The Potential of Using Quantum Theory to Build Models of Cognition

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Abstract

Quantum cognition research applies abstract, mathematical principles of quantum theory to inquiries in cognitive science. It differs fundamentally from alternative speculations about quantum brain processes. This topic presents new developments within this research program. In the introduction to this topic, we try to answer three questions: Why apply quantum concepts to human cognition? How is quantum cognitive modeling different from traditional cognitive modeling? What cognitive processes have been modeled using a quantum account? In addition, a brief introduction to quantum probability theory and a concrete example is provided to illustrate how a quantum cognitive model can be developed to explain paradoxical empirical findings in psychological literature.

Keywords: Quantum probability; Classical probability; Cognitive process; Compatibility; Entanglement; Non-Boolean logic

With astonishing counterintuitive ramifications, quantum theory is the best empirically confirmed scientific theory in human history. It is "essential to every natural science" and its practical applications, such as the laser and the transistor, form the direct basis of at least one-third of our current economy (Rosenblum & Kuttner, 2006, p. 81). However, applying quantum theory to human cognition is not merely a simple extension of a most successful scientific theory. Rather, this endeavor is driven by a myriad of puzzling findings and stubborn challenges in psychological literature, by deep resonations between basic notions of quantum theory and psychological conceptions and intuitions, and by the

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exhibited potential of the theory to provide coherent and mathematically principled explanations for the puzzles and challenges in human cognitive research (Busemeyer & Bruza, 2012).

Given the still nascent status of quantum cognition research, it is important to note that it differs from the approaches which treat (parts of) the brain literally as material quantum systems or a quantum computer (e.g., Beck & Eccles, 1992; Hameroff & Penrose, 1996; Stapp, 1993; Vitiello, 1995). In contrast, our approach applies abstract, mathematical principles of quantum theory to inquiries in cognitive science. In fact, to convey the idea that researchers in this area are not doing quantum mechanics, various modifiers have been proposed to describe the approach, such as cognitive models based on quantum structure (Aerts, 2009), quantum-like models (Khrennikov, 2010), and generalized quantum models (Atmanspacher, Römer, & Walach, 2002).

This topic presents new developments within the quantum cognition modeling research program. In the introduction to this special issue, we try to answer three questions: Why apply quantum concepts to human cognition? How is quantum cognitive modeling different from traditional cognitive modeling? What cognitive processes have been modeled using a quantum account? In addition, a brief introduction to quantum probability theory can be found in the Appendix A.

1. Why use quantum theory to build models of human cognition?

Quantum physics was created to explain puzzling findings that were impossible to understand using traditional classical physical theories. In the process of creating quantum mechanics, physicists also had to accept basically new ways of thinking that ultimately included a novel understanding of probabilities. Currently, we see a similar development happening in areas of cognitive science. Previously, almost all cognitive modeling research has relied on principles derived from classical probability theory and mathematical principles of classical mechanics. Although tremendous progress has been achieved in understanding cognition in this way, empirical findings have accumulated which seem puzzling within the framework of classical probability theory. For example, in decision research, to explain all kinds of decision fallacies and bias, various tool boxes of heuristics had to be proposed to bypass or patch the classical theoretical framework (e.g., Gigerenzer & Todd, 1999; Payne, Bettman, & Johnson, 1993; Tversky & Kahneman, 1974). In comparison, quantum theory provides a unified and principled theoretical framework accounting for such paradoxical findings (e.g., Busemeyer & Bruza, 2012; Busemeyer, Pothos, Franco, & Trueblood, 2011). In particular, five enduring challenges in modeling cognition can be re-examined in a new light from quantum theory.

First, the challenge of formalizing psychological concepts of conflict, ambiguity, and uncertainty. Traditional cognitive models assume that at each moment, a person is in a definite (technically, non-dispersive) state with respect to certain judgment and cognition; however, the person's true state is unknown to the modeler at each moment, and thus the model can only assign a probability to a cognitive response with some value at each

moment. This type of models is stochastic only because the modeler does not know exactly what trajectory (i.e., the definite state at each time point) a person is following. In this sense, the cognitive system is modeled as if its state follows a well-defined trajectory in its state space. In contrast, a quantum account allows a person to be in an indefinite (technically, dispersive) state, called a *superposition* state, at each moment in time. Strictly speaking, this means that one cannot assume that psychological states are characterized by definite values to be registered by a psychological measurement at each moment in time. To be in a superposition state means that all possible definite values within the superposition have potential for being expressed at each moment (Heisenberg, 1958). A superposition state provides an intrinsic representation of the conflict, ambiguity, or uncertainty that people experience in cognitive processes (Blutner, Bruza, & Pothos, 2013; Brainerd, Wang, & Reyna, 2013; Wang & Busemeyer, 2013). In this sense, quantum modeling allows us to formalize the state of a cognitive system moving across time in its state space (Busemeyer, Wang, & Townsend, 2006, Atmanspacher & Filk, 2013; Fuss & Navarro, 2013) until a decision is reached, at which time the state collapses to a definite value.

