The Skew-Normal Approximation of the Binomial Distribution

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1 Introduction

One of the most basic distributions in statistics is the binomial, $X \sim Bin(n, p)$, with pdf

$$f_X(x) = \binom{n}{x} p^x q^{n-x}$$

Calculating the binomial cdf, $F_X(x) = P(X \leq x) = \sum_{k=1}^{x} f_X(k)$, by hand is manageable for small n but quickly becomes cumbersome as n grows even mediumly large. A common strategy is to use the normal distribution as an approximation:

$$F_X(x) \approx \Phi\left(\frac{x + 0.5 - \mu}{\sigma}\right) \tag{1}$$

where Φ is the standard normal cdf and $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

This approximation works well when either n is very large (invoking the Central Limit Theorem) or when p is close to 0.5 (making X roughly symmetric). However, when n is not large and p is close to 0 or 1, the binomial distribution is skewed and the normal approximation can be inaccurate; sometimes, as demonstrated by Schader and Schmid (1989), rather substantially. In these cases, the skew-normal approximation can provide an alternate – and considerably more accurate – method of approximation.

2 The Skew-Normal

The skew-normal distribution is similar to the normal but with an added parameter for skew, allowing it to lean asymmetrically to the left or right. In this section, we'll acquaint ourselves with some of its basic properties.

2.1 Basics

Let Y be a skew-normal distribution, with location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$, and shape parameter $\lambda \in \mathbb{R}$; we will denote it $SN(\mu, \sigma, \lambda)$. Then Y has pdf

$$f_Y(x) = \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot \Phi\left(\frac{\lambda(x-\mu)}{\sigma}\right) \tag{2}$$

where ϕ is the standard normal pdf and Φ is the standard normal cdf. Some other basic properties of Y, given by Pewsey (2000), are

$$\begin{aligned} E(Y) &= \mu + b\delta\sigma \\ E(Y^2) &= \mu^2 + 2b\delta\mu\sigma + \sigma^2 \\ E(Y^3) &= \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3 \\ Var(Y) &= \sigma^2(1 - b^2\delta^2) \end{aligned} \tag{3}$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.

The $SN(0, 1, \lambda)$ distribution is called the standard skew-normal; its pdf is

$$f_Z(x) = 2 \cdot \phi(x) \cdot \Phi(\lambda x) \tag{4}$$

Similar to the normal and standard normal, $Z = \frac{Y-\mu}{\sigma}$ and $Y = \sigma Z + \mu$.

A natural question to ask is how the skew-normal relates to the normal. Fortunately, the connection is very intuitive: When $\lambda = 0$, Equation (2) reverts to the normal pdf:

$$
\begin{aligned}
f_Y(x|\lambda = 0) &= \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot \Phi(0) \\
&= \frac{2}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \cdot 0.5 \\
&= \frac{1}{\sigma} \cdot \phi\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} \ \cdot \ \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \ \cdot \ \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

2.2 Four Properties

The following four properties of the skew-normal, given by Chang et al. (2008), help shed some light on our enigmatic new distribution:

Property 1. *If $Z \sim SN(0, 1, \lambda)$, then $(-Z) \sim SN(0, 1, -\lambda)$.*

Proof. The standard normal pdf is an even function: $\phi(-x) = \frac{1}{\sqrt{2\pi}} \ e^{-(-x)^2/2} = \frac{1}{\sqrt{2\pi}} \ e^{-x^2/2} = \phi(x)$. But the standard normal cdf, $\Phi(x) = \int_{-\infty}^{\infty} \phi(x),$ is not, being 0 near $-\infty$ and 1 near ∞. Thus,

$$
\begin{aligned}
f_{(-Z)}(x) &= f_Z(-x) \\
&= 2 \cdot \phi(-x) \cdot \Phi(-\lambda x) \\
&= 2 \cdot \phi(x) \cdot \Phi(-\lambda x)
\end{aligned}
$$

which is the pdf of $SN(0, 1, -\lambda)$. $\mathcal{Q.E.D.}$

Property 2. *If $Z \sim SN(0, 1, \lambda)$, then $Z^2 \sim \chi_1^2$ (chi-square with 1 degree of freedom).*

Proof. To prove our result, we make use of Lemma 1 in Azzalini (2005):

Lemma 2.1. *If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one-dimensional distribution function such that G' exists and is a density symmetric about 0, then*

$$
f(z) = 2 \cdot f_0(z) \cdot G\{w(z)\} \quad (-\infty < z < \infty) \tag{5}
$$

is a density function for any odd function $w(\cdot)$.

This lemma provides a very useful corollary:

Corollary 2.1 (Perturbation Invariance). *If $Y \sim f_0$ and $Z \sim f$, then $|Y| \stackrel{d}{=} |Z|$, where the notation $\stackrel{d}{=}$ denotes equality in distribution.*

Let $f_0 = \phi$ and $G = \Phi$. Then, $f_Z(z) = 2 \cdot \phi(z) \cdot \Phi(\lambda z)$ conforms to equation (5), and we can conclude that *phi* and Z are equal in distribution.

