1 Developing an Approximation

Now that we have gotten to know our new distribution a little better, we can use it to develop an approximation for the binomial.

Let $B \sim Bin(n, p)$ and $Y \sim SN(\mu, \sigma^2, \lambda)$. We will find estimates for μ, σ, and λ by comparing the first, second, and third moments about the mean of B and Y.

1.1 The Moments of the Binomial

Let's start with the binomial. The first two moments are simply the mean and variance, which we can state from memory:

$$E(B) = np, \quad Var(B) = np(1-p)$$

Having these, we can easily find

$$E(B^2) = Var(B) + [E(B)]^2 = np(1-p) + n^2p^2 = np - np^2 + n^2p^2$$

which we will need for the third moment. We will also need $E(B^3)$, which we will get via the third factorial moment:

$$E[B(B-1)(B-2)] = \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$$

Notice that the first three terms of our sum are zero, so we can rewrite our sum beginning at $x = 3$:

$$\begin{aligned}
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\; p^x q^{n-x} \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\; p^x q^{n-x} \\
&= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\; p^{x-3} q^{n-x}
\end{aligned}$$

Let $y = x - 3$. Then $x = y + 3$, and $x = 3, x = n \;\Rightarrow\; y = 0, y = n - 3$.

$$
\begin{aligned}
&= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\; p^y q^{n-(y+3)} \\
&= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\; p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,p) \text{ summed from 0 to } n-3\text{] = 1}} \\
&= n(n-1)(n-2)p^3 \\
&= n^3p^3 - 3n^2p^3 + 2np^3
\end{aligned}
$$

Further expanding the left side and solving for $E(B^3)$,

$$
\begin{aligned}
E\left[B^3 - 3B^2 + 2B\right] &= n^3p^3 - 3n^2p^3 + 2np^3 \\
E(B^3) - 3E(B^2) + 2E(B) &= \\
E(B^3) - 3(np - np^2 + n^2p^2) + 2np &= \\
\Rightarrow \qquad E(B^3) &= n^3p^3 - 3n^2p^3 + 2np^3 + 3np - 3np^2 + 3n^2p^2 - 2np \\
&= n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np
\end{aligned}
$$

Now, finally, we have all the building blocks necessary to obtain the third moment:

$$
\begin{aligned}
E\left([B - E(B)]^3\right) &= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) - 3(np - np^2 + n^2p^2)(np) + 2(np)^3 \\
&= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3} \\
&= 2np^3 - 3np^2 + np \\
&= np(p-1)(2p-1)
\end{aligned}
$$

Our hard-earned results, restated for convenience:

$$
\begin{aligned}
E(B) &= np \\
E([B - E(B)]^2) &= np(1-p) \\
E([B - E(B)]^3) &= np(p-1)(2p-1)
\end{aligned}
\tag{1}
$$

1.2 The Moments of the Skew Normal

Now we'll take a look at the moments of the skew normal. Equation (**??**) takes care of the mean and variance; again the third moment is a little more complicated:

$$
\begin{aligned}
E([Y-E(Y)]^3) &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) \\
&\quad + 2\,(\mu + b\delta\sigma)^3 \\
&= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} \\
&\quad - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\
&= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3 \\
&= b\delta^3\sigma^3(2b^2 - 1)
\end{aligned}
$$

We restate our results:

$$
\begin{aligned}
E(Y) = \mu + b\delta\sigma \qquad &= \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \\
E([Y-E(Y)]^2) = \sigma^2(1 - b^2\delta^2) \quad &= \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \\
E([Y-E(Y)]^3) = b\delta^3\sigma^3(2b^2-1) &= \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right)
\end{aligned}
\tag{2}
$$

1.3 Solving for μ, σ, λ

Now we can set the moments of our two distributions equal to each other and, taking n and p as constants, solve for μ, σ and λ.

$$
np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{3a}
$$

$$
np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{3b}
$$

$$
np(p-1)(2p-1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right) \tag{3c}
$$

To get λ, we divide the cube of (3b) by the square of (3c):

$$\frac{\sigma^6\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6\cdot\frac{2}{\pi}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6\left(\frac{4}{\pi}-1\right)^2}=\frac{n^3p^3(1-p)^3}{n^2p^2(p-1)^2(2p-1)^2}$$

$$\Rightarrow\quad\frac{\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2}=\frac{np(1-p)}{(1-2p)^2} \tag{4}$$