Second, the challenge of formalizing the cognitive system's sensitivity to measurements. Traditional cognitive models assume that what we measure and record at a particular moment reflects the state of the cognitive system as it existed immediately before we measure it. For example, the answer to a judgment question simply reflects the state regarding this question just before we asked it. One of the more provocative lessons learned from quantum theory is that a measurement of a system both creates and records a property of the system (Peres, 1998). Immediately before asking a question, a quantum system can be in a superposition state. The answer we obtain from the system is constructed from the *interaction* of its state and the question that we ask (Bohr, 1958). This interaction creates a definite or sharply defined state out of a superposition state. The quantum principle of constructing a reality from an interaction between the person's indefinite state and the question being asked is consistent with the constructive view in psychological research (e.g., Feldman & Lynch, 1988; Schwarz, 2007) and the idea that choice can alter preference (e.g., Ariely & Norton, 2008; Sharot, Velasquez, & Dolan, 2010). In fact, it matches psychological intuition better for complex judgments, decisions, and other cognitive measurements than the assumption that the measured answer is simply a registration of a pre-existing state or retrieves a pre-existing cognitive property.

Third, the challenge of formalizing order effects of cognitive measurements. The change in cognitive states that results from answering one question can cause a person to respond differently to subsequent questions (Moore, 2002). In other words, the first cognitive measurement changes the context of the next measurement, and the order of measurements becomes important. Such order effects have been recognized and described in psychological and public opinion research (e.g., Moore, 2002). However, to develop formal cognitive models, this means that we cannot define a joint probability of answers simultaneously to a conjunction of questions. Instead, we can only assign a probability to the sequence of answers. In quantum physics, order-dependent measurements are

non-commutative operations (Atmanspacher & Römer, 2012; Busemeyer & Wang, 2010; Wang & Busemeyer, 2013). Many of the mathematical properties of quantum theory arise from developing a probabilistic model for non-commutative measurements, including Heisenberg's uncertainty principle proposed in 1927 (Heisenberg, 1958). Measurement order effects, such as question order effects, are a major concern for attitude, judgment and decision research, which seeks a theoretical understanding of the order effects similar to that achieved in quantum theory (e.g., Feldman & Lynch, 1988; Busemeyer & Wang, 2010; Wang & Busemeyer, 2013).

Fourth, the challenge of understanding violations of classical probability laws in cognitive and decision studies. Human cognition and judgment do not always obey classical laws of logic and probability (for a review, see Busemeyer & Bruza, 2012; Busemeyer, Pothos, Franco, & Trueblood, 2011). Classical probabilities used in current cognitive and decision models are derived from the Kolmogorov axioms (Kolmogorov, 1933), which assign probabilities to events defined as sets. The family of sets in the Kolmogorov theory obeys the Boolean axioms of logic. One important axiom of Boolean logic is the distributive axiom. From this distributive axiom, the law of total probability can be derived, which provides the foundation for inferences with Bayes nets. However, the law of total probability is found to be violated in many psychological experiments (e.g., Tversky & Shafir, 1992). Quantum probability theory is derived from von Neumann (1932) axioms, which assign probabilities to events defined as subspaces of a vector space. Representing events by subspaces entails a logic of subspaces and does not obey the distributive axiom of Boolean logic (Hughes, 1989) (although they form a partial Boolean algebra structure). This implies that quantum models do not always obey the law of total probability (Busemeyer & Wang, 2007; Busemeyer, Wang, & Lambert-Mogiliansky, 2009; Khrennikov, 2010; Pothos & Busemeyer, 2009). Essentially, quantum logic is a generalization of classical logic, and quantum probability is a generalized probability theory. As suggested by the accumulating empirical findings that violate classical probabilities principles, it is plausible that classical probability theory may be too restrictive to explain human cognition and judgment (Aerts, Durt, Grib, Van Bogaert, & Zapatrin, 1993).

