

# THE SKEW-NORMAL APPROXIMATION OF THE BINOMIAL DISTRIBUTION

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# INTRODUCTION

## DEFINITION (BINOMIAL)

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and cdf

$$F_X(x) = P(X \leq x) = \sum_{k=0}^x f_X(k).$$

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For example ...

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When  $n = 25$ ,

$$\begin{aligned} F(12) = & \binom{25}{12} p^{12} q^{13} + \binom{25}{11} p^{11} q^{14} + \binom{25}{10} p^{10} q^{15} + \binom{25}{9} p^9 q^{16} \\ & + \binom{25}{8} p^8 q^{17} + \binom{25}{7} p^7 q^{18} + \binom{25}{6} p^6 q^{19} + \binom{25}{5} p^5 q^{20} \\ & + \binom{25}{4} p^4 q^{21} + \binom{25}{3} p^3 q^{22} + \binom{25}{2} p^2 q^{23} + \binom{25}{1} p^1 q^{24} \\ & + \binom{25}{0} p^0 q^{25} \end{aligned}$$

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A common technique is to use the normal distribution as an approximation:

$$F_X(x) \approx \Phi\left(\frac{x + 0.5 - \mu}{\sigma}\right),$$

where  $\mu = np$ ,  $\sigma = \sqrt{np(1-p)}$ , and  $\Phi$  is the standard normal cdf.

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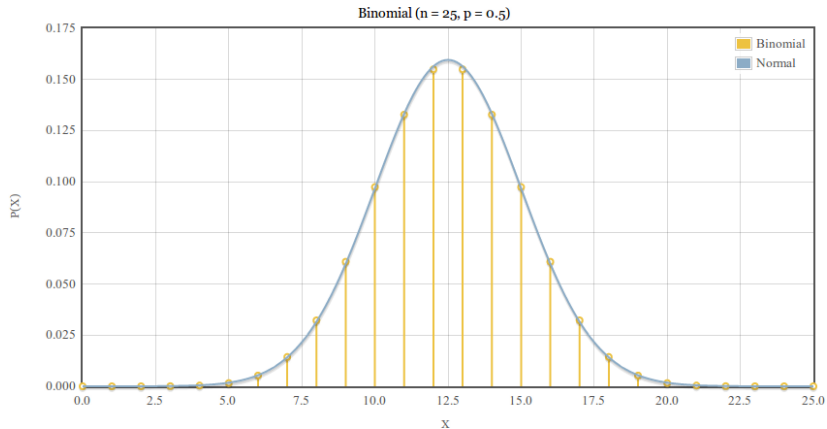
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When does this work well? ... In a nutshell, when the binomial is symmetric.

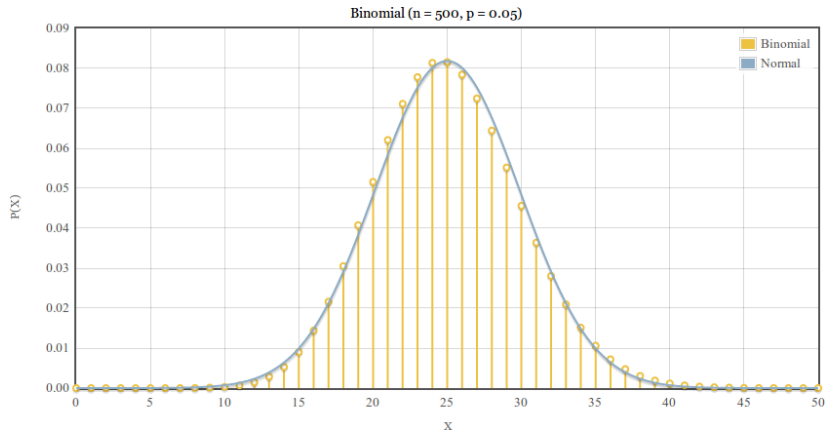
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The binomial is symmetric when  $p = 0.5$



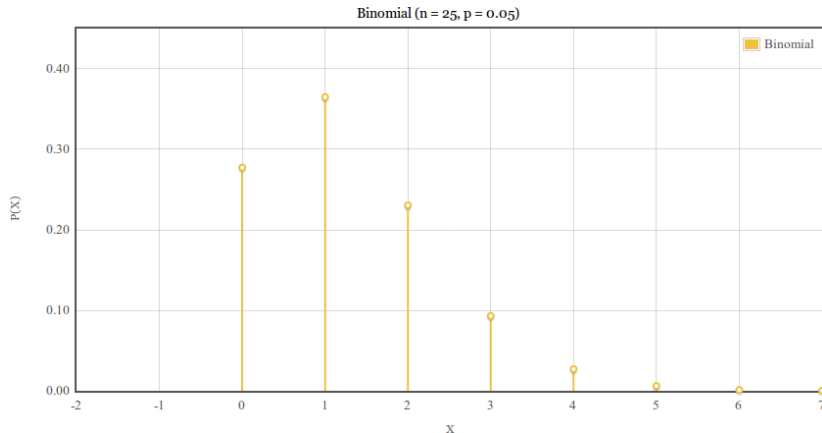
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The binomial is symmetric when  $p = 0.5$  or  $n$  is very large.



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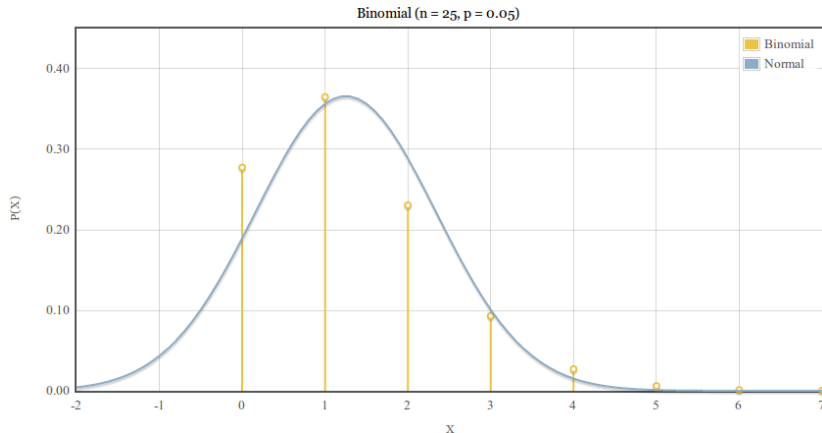
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the binomial is very skewed ...

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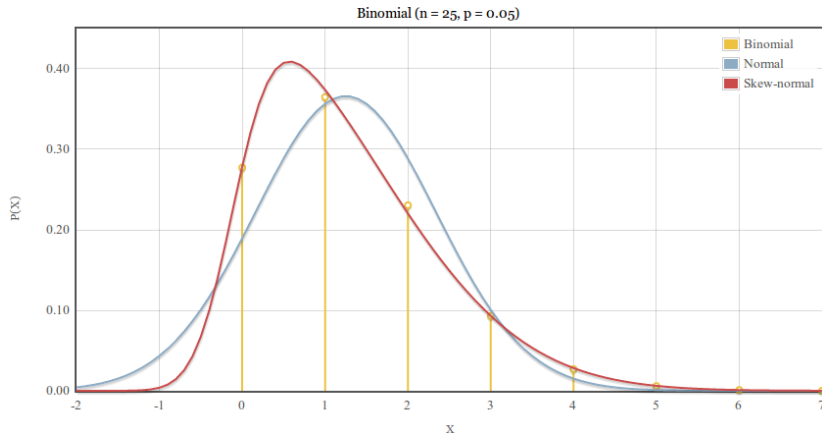


and the normal approximation doesn't work very well.