We will now show that $\phi^2 \sim \chi_1^2$:

$$
\begin{aligned}
M_{\phi^2}(t) &= E[e^{tx^2}] \\
&= \int_{-\infty}^{\infty} e^{tx^2} \left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right] dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{tx^2 - x^2/2} \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2}(1-2t)} \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\sqrt{1-2t}\, x)^2} \, dx
\end{aligned}
$$

Let $u = (\sqrt{1-2t})\, x$; then $du = (\sqrt{1-2t})\, dx$, $\;dx = \frac{du}{\sqrt{1-2t}}$, and our limits become $x \to -\infty, x \to \infty \;\Rightarrow\; u \to -\infty, u \to \infty$.

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-u^2/2} \left(\frac{1}{\sqrt{1-2t}} du \right) \\
&= \frac{1}{\sqrt{1-2t}} \underbrace{\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-u^2/2} \, du \right)}_{\phi(u) \text{ integrated over } (-\infty,\infty) = 1} \\
&= \frac{1}{\sqrt{1-2t}}
\end{aligned}
$$

which is the MGF of the χ_1^2. Since Z is equal in distribution to ϕ, we can also conclude that $Z^2 \sim \chi_1^2$. $\mathcal{Q.E.D.}$

Property 3. *As $\lambda \to \pm\infty$, $SN(0, 1, \lambda)$ tends to the half normal distribution, $\pm|N(0, 1)|$.*

To prove our theorem, it is helpful to formally define the half normal distribution:

Lemma 3.1. *Let $X \sim N(0, \sigma^2)$. Then the distribution of $|X|$ is a half-normal random variable with parameter σ and*

$$
f_{|X|}(x) = \begin{cases} 0 & \textit{when } -\infty < x \leq 0 \\ 2 \cdot f_X(x) & \textit{when } 0 < x < \infty \end{cases}
$$

Proof. Let $X \sim N(0, \sigma^2)$, defined over $A = (-\infty, \infty)$. Define

$$Y = |X| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

Y is not one-to-one over A. However, we can partition A into disjoint subsets $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$, and $A_3 = \{0\}$ such that $A = A_1 \cup A_2 \cup A_3$ and Y is one-to-one over each A_i. We can then transform each piece separately using Theorem 6.3.2 from Bain and Engelhardt (1992):

On A_1: $y = -x \;\Rightarrow\; x = -y$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |-1| = 1$, yielding

$$\begin{aligned} f_Y(y) &= f_X(x) \cdot \mathbb{J} \\ &= f_X(-y) \cdot 1 \\ &= \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(-y)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{y^2}{2\sigma^2}} \\ &= f_X(y) \end{aligned}$$

over the domain $A_1 : -\infty < x < 0 \;\Rightarrow\; -\infty < -y < 0 \;\Rightarrow\; 0 < y < \infty : B_1$.

Similarly, on A_2: $y = x \;\Rightarrow\; x = y$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |1| = 1$, yielding

$$\begin{aligned} f_Y(y) &= f_X(x) \cdot \mathbb{J} \\ &= f_X(y) \cdot 1 \\ &= f_X(y) \end{aligned}$$

over the domain $A_2 : 0 < x < \infty \;\Rightarrow\; 0 < y < \infty : B_2$.

On A_3, we have $x = 0 \;\Rightarrow\; y = 0$ and $\mathbb{J} = \left|\frac{dx}{dy}\right| = |0| = 0$, yielding $f_Y(y) = f_X(x) \cdot \mathbb{J} = f_X(x) \cdot 0 = 0$.

Now, by Theorem 6.3.10 from Bain and Engelhardt (1992)

$$\begin{aligned} f_Y(y) &= \{f_Y(y) \text{ over } A_1\} + \{f_Y(y) \text{ over } A_2\} \\ &= f_X(y) + f_X(y) \\ &= 2 \cdot f_X(y) \end{aligned} \tag{6}$$

over $B = B_1 \cup B_2 = (0, \infty)$, and 0 otherwise. $\mathcal{Q.E.D.}$

With Lemma 3.1, we can easily show our property:

Proof of Property 3. Let $Z \sim SN(0, 1, \lambda)$. Recall that $f_Z(x) = 2 \cdot \phi(x) \cdot \Phi(\lambda x)$.

Consider $\lim_{\lambda \to \infty} f_X(x)$. When x is negative, $\lambda x \to -\infty$ and thus $\Phi(\lambda x) \to 0$. When x is positive, however, $\lambda x \to \infty$ and $\Phi(\lambda x) \to 1$. Thus

$$\lim_{\lambda \to \infty} 2 \cdot \phi(x) \cdot \Phi(\lambda x) \;=\; \begin{cases} 0 & \text{when } x \leq 0 \\ 2 \cdot \phi(x) & \text{when } x > 0 \end{cases} \;=\; |N(0,1)| \tag{7}$$

In $\lim_{\lambda \to -\infty} f_X(x)$, the signs are reversed. When x is negative, $\lambda x \to \infty$ and $\Phi(\lambda x) \to 1$. When x is positive, $\lambda x \to -\infty$ and $\Phi(\lambda x) \to 0$. Thus,

$$\lim_{\lambda \to -\infty} 2 \cdot \phi(x) \cdot \Phi(\lambda x) \;=\; \begin{cases} 2 \cdot \phi(x) & \text{when } x < 0 \\ 0 & \text{when } x \geq 0 \end{cases} \;=\; -|N(0,1)| \tag{8}$$

$\mathcal{Q.E.D.}$

Property 4. *The MGF of $SN(0, 1, \lambda)$ is*

$$M(t|\lambda) = 2 \cdot \Phi(\delta t) \cdot e^{t^2/2} \tag{9}$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $t \in (-\infty, \infty)$.