Fifth, the challenge of understanding non-decomposability of cognition. Conventionally in cognitive science, concepts and processes are assumed to be "decomposable" so that the whole can be understood in terms of its constituent components. This decomposability is reflected in the common assumption that there exists a complete joint probability distribution across all measurements on the cognitive system, from which one can reconstruct a pairwise joint distribution for any pair of measurements. However, psychological literature has supplied ample findings indicating otherwise. For example, facial cognition is hallmarked by strong reliance on holistic properties of faces (e.g., Farah, Wilson, Drain, & Tanaka, 1998; Wenger & Townsend, 2001) and perception of ambiguous figures has been argued to be a holistic process (e.g., Atmanspacher & Filk, 2013; Long & Toppino, 2004). Conceptual combinations cannot be decomposed into meaningful parts (Aerts, Gabora, & Sozzo, 2013; Aerts et al., 1993; Bruza, Kitto, Nelson, & McEvoy, 2009; Blutner et al., 2013). In contrast to classical models, quantum models allow us to describe

cognitive systems as non-decomposable, and it may be the case that pairwise probabilities cannot be derived from a common joint probability distribution. This suggests a radically new type of correlations beyond those in classical modeling, known as *entanglement* correlations in quantum theory.

2. How do quantum models differ from traditional models?

Four groups of key concepts and principles of quantum theory best illustrate how quantum cognitive models differ from traditional cognitive models in the study of cognitive processes. These four groups closely relate to each other but emphasize different features of quantum theory. An illustrative example using some of the key concepts and principles can be found in the Appendix A.

First, dispersive states and quantum probabilities. When Heisenberg and Schrödinger presented their different versions of the new quantum mechanics in terms of "matrix mechanics" and "wave mechanics" in the mid-1920s, their work enabled enormous, rapid progress in understanding the behavior of the micro-world. In three articles during 1927–1929 (reprinted in 1962), von Neumann rigorously proved the equivalence of matrix and wave mechanics, and introduced the mathematical framework of Hilbert space and operator calculus to develop a *statistical* ensemble formalism for quantum theory. It is summarized in his monograph, *Mathematical Foundations of Quantum Mechanics* (1932), which is still regarded as the standard reference for orthodox quantum theory. Von Neumann's approach is statistical because the predictions of the theory refer to ensembles of experimental results so that only probabilistic predictions are possible for individual results.

It is crucial to note that the rules for quantum probabilities differ from those of classical probabilities. Not only mixed ensemble states but also pure quantum superposition states are dispersive: They cannot be represented point-wise, as we are used to doing in classical point mechanics. Born's rule (Born, 1926), which defines the probability to measure a certain quantum state by the squared magnitude of its amplitude, therefore entails interference terms that are absent in classical probability theory. This is what leads to the non-additive nature of quantum probabilities. Furthermore, since states are changed by measurements, quantum probabilities for transitions between states depend decisively on prior measurements.

States of individual quantum systems are called pure states, which are represented as vectors in a vector space (the Hilbert space). Pure states are dispersive, and every superposition of pure states is another pure state. Another class of dispersive states is mixed states, which are states of statistical ensembles of pure states with different probabilities. They are also called statistical states. While dispersive pure states are considered the source of a "genuine" (ontic) randomness in quantum theory, the dispersion of mixed states is an ensemble property, often interpreted as expressing (epistemic) incomplete knowledge about the exact state of the system (beim Graben & Atmanspacher, 2006).

Second, entanglement, non-local correlations, and Bell inequalities. A specific form of quantum superposition is called quantum entanglement. This term was coined by

Schrödinger (1935), in connection with the seminal paper by Einstein, Podolsky, and Rosen (1935). Entangled states exhibit acausal non-local correlations between the results of measurements on spatially separated subsystems of a system. Entangled systems can arise due to all kinds of interactions and entanglement is a generic feature of the quantum world. Since these correlations are ubiquitous for entangled systems, such systems are also called non-separable. Non-local correlations are the basis of what is often referred to as quantum holism.

Ironically, Einstein et al. (1935), who demonstrated that quantum theory yields such non-local correlations, concluded that they are so absurd ("spooky action at a distance") that quantum theory must miss essential elements of reality. John Bell (1964) proposed how this issue can be settled empirically by measuring the size of the correlations between pairs of variables of entangled quantum systems. Bell's inequalities define a bound for classical correlations which is predicted to be violated for entangled quantum systems. Today, such non-local quantum correlations are well established empirically in physics, and fascinating applications of them have been developed (Vedral, 2008).

