Engineering Mathematics – III

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UNIT- I

FOURIER SERIES

CONTENTS:

- **Introduction**
- **Periodic function**
- **Trigonometric series and Euler's formulae**
- **Fourier series of period** 2π
- **Fourier series of even and odd functions**
- **Fourier series of arbitrary period**
- **Half range Fourier series**
- **Complex form of Fourier series**
- **Practical Harmonic Analysis0**

UNIT- I

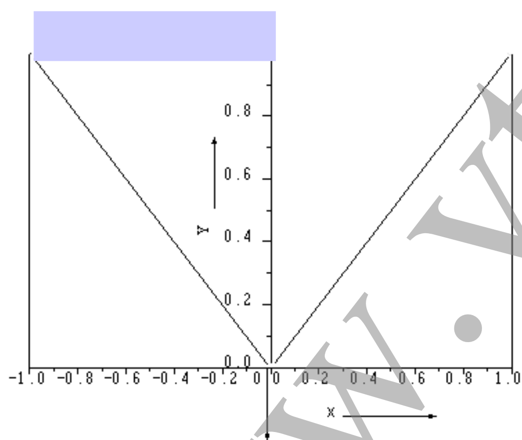
FOURIER SERIES

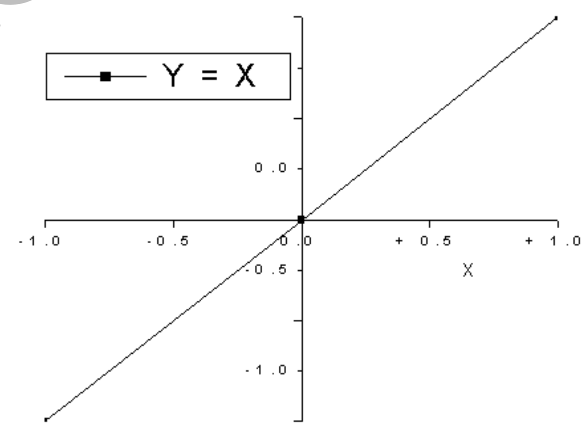
DEFINITIONS :

A function y = f(x) is said to be even, if f(-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y=f(x) is said to be odd, if f(-x) = - f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = $|x|$ in [-1,1] is even as f(-x) = $|-x| = |x|$ = f(x) and the function f(x) = x in [-1,1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below :





Graph of f(x) = $|x|$ Graph of f(x) = x

Note that the graph of f(x) = $|x|$ is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then

   - h(x) = f(x) x g(x) is odd
   - h(x) = f(x) x f(x) is even
   - h(x) = g(x) x g(x) is even

For example,
1. $h(x) = x^2 \cos x$ is even, since both x^2 and $\cos x$ are even functions
2. $h(x) = x\sin x$ is even, since x and $\sin x$ are odd functions
3. $h(x) = x^2 \sin x$ is odd, since x^2 is even and $\sin x$ is odd.

2. If f(x) is even, then $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

3. If f(x) is odd, then $$\int_{-a}^{a} f(x)dx = 0$$

For example,
$$\int_{-a}^{a} \cos x dx = 2\int_{0}^{a} \cos x dx, \quad \text{as cosx is even}$$

and $$\int_{-a}^{a} \sin x dx = 0, \text{ as sinx is odd}$$

PERIODIC FUNCTIONS :-

A periodic function has a basic shape which is repeated over and over again. The fundamental range is the time (or sometimes distance) over which the basic shape is defined. The length of the fundamental range is called the period.