# INTRODUCTION

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Introducing ... the skew-normal distribution.

# OUTLINE

Today's itinerary:

1. Skew-Normal distribution – basic properties

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1. Skew-Normal distribution – basic properties
2. Method of Moments – derive an approximation
3. Accuracy – examine the accuracy of our approximation

# THE SKEW-NORMAL DISTRIBUTION: FOUNDATIONS

## DEFINITION (SKEW-NORMAL)

Let  $Y$  be a skew-normal distribution, with location parameter  $\mu \in \mathbb{R}$ , scale parameter  $\sigma > 0$ , and shape parameter  $\lambda \in \mathbb{R}$ . Then  $Y$  has pdf

$$f(x|\mu, \sigma, \lambda) = \frac{2}{\sigma} \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \cdot \Phi\left(\frac{\lambda(x - \mu)}{\sigma}\right), \quad x \in \mathbb{R},$$

where  $\phi$  is the standard normal pdf and  $\Phi$  is the standard normal cdf.

We write  $Y \sim SN(\mu, \sigma, \lambda)$ .

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## LEMMA

*If  $f_0$  is a one-dimensional probability density function symmetric about 0, and  $G$  is a one-dimensional distribution function such that  $G'$  exists and is a density symmetric about 0, then*

$$f(z) = 2 \cdot f_0(z) \cdot G\{w(z)\} \quad (-\infty < z < \infty)$$

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- ▶ Kernel
- ▶ CDF

# THE SKEW-NORMAL DISTRIBUTION: FOUNDATIONS

Basic properties:

$$E(Y) = \mu + b\delta\sigma$$

$$E(Y^2) = \mu^2 + 2b\delta\mu\sigma + \sigma^2$$

$$E(Y^3) = \mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3$$

$$\text{Var}(Y) = \sigma^2(1 - b^2\delta^2)$$

$$\text{where } b = \sqrt{\frac{2}{\pi}} \text{ and } \delta = \frac{\lambda}{\sqrt{1 + \lambda^2}}. \text{ (?)}$$

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which is the pdf of the normal distribution  $(\mu, \sigma)$ .



# THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

## DEFINITION (STANDARD SKEW-NORMAL)

The  $SN(0, 1, \lambda)$  distribution is called the standard skew-normal and has pdf

$$f_Z(x|\lambda) = 2 \cdot \phi(x) \cdot \Phi(\lambda x), \quad x \in \mathbb{R}.$$

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Similar to the normal and standard normal,  $Z = \frac{Y - \mu}{\sigma}$  and  $Y = \sigma Z + \mu$ .

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## PROPERTY (1)

*If  $Z \sim SN(0, 1, \lambda)$ , then  $(-Z) \sim SN(0, 1, -\lambda)$ .*

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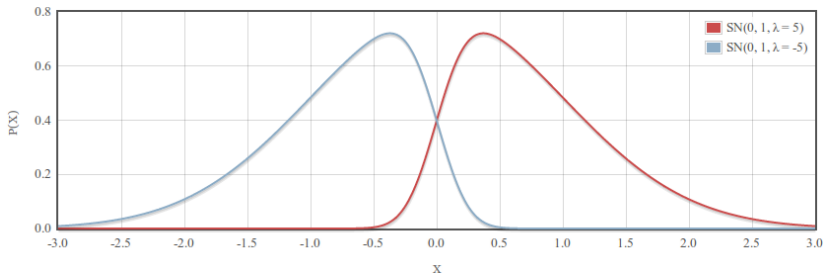
which is the pdf of  $SN(0, 1, -\lambda)$ .

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# THE SKEW-NORMAL DISTRIBUTION: THE STANDARD SKEW-NORMAL

*Property 1:*  $-SN(0, 1, \lambda) \sim SN(0, 1, -\lambda)$



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## PROPERTY (2)

*If  $Z \sim SN(0, 1, \lambda)$ , then  $Z^2 \sim \chi_1^2$  (chi-square with 1 degree of freedom).*

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Lemma 1 comes with a handy result, (? , page 161):

If  $Y \sim f_0$  and  $Z \sim f$ , then  $|Y| \stackrel{d}{=} |Z|$ , where the notation  $\stackrel{d}{=}$  denotes equality in distribution.

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Let  $X \sim N(0, 1)$ . Since  $X^2 \sim \chi_1^2$  and  $|X| \stackrel{d}{=} |Z|$ , then  $Z^2 \sim \chi_1^2$ .  
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## PROPERTY (3)

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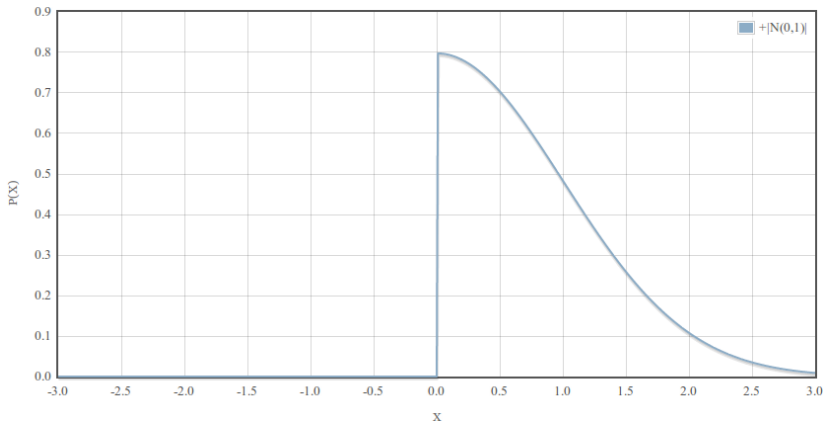
As  $\lambda \rightarrow \pm\infty$ ,  $SN(0, 1, \lambda)$  tends to the half normal distribution,  $\pm|N(0, 1)|$ .