The usual way of formulating Bell inequalities refers to measurements of pairs of spatially separated variables. However, it is also possible to derive temporal Bell inequalities (Leggett & Garg, 1985) for a single variable separated in time (i.e., at different time points). Violations of temporal Bell inequalities would prove that a system states cannot be sharply localized in time, as would always have to be in classical theory. So far, violations of temporal Bell inequalities have not been found empirically because empirically, it is very difficult to introduce a measurement at a particular time that does not destroy a key condition for violating the temporal Bell inequalities, namely, that the variable is in a superposition state. (Once a measurement has been made, the state is no longer a superposition state.)

Quantum entanglement is ubiquitous and is at the foundation of quantum theory. It reveals that a key challenge for understanding quantum systems is how to decompose them into subsystems, rather than how to compose them from elementary constituents. The decomposition of an entangled system leads to subsystems that are correlated "non-locally" (i.e., without direct causal interactions). As an interesting twist in the history of science, quantum theory, which was originally developed to complete the atomistic program of the 19th century physics, tells us that this atomistic picture has to be turned upside down into a fundamentally holistic worldview.

Third, non-commuting observables and complementarity. Another distinction between classical and quantum behavior is whether observables (i.e., measurements) are commuting or non-commuting. While systems studied in classical physics are restricted to commuting observables, quantum systems can have both commuting (e.g., mass and charge) and non-commuting observables (e.g., position and momentum). As discussed earlier, the non-commutativity of observables A and B can be characterized as $AB \neq BA$. This expresses an order effect unknown for classical observables, which obey the law of commutativity, $AB = BA$. Any observable can be decomposed into its eigenstates. If observables A and B do not commute, they are incompatible and have no complete orthonormal system of common eigenstates—that is, they may share some but not all eigenstates. A

and B are maximally incompatible (or complementary) if they have no single eigenstate in common.

As observables in quantum theory play a double role as codifications of properties of a system and operations on the state of a system, they can also be considered "actions." In this sense, non-commutativity means that the action sequence of B and then A on an initial state leads to a result different from what is produced by reversing the sequence of the actions. The uncertainty relations introduced by Heisenberg (1927) are a direct consequence of this novel feature of non-commuting observables in quantum theory.

As mentioned above, maximally incompatible observables are said to be complementary. Bohr (1928) introduced the concept of *complementarity*, and originally used it to address the ambiguity of wave-like and particle-like behavior of quantum systems. It is another interesting twist in the history of science that Bohr actually imported this concept from psychology, where it had been coined by William James (1890a, p. 206). Edgar Rubin, a Danish psychologist, acquainted Bohr with the idea (Holton, 1970). Bohr's attempt to understand the wave-like and particle-like manifestations of quantum systems as complementary may sound simple, but it has profound and far-reaching implications. In its narrow meaning, complementarity refers to an incompatibility of measurements that do not commute, that is, their sequence matters for the final result. More generally, complementarity refers to descriptions that mutually exclude each other, but are jointly necessary to describe a situation exhaustively. In this sense, complementarity refers to incompatible aspects of a system, which cannot be combined in a single Boolean description.

Fourth, non-Boolean and partial Boolean logic, and complementarity. In the mid-1930s, von Neumann realized a number of severe limitations of his original framework of quantum theory and proposed two generalizations. One was an algebraic approach, which expressed quantum theory in terms of generally *non-commutative operator algebras* (Murray & von Neumann, 1936, 1937). This led to the development of algebraic quantum theory and algebraic quantum field theory. The other alternative allowed a formalization of quantum measurements in terms of a *non-classical (non-Boolean) logic* of propositions (Birkhoff & von Neumann, 1936). This led to the field of quantum logic. In contrast to a two-valued (yes/no) logic of classical binary alternatives, where the complement of a proposition is identical with its negation, the complement of a quantum proposition typically deviates from its negation. A nice colloquial example is due to William James (1890b, p. 284): "The true opposites of belief… are doubt and inquiry, not disbelief." In the Boolean sense, the complement and negation of belief is disbelief; in the quantum non-Boolean sense, doubt and inquiry are complements but not negations of belief.

Quantum logic entails a violation of the Boolean law of the excluded middle or *tertium non-datur*. In an early lattice theoretical formulation of quantum logic (Birkhoff & von Neumann, 1936), this is expressed as a violation of the law of distributivity with respect to the complement. For two propositions A and B, and the complement of A (noted as A′), the distributive inequality reads: $B \wedge (A \vee A') \neq (B \wedge A) \vee (B \wedge A')$. The significance of the notion of the complement in non-Boolean logic underlines its tight relation to the quantum concept of complementarity.

