Deriving λ with the Method of Moments

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1 Method of Moments

Let $B \sim Bin(n, p)$ and $Y \sim SN(\xi, \omega^2, \lambda)$. We will find approximations for ξ, ω, and λ by comparing the first, second, and third moments about the mean of B and Y.

1.1 The Moments of the Binomial

For B, our binomial, the first two moments, the mean and variance, are straightforward

$$\begin{aligned} E(B) &= np \\ Var(B) &= np(1-p) \end{aligned} \tag{1}$$

From here, we can easily find

$$E(B^2) = Var(B) + [E(B)]^2 = np(1-p) + n^2p^2 = np - np^2 + n^2p^2 \tag{2}$$

which we will need for the third moment. We will also need $E(B^3)$, for which we take a quick detour through the third factorial moment:

$$\begin{aligned} E[B(B-1)(B-2)] &= \sum_{x=0}^{n} x(x-1)(x-2) \cdot \left\{ \binom{n}{x} p^x q^{n-x} \right\} \\ &= \sum_{x=3}^{n} x(x-1)(x-2) \cdot \frac{n!}{x!\,(n-x)!}\; p^x q^{n-x} \\ &= \sum_{x=3}^{n} \frac{n!}{(x-3)!\,(n-x)!}\; p^x q^{n-x} \\ &= \sum_{x=3}^{n} n(n-1)(n-2)p^3 \cdot \frac{(n-3)!}{(x-3)!\,(n-x)!}\; p^{x-3} q^{n-x} \end{aligned}$$

Let $y = x - 3$. Then $x = y + 3$, and $x = 3 \rightarrow y = 0$ and $x = n \rightarrow y = n - 3$.

$$
\begin{aligned}
&= n(n-1)(n-2)p^3 \cdot \sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,(n-(y+3))!}\; p^y q^{n-(y+3)} \\
&= n(n-1)(n-2)p^3 \cdot \underbrace{\sum_{y=0}^{n-3} \frac{(n-3)!}{y!\,((n-3)-y)!}\; p^y q^{(n-3)-y}}_{\text{[pdf of } Bin(n-3,p) \text{ summed from 0 to } n-3] = 1} \\
&= n(n-1)(n-2)p^3 \\
&= n^3p^3 - 3n^2p^3 + 2np^3
\end{aligned}
$$

Further expanding the left side and solving for $E(B^3)$,

$$
\begin{aligned}
E[B^3 - 3B^2 + 2B] &= n^3p^3 - 3n^2p^3 + 2np^3 \\
E(B^3) - 3E(B^2) + 2E(B) &= \\
E(B^3) - 3(np - np^2 + n^2p^2) + 2np &= \\
E(B^3) &= n^3p^3 - 3n^2p^3 + 2np^3 + 3np - 3np^2 + 3n^2p^2 - 2np \\
&= n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np \qquad (3)
\end{aligned}
$$

With these results (and a bit of elbow grease), we can easily obtain the third moment:

$$
\begin{aligned}
E([B - E(B)]^3) &= E(B^3 - 3B^2E(B) + 3B[E(B)]^2 - [E(B)]^3) \\
&= E(B^3) - 3E(B^2)E(B) + 3E(B)[E(B)]^2 - [E(B)]^3 \\
&= E(B^3) - 3E(B^2)E(B) + 2[E(B)]^3 \\
&= (n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np) - 3np(np - np^2 + n^2p^2) + 2n^3p^3 \\
&= n^3p^3 - 3n^2p^3 + 2np^3 - 3np^2 + 3n^2p^2 + np - 3n^2p^2 + 3n^2p^3 - 3n^3p^3 + 2n^3p^3 \\
&= 2np^3 - 3np^2 + np \\
&= np(p-1)(2p-1) \qquad (4)
\end{aligned}
$$

For our future convenience, we'll restate our three moments here:

$$
\begin{aligned}
E(B) &= np \\
E([B - E(B)]^2) &= np(1-p) \qquad (5) \\
E([B - E(B)]^3) &= np(p-1)(2p-1)
\end{aligned}
$$

1.2 The Moments of the Skew Normal

Now we'll take a look at the moments of the skew normal. According to Equation 1 in Pewsey (2000)

$$
\begin{aligned}
E(Y) &= \xi + \omega b\delta \\
E(Y^2) &= \xi^2 + 2\xi\omega b\delta + \omega^2 \\
Var(Y) &= \omega^2(1 - b^2\delta^2) \\
E(Y^3) &= \xi^3 + 3b\xi^2\omega\delta + 3\xi\omega^2 + 3b\omega^3\delta - b\omega^3\delta^3
\end{aligned}
\tag{6}
$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.