Let  $X \sim |N(0, 1)|$ . Then

$$f_X(x) = \begin{cases} 0 & \text{when } -\infty < x \leq 0 \\ 2\phi & \text{when } 0 < x < \infty \end{cases}.$$

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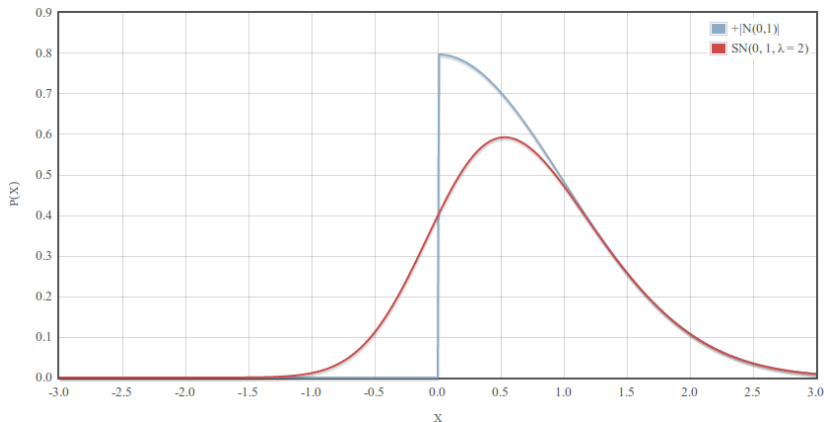
*Property 3:*  $SN(0, 1, \lambda) \rightarrow +|N(0, 1)|$  as  $\lambda \rightarrow \infty$ :





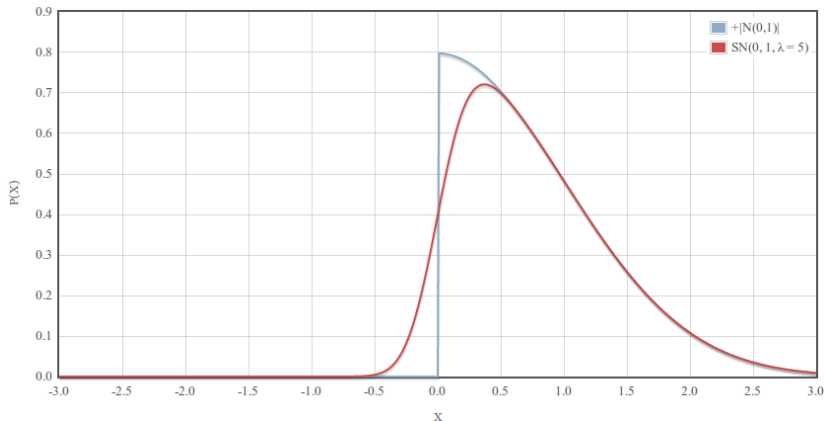
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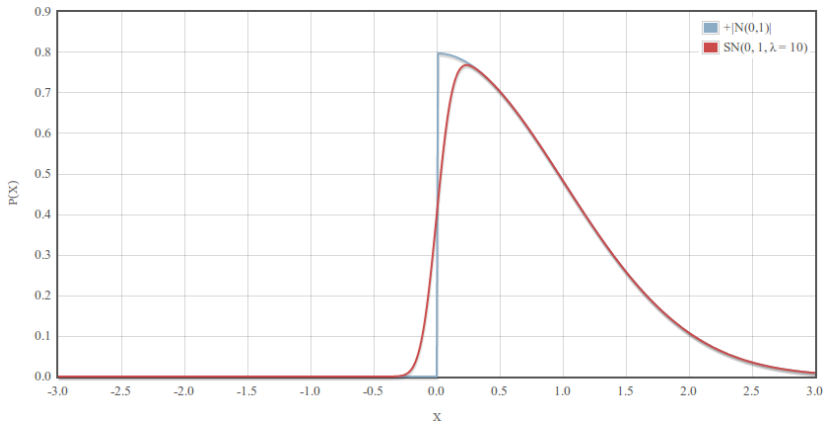
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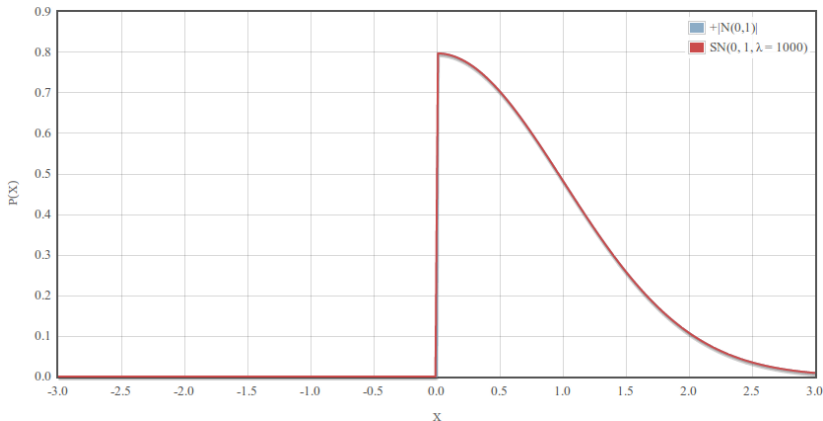
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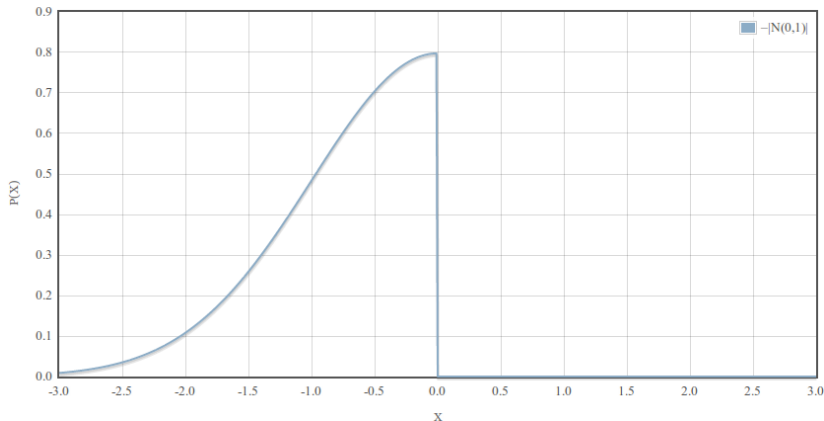
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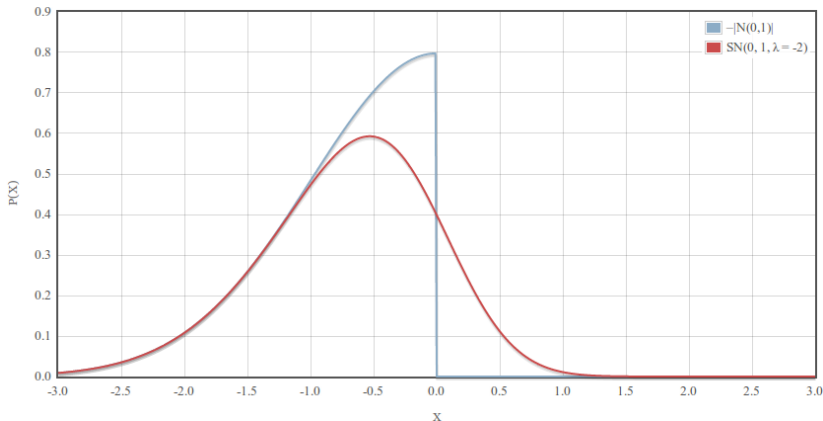
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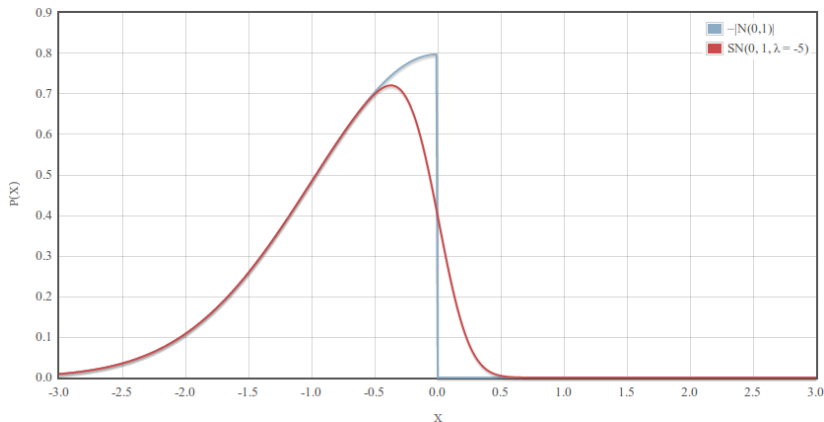
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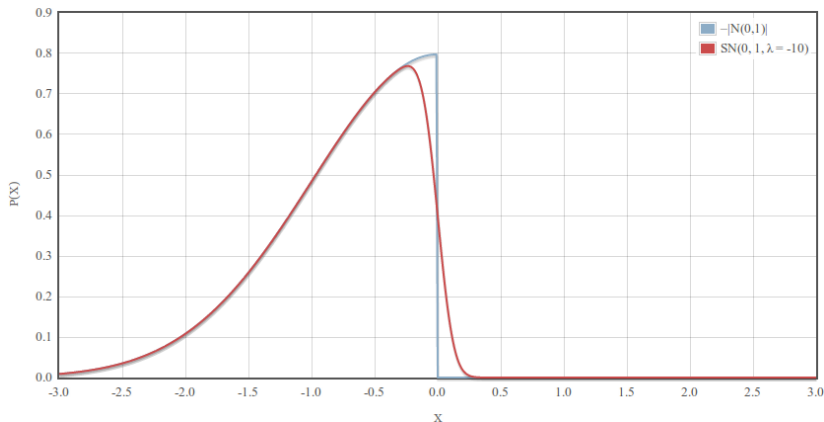
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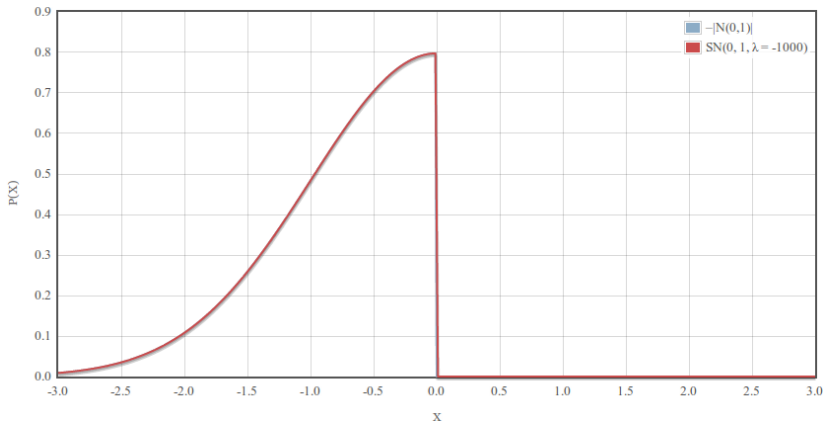
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## PROPERTY (4)