Proof. According to Equation 5 in Azzalini (2005), the MGF of $SN(\mu, \sigma^2, \lambda)$ is

$$M(t) \;=\; E\{e^{tY}\} \;=\; 2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot \Phi(\delta \sigma t)$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}} \in (-1, 1)$. It follows that the MGF of the $SN(0, 1, \lambda)$ is

$$2 \cdot \exp\left(0 \cdot t + \frac{1 \cdot t^2}{2}\right) \cdot \Phi(\delta \cdot 1 \cdot t) \;=\; 2 \cdot e^{t^2/2} \cdot \Phi(\delta t)$$

$\mathcal{Q.E.D.}$

3 Developing an Approximation

Now that we have gotten to know our new distribution a little better, we can use it to develop an approximation for the binomial.

Let $B \sim Bin(n, p)$ and $Y \sim SN(\mu, \sigma^2, \lambda)$. We will find estimates for μ, σ, and λ by comparing the first, second, and third moments about the mean of B and Y.

3.1 The Moments of the Binomial

Let's start with the binomial. The first two moments are simply the mean and variance, which we can state from memory:

$$E(B) = np, \quad Var(B) = np(1-p)$$

Having these, we can easily find

$$E(B^2) = Var(B) + [E(B)]^2 = np(1-p) + n^2p^2 = np - np^2 + n^2p^2$$

which we will need for the third moment. We will also need $E(B^3)$, which we will get via the third factorial moment:

$$E[B(B-1)(B-2)] = \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\}$$

Notice that the first three terms of our sum are zero, so we can rewrite our sum beginning at $x = 3$:

$$\begin{aligned}
&= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\; p^x q^{n-x} \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\; p^x q^{n-x} \\
&= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\; p^{x-3} q^{n-x}
\end{aligned}$$

Let $y = x - 3$. Then $x = y + 3$, and $x = 3, x = n \Rightarrow y = 0, y = n - 3$.

$$\begin{aligned}
&= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\ p^y q^{n-(y+3)} \\
&= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\ p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,p) \text{ summed from 0 to } n-3] = 1} \\
&= n(n-1)(n-2)p^3 \\
&= n^3p^3 - 3n^2p^3 + 2np^3
\end{aligned}$$

Further expanding the left side and solving for $E(B^3)$,

$$\begin{aligned}
E\left[B^3 - 3B^2 + 2B\right] &= n^3p^3 - 3n^2p^3 + 2np^3 \\
E(B^3) - 3E(B^2) + 2E(B) &= \\
E(B^3) - 3(np - np^2 + n^2p^2) + 2np &= \\
\Rightarrow \qquad E(B^3) &= n^3p^3 - 3n^2p^3 + 2np^3 + 3np - 3np^2 + 3n^2p^2 - 2np \\
&= n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np
\end{aligned}$$

Now, finally, we have all the building blocks necessary to obtain the third moment:

$$\begin{aligned}
E\left([B - E(B)]^3\right) &= E\left(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3\right) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) - 3(np - np^2 + n^2p^2)(np) + 2(np)^3 \\
&= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 - 3np^2 + \cancel{3n^2p^2} + np - \cancel{3n^2p^2} + \cancel{3n^2p^3} - \cancel{3n^3p^3} + \cancel{2n^3p^3} \\
&= 2np^3 - 3np^2 + np \\
&= np(p-1)(2p-1)
\end{aligned}$$

Our hard-earned results, restated for convenience:

$$\begin{aligned}
E(B) &= np \\
E([B - E(B)]^2) &= np(1-p) \\
E([B - E(B)]^3) &= np(p-1)(2p-1)
\end{aligned} \tag{10}$$

3.2 The Moments of the Skew Normal

Now we'll take a look at the moments of the skew normal. Equation (3) takes care of the mean and variance; again the third moment is a little more complicated:

$$\begin{aligned}
E([Y-E(Y)]^3) &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) \\
&\quad + 2\,(\mu + b\delta\sigma)^3 \\
&= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} \\
&\quad - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\
&= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3 \\
&= b\delta^3\sigma^3(2b^2 - 1)
\end{aligned}$$

We restate our results:

$$\begin{aligned}
E(Y) &= \mu + b\delta\sigma & &= \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \\
E([Y-E(Y)]^2) &= \sigma^2(1 - b^2\delta^2) & &= \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \\
E([Y-E(Y)]^3) &= b\delta^3\sigma^3(2b^2-1) & &= \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right)
\end{aligned} \tag{11}$$

3.3 Solving for μ, σ, λ

Now we can set the moments of our two distributions equal to each other and, taking n and p as constants, solve for μ, σ and λ.