Again, our first two moments are already taken care of. The third is a little more complicated:

$$
\begin{aligned}
E([Y - E(Y)]^3) &= E(Y^3) - 3E(Y^2)E(Y) + 2[E(Y)]^3 \\
&= (\xi^3 + 3b\xi^2\omega\delta + 3\xi\omega^2 + 3b\omega^3\delta - b\omega^3\delta^3) - 3(\xi^2 + 2\xi\omega b\delta + \omega^2)(\xi + \omega b\delta) \\
&\quad + 2(\xi + \omega b\delta)^3 \\
&= \xi^3 + 3b\xi^2\omega\delta + 3\xi\omega^2 + 3b\omega^3\delta - b\omega^3\delta^3 - 3\xi^3 - 9b\xi^2\omega\delta - 6b^2\xi\omega^2\delta^2 - 3\xi\omega^2 \\
&\quad - 3b\omega^3\delta + 2\xi^3 + 6b\xi^2\omega\delta + 6b^2\xi\omega^2\delta^2 + 2b^3\omega^3\delta^3 \\
&= 2\omega^3 b^3\delta^3 - b\omega^3\delta^3 \\
&= b\omega^3\delta^3(2b^2 - 1)
\end{aligned}
\tag{7}
$$

We restate our results:

$$
\begin{aligned}
E(B) &= \xi + b\omega\delta \\
E([B - E(B)]^2) &= \omega^2(1 - b^2\delta^2) \\
E([B - E(B)]^3) &= b\omega^3\delta^3(2b^2 - 1)
\end{aligned}
\tag{8}
$$

1.3 Curiosity

As a curiosity, I was unable to get Pewsey and Azzalini to agree with each other on $E(Z^3)$. According to Pewsey (2000),

$$
E(Y^3) = \xi^3 + 3b\xi^2\omega\delta + 3\xi\omega^2 + 3b\omega^3\delta - b\omega^3\delta^3 \tag{9}
$$

where $b = \sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}} \in (-1, 1)$. Since $Y = \xi + \omega Z$, by the linearity of expected value, we also have

$$\begin{aligned}
E(Y^3) &= E[(\xi + \omega Z)^3] \\
&= E[\xi^3 + 3\xi^2 \omega Z + 3\xi \omega^2 Z^2 + \omega^3 Z^3] \\
&= \xi^3 + 3\xi^2 \omega \; E(Z) + 3\xi \omega^2 \; E(Z^2) + \omega^3 \; E(Z^3) \\
&= \xi^3 + 3\xi^2 \omega \; b\delta + 3\xi\omega^2 + \omega^3 \; E(Z^3) \qquad (10)
\end{aligned}$$

By comparing equations 10 and 11 and eliminating terms, we arrive at

$$\begin{aligned}
\omega^3 \; E(Z^3) &= 3b\omega^3\delta - b\omega^3\delta^3 \\
\Rightarrow \quad E(Z^3) &= 3b\delta - b\delta^3 \\
&= b\delta(3 - \delta^2) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}} \cdot \left(3 - \frac{\lambda^2}{1+\lambda^2}\right) \qquad (11)
\end{aligned}$$

However, according to equation (6.5?) in Azzalini (2005),

$$E(Z^r) = \begin{cases} 1 \times 3 \times \cdots \times (r-1) & \text{if r is even} \\ \dfrac{\sqrt{2}\;(2k+1)!\;\lambda}{\sqrt{\pi}\;(1+\lambda^2)^{k+1/2}\;2^k} \displaystyle\sum_{m=0}^{k} \frac{m!\;(2\lambda)^{2m}}{(2m+1)!\;(k-m)!} & \text{if } r = 2k+1 \text{ and } k = 0, 1, ... \end{cases}$$

So, for $E(Z^3)$, we have $r = 2k + 1 = 3$ and $k = 1$:

$$\begin{aligned}
E(Z^3) &= \frac{\sqrt{2} \cdot 3! \cdot \alpha}{\sqrt{\pi} \cdot (1+\alpha^2)^{3/2} \cdot 2} \sum_{m=0}^{1} \frac{m!\;(2\alpha)^{2m}}{(2m+1)!\;(1-m)!} \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\alpha}{(1+\alpha^2)^{3/2}} \cdot \left(\frac{0!(2\alpha)^0}{1!1!} + \frac{1!(2\alpha)^2}{3!0!}\right) \\
&= \frac{3\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\alpha}{(1+\alpha^2)^{3/2}} \cdot \left(1 + \frac{2}{3}\alpha^2\right) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{(\sqrt{1+\alpha^2})^3} \cdot (3 + 2\alpha^2) \qquad (12)
\end{aligned}$$