*The moment generating function of  $SN(0, 1, \lambda)$  is*

$$M(t|\lambda) = 2 \cdot \Phi(\delta t) \cdot e^{t^2/2},$$

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According to Equation 5 in ?, the mgf of  $SN(\mu, \sigma, \lambda)$  is

$$M(t) = E\{e^{tY}\} = 2 \cdot \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \cdot \Phi(\delta \sigma t).$$

Our result follows.

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2. Set them equal to each other.
3. Take  $n$  and  $p$  to be constants; solve for  $\mu$ ,  $\sigma$ , and  $\lambda$ .

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

Let's start with the binomial ...

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$$E(B) = np$$

$$E([B - E(B)]^2) = \text{Var}(B) = np(1 - p)$$

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The third one takes some elbow grease.

First we'll need to find  $E(B^2)$  and  $E(B^3)$ .

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

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$$E(B^2) = \text{Var}(B) + [E(B)]^2$$



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$$\begin{aligned} E(B^2) &= \text{Var}(B) + [E(B)]^2 \\ &= np(1-p) + n^2p^2 \end{aligned}$$

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

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$$\begin{aligned} E(B^2) &= \text{Var}(B) + [E(B)]^2 \\ &= np(1-p) + n^2p^2 \\ &= np - np^2 + n^2p^2. \end{aligned}$$

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

We will get  $E(B^3)$  via the third factorial moment,  $E[B(B-1)(B-2)]$ .

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

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$$\begin{aligned} &E[B(B-1)(B-2)] \\ &= \sum_{x=0}^n x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \end{aligned}$$

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$$= n(n-1)(n-2)p^3$$

$$= n^3 p^3 - 3n^2 p^3 + 2np^3$$

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Left side:  $E(B^3) + 3np^2 - 3n^2p^2 - np$

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$$\Rightarrow E(B^3) = n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np$$

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# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE BINOMIAL

Let's restate our results:

$$E(B) = np,$$

$$E([B - E(B)]^2) = np(1 - p),$$

$$E([B - E(B)]^3) = np(p - 1)(2p - 1)$$

# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE SKEW-NORMAL

Now lets move on to skew-normal ...

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Again, the first and second central moments are the mean and variance.

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$$E(Y) = \mu + b\delta\sigma$$
$$Var(Y) = \sigma^2(1 - b^2\delta^2)$$

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Again, the third one is a little harder:



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Again, the third one is a little harder:

$$\begin{aligned} E([Y - E(Y)]^3) \\ = E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \end{aligned}$$

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Again, the third one is a little harder:

$$\begin{aligned} & E([Y - E(Y)]^3) \\ &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\ &= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\ &\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2(\mu + b\delta\sigma)^3 \end{aligned}$$

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Again, the third one is a little harder:

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Again, the third one is a little harder:

$$\begin{aligned} E([Y - E(Y)]^3) &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\ &= (\mu^3 + 3b\delta\mu^2\sigma + 3\mu\sigma^2 + 3b\delta\sigma^3 - b\delta^3\sigma^3) \\ &\quad - 3(\mu^2 + 2b\delta\mu\sigma + \sigma^2)(\mu + b\delta\sigma) + 2(\mu + b\delta\sigma)^3 \\ &= \cancel{\mu^3} + \cancel{3b\delta\mu^2\sigma} + \cancel{3\mu\sigma^2} + \cancel{3b\delta\sigma^3} - b\delta^3\sigma^3 - \cancel{3\mu^3} - \cancel{3b\delta\mu^2\sigma} \\ &\quad - \cancel{6b\delta\mu^2\sigma} - \cancel{6b^2\delta^2\mu\sigma^2} - \cancel{3\mu\sigma^2} - \cancel{3b\delta\sigma^3} + \cancel{2\mu^3} + \cancel{6b\delta\mu^2\sigma} \\ &\quad + \cancel{6b^2\delta^2\mu\sigma^2} + 2b^3\delta^3\sigma^3 \\ &= 2b^3\delta^3\sigma^3 - b\delta^3\sigma^3 \end{aligned}$$

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Again, the third one is a little harder:

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# METHOD OF MOMENTS: CENTRAL MOMENTS OF THE SKEW-NORMAL

Our results, restated:

$$E(Y) = \mu + b\delta\sigma = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}}$$

$$E([Y - E(Y)]^2) = \sigma^2(1 - b^2\delta^2) = \sigma^2 \left( 1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1 + \lambda^2} \right)$$

$$E([Y - E(Y)]^3) = b\delta^3\sigma^3(2b^2 - 1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right)^3 \left( \frac{4}{\pi} - 1 \right)$$

# METHOD OF MOMENTS: DERIVING AN APPROXIMATION

We're finally ready to derive our approximation!