$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{12a}$$

$$np(1-p) = \sigma^2 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right) \tag{12b}$$

$$np(p-1)(2p-1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^3 \left(\frac{4}{\pi} - 1\right) \tag{12c}$$

To get λ, we divide the cube of (12b) by the square of (12c):

$$
\begin{aligned}
&\frac{\sigma^6\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6\cdot\frac{2}{\pi}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6\left(\frac{4}{\pi}-1\right)^2}=\frac{n^3p^3(1-p)^3}{n^2p^2(p-1)^2(2p-1)^2}\\
\Rightarrow\quad &\frac{\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi}\left(\frac{\lambda^2}{1+\lambda^2}\right)^3\left(\frac{4}{\pi}-1\right)^2}=\frac{np(1-p)}{(1-2p)^2}
\end{aligned}
\tag{13}
$$

The above equation (13) is a rational expression in λ^2 that can be solved with either a considerable amount of manual labor or, more efficiently, with a computer algebra system. Once we have λ^2, then λ is simply either the positive or negative square root, as determined by the sign of $(1-2p)$. This can be explained with a little assistance from Property 3: When $p \to 0$, the binomial skews left and converges toward the positive half normal, which by (7) corresponds to a positive λ. When $p \to 1$, the binomial skews right and converges toward the negative half normal, which by (8) corresponds to a negative λ. When $p = 0.5$, the binomial is symmetric and λ is 0, eliminating the need for a sign. Thus:

$$
\lambda = \{\text{sign of } (1-2p)\}\sqrt{\lambda^2} \tag{14}
$$

Having secured λ, we can find σ using (12b):

$$
np(1-p)=\sigma^2\left(1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}\right)\quad\Rightarrow\quad\sigma=\sqrt{\frac{np(1-p)}{1-\frac{2}{\pi}\cdot\frac{\lambda^2}{1+\lambda^2}}} \tag{15}
$$

And with both λ and σ, a simple rearrangement of (12a) yields μ:

$$
np=\mu+\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}}\quad\Rightarrow\quad\mu=np-\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}} \tag{16}
$$

When $p = 0.5$, we would expect the binomial to be perfectly symmetrical and therefore $\mu = np = n/2$. From (16), this implies that $\sigma\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\lambda}{\sqrt{1+\lambda^2}} = 0 \Rightarrow$ either $\sigma = 0$ or $\lambda = 0$. Since the former is impossible, we must conclude the latter, which brings us back to the normal distribution.

3.4 Restrictions

To obtain an estimate for λ, we must put a few restrictions on n and p.

If we let $u = \frac{\lambda^2}{1+\lambda^2}$ and $v = 1/u$, we can rewrite the left hand side of (13) as

$$
\begin{aligned}
&\left(1-\frac{2}{\pi}u\right)^3 \Big/ \frac{2}{\pi}u^3\left(\frac{4}{\pi}-1\right)^2 \\
&\left(1-\frac{2}{\pi}u\right)^3 \cdot v^3 \cdot \frac{\pi}{2} \cdot \left(\frac{\pi}{4-\pi}\right)^2 \\
&\left[v\left(1-\frac{2}{\pi}u\right)\right]^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) \\
&\left(v-\frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) \qquad (17)
\end{aligned}
$$

we can see that it is increasing in v, which is always ≥ 1. Therefore:

$$
\min_v \{\text{Eq. 17}\} = \{\text{Eq. 17}\}|_{v=1} = \left(1-\frac{2}{\pi}\right)^3 \left(\frac{\pi^3}{2(4-\pi)^2}\right) = 1.009524 \approx 1 \qquad (18)
$$

This means that the right hand side of (13), which is supposed to be equal to the left hand side of (13), can't ever be less than 1. Unfortunately, it sometimes is; in particular, $\frac{np(1-p)}{(1-2p)^2} \to 0$ when $p \to 0$ or $p \to 1$. So if we want a solution, we must restrict n and p such that

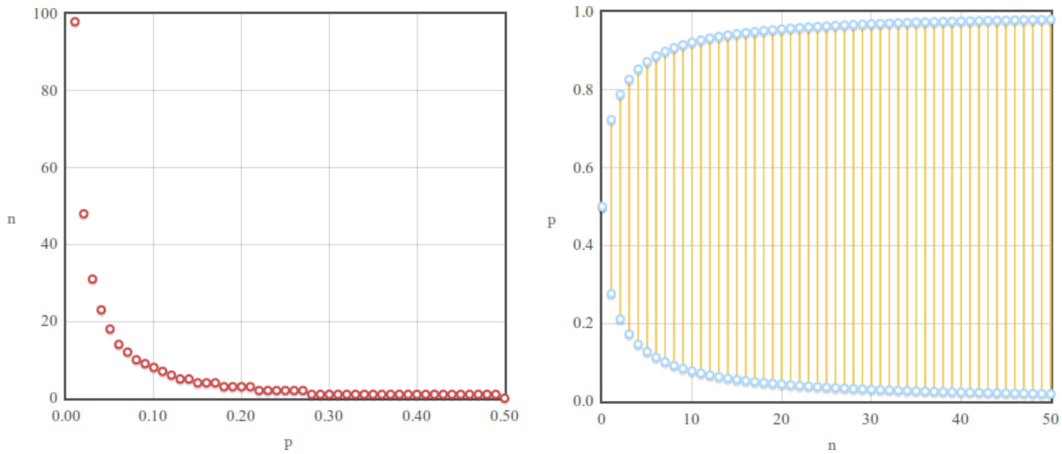
$$
\begin{aligned}
&\{\text{right hand side of (13)}\} \geq \{\text{min of left hand side of (13)}\} \\
&\frac{np(1-p)}{(1-2p)^2} \geq 1 \\
&np(1-p) \geq (1-2p)^2 \qquad (19)
\end{aligned}
$$