The non-commutativity of quantum observables translates directly into the non-distributivity of the lattice of quantum propositions. Both formal criteria express the idea that non-commuting observables and corresponding propositions are incompatible with one another. In the broader sense of complementarity mentioned above, such subtle grades of incompatibility are hard to assess, but it is still possible to talk about the compatibility or incompatibility of propositions. The framework of propositional logic is more general than the algebraic framework insofar as it can be applied to situations in which operators in the sense of quantum theory are not defined. For systems exhibiting both classical and quantum behavior, a proper logical framework is partial Boolean logic. Partial Boolean logic involves "locally" Boolean sublattices which are pasted together in a globally non-Boolean way (Primas, 2007). Such partial Boolean structures correspond to algebras of observables that combine both non-commuting and commuting observables.

The above characterization, condensed and abstract, is intended to underline how general quantum principles can be formulated. The Appendix A provides a concrete example which illustrates how a quantum cognitive model can be developed to explain paradoxical empirical findings in psychological literature.

3. Current quantum cognitive models

The tightly connected key features of quantum theory, as sketched in the preceding sections, illustrate the basic procedures for using quantum theory to understand human cognition. As mentioned, some of these key conceptions, most notably complementarity, were proposed by psychologists before they proved to be essential for quantum physics. However, in quantum physics, these conceptions were rigorously formalized to enable precise empirical predictions and testing. The basic notions of quantum theory not only represent essential features of how nature is organized but also of how our fields of knowledge are organized, which is a key focus of cognitive science. Some of the founding fathers of quantum theory, most ardently Niels Bohr, were convinced that the theory's central notions would prove meaningful outside of physics, in areas such as psychology and philosophy (Murdoch, 1987; Pais, 1991).

The first steps to utilize Bohr's proposal were made in the 1990s by Aerts and colleagues (e.g., Aerts & Aerts, 1994; Aerts et al., 1993) and Bordley and colleagues (Bordley, 1998; Bordley & Kadane, 1999). They used non-distributive propositional lattices to address quantum-like behavior in non-quantum systems. Alternative approaches have been initiated by Khrennikov (1999), focusing on non-classical probabilities, and Atmanspacher et al. (2002), applying an algebraic framework based on non-commuting operations with prolific results. Since 2006, Busemeyer and collaborators have successfully used quantum probability theory to address concrete and long-standing paradoxical problems in decision and cognition research (Busemeyer, Matthew, & Wang, 2006a; Busemeyer, Wang, & Townsend, 2006b; Busemeyer & Wang, 2010; see Busemeyer & Bruza, 2012; Busemeyer et al., 2011, for reviews). Two other more recent lines of thinking are due to Primas (2007), addressing complementary descriptions with partial

Boolean algebras, and Filk and von Müller (2009), proposing additional links between basic concepts in quantum physics and psychology.

Meanwhile, there are several groups worldwide working on various topics relating to quantum cognition. Since 2007, annual international conferences of Quantum Interaction have fostered the exchange of new results and ideas. Also since 2007, regular tutorials and workshops on building cognitive models using quantum probability and dynamics have been held at annual meetings of the Cognitive Science Society and the Society for Mathematical Psychology. A special issue of the *Journal of Mathematical Psychology* on quantum cognition (Bruza, Busemeyer, & Gabora, 2009) was devoted to new developments. In 2011, the annual meeting of the Cognitive Science Society hosted a symposium addressing some of the latest work. The popular science magazine *New Scientist* presented a cover story, *Your Quantum Mind*, covering some aspects of the quantum cognition program on September 3, 2011. An in-depth exchange of different formal approaches and research directions took place at an intense expert roundtable meeting in Switzerland in July 2012. Also in 2012, a book on *Quantum Models of Cognition and Decision* by Busemeyer and Bruza was published by Cambridge University Press. So far, the rapid development of the research program on quantum cognition has focused on the following six areas.

Decision processes. An early precursor is due to Aerts and Aerts (1994) and Bordley (1998). Detailed applications were developed by Busemeyer, Wang, et al., (2006b, 2009), Lambert-Mogiliansky, Zamir, and Zwirn (2009), LaMura (2009), Pothos and Busemeyer (2009, 2013), Yukalov and Sornette (2009), and Fuss and Navarro (2013). The key idea is to define probabilities for decision outcomes and decision times in terms of quantum probability amplitudes in a Hilbert space model. Long-standing riddles in decision-making literature, including the disjunction effect (Tversky & Shafir, 1992), were explained using quantum interference (Busemeyer et al., 2006a; Khrennikov & Haven, 2009; Pothos & Busemeyer, 2009; Yukalov & Sornette, 2009).

