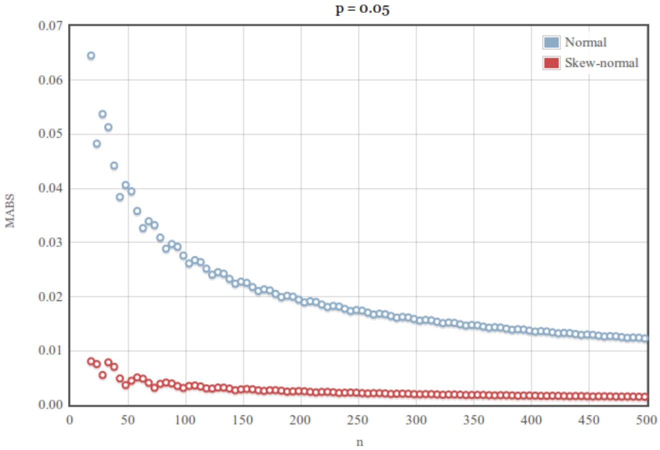
Demonstrating Improved Accuracy: MABS

MABS as a function of p, with fixed $n = 100$:



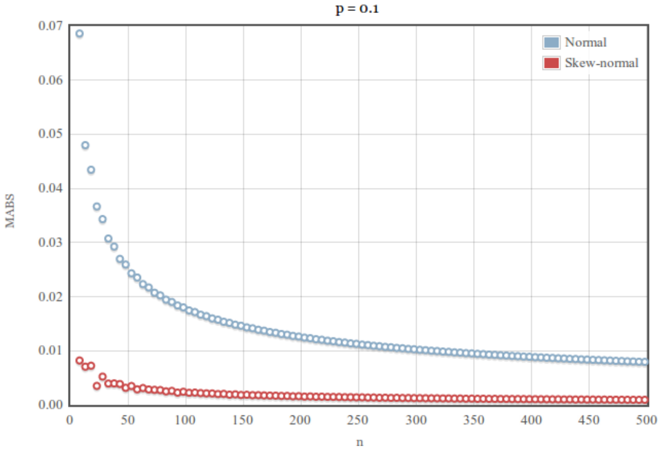
Demonstrating Improved Accuracy: MABS

MABS as a function of n, with fixed $p = 0.05$:



Demonstrating Improved Accuracy: MABS

MABS as a function of n, with fixed $p = 0.1$:



RESOURCES

Estimations of $SN(\mu, \sigma, \lambda)$ for $Bin(n, p)$

		n				
		25	50	100	250	500
	0.05	(-0.11, 1.74, 4.56)	(0.79, 2.30, 2.54)	(2.85, 3.06, 1.86)	(9.58, 4.52, 1.38)	(21.32, 6.11, 1.15)
	0.10	(0.89, 2.20, 2.31)	(2.97, 2.94, 1.74)	(7.44, 3.94, 1.40)	(21.53, 5.88, 1.10)	(45.62, 8.01, 0.94)
	0.15	(2.02, 2.49, 1.79)	(5.32, 3.34, 1.43)	(12.25, 4.51, 1.19)	(33.77, 6.77, 0.96)	(70.30, 9.27, 0.82)
	0.20	(3.23, 2.67, 1.50)	(7.76, 3.61, 1.24)	(17.18, 4.89, 1.04)	(46.18, 7.39, 0.85)	(95.18, 10.16, 0.74)
	0.25	(4.49, 2.79, 1.29)	(10.28, 3.78, 1.09)	(22.20, 5.15, 0.93)	(58.71, 7.83, 0.76)	(120.22, 10.80, 0.67)
	0.30	(5.80, 2.85, 1.12)	(12.86, 3.88, 0.95)	(27.31, 5.32, 0.82)	(71.34, 8.12, 0.68)	(145.39, 11.24, 0.60)
	0.35	(7.17, 2.86, 0.96)	(15.50, 3.92, 0.83)	(32.49, 5.39, 0.72)	(84.09, 8.28, 0.60)	(170.70, 11.50, 0.53)
	0.40	(8.59, 2.83, 0.80)	(18.23, 3.89, 0.70)	(37.76, 5.39, 0.61)	(96.96, 8.32, 0.51)	(196.18, 11.60, 0.45)
	0.45	(10.12, 2.73, 0.61)	(21.08, 3.79, 0.53)	(43.21, 5.29, 0.47)	(110.07, 8.23, 0.40)	(221.93, 11.54, 0.35)
p	0.50	(12.50, 2.50, 0.00)	(25.00, 3.54, 0.00)	(50.00, 5.00, 0.00)	(125.00, 7.91, 0.00)	(250.00, 11.18, 0.00)
	0.55	(14.88, 2.73, -0.61)	(28.92, 3.79, -0.53)	(56.79, 5.29, -0.47)	(139.93, 8.23, -0.40)	(278.07, 11.54, -0.35)
	0.60	(16.41, 2.83, -0.80)	(31.77, 3.89, -0.70)	(62.24, 5.39, -0.61)	(153.04, 8.32, -0.51)	(303.82, 11.60, -0.45)
	0.65	(17.83, 2.86, -0.96)	(34.50, 3.92, -0.83)	(67.51, 5.39, -0.72)	(165.91, 8.28, -0.60)	(329.30, 11.50, -0.53)
	0.70	(19.20, 2.85, -1.12)	(37.14, 3.88, -0.95)	(72.69, 5.32, -0.82)	(178.66, 8.12, -0.68)	(354.61, 11.24, -0.60)
	0.75	(20.51, 2.79, -1.29)	(39.72, 3.78, -1.09)	(77.80, 5.15, -0.93)	(191.29, 7.83, -0.76)	(379.78, 10.80, -0.67)
	0.80	(21.77, 2.67, -1.50)	(42.24, 3.61, -1.24)	(82.82, 4.89, -1.04)	(203.82, 7.39, -0.85)	(404.82, 10.16, -0.74)
	0.85	(22.98, 2.49, -1.79)	(44.68, 3.34, -1.43)	(87.75, 4.51, -1.19)	(216.23, 6.77, -0.96)	(429.70, 9.27, -0.82)
	0.90	(24.11, 2.20, -2.31)	(47.03, 2.94, -1.74)	(92.56, 3.94, -1.40)	(228.47, 5.88, -1.10)	(454.38, 8.01, -0.94)
	0.95	(25.11, 1.74, -4.56)	(49.21, 2.30, -2.54)	(97.15, 3.06, -1.86)	(240.42, 4.52, -1.38)	(478.68, 6.11, -1.15)

Resources

All values in my project were computed using a Python library, which is freely available online:

http://github.com/joycetipping/skew-normal-capstone/

Bibliography I