Here, two scenarios arise. The first is when we have a fixed p and wish to find the minimum n necessary to derive a skew-normal approximation. From (19), this is very simple:

$$
n \geq \frac{(1-2p)^2}{p(1-p)} \qquad (20)
$$

Figure 1a shows the least sample size required to estimate λ as a function of p. As expected, it is quite large when p is small and rapidly goes to 0 as p increases; for example, when $p = 0.01$, n must be ≥ 98, but at $p = 0.2$, n need only be ≥ 3, a trivial requirement to meet.

The second scenario is when n is fixed and we wish to solve for p. In this case, we return to (19) for further factoring



(a) Least possible n, given a fixed p

(b) Range of possible p, given a fixed n

Figure 1: Restrictions on n and p for estimating λ

$$
\begin{aligned}
np - np^2 &\geq 1 - 4p + 4p^2 \\
1 - 4p + 4p^2 - np + np^2 &\leq 0 \\
(n+4)p^2 - (n+4)p + 1 &\leq 0
\end{aligned}
\tag{21}
$$

and apply the quadratic formula with $a = n + 4$, $b = -(n+4)$, and $c = 1$:

$$
\begin{gathered}
\frac{(n+4) \pm \sqrt{(n+4)^2 - 4 \cdot (n+4) \cdot 1}}{2(n+4)} \\
\frac{(n+4) \pm \sqrt{n^2 + 8n + 16 - 4n - 16}}{2(n+4)} \\
\frac{(n+4) \pm \sqrt{n^2 + 4n}}{2(n+4)} \\
\frac{n+4}{2(n+4)} \pm \frac{1}{2}\sqrt{\frac{n(n+4)}{(n+4)^2}} \\
\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{n}{n+4}}
\end{gathered}
$$

Let $r_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}}$ and $r_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}}$. (Note that $r_1 < r_2$.) Now we can rewrite (21) as

$$
(p - r_1)(p - r_2) \leq 0
$$

Examining the left hand side, when $p < r_1$, both terms are negative and so their product is positive; when $p > r_2$, both terms are positive, again leading the product to be positive. Therefore, our solution lies where $r_1 \leq p \leq r_2$, or more explicitly:

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}} \;\leq\; p \;\leq\; \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}} \tag{22}$$

Shown in figure 1b as a function of n, this interval grows quickly as n increases, and for sufficiently large n, it becomes almost $(0, 1)$. For example, when $n = 100$, our interval is $(0.00971, 0.99029)$; when $n = 500$, it is (0.00199, 0.99801).

For those unfortunate combinations of n and p so extreme that our skew-normal approximation will not work, our authors suggest a Poission approximation.

4 Demonstrating Improved Accuracy

Now comes the time to justify our efforts by comparing the accuracy of our skew-normal approximation to that of the normal.

4.1 Visual Comparison

The first and most obvious way of judging accuracy is by visual inspection. Figures 2, 3, and 4 compare the binomial, normal, and skew-normal at small values of p for $n = 25$, $n = 50$, and $n = 100$, respectively. It is not hard to see that, especially at very small n and p, our skew-normal curve follows the shape of the binomial much more closely.

4.2 Maximal Absolute Error

A more numerical method would be to compare the maximal absolute errors of our two approximations, defined by Schader and Schmid (1989) as

$$\text{MABS}(n, p) \;=\; \max_{k \in \{0,1,\ldots,n\}} \left| F_{B(n,p)}(k) - F_{\text{appr}(n,p)}(k + 0.5) \right| \tag{23}$$

where $F_{B(n,p)}$ is the cdf of the binomial and $F_{\text{appr}(n,p)}$ is the cdf of either the normal or skew-normal approximation; the 0.5 is a continuity correction.

Figures 5 and 6 shows the MABS of the skew-normal and normal approximations as a function of p and n, respectively. Again, the skew-normal outperforms the normal considerably in the extreme ranges, with the two approximations converging as $n \to \infty$ or $p \to 0.5$.