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We start by setting the central moments of the binomial equal to the central moments of the skew-normal.

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$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}} \quad (1a)$$

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$$np(1 - p) = \sigma^2 \left( 1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1 + \lambda^2} \right) \quad (1b)$$

$$np(p - 1)(2p - 1) = \sigma^3 \sqrt{\frac{2}{\pi}} \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right)^3 \left( \frac{4}{\pi} - 1 \right) \quad (1c)$$

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To get  $\lambda$ , divide the cube of (1b) by the square of (1c):

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$$\frac{\sigma^6 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6 \cdot \frac{2}{\pi} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6 \left(\frac{4}{\pi} - 1\right)^2} = \frac{n^3 p^3 (1-p)^3}{n^2 p^2 (p-1)^2 (2p-1)^2}$$

(2)

# METHOD OF MOMENTS: DERIVING AN APPROXIMATION

To get  $\lambda$ , divide the cube of (1b) by the square of (1c):

$$\begin{aligned} \frac{\sigma^6 \left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\sigma^6 \cdot \frac{2}{\pi} \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^6 \left(\frac{4}{\pi} - 1\right)^2} &= \frac{n^3 p^3 (1-p)^3}{n^2 p^2 (p-1)^2 (2p-1)^2} \\ \Rightarrow \frac{\left(1 - \frac{2}{\pi} \cdot \frac{\lambda^2}{1+\lambda^2}\right)^3}{\frac{2}{\pi} \left(\frac{\lambda^2}{1+\lambda^2}\right)^3 \left(\frac{4}{\pi} - 1\right)^2} &= \frac{np(1-p)}{(1-2p)^2} \end{aligned} \quad (2)$$

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Why?

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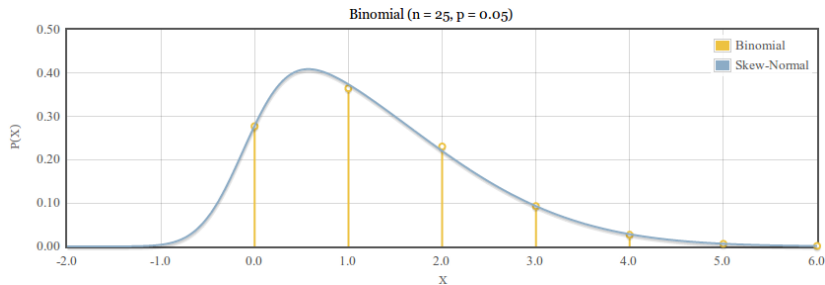
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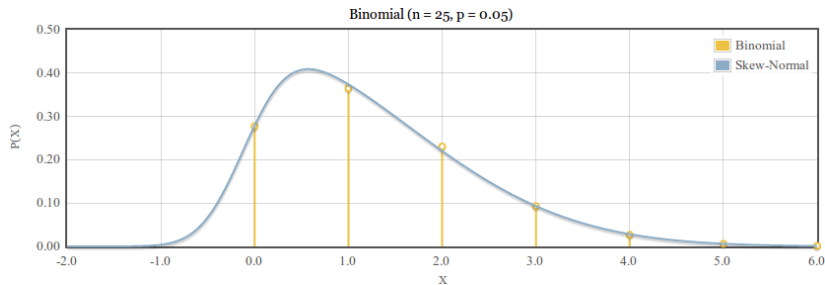
Why? Recall Property 3 ...

# METHOD OF MOMENTS: DERIVING AN APPROXIMATION



When  $p < 0.5$ :

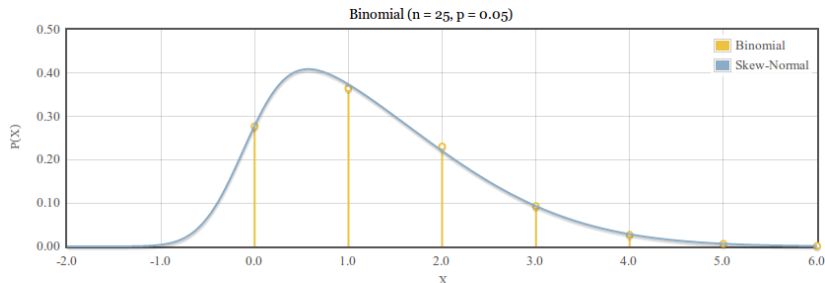
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When  $p < 0.5$ :

- ▶ The binomial skews right (weight shifts left) and approaches  $+|N(0, 1)| \rightarrow \lambda$  is positive

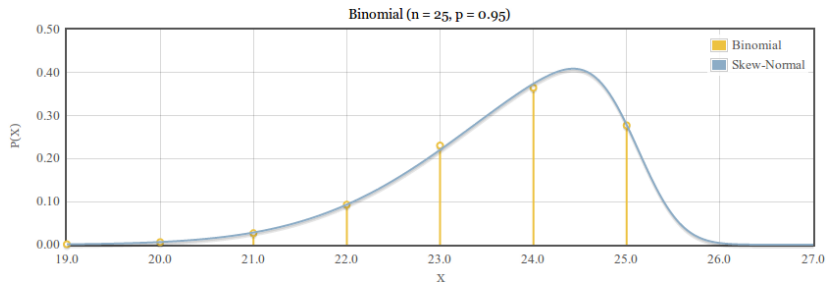
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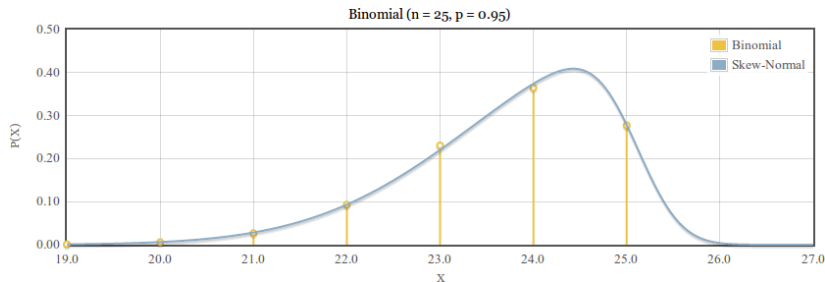
- ▶ The binomial skews right (weight shifts left) and approaches  $+|N(0, 1)| \rightarrow \lambda$  is positive
- ▶  $(1 - 2p)$  is positive

# METHOD OF MOMENTS: DERIVING AN APPROXIMATION



When  $p > 0.5$ :