Ambiguous perception. A good example is bistable perception, which concerns alternating views of ambiguous figures, such as the Necker cube. Atmanspacher, Filk, and Römer (2004) and Atmanspacher and Filk (2010) developed a detailed model describing a number of psychophysical features of bistable perception that have been experimentally demonstrated. In addition, Atmanspacher and Filk (2010, 2013) predicted that particular distinguished states in bistable perception may violate the temporal Bell inequalities—a litmus test for quantum behavior. Other research applying quantum theory to perception of ambiguous figures has been carried out by Conte et al. (2009).

Semantic networks. The difficult issue of meaning in natural languages is often explored in terms of semantic networks. Gabora and Aerts (2002) described the contextual manner in which concepts are evoked, used, and combined to generate meaning. Their ideas about concept association in an evolutionary context were further developed in recent work (Gabora & Aerts, 2009; Aerts et al., 2013). Bruza et al. (2009b) explored meaning relations in terms of entanglement-style features in quantum representations of the human mental lexicon, and proposed corresponding experimental tests. Related work is due to Blutner and colleagues (Blutner, 2009; Blutner et al., 2013).

Probability judgments. Many paradoxical phenomena of human judgment have resisted classical probability explanations for decades and have thus been labeled as examples of human "irrationality." For example, people sometimes judge the probability of event A and B to be greater than the probability of event A alone, which is called the conjunction fallacy (Kahneman, Slovic, & Tversky, 1982; Tversky & Kahneman, 1983). The Linda problem, as discussed in Appendix A, is a case in point. Sometimes people judge the probability of A or B to be less than the probability of A alone, which is called the disjunction fallacy (Carlson & Yates, 1989). Busemeyer et al. (2011) developed a single quantum model to account for these probability judgment "fallacies." In addition, Trueblood and Busemeyer (2011) developed a quantum inference model to explain order effects which can arise in sequential probability judgment tasks.

Order effects of cognitive measurements. Question order effects in surveys have been recognized for a long time (Schwarz & Sudman, 1992) but are still insufficiently understood today. In this introduction, we have discussed how quantum theory was developed in physics to address the puzzling findings that the order of measurements can affect the final observed probabilities. Thus, it seems natural to apply quantum probability theory to understanding the effects of the order of measurements in human cognitive behavior (Busemeyer & Wang, 2010; Wang & Busemeyer, 2013). To rigorously test the idea, Wang and Busemeyer (2013) derive an *a priori*, exact, quantitative prediction about order effects of attitude questions from quantum theory, which is used to empirically test the quantum question order model. Furthermore, Atmanspacher and Römer (2012) predicted novel kinds of order effects and indicated potential limitations of Hilbert space representations.

Memory. Findings that are puzzling from a classical probability perspective exist in more basic human cognitive processes as well, such as basic memory processes. One paradoxical phenomenon is called the episodic overdistribution effect, which can be explained using a simple quantum model involving the superposition of memory traces (Brainerd et al., 2013). Related to memory phenomena, Bruza and colleagues (2009b) developed a quantum model to explain the entanglement-like behavior observed in human mental lexicon and word association (Nelson, McEvoy, & Pointer, 2003).

These active research areas illustrate the potential of using the set of coherent concepts of quantum theory to formalize a large variety of cognitive phenomena. These phenomena often seem puzzling from the classical probability framework. Perhaps in contrast to the common impression of being mysterious, quantum theory is inherently consistent with deeply rooted psychological conceptions and intuitions. It offers a fresh conceptual framework for explaining empirical puzzles of cognition and provides a rich new source of alternative formal tools for cognitive modeling.

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Appendix A: Quantum probability theory

Quantum probability theory is unfamiliar to most social and behavioral scientists, so a brief but general overview and comparison with the more familiar classical probability theory is provided. To keep it simple, we assume finite spaces, although both probability theories can be extended to infinite spaces. More details about these principles can be found in Griffiths (2003) and Busemeyer and Bruza (2012). In addition, detailed quantum cognition modeling examples and MATLAB codes are offered by Busemeyer and Bruza (2012).