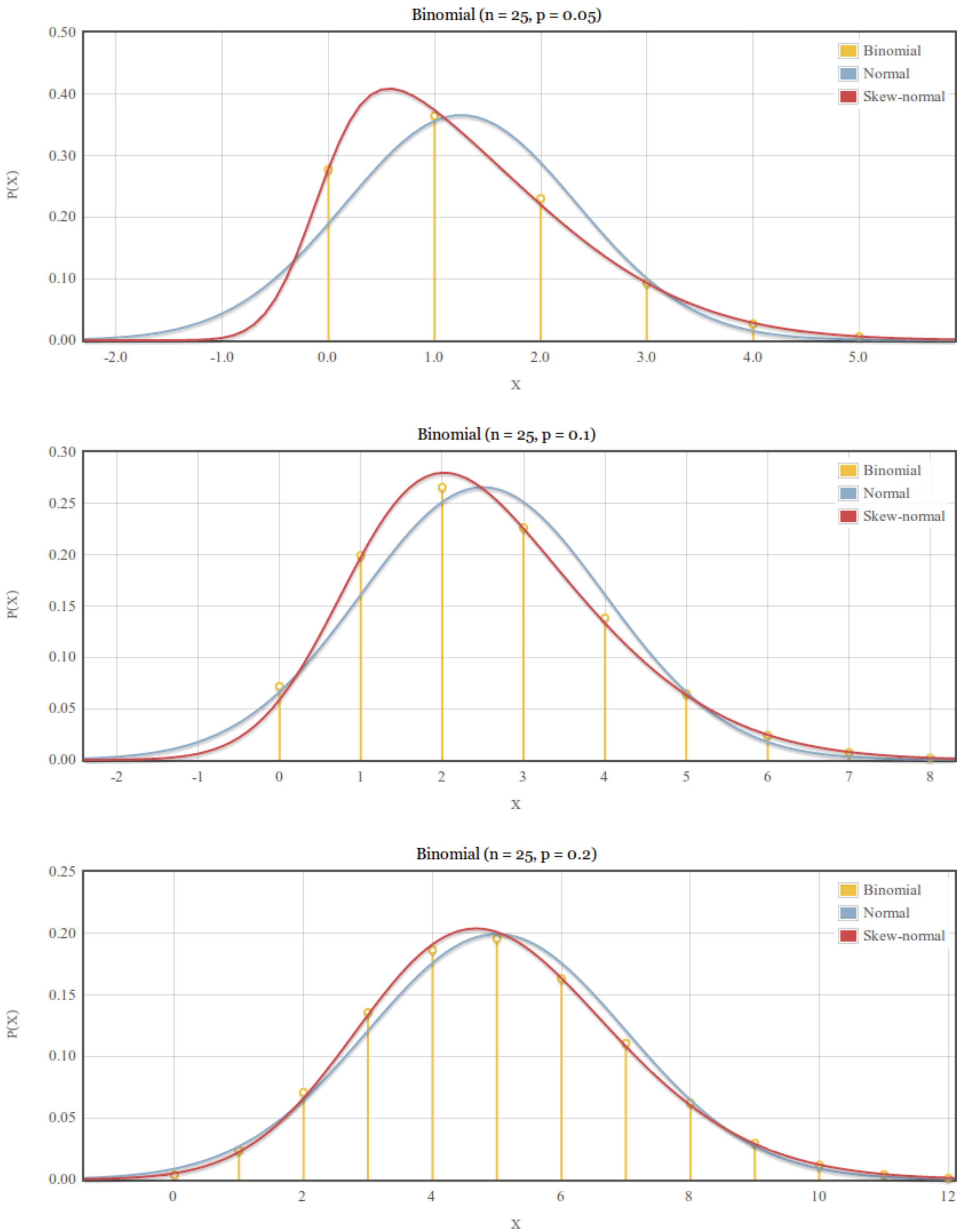


Figure 2: Binomial, normal, and skew-normal, $n = 25$

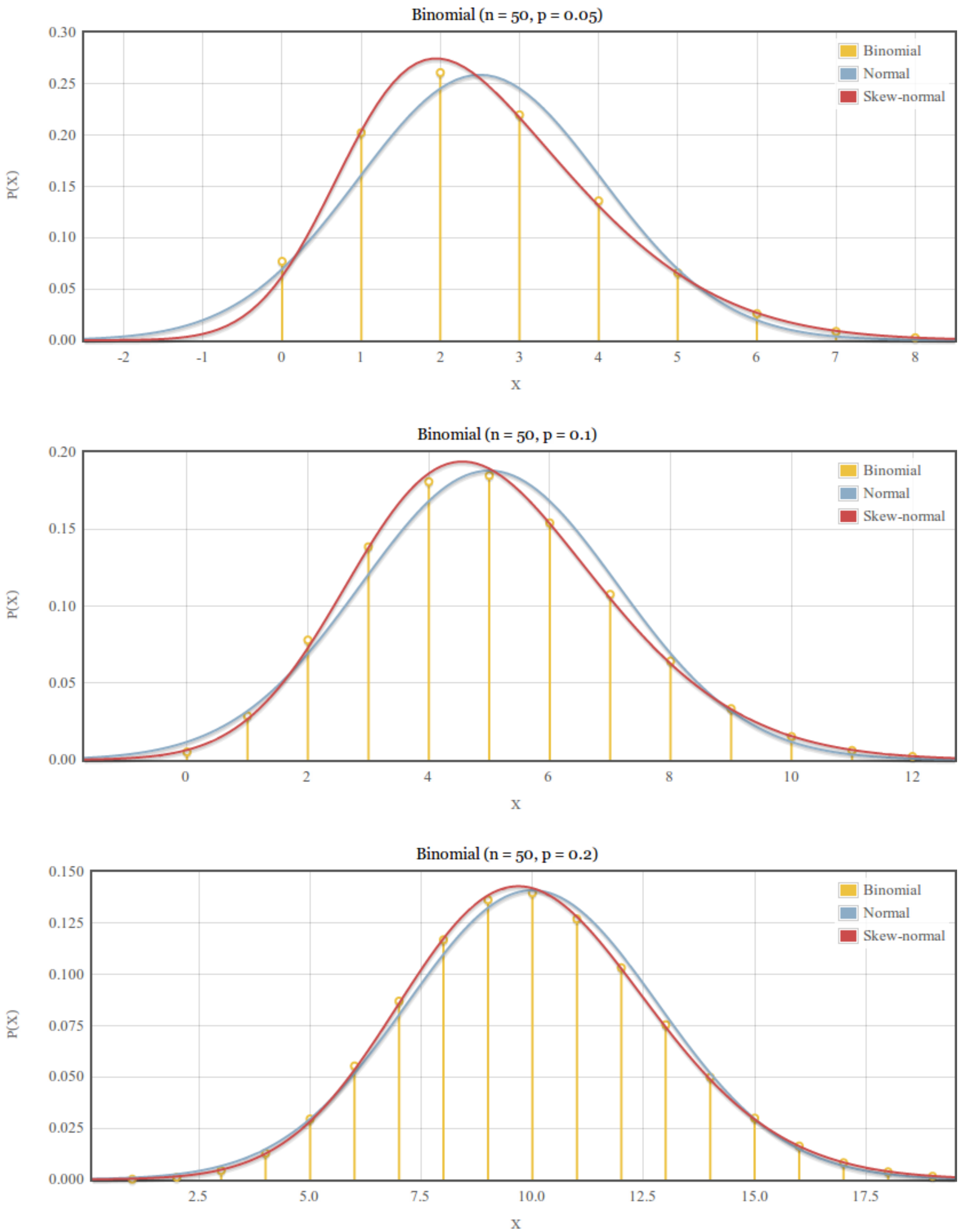


Figure 3: Binomial, normal, and