The Properties of the Standard Skew Normal

The four properties of the skew normal listed in section 2 of "A Note on Improved Approx ...".

1 Property 1

Theorem. *If $Z \sim SN(0, 1, \lambda)$, then $(-Z) \sim SN(0, 1, -\lambda)$.*

Proof. The standard normal pdf is an even function: $\phi(-x) = \frac{1}{\sqrt{2\pi}} e^{-(-x)^2/2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \phi(x)$. The standard normal cdf, however, $\Phi(x) = \int_{-\infty}^{\infty} \phi(x)$, is not, being 0 near $-\infty$ and 1 near ∞. Thus,

$$\begin{aligned} f_{(-Z)}(x) &= f_Z(-x) \\ &= 2 \cdot \phi(-x) \cdot \Phi(-\lambda x) \\ &= 2 \cdot \phi(x) \cdot \Phi(-\lambda x) \end{aligned}$$

which is the pdf of $SN(0, 1, -\lambda)$. $\mathcal{Q.E.D.}$

2 Property 2

Theorem. *As $\lambda \to \pm\infty$, $SN(0, 1, \lambda)$ tends to the half normal distribution.*

To prove our theorem, it is helpful to formally define the half normal distribution:

Lemma. *Let $X \sim N(0, \sigma^2)$. Then the distribution of $|X|$ is a half-normal random variable with parameter σ and*

$$f_{|X|}(x) = \begin{cases} 2 \cdot f_{N(0,\sigma^2)}(x) & \textit{when } 0 < x < \infty \\ 0 & \textit{everywhere else} \end{cases}$$

Proof. Let $X \sim N(0, \sigma^2)$, defined over $A = (-\infty, \infty)$.

Define

$$Y = |X| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

Y is not one-to-one over A. However, we can partition A into disjoint subsets $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$, and $A_3 = \{0\}$ such that $A = A_1 \cup A_2 \cup A_3$ and Y is one-to-one over each A_i. We can then transform each piece separately using Theorem 6.3.2:

On A_1: $y = -x \;\longrightarrow\; x = -y$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |-1| = 1$, yielding

$$\begin{aligned} f_Y(y) &= f_x(x) \cdot \mathbb{J} \\ &= f_x(-y) \cdot 1 \\ &= \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{(-y)^2}{2\sigma}} \\ &= \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{y^2}{2\sigma}} \\ &= f_{N(0,\sigma^2)}(y) \end{aligned}$$

over the domain $A_1 : -\infty < x < 0 \;\longrightarrow\; -\infty < -y < 0 \;\longrightarrow\; 0 < y < \infty : B_1$.

Similarly, on A_2: $y = x \;\longrightarrow\; x = y$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |1| = 1$, yielding

$$\begin{aligned} f_Y(y) &= f_x(x) \cdot \mathbb{J} \\ &= f_x(y) \cdot 1 \\ &= \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{y^2}{2\sigma}} \\ &= f_{N(0,\sigma^2)}(y) \end{aligned}$$

over the domain $A_2 : 0 < x < \infty \;\longrightarrow\; 0 < y < \infty : B_2$.

On A_3, we have $x = 0, y = 0$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |0| = 0$, yielding $f_Y(y) = f_X(x) \cdot \mathbb{J} = f_X(x) \cdot 0 = 0$.

Then, by Theorem 6.3.10,

$$\begin{aligned} f_Y(y) &= \{f_Y(y) \text{ over } A_1\} + \{f_Y(y) \text{ over } A_2\} \\ &= f_{N(0,\sigma^2)}(y) + f_{N(0,\sigma^2)}(y) \\ &= 2 \cdot f_{N(0,\sigma^2)}(y) \end{aligned}$$

over $B = B_1 \cup B_2 = (0, \infty)$, and 0 otherwise. $\mathcal{Q.E.D.}$

With this result, we can easily show our property:

Proof. Let $X \sim SN(0, 1, \lambda)$. Recall that $f_X(x) = 2 \cdot \phi(x) \cdot \Phi(\lambda x)$.