# METHOD OF MOMENTS: DERIVING AN APPROXIMATION

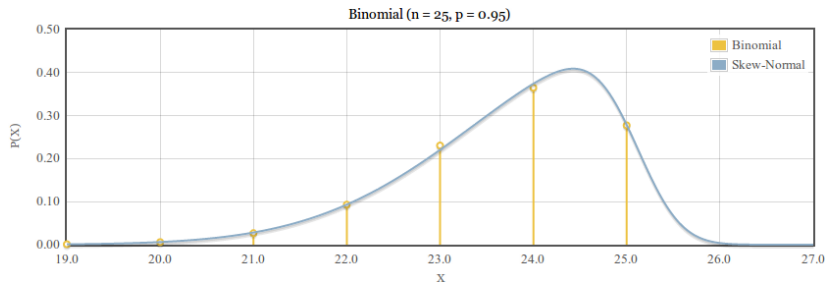


When  $p > 0.5$ :

- ▶ The binomial skews left (weight shifts right) and approaches  $-|N(0, 1)| \rightarrow \lambda$  is negative



# METHOD OF MOMENTS: DERIVING AN APPROXIMATION



When  $p > 0.5$ :

- ▶ The binomial skews left (weight shifts right) and approaches  $-|N(0, 1)| \rightarrow \lambda$  is negative
- ▶  $(1 - 2p)$  is negative

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$$np = \mu + \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}} \Rightarrow \mu = np - \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1 + \lambda^2}}$$

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$$\sigma = \sqrt{\frac{np(1-p)}{1 - \frac{2}{\pi} \cdot \frac{0^2}{1+0^2}}} = \sqrt{\frac{np(1-p)}{1}} = \sqrt{np(1-p)}$$

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$$\mu = np - \sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{0}{\sqrt{1+0^2}} = np - 0 = np$$



# METHOD OF MOMENTS: RESTRICTIONS

Unfortunately, though better than the normal approximation, our skew-normal method isn't universal.

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To be able to solve for  $\lambda$ , we must restrict  $n$  and  $p$  by the following equation:

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From (3), we can answer two questions:

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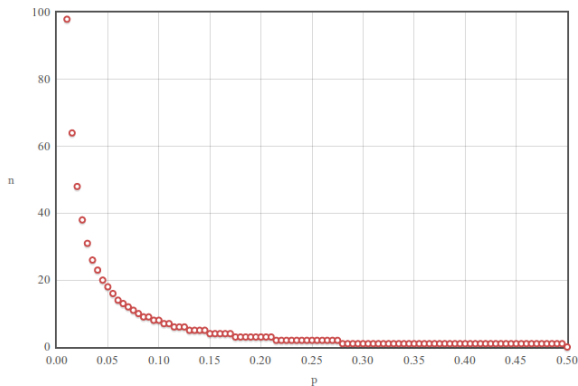
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$$n \geq \frac{(1 - 2p)^2}{p(1 - p)}$$

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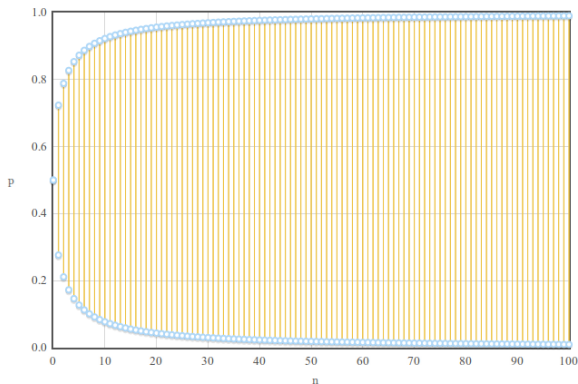
Two: Given  $n$ , what is the range of possible  $p$ 's?

$$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{n}{n+4}} \leq p \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{n+4}}$$

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We have an approximation!!

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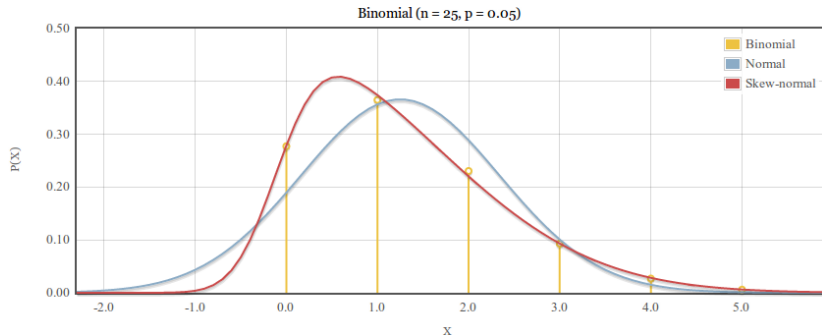
We have an approximation!!

But is it more accurate? ... Answer: Yes!

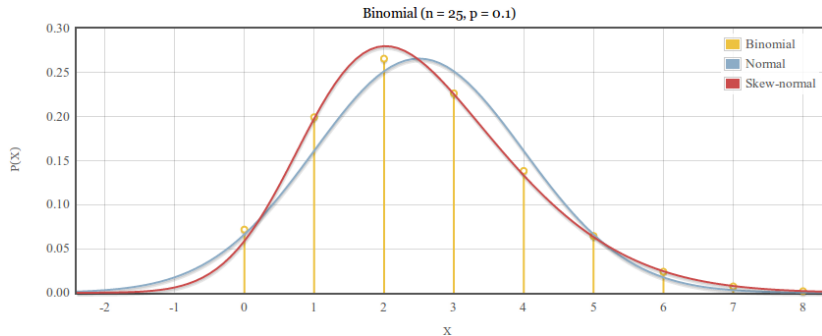
# DEMONSTRATING IMPROVED ACCURACY: VISUAL

The easiest way of gauging accuracy is by visual inspection.

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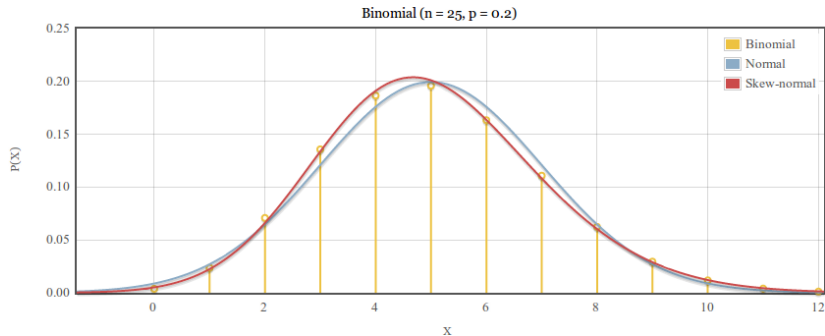


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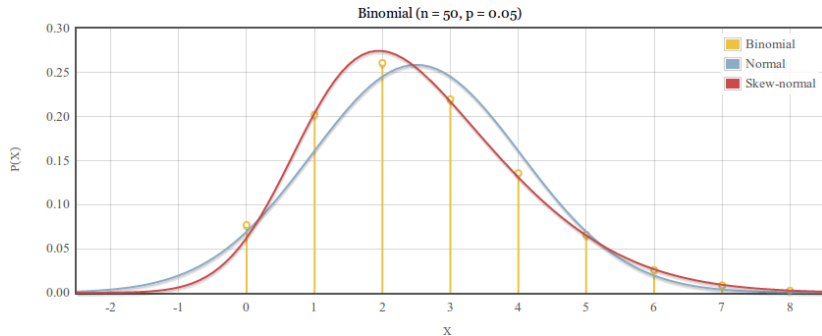