A.1. Comparing classical and quantum probability theories

Classical theory begins by postulating a set called the *sample space*, Ω, which is a set of elements that contains all the events, and in the finite case this set has cardinality *N*. *Quantum* theory begins by postulating a *vector space* (technically, a Hilbert space), *V*, which contains all the events, and in the finite case this vector space has dimension *N*.

Classical theory is based on the premise that an event, such as A, is a *subset* A $\subseteq \Omega$ of the sample space. *Quantum* theory is based on the premise that an event, such as A, is a *subspace* A $\subseteq V$ of the vector space. Corresponding to each subspace A is a projector, P_A, that projects points in *V* onto the subspace A.

Classical theory postulates that a *state* is represented by a function p: 2^{Ω} [0,1], which assigns probabilities to events. In other words, p(A) is the probability assigned to event A $\in \Omega$, and if A $\cap$ B $= \varnothing$, then p(A $\cup$ B) $= p$(A) $+ p$(B). *Quantum* theory postulates that a *state* is represented by a unit length vector $\psi \in V$, which assigns probabilities to events: p(A) $= \|P_A\psi\|^2$ and if A $\cap$ B $= \varnothing$ then p(A $\cup$ B) $= p$(A) $+ p$(B).

Classical theory defines a conditional state, p_A, that is a conditional probability function, as follows: If event A is observed, then $p_A(B) = p(B|A) = p(A \cap B)/p(A)$. Bayes' rule follows from this definition. *Quantum* theory defines a conditional state, ψ_A, as follows: If event A is observed then $\psi_A = P_A\psi/\sqrt{p(A)}$, so that $p(B|A) = \|P_B P_A \psi\|^2/p(A)$.

According to *classical* theory, if A, B are two events in Ω, then we can always define the *intersection* event $A \cap B = B \cap A$, and $p(A \cap B) = p(A)p(B|A) = p(B)p(A|B) = p(B \cap A)$, so the order of events does not matter. According to *quantum* theory, if A, B are two events in V, then we can define the *sequence* of events A and then B, denoted (A, B); and $p(A, B) = p(A)p(B|A) = \|P_B P_A \psi\|^2$ and the order of the events can matter. The intersection event, $A \cap B = B \cap A$, only exists in quantum theory if $P_B P_A = P_A P_B$, that is, the projectors commute, and in such cases, there is not any order effects (see Griffiths, 2003, p. 53; Niestegge, 2008, pp. 247). Commutativity is a key point where the two theories diverge.

Classical probability theory can be dynamically extended to sequences of events across time by a Markov process that evolves according to a transition operator derived from the Kolmogorov forward equation. *Quantum* theory can be extended to sequences of events across time by a quantum process that evolves according to a unitary operator derived from the Schrödinger equation.

A.2. An example of using quantum probability theory to model cognitive processes

The above summary is abstract and general. Next, a simple and concrete example is provided to illustrate how a quantum cognitive model can be developed to explain paradoxical empirical findings in psychological literature.

Classical probability theory assumes that events are as subsets of a sample space. This implies that the probability of an event A can never be less than the probability of the conjunction of A with another event: $p(A) \geq p(A \cap B)$. However, violations were discovered (e.g., Tversky & Kahneman, 1973, 1983). Consider the famous Linda problem (Tversky & Kahneman, 1983). Participants were shown short stories describing a hypothetical person, Linda, who was described much like a feminist. Then, they were asked to evaluate the relative probability of statements about Linda. In one statement, Linda was said to be a bank teller. In another statement, Linda was said to be a bank teller and a feminist. According to classical probability theory, the second statement is a conjunction involving the first statement and another property for Linda, so it can never be more probable than the first. However, empirical data showed that participants considered the conjunction to be more probable. This violation of classical principles, known as the conjunction fallacy, has been consistently replicated with experimental variations (Gavanski & Roskos-Ewoldsen, 1991; Wedell & Moro, 2008). Tversky and Kahneman (1983) explained their finding using a representativeness heuristic, which bypasses (classical) probabilities.