The above equation (4) is a rational expression in λ^2 that can be solved with either a considerable amount of manual labor or, more efficiently, with a computer algebra system. Once we have λ^2, then λ is simply either the positive or negative square root, as determined by the sign of $(1-2p)$. This can be explained with a little assistance from Property **??**: When $p \to 0$, the binomial skews left and converges toward the positive half normal, which by (**??**) corresponds to a positive λ. When $p \to 1$, the binomial skews right and converges toward the negative half normal, which by (**??**) corresponds to a negative λ. When $p = 0.5$, the binomial is symmetric and λ is 0, eliminating the need for a sign. Thus:

$$\lambda = \{\text{sign of } (1-2p)\}\sqrt{\lambda^2} \tag{5}$$

Having secured λ, we can find σ using (3b):

$$np(1-p)=\sigma^2\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)\quad\Rightarrow\quad\sigma=\sqrt{\frac{np(1-p)}{1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}}} \tag{6}$$

And with both λ and σ, a simple rearrangement of (3a) yields μ:

$$np=\mu+\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}}\quad\Rightarrow\quad\mu=np-\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}} \tag{7}$$

When $p = 0.5$, we would expect the binomial to be perfectly symmetrical and therefore $\mu = np = n/2$. From (7), this implies that $\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}} = 0 \;\Rightarrow\;$ either $\sigma = 0$ or $\lambda = 0$. Since the former is impossible, we must conclude the latter, which brings us back to the normal distribution.

1.4 Restrictions

To obtain an estimate for λ, we must put a few restrictions on n and p.

If we let $u = \frac{\lambda^2}{1+\lambda^2}$ and $v = 1/u$, we can rewrite the left hand side of (4) as

$$
\begin{aligned}
\frac{\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2} &= \left(1-\frac{2}{\pi}u\right)^3 \Big/ \frac{2}{\pi}u^3\left(\frac{4}{\pi}-1\right)^2 \\
&= \left(1-\frac{2}{\pi}u\right)^3 \cdot v^3 \cdot \frac{\pi}{2} \cdot \left(\frac{\pi}{4-\pi}\right)^2 \\
&= \left[v\left(1-\frac{2}{\pi}u\right)\right]^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) \\
&= \left(v-\frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) \qquad = g(v).
\end{aligned}
$$

We can see that $g(v)$ is increasing in v, which is always ≥ 1. Therefore:

$$
\min_v g(v) = g(v)|_{v=1} = \left(1-\frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) = 1.009524 \approx 1, \tag{8}
$$

which means that the right hand side of (4), which is supposed to be equal to the left hand side of (4), can't ever be less than 1. Unfortunately, it sometimes is; in particular, $\frac{np(1-p)}{(1-2p)^2} \to 0$ when $p \to 0$ or $p \to 1$. So if we want a solution, we must restrict n and p such that

$$
\begin{aligned}
\{\text{right hand side of (4)}\} &\geq \{\text{min of left hand side of (4)}\} \\
\frac{np(1-p)}{(1-2p)^2} &\geq 1 \\
np(1-p) &\geq (1-2p)^2
\end{aligned} \tag{9}
$$

Here, two scenarios arise. The first is when we have a fixed p and wish to find the minimum n necessary to derive a skew-normal approximation. From (**??**), this is very simple:

$$
n \geq \frac{(1-2p)^2}{p(1-p)} \tag{10}
$$

Figure **??** shows the least sample size required to estimate λ as a function of p. As expected, it is quite large when p is small and rapidly goes to 0 as p increases; for example, when $p = 0.01$, n must be ≥ 98, but at $p = 0.2$, n need only be ≥ 3, a trivial requirement to meet.

The second scenario is when n is fixed and we wish to solve for p. In this case, we return to (**??**) for further factoring:

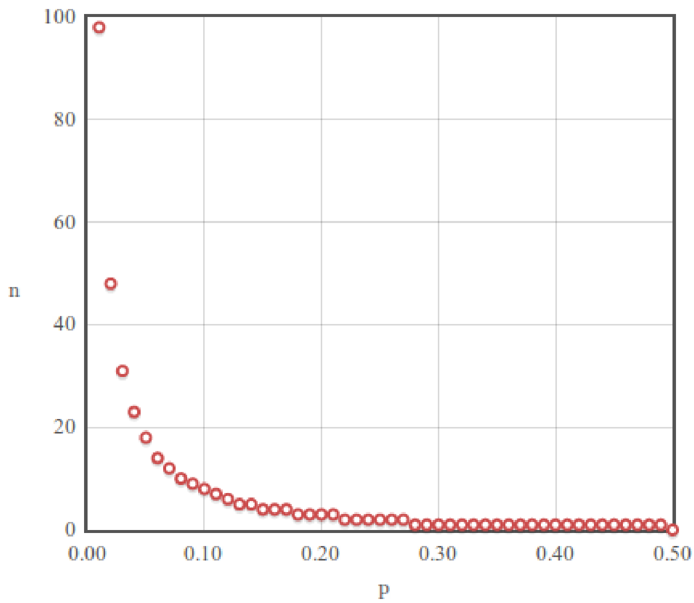

Figure 1: Least possible n, given a fixed p

$$
\begin{aligned}
np - np^2 &\geq 1 - 4p + 4p^2 \\
1 - 4p + 4p^2 - np + np^2 &\leq 0 \\
(n+4)p^2 - (n+4)p + 1 &\leq 0
\end{aligned}
\tag{11}
$$