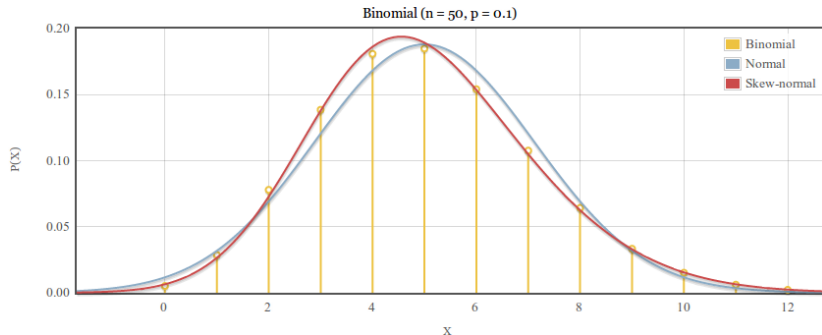
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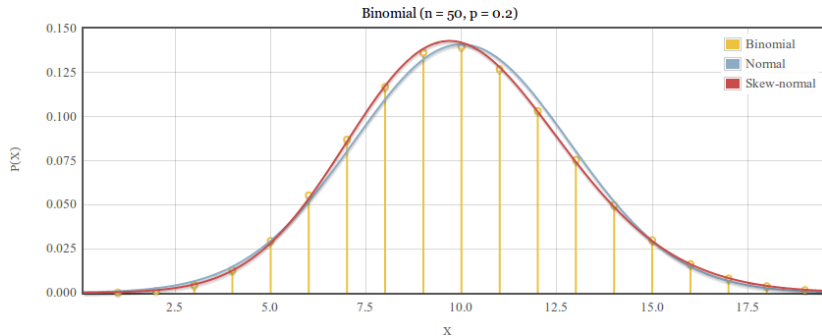
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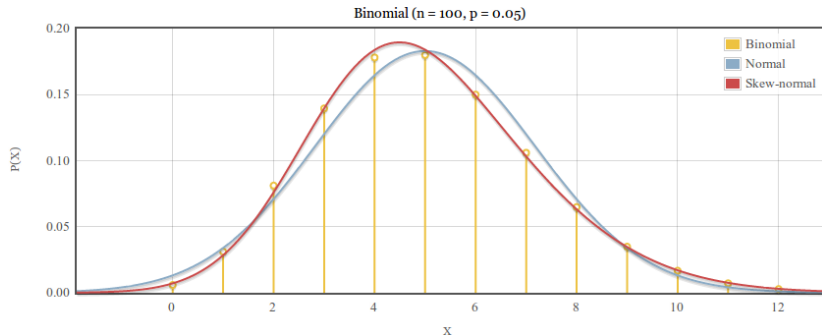
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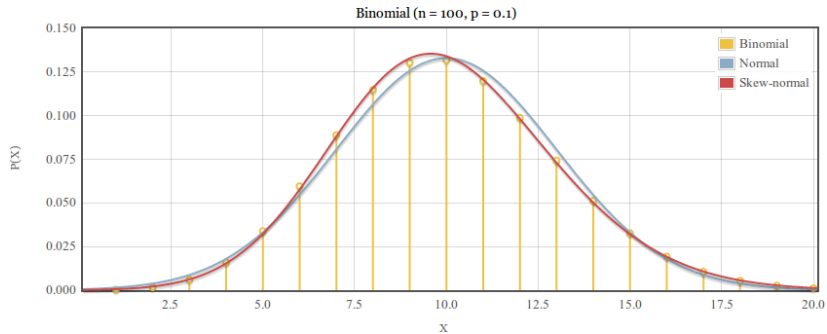
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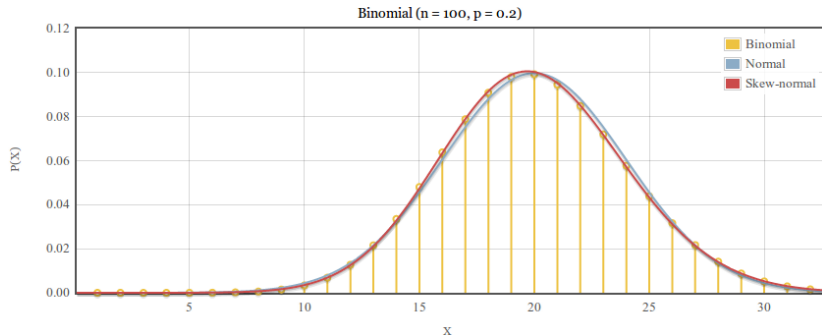
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# DEMONSTRATING IMPROVED ACCURACY: MABS

A more numerical way of gauging accuracy is the *MABS*, defined by ? as

$$\text{MABS}(n, p) = \max_{k \in \{0, 1, \dots, n\}} \left| F_{B(n, p)}(k) - F_{\text{appr}(n, p)}(k + 0.5) \right|.$$



# DEMONSTRATING IMPROVED ACCURACY: MABS

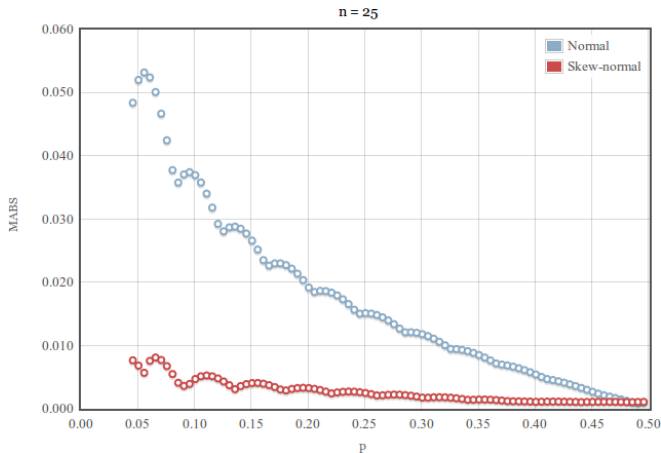
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(In English: The biggest vertical distance between the binomial cdf and the approximation curve.)