A general periodic function f(x) of period T satisfies the condition

$$f(x+T) = f(x)$$

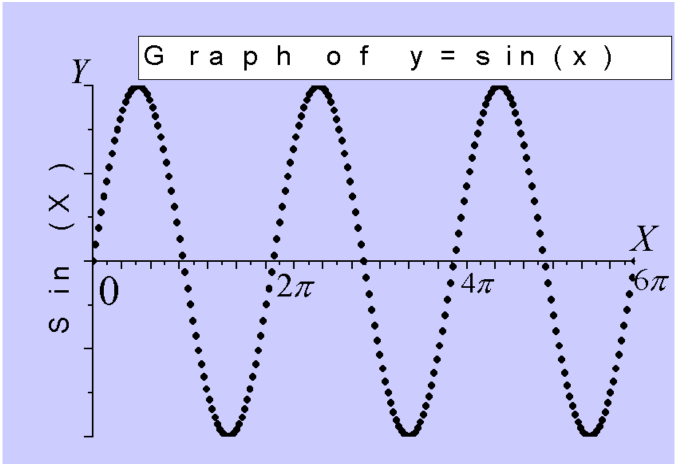
Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that
$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots.. = f(x+nT)$$

Thus,
$$f(x) = f(x+nT), n=1,2,3,\ldots..$$

The function $f(x) = \sin x$ is periodic of period 2π since
$$\text{Sin}(x+2n\pi) = \sin x, \quad n=1,2,3,\ldots\ldots..$$

The graph of the function is shown below :

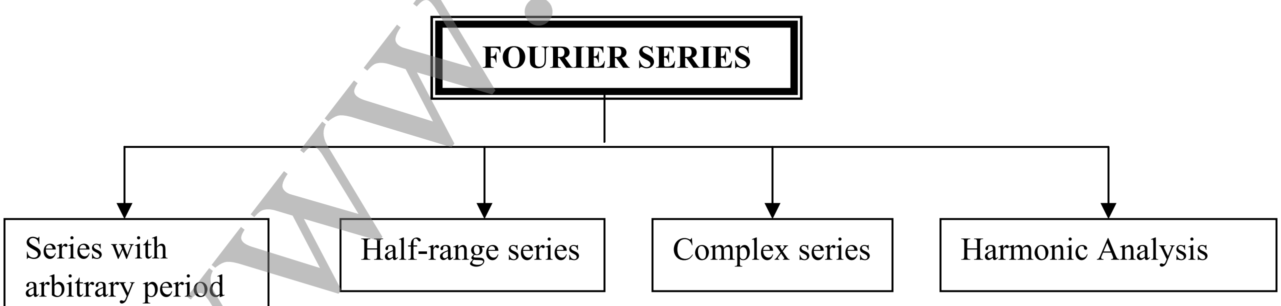


Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. It may be verified that a linear combination of periodic functions is also periodic.

FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet's conditions :

1. f(x) is defined in an interval (a,a+2l), and f(x+2l) = f(x) so that f(x) is a periodic function of period 2l.
2. f(x) is continuous or has only a finite number of discontinuities in the interval (a,a+2l).
3. f(x) has no or only a finite number of maxima or minima in the interval (a,a+2l).

Also, let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \quad n = 1,2,3,\ldots\ldots \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \quad n = 1,2,3,\ldots\ldots \quad (3)$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval (a,a+2l). Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x \ldots\ldots \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where f(x^+) and f(x^-) are the values of f(x) immediately to the right and to the left of f(x) respectively.

Particular Cases

Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2l). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx \qquad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx,$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x dx, \qquad n = 1,2,......\infty \qquad (6)$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2*l*).

If we set *l*=π, then f(x) is defined over the interval (0,2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad n=1,2,.....\infty \qquad (7)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \qquad n=1,2,.....\infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (8)$$

Case (ii)

Suppose a=-*l*. Then f(x) is defined over the interval (-*l* , *l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \qquad (9)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x dx,$$

n =1,2,…… ∞

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l* , *l*).