Consider $\lim_{\lambda \to \infty} f_X(x)$. When x is negative, $\lambda x \to -\infty$ and thus $\Phi(\lambda x) \to 0$. When x is positive, however, $\lambda x \to \infty$ and $\Phi(\lambda x) \to 1$. Thus we have

$$\lim_{\lambda \to \infty} 2 \cdot \phi(x) \cdot \Phi(\lambda x) = \begin{cases} 0 & \text{when } x \leq 0 \\ 2 \cdot \phi(x) & \text{when } x > 0 \end{cases}$$

In $\lim_{\lambda \to -\infty} f_X(x)$, the signs are reversed. When x is negative, $\lambda x \to \infty$ and $\Phi(\lambda x) \to 1$. When x is positive, $\lambda x \to -\infty$ and $\Phi(\lambda x) \to 0$. Thus,

$$\lim_{\lambda \to -\infty} 2 \cdot \phi(x) \cdot \Phi(\lambda x) = \begin{cases} 2 \cdot \phi(x) & \text{when } x < 0 \\ 0 & \text{when } x \geq 0 \end{cases}$$

$\mathcal{Q.E.D.}$

3 Property 3

Theorem. *If $Z \sim SN(0, 1, \lambda)$, then $Z^2 \sim \chi^2_1$ (chi-square with 1 degree of freedom).*

Proof. To prove our result, we make use of a lemma in Azzalini (2005):

Lemma 1. *If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one-dimensional distribution function such that G' exists and is a density symmetric about 0, then*

$$f(z) = 2 \cdot f_0(z) \cdot G\{w(z)\} \quad (-\infty < z < \infty) \tag{3.1}$$

is a density function for any odd function $w(\cdot)$.

Notice that $\phi(x)$ is a one-dimensional probability density function symmetric about 0, and $\Phi(x)$ is a one-dimensional distribution function such that Φ' exists and is a density symmetric about 0. Furthermore, λx is an odd function. Thus, $f_Z(z) = 2 \cdot \phi(z) \cdot \Phi(\lambda z)$ conforms to equation 3.1. With that in mind, the corollary to this lemma provides a very useful result:

Corollary (Perturbation Invariance)**.** *If $Y \sim f_0$ and $Z \sim f$, then $|Y| \overset{d}{=} |Z|$, where the notation $\overset{d}{=}$ denotes equality in distribution.*

Thus, we can treat ϕ and Z as being equal in distribution. We will now show that $\phi^2 \sim \chi_1^2$:

$$\begin{aligned}
M_{\phi^2}(t) &= E[e^{tx^2}] \\
&= \int_{-\infty}^{\infty} e^{tx^2} \left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right] dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx^2 - x^2/2} \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1-2t)} \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{1-2t}\, x)^2} \, dx \qquad (3.2)
\end{aligned}$$

Let $u = (\sqrt{1-2t})\ x$; then we have $du = \sqrt{1-2t}$, $dx = \frac{du}{\sqrt{1-2t}}$, and our limits become $x \to -\infty \Rightarrow u \to -\infty$ and $x \to \infty \Rightarrow u \to \infty$. Now we can rewrite equation 3.2 as

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left(\frac{1}{\sqrt{1-2t}}\right) du \\
&= \frac{1}{\sqrt{1-2t}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du\right) \qquad (3.3)
\end{aligned}$$

Notice that $\frac{1}{\sqrt{2\pi}}\ e^{-u^2/2}$ is the pdf of the standard normal, which integrated over $(-\infty, \infty)$ equals 1. Thus equation 3.3 reduces to $\frac{1}{\sqrt{1-2t}} = (1-2t)^{-1/2}$, which is the MGF of the χ_1^2.

Since Z is equal in distribution to ϕ, we can also conclude that $Z^2 \sim \chi_1^2$. $\mathcal{Q.E.D.}$

4 Property 4

Theorem. *The MGF of $SN(0, 1, \lambda)$ is $M(t|\lambda) = 2 \cdot \Phi(\delta t) \cdot e^{t^2/2}$ where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $t \in (-\infty, \infty)$.*

According to equation 5 in Azzalini (2005), the MGF of $SN(\mu, \sigma^2, \lambda)$ is

$$M(t) = E\{e^{tY}\} = 2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot \Phi(\delta \sigma t) \qquad (4.1)$$

where $\delta = \frac{\lambda}{1+\lambda^2} \in (-1, 1)$.

It follows that the MGF of the $SN(0, 1, \lambda)$ is

$$2 \cdot \exp\left(0 \cdot t + \frac{1 \cdot t^2}{2}\right) \cdot \Phi(\delta \cdot 1 \cdot t) = 2 \cdot e^{t^2/2} \cdot \Phi(\delta t) \qquad (4.2)$$

where $\delta = \frac{\lambda}{1+\lambda^2} \in (-1, 1)$.