skew-normal, $n = 50$

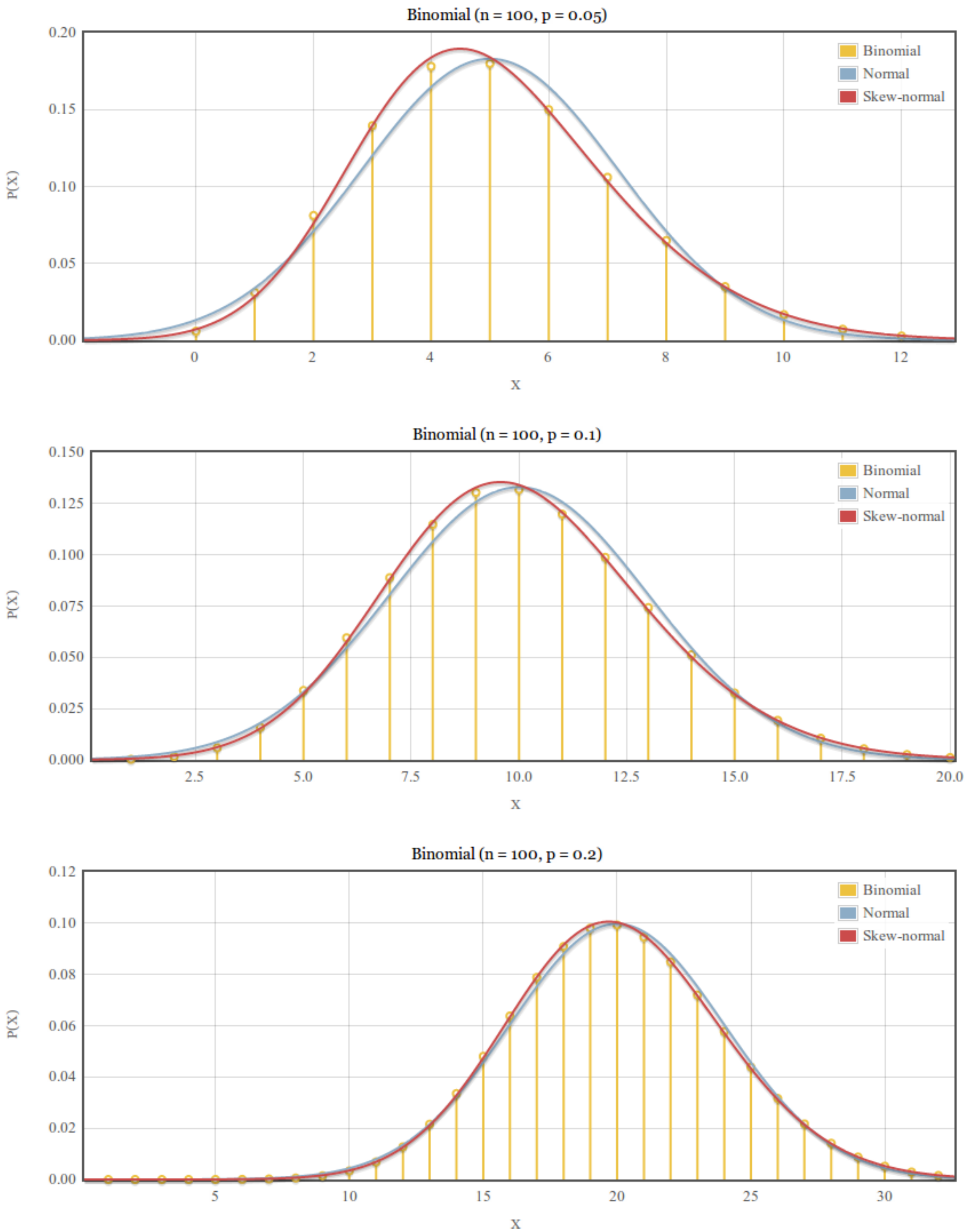


Figure 4: Binomial, normal, and skew-normal, $n = 100$

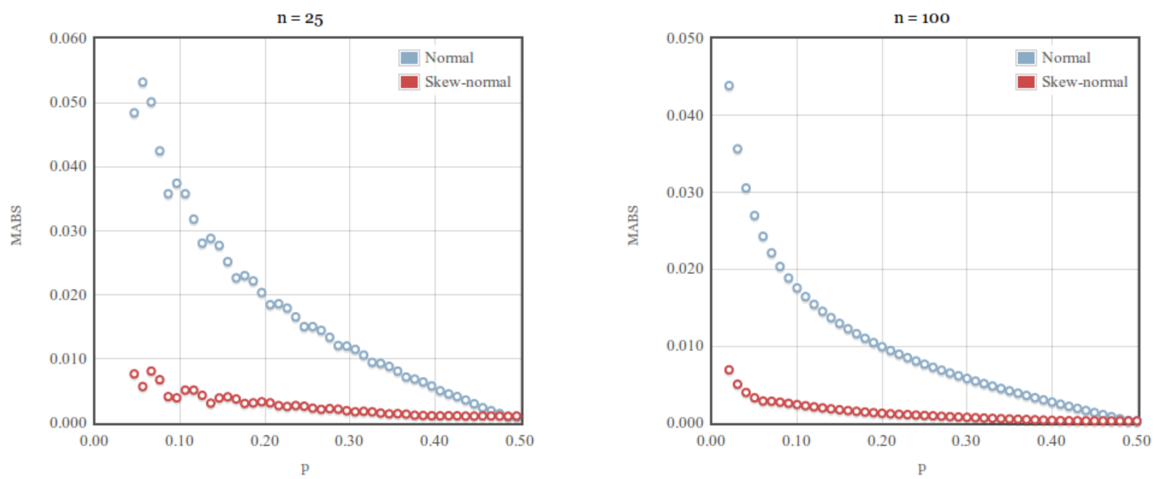