If we set *l* = π, then f(x) is defined over the interval (-π, π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad n=1,2,\ldots..\infty \qquad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n=1,2,\ldots..\infty$$

Putting $l = \pi$ in (5), we get f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

Some useful results :

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in
evaluating the Fourier coefficients.

$$\int uv dx = uv_1 - u' v_2 + u'' v_3 + \ldots\ldots$$

Here u', u'' ,….. are the successive derivatives of u and

$$v_1 = \int v dx, v_2 = \int v_1 dx, \ldots\ldots$$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nx dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

3. If 'n' is integer, then
sin nπ = 0 , cosnπ = $(-1)^n$, sin2nπ = 0, cos2nπ=1

Problems

1. **Obtain the Fourier expansion of**

f(x) = $\frac{1}{2}(\pi - x)$ **in -π < x < π**

We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

2. **Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that**

$$\operatorname{cosech}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus, $f(x) = \dfrac{\sinh a\pi}{a\pi} + \dfrac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{a^2+n^2}\cos nx + \dfrac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\dfrac{n(-1)^n}{a^2+n^2}\sin nx$

For x=0, a=1, the series reduces to

$$f(0) = 1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

or

$$1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

or $$1 = \frac{2\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi \operatorname{cosech}\pi = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$ This is the desired deduction.

3. Obtain the Fourier expansion of $f(x) = x^2$ over the interval $(-\pi, \pi)$. Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \infty$$

The function f(x) is even. Hence $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or $$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx, \quad \text{since f(x)cosnx is even}$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x)sinnx is odd.

Thus $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

4. **Obtain the Fourier expansion of**

$$f(x)=\begin{cases} x, 0\le x\le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$$

Deduce that $\dfrac{\pi^2}{8} = 1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+......$

Here, $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \dfrac{2}{\pi}\int_0^{\pi} f(x)dx$

$$=\frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$$

since f(x)cosnx is even.

$$=\frac{2}{\pi}\int_0^{\pi} x\cos nxdx$$

$$=\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0,$$ since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n-1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n-1\right]\cos n\pi$$

or $$\pi = \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

This is the series as required.

5. **Obtain the Fourier expansion of**

$$\mathbf{f(x) =} \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

Deduce that $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is

$$\text{f(x)} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1 - 2(-1)^n\right]}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0.

Hence $$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or\,\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

6. **Obtain the Fourier series of f(x) = 1-x^2 over the interval (-1,1).**

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here $$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_0^1 (1-x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)dx$$ as f(x) cos(nπx) is even

$$= 2\int_0^1 (1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$ =0, since f(x)sin(nπx) is odd.

The Fourier series of f(x) is

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

7. **Obtain the Fourier expansion of**

$$\mathbf{f(x)} = \begin{cases} 1+\dfrac{4x}{3} in -\dfrac{3}{2} < x \le 0 \\ 1-\dfrac{4x}{3} in 0 \le x < \dfrac{3}{2} \end{cases}$$

Deduce that $\dfrac{\pi^2}{8} = 1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+......$

The period of f(x) is $\dfrac{3}{2}-\left(\dfrac{-3}{2}\right)=3$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)-\left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\Bigg|_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also, $b_n = \dfrac{1}{3}\displaystyle\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$

Thus

$$\text{f(x)} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

putting x=0, we get

$$\text{f(0)} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period $2l$ may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval $(0,l)$ which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose $f(x) = \varphi(x)$ is given in the interval $(0,l)$. Then we define $f(x) = -\varphi(-x)$ in $(-l,0)$. Hence f(x) becomes an odd function in $(-l , l)$. The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over $(0,l)$.

Putting $l=\pi$ in (11), we obtain the half-range sine series of f(x) over $(0,\pi)$ given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \quad .....given \\ \phi(-x) & \text{in } (-l,0) \quad .....\text{in order to make the function even.} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over (0,l)

Putting l = π in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \quad n = 1,2,3,\ldots..$$

Problems :

1. **Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).**

We have,

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1 - (-1)^n\right]\sin nx$$

2. **Obtain the cosine series of**

$$f(x) = \begin{cases} x, 0 < x < \dfrac{\pi}{2} \\ \pi - x, \dfrac{\pi}{2} < x < \pi \end{cases} \qquad over(0,\pi)$$

$$Solution: \quad a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

$$\text{f(x)} = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

3. **Obtain the half-range cosine series of f(x) = c-x in 0<x<c**

Here

$$a_0 = \frac{2}{c}\int_0^c (c - x)\,dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c - x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n = \frac{2c}{n^2\pi^2}\left[1 - (-1)^n\right]$$

