Deriving λ with the Method of Moments

Let $Y \sim SN(\xi, \omega^2, \alpha)$. Find $E(Y^3)$. Find MME's for μ, σ^2, λ.

First, we'll examine the moments $E(Y)$, $E(Y^2)$, $E(Y^3)$. According to equation (6.5?) in Azzalini (2005),

$$E(Y) = \xi + \omega\mu_z, \quad Var(Y) = \omega^2(1 - \mu_z^2) \tag{1}$$

where $\mu_z = \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}$. From that we have

$$\begin{aligned} E(Y^2) &= Var(Y) + E(Y)^2 \\ &= \omega^2(1 - \mu_z^2) + (\xi + \omega\mu_z)^2 \\ &= \xi^2 + 2\omega\xi\mu_z + \omega^2 \end{aligned} \tag{2}$$

At this point, it is handy to note that, taking $\xi = 0$ and $\omega = 1$ in the above equations, we find that

$$E(Z) = \mu_z, \quad E(Z^2) = 1, \quad Var(Z) = 1 - \mu_z^2 \tag{3}$$

To get $E(Y^3)$, we must first find $E(Z^3)$ where $Z \sim SN(0, 1, \alpha)$. According to equation (6.5?) in Azzalini (2005),

$$E(Z^r) = \begin{cases} 1 \times 3 \times \cdots \times (r-1) & \text{if r is even} \\ \dfrac{\sqrt{2}\ (2k+1)!\ \alpha}{\sqrt{\pi}\ (1+\alpha^2)^{k+1/2}\ 2^k} \displaystyle\sum_{m=0}^{k} \dfrac{m!\ (2\alpha)^{2m}}{(2m+1)!\ (k-m)!} & \text{if } r = 2k+1 \text{ and } k = 0, 1, ... \end{cases}$$

So, for $E(Z^3)$, we have $r = 2k + 1 = 3$ and $k = 1$:

$$
\begin{aligned}
E(Z^3) &= \frac{\sqrt{2}\cdot 3!\cdot \alpha}{\sqrt{\pi}\cdot(1+\alpha^2)^{3/2}\cdot 2}\ \sum_{m=0}^{1}\frac{m!\ (2\alpha)^{2m}}{(2m+1)!\ (1-m)!} \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\alpha}{(1+\alpha^2)^{3/2}}\cdot\left(\frac{0!(2\alpha)^0}{1!1!}+\frac{1!(2\alpha)^2}{3!0!}\right) \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\alpha}{(1+\alpha^2)^{3/2}}\cdot\left(1+\frac{2}{3}\alpha^2\right) \\
&= \sqrt{\frac{2}{\pi}}\cdot\frac{\alpha}{(\sqrt{1+\alpha^2})^3}\cdot(3+2\alpha^2) \qquad (4)
\end{aligned}
$$

Since $Y=\xi+\omega Z$, we have $Y^3=(\xi+\omega Z)^3=\omega^3Z^3+3\omega^2\xi Z^2+3\omega\xi^2Z+\xi^3$, and by the linearity of expected value,

$$
\begin{aligned}
E(Y^3) &= E[(\xi+\omega Z)^3] \\
&= E[\omega^3Z^3+3\omega^2\xi Z^2+3\omega\xi^2Z+\xi^3] \\
&= \omega^3\ E(Z^3)+3\omega^2\xi\ E(Z^2)+3\omega\xi^2\ E(Z)+\xi^3 \\
&= \omega^3\ E(Z^3)+3\omega^2\xi+3\omega\xi^2\ \mu_z+\xi^3 \qquad (5)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\mu_1' &= E(Y)\ = \xi+\omega\mu_z \\
\mu_2' &= E(Y^2) = \xi^2+2\mu_z\omega\xi+\omega^2 \qquad (6) \\
\mu_3' &= E(Y^3) = \omega^3\ E(Z^3)+3\omega^2\xi\ E(Z^2)+3\omega\xi^2\ E(Z)+\xi^3
\end{aligned}
$$

Also:

$$
\begin{aligned}
M_1' &= \bar{Y} \\
M_2' &= \frac{1}{n}\sum_{i=1}^{n}Y_i^2 \qquad (7) \\
M_3' &= \frac{1}{n}\sum_{i=1}^{n}Y_i^3
\end{aligned}
$$

So we have:

$$
\begin{cases}
\mu_1' = M_1' \\
\mu_2' = M_2' \\
\mu_3' = M_3'
\end{cases}
\quad \Rightarrow \quad
\begin{cases}
\bar{Y} & = \xi + \omega \mu_z \\
\dfrac{1}{n} \displaystyle\sum_{i=1}^{n} Y_i^2 & = \xi^2 + 2\mu_z \omega \xi + \omega^2 \\
\dfrac{1}{n} \displaystyle\sum_{i=1}^{n} Y_i^3 & = \omega^3 \ E(Z^3) + 3\omega^2 \xi \ E(Z^2) + 3\omega \xi^2 \ E(Z) + \xi^3
\end{cases}
$$