Figure 5: MABS as a function of p

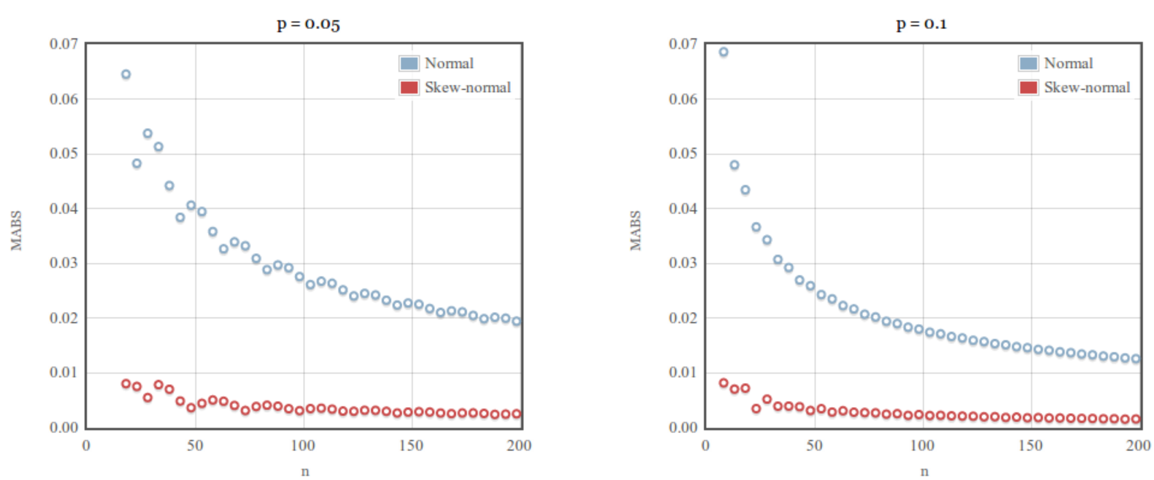


Figure 6: MABS as a function of n

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