We then apply the quadratic formula with $a = n + 4$, $b = -(n + 4)$, and $c = 1$:

$$
\begin{aligned}
p &= \frac{(n+4) \pm \sqrt{(n+4)^2 - 4 \cdot (n+4) \cdot 1}}{2(n+4)} \\
&= \frac{(n+4) \pm \sqrt{n^2 + 8n + 16 - 4n - 16}}{2(n+4)} \\
&= \frac{(n+4) \pm \sqrt{n^2 + 4n}}{2(n+4)} \\
&= \frac{n+4}{2(n+4)} \pm \frac{1}{2}\sqrt{\frac{n(n+4)}{(n+4)^2}} \\
&= \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{n}{n+4}}
\end{aligned}
$$

Let $r_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}}$ and $r_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}}$. (Note that $r_1 < r_2$.) Now we can rewrite (**??**) as

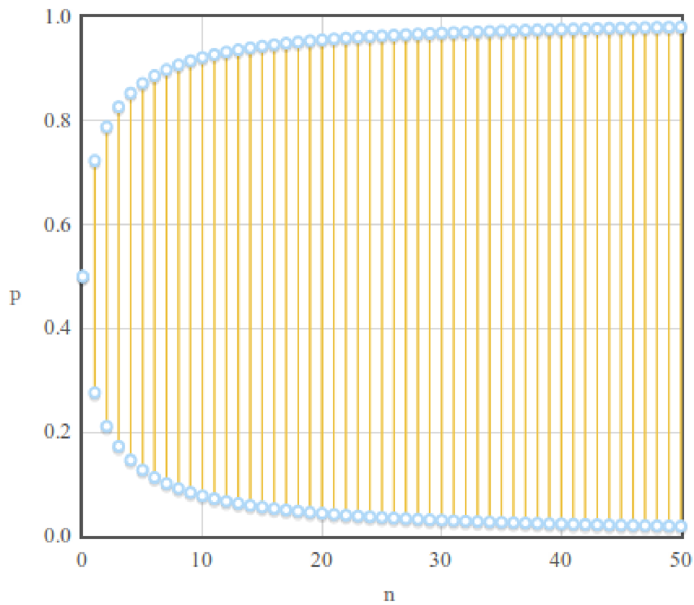

Figure 2: Range of possible p, given a fixed n

$$(p - r_1)(p - r_2) \leq 0$$

Examining the left hand side, when $p < r_1$, both terms are negative and so their product is positive; when $p > r_2$, both terms are positive, again leading the product to be positive. Therefore, our solution lies where $r_1 \leq p \leq r_2$, or more explicitly:

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}} \;\leq\; p \;\leq\; \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}} \tag{12}$$

Shown in figure **??** as a function of n, this interval grows quickly as n increases, and for sufficiently large n, it becomes almost $(0, 1)$. For example, when $n = 100$, our interval is $(0.00971, 0.99029)$; when $n = 500$, it is (0.00199, 0.99801).

For those unfortunate combinations of n and p so extreme that our skew-normal approximation will not work, our authors suggest a Poission approximation.

A Solving for λ

Hoping that it would provide some insight into (5), I expanded (4) after moving all terms to the left hand side:

$$\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3 \Bigg/ \left[\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2\right] - \frac{np(1-p)}{(1-2p)^2}$$

$$\left(\frac{\pi(1+\lambda^2)-2\lambda^2}{\pi(1+\lambda^2)}\right)^3 \Bigg/ \left(\frac{2(4-\pi)^2\lambda^6}{\pi^3(1+\lambda^2)^3}\right) - \frac{np(1-p)}{(1-2p)^2}$$

$$\left(\frac{\pi^3(1+\lambda^2)^3-3\pi^2(1+\lambda^2)^2\cdot 2\lambda^2+3\pi(1+\lambda^2)\cdot 4\lambda^4-8\lambda^6}{\cancel{\pi^3(1+\lambda^2)^3}}\right)\cdot\left(\frac{\cancel{\pi^3(1+\lambda^2)^3}}{2(4-\pi)^2\lambda^6}\right) - \frac{np(1-p)}{(1-2p)^2}$$

$$\frac{\pi^3(1+3\lambda^2+3\lambda^4+\lambda^6)-6\pi^2\lambda^2(1+2\lambda^2+4\lambda^4)+12\pi\lambda^4(1+\lambda^2)-8\lambda^6}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$

$$\frac{\pi^3+3\pi^3\lambda^2+3\pi^3\lambda^4+\pi^3\lambda^6-6\pi^2\lambda^2-12\pi^2\lambda^4-24\pi^2\lambda^6+12\pi\lambda^4+12\pi\lambda^6-8\lambda^6}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$

$$\frac{\lambda^6(\pi^3-24\pi^2+12\pi-8)+\lambda^4(3\pi^3-12\pi^2+12\pi)+\lambda^2(3\pi^3-6\pi^2)+\pi^3}{2(4-\pi)^2\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$