The cosine series is given by

$$\text{f(x)} = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1 - (-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

COMPLEX FORM OF FOURIER SERIES

The standard form of Fourier series of f(x) over the interval (α, α+2*l*) is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

If we use Euler formulae

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

then we obtain a complex exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi}{l}\right)x} \quad \ldots\ldots\ldots.(*)$$

where

$$c_n = \frac{1}{2l}\int_{\alpha}^{\alpha+2l} f(x) e^{-i\left(\frac{n\pi}{l}\right)x} dx \qquad n = 0, \pm1, \pm2, \pm3, \ldots\ldots$$

Note:-(1)

Putting $\alpha = -l$ in (*), we get Fourier series valid in $(-l, l)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi}{l}\right)x} \quad \ldots\ldots\ldots.(**)$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-i\left(\frac{n\pi}{l}\right)x} dx$$

Note:-(2)

Putting α = 0 and $l = \pi$ in (*), we get Fourier series valid in $(0, 2\pi)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Note:-(3)

Putting $l = \pi$ in (**), we get Fourier series valid in $(-\pi, \pi)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{0}^{2\pi} f(x) e^{-inx} dx$$

Problems:

1. **Obtain the complex Fourier series of the function f(x) defined by f(x) = x over the interval**
(-π , π).
Here f(x) is defined over the interval (-π , π). Hence complex Fourier series of f(x) is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (1)$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

n = 0, ±1, ±2, ±3, …….

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx$$

Integrating by parts and substituting the limits, we get

$$c_n = \frac{i(-1)^n}{n}, \qquad n \neq 0$$

Using this in (1), we get

$$f(x) = i\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} e^{inx}, \qquad n \neq 0$$

This is the complex form of the Fourier series of the given function

2. **Obtain the complex Fourier series of the function f(x) = e^{ax} over the interval (-π , π).**

As the interval is again (-π , π) we find c_n which is given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

Hence the complex Fourier series for f(x) is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$= \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{a^2+n^2} e^{inx}$$

HARMONIC ANALYSIS

The Fourier series of a **known** function f(x) in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when f(x) is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result :
The mean value of a continuous function f(x) over the interval (a,b) denoted by **[f(x)]** is defined as
The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$[f(x)] = \frac{1}{b-a}\int_a^b f(x)dx \qquad a_0 = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)dx\right] = 2[f(x)].$$

$$a_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$b_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]$$

Using these in (5), we obtain the Fourier series of f(x). The term $a_1\cos x+b_1\sin x$ is called the first harmonic or fundamental harmonic, the term $a_2\cos 2x+b_2\sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2+b_1^2}$ and that of second harmonic is $\sqrt{a_2^2+b_2^2}$ and so on.

Problems:

1. **Find the first two harmonics of the Fourier series of f(x) given the following table :**

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of y = f(x) are spread over the interval $0\le x \le 2\pi$ and $f(0) = f(2\pi) = 1.0$. Hence the function is periodic and so we omit the last value $f(2\pi) = 0$. We prepare the following table to compute the first two harmonics.

x^0	y = f(x)	cosx	cos2x	sinx	sin2x	ycosx	ycos2x	ysinx	ysin2x
0	1.0	1	1	0	0	1	1	0	0

60	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	3.1176	-0.1732

We have

$$a_n = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right] = 2[y\cos nx]$$
$$b_n = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right] = 2[y\sin nx]$$
as the length of interval= $2l = 2\pi$ or $l=\pi$

Putting, n=1,2, we get

$$a_1 = 2[y\cos x] = \frac{2\sum y\cos x}{6} = \frac{2(1.1)}{6} == -0.367$$
$$a_2 = 2[y\cos 2x] = \frac{2\sum y\cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = [y\sin x] = \frac{2\sum y\sin x}{6} = 1.0392$$
$$b_2 = [y\sin 2x] = \frac{2\sum y\sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1\cos x + b_1\sin x$ and $a_2\cos 2x + b_2\sin 2x$. That is ($-0.367\cos x + 1.0392\sin x$) and ($-0.1\cos 2x - 0.0577\sin 2x$)