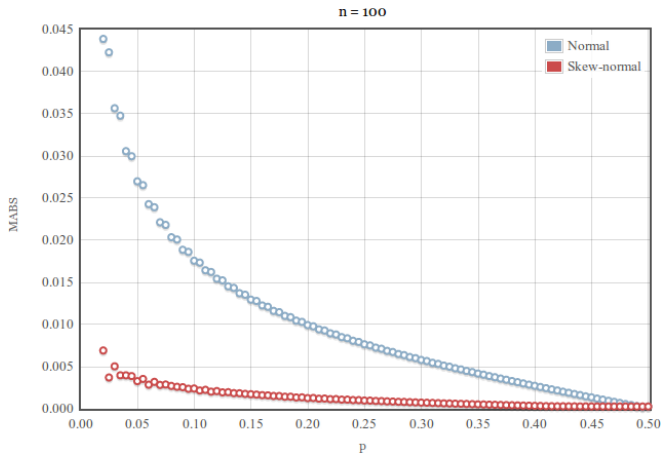
# DEMONSTRATING IMPROVED ACCURACY: MABS

MABS as a function of  $p$ , with fixed  $n = 25$ :



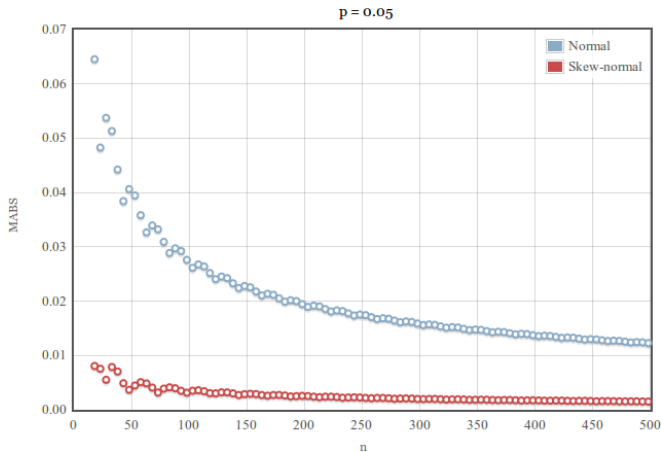
# DEMONSTRATING IMPROVED ACCURACY: MABS

MABS as a function of  $p$ , with fixed  $n = 100$ :



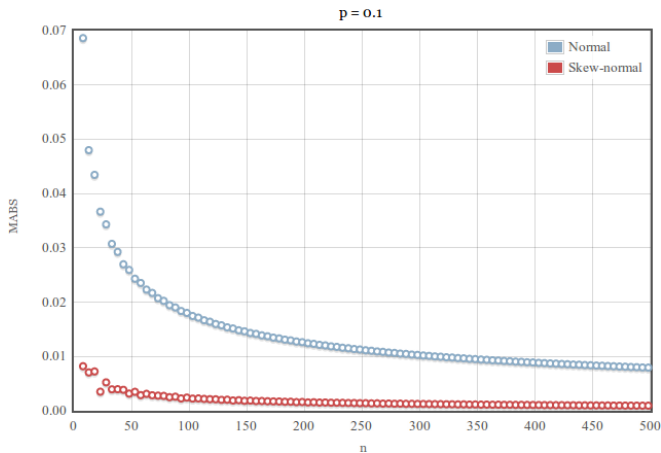
# DEMONSTRATING IMPROVED ACCURACY: MABS

MABS as a function of  $n$ , with fixed  $p = 0.05$ :



# DEMONSTRATING IMPROVED ACCURACY: MABS

MABS as a function of  $n$ , with fixed  $p = 0.1$ :



# RESOURCES

Estimations of  $SN(\mu, \sigma, \lambda)$  for  $Bin(n, p)$

	$n$				
	25	50	100	250	500
0.05	(-0.11, 1.74, 4.56)	(0.79, 2.30, 2.54)	(2.85, 3.06, 1.86)	(9.58, 4.52, 1.38)	(21.32, 6.11, 1.15)
0.10	(0.89, 2.20, 2.31)	(2.97, 2.94, 1.74)	(7.44, 3.94, 1.40)	(21.53, 5.88, 1.10)	(45.62, 8.01, 0.94)
0.15	(2.02, 2.49, 1.79)	(5.32, 3.34, 1.43)	(12.25, 4.51, 1.19)	(33.77, 6.77, 0.96)	(70.30, 9.27, 0.82)
0.20	(3.23, 2.67, 1.50)	(7.76, 3.61, 1.24)	(17.18, 4.89, 1.04)	(46.18, 7.39, 0.85)	(95.18, 10.16, 0.74)
0.25	(4.49, 2.79, 1.29)	(10.28, 3.78, 1.09)	(22.20, 5.15, 0.93)	(58.71, 7.83, 0.76)	(120.22, 10.80, 0.67)
0.30	(5.80, 2.85, 1.12)	(12.86, 3.88, 0.95)	(27.31, 5.32, 0.82)	(71.34, 8.12, 0.68)	(145.39, 11.24, 0.60)
0.35	(7.17, 2.86, 0.96)	(15.50, 3.92, 0.83)	(32.49, 5.39, 0.72)	(84.09, 8.28, 0.60)	(170.70, 11.50, 0.53)
0.40	(8.59, 2.83, 0.80)	(18.23, 3.89, 0.70)	(37.76, 5.39, 0.61)	(96.96, 8.32, 0.51)	(196.18, 11.60, 0.45)
0.45	(10.12, 2.73, 0.61)	(21.08, 3.79, 0.53)	(43.21, 5.29, 0.47)	(110.07, 8.23, 0.40)	(221.93, 11.54, 0.35)
0.50	(12.50, 2.50, 0.00)	(25.00, 3.54, 0.00)	(50.00, 5.00, 0.00)	(125.00, 7.91, 0.00)	(250.00, 11.18, 0.00)
0.55	(14.88, 2.73, -0.61)	(28.92, 3.79, -0.53)	(56.79, 5.29, -0.47)	(139.93, 8.23, -0.40)	(278.07, 11.54, -0.35)
0.60	(16.41, 2.83, -0.80)	(31.77, 3.89, -0.70)	(62.24, 5.39, -0.61)	(153.04, 8.32, -0.51)	(303.82, 11.60, -0.45)
0.65	(17.83, 2.86, -0.96)	(34.50, 3.92, -0.83)	(67.51, 5.39, -0.72)	(165.91, 8.28, -0.60)	(329.30, 11.50, -0.53)
0.70	(19.20, 2.85, -1.12)	(37.14, 3.88, -0.95)	(72.69, 5.32, -0.82)	(178.66, 8.12, -0.68)	(354.61, 11.24, -0.60)
0.75	(20.51, 2.79, -1.29)	(39.72, 3.78, -1.09)	(77.80, 5.15, -0.93)	(191.29, 7.83, -0.76)	(379.78, 10.80, -0.67)
0.80	(21.77, 2.67, -1.50)	(42.24, 3.61, -1.24)	(82.82, 4.89, -1.04)	(203.82, 7.39, -0.85)	(404.82, 10.16, -0.74)
0.85	(22.98, 2.49, -1.79)	(44.68, 3.34, -1.43)	(87.75, 4.51, -1.19)	(216.23, 6.77, -0.96)	(429.70, 9.27, -0.82)
0.90	(24.11, 2.20, -2.31)	(47.03, 2.94, -1.74)	(92.56, 3.94, -1.40)	(228.47, 5.88, -1.10)	(454.38, 8.01, -0.94)
0.95	(25.11, 1.74, -4.56)	(49.21, 2.30, -2.54)	(97.15, 3.06, -1.86)	(240.42, 4.52, -1.38)	(478.68, 6.11, -1.15)

# RESOURCES

All values in my project were computed using a Python library, which is freely available online:

<http://github.com/joycetipping/skew-normal-capstone/>

# BIBLIOGRAPHY I