Taking $c_1 = \pi^3 - 24\pi^2 + 12\pi - 8$, $c_2 = 3\pi^3 - 12\pi^2 + 12\pi$, $c_3 = 3\pi^3 - 6\pi^2$, $c_4 = \pi^3$, and $c_5 = 2(4-\pi)^2$, we can simplify this to

$$\frac{c_1\lambda^6+c_2\lambda^4+c_3\lambda^2+c_4}{c_5\lambda^6} - \frac{np(1-p)}{(1-2p)^2}$$

$$\frac{c_1(1-2p)^2\lambda^6+c_2(1-2p)^2\lambda^4+c_3(1-2p)^2\lambda^2+c_4(1-2p)^2-c_5\ np(1-p)\lambda^6}{c_5(1-2p)^2\lambda^6}$$

$$\frac{\left[c_1(1-2p)^2-c_5\ np(1-p)\right]\lambda^6+\left[c_2(1-2p)^2\right]\lambda^4+\left[c_3(1-2p)^2\right]\lambda^2+c_4(1-2p)^2}{c_5(1-2p)^2\ \lambda^6}$$

B Curiosity

As a curiosity, I was unable to get Pewsey and Azzalini to agree with each other on $E(Z^3)$. According to Pewsey (2000),

$$E(Y^3) = \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 \tag{13}$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}} \in (-1, 1)$. Since $Y = \mu + \sigma Z$, by the linearity of expected value, we also have

$$
\begin{aligned}
E(Y^3) &= E\left[(\mu+\sigma Z)^3\right] \\
&= E(\mu^3 + 3\mu^2\sigma Z + 3\mu\sigma^2 Z^2 + \sigma^3 Z^3) \\
&= \mu^3 + 3\mu^2\sigma\ E(Z) + 3\mu\sigma^2\ E(Z^2) + \sigma^3\ E(Z^3) \\
&= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + \sigma^3\ E(Z^3) \qquad (14)
\end{aligned}
$$

By comparing equations (**??**) and (**??**) and eliminating terms, we arrive at

$$
\begin{aligned}
\sigma^3\ E(Z^3) &= 3b\delta\sigma^3 - b\delta^3\sigma^3 \\
\Rightarrow \quad E(Z^3) &= 3b\delta - b\delta^3 \\
&= b\delta(3-\delta^2) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \cdot \left(3 - \frac{\lambda^2}{1+\lambda^2}\right) \qquad (15)
\end{aligned}
$$

However, according to equation (6.5?) in Azzalini (2005),

$$
E(Z^r) = \begin{cases} 1 \times 3 \times \cdots \times (r-1) & \text{if r is even} \\ \dfrac{\sqrt{2}\ (2k+1)!\ \lambda}{\sqrt{\pi}\ (1+\lambda^2)^{k+1/2}\ 2^k} \displaystyle\sum_{m=0}^{k} \frac{m!\ (2\lambda)^{2m}}{(2m+1)!\ (k-m)!} & \text{if } r = 2k+1 \text{ and } k = 0, 1, \ldots \end{cases} \qquad (16)
$$

So, for $E(Z^3)$, we have $r = 2k+1 = 3$ and $k = 1$:

$$
\begin{aligned}
E(Z^3) &= \frac{\sqrt{2}\cdot 3! \cdot \lambda}{\sqrt{\pi}\cdot(1+\lambda^2)^{3/2}\cdot 2} \sum_{m=0}^{1} \frac{m!\ (2\lambda)^{2m}}{(2m+1)!\ (1-m)!} \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\lambda}{(1+\lambda^2)^{3/2}} \cdot \left(\frac{0!(2\lambda)^0}{1!1!} + \frac{1!(2\lambda)^2}{3!0!}\right) \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\lambda}{(1+\lambda^2)^{3/2}} \cdot \left(1 + \frac{2}{3}\lambda^2\right) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{(\sqrt{1+\lambda^2})^3} \cdot (3+2\lambda^2) \qquad (17)
\end{aligned}
$$

Unfortunately, equations (**??**) and (**??**) do not really line up.