2. **Express y as a Fourier series upto the third harmonic given the following values :**

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The values of y at x=0,1,2,3,4,5 are given and hence the interval of x should be $0 \le x < 6$.
The length of the interval = 6-0 = 6, so that $2l = 6$ or $l = 3$.
The Fourier series upto the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right)$$

or

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta) \ldots\ldots\ldots \quad (1)$$

We prepare the following table using the given values :

x	$\theta = \frac{\pi x}{3}$	y	ycosθ	ycos2θ	ycos3θ	ysinθ	ysin2θ	ysin3θ
0	0	04	4	4	4	0	0	0
1	60^0	08	4	-4	-8	6.928	6.928	0
2	120^0	15	-7.5	-7.5	15	12.99	-12.99	0
3	180^0	07	-7	7	-7	0	0	0
4	240^0	06	-3	-3	6	-5.196	5.196	0
5	300^0	02	1	-1	-2	-1.732	-1.732	0
Total		**42**	**-8.5**	**-4.5**	**8**	**12.99**	**-2.598**	**0**

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14$$

$$a_1 = 2[y\cos\theta] = \frac{2}{6}(-8.5) = -2.833$$

$$b_1 = 2[y\sin\theta] = \frac{2}{6}(12.99) = 4.33$$

$$a_2 = 2[y\cos 2\theta] = \frac{2}{6}(-4.5) = -1.5$$

$$b_2 = 2[y\sin 2\theta] = \frac{2}{6}(-2.598) = -0.866$$

$$a_3 = 2[y\cos 3\theta] = \frac{2}{6}(8) = 2.667$$

$$b_3 = 2[y\sin 3\theta] = 0$$

Usingthesein(1),we get

$$y = 7 - 2{,}833\cos\left(\frac{\pi x}{3}\right) + (4.33)\sin\left(\frac{\pi x}{3}\right) - 1.5\cos\left(\frac{2\pi x}{3}\right) - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos\pi x$$

This is the required Fourier series upto the third harmonic.

3. **The following table gives the variations of a periodic current A over a period T :**

t(secs)	**0**	**T/6**	**T/3**	**T/2**	**2T/3**	**5T/6**	**T**
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at t=0 and t=T are the same. Hence A(t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$. We have

$$a_0 = 2[A]$$

$$a_1 = 2\left[A\cos\left(\frac{2\pi}{T}\right)t\right] = 2[A\cos\theta] \qquad (1)$$

$$b_1 = 2\left[A\sin\left(\frac{2\pi}{T}\right)t\right] = 2[A\sin\theta]$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cosθ	sinθ	Acosθ	Asinθ
0	0	1.98	1	0	1.98	0
T/6	60^0	1.30	0.5	0.866	0.65	1.1258
T/3	120^0	1.05	-0.5	0.866	-0.525	0.9093
T/2	180^0	1.30	-1	0	-1.30	0
2T/3	240^0	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300^0	-0.25	0.5	-0.866	-0.125	0.2165
Total		**4.5**			**1.12**	**3.0137**

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5$$
$$a_1 = \frac{2\sum A\cos\theta}{6} = \frac{1.12}{3} = 0.3733$$
$$b_1 = \frac{2\sum A\sin\theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion upto the first harmonic is

$$A = \frac{a_0}{2} + a_1\cos\left(\frac{2\pi t}{T}\right) + b_1\sin\left(\frac{2\pi t}{T}\right)$$
$$= 0.75 + 0.3733\cos\left(\frac{2\pi t}{T}\right) + 1.0046\sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2} = 1.0717$.