Quantum probability theory does not rely on a heuristic explanation, and instead allows a mathematically principled explanation. According to quantum probability theory, the conjunction fallacy is a natural consequence of the context- and order-dependence of probabilistic assessments. Assume that the possible characterizations of Linda as a bank teller and a feminist represent incompatible events because making a decision regarding
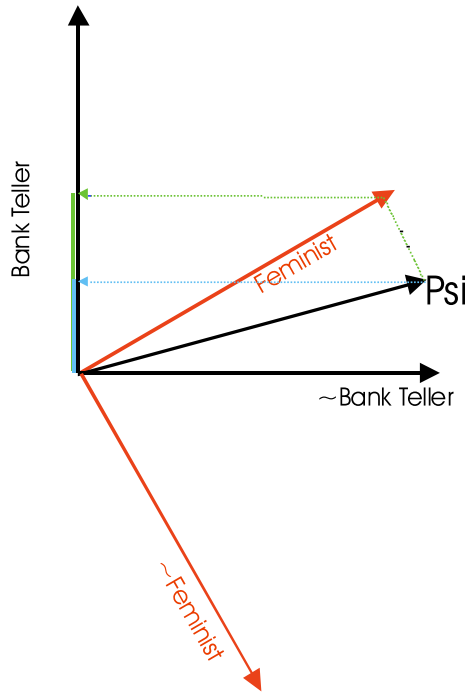
Fig. 1. An illustration of a quantum model of the conjunction fallacy.

one introduces thoughts about the other. Incompatible events can be represented as sub-spaces at oblique angles, as shown in Fig. 1. For simplicity in the illustration, we represent the story information in a two-dimensional space, with the possibilities of bank teller and feminist corresponding to one-dimensional subspaces (i.e., rays). The event that Linda is not a bank teller is represented by a ray orthogonal to the ray representing the event that Linda is a bank teller. Also, the feminist ray is approximately at 45 degrees relative to the bank teller ray. (Note this angle could be varied by rotating the feminist ray further away from or closer to the bank teller ray, but we leave the angle at 45 degrees for simplicity in this illustration). This means that if a person is a feminist, there is an equal chance of her being a bank teller and not being a bank teller. Also note that, using this representation, we cannot concurrently assess the conjunction that Linda is a bank teller *and* a feminist: If we are certain she is a feminist, then the state vector is aligned with the feminist ray, which implies uncertainty regarding the bank teller question, and vice versa. Thus, the evaluation of two incompatible properties in quantum theory has to be a *sequential* operation. Busemeyer et al. (2011) proposed that the evaluation of *A* and then *B* corresponds to a sequential projection from an initial belief state ψ to the *A* subspace first and then to the *B* subspace. That is, $p(A\ and\ then\ B) = |P_B P_A\ \psi|^2$. This is analogous to the classical definition of conjunction: $p(B|A) \cdot p(A) = \frac{|P_B P_A \psi|^2}{|P_A \psi|^2} |P_A \psi|^2 = |P_B P_A \psi|^2$, expect that here the order of operations is critical.

Next, we need to consider where to place the initial belief state vector, ψ. The initial background information on Linda provided by Tversky and Kahneman (1983) suggests

that she is unlikely to be a bank teller (thus, the state vector is reasonably far from the bank teller ray) and likely to be a feminist (thus, the state vector is near the feminist ray). Finally, for conjunctive statements, we assume that the more plausible outcomes are evaluated first (except when there are strong constraints from the syntactic on the order of the premises). Thus, assessing the conjunction that Linda is a feminist and a bank teller involves projecting onto the feminist ray and then projecting onto the bank teller ray; the squared length of the final projection (the green line in Fig. 1) is the probability of this conjunction. In contrast, projecting straight onto the bank teller ray is associated with a projection of smaller length (the aqua line in Fig. 1). Hence, the quantum model correctly predicts that the probability that Linda is a bank teller is smaller than the conjunction.

The quantum model provides an "availability" type of intuitive interpretation for the empirical results of Tversky and Kahneman (1983): Immediately, after participants read the Linda story, it is nearly impossible to think that Linda can be a bank teller. However, once participants have accepted the highly likely event that Linda is a feminist, then from this feminist point of view, it becomes a bit more likely that Linda is also a bank teller. In a way, accepting Linda as a feminist abstracts away some of the initial information about Linda, but it makes other information available, notably information about various professions that feminists can have. Perhaps being a bank teller is not the most likely profession for a feminist, but it is not entirely unlikely either. It is important to note that a simple formalization along the above lines can explain not only the conjunction fallacy but also a range of other probability judgment "errors" found in literature, including the disjunction fallacy (Busemeyer et al., 2011). The Linda example shows how a finding which seems paradoxical in classical theory becomes intuitive in quantum theory. The quantum explanation is mathematically principled and does not rely on tool boxes of heuristics. See Busemeyer et al. (2011) and Busemeyer and Bruza (2012) for more detailed model description and testing.