UNIT-II

FOURIER TRANSFORMS

CONTENTS:

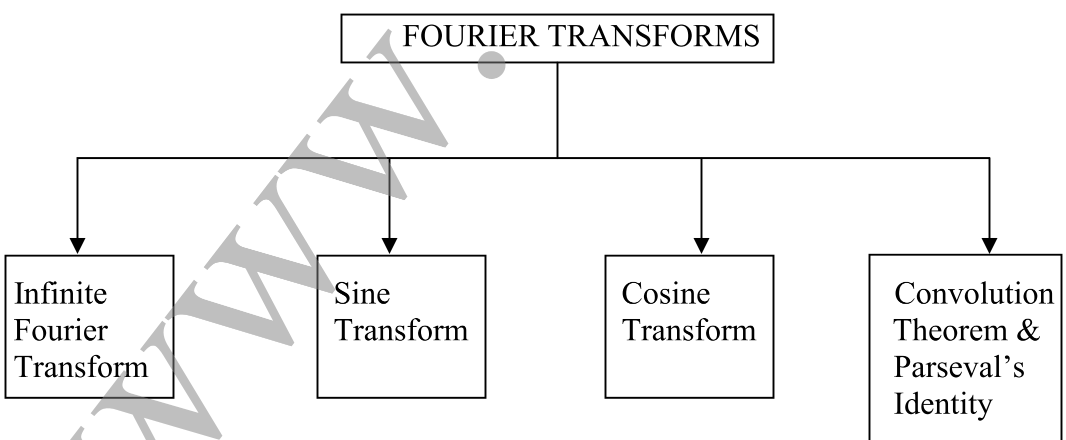
- **Introduction**
- **Finite Fourier transforms and Inverse finite Fourier transforms**
- **Infinite Fourier transform (Complex Fourier transform) and Inverse Fourier transforms**
- **Properties [Linearity, Change of scale, Shifting and Modulation**

- **Fourier cosine and Fourier sine transforms & Inverse Fourier cosine and sine transforms**

FOURIER TRANSFORMS

Introduction

Fourier Transform is a technique employed to solve ODE's, PDE's,IVP's, BVP's and Integral equations.
The subject matter is divided into the following sub topics :



Infinite Fourier Transform

Let f(x) be a real valued, differentiable function that satisfies the following conditions:

1) f(x) and its derivative $f'(x)$ are continuous, or have only a finite number of simple discontinuities in every finite interval, and

2) the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

Also, let α be non - zero real parameter. The infinite Fourier Transform of f(x) is defined by

$$\hat{f}(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

provided the integral exists.

The infinite Fourier Transform is also called complex Fourier Transform or just the Fourier Transform. The inverse Fourier Transform of $\hat{f}(\alpha)$ denoted by $F^{-1}[\hat{f}(\alpha)]$ is defined by

$$F^{-1}[\hat{f}(\alpha)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$$

Note : The function f(x) is said to be self reciprocal with respect to Fourier transform if $\hat{f}(\alpha) = f(\alpha)$.

Basic Properties

1. **Linearity Property**

For any two functions f(x) and $\phi(x)$ (whose Fourier Transforms exist) and any two constants a and b,

$$F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$$

Proof :

By definition, we have

$$\begin{aligned} F[af(x) + b\phi(x)] &= \int_{-\infty}^{\infty} [af(x) + b\phi(x)]\, e^{i\alpha x} dx \\ &= a\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx + b\int_{-\infty}^{\infty} \phi(x)\, e^{i\alpha x} dx \\ &= aF[f(x)] + bF[\phi(x)] \end{aligned}$$

This is the desired property.

In particular, if a = b = 1, we get

$$F[f(x) + \phi(x)] = F[f(x)] + F[\phi(x)]$$

Again if a= -b = 1, we get

$$F[f(x) - \phi(x)] = F[f(x)] - F[\phi(x)]$$

2. **Change of Scale Property**

If $\hat{f}(\alpha) = F[f(x)]$, then for any non - zero constant a, we have

$$F[f(x)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$$

Proof : By definition, we have

$$F[f(ax)] = \int_{-\infty}^{\infty} [f(ax)] e^{i\alpha x} dx \qquad (1)$$

Suppose a > 0. let us set ax = u. Then expression (1) becomes

$$F[f(ax)] = \int_{-\infty}^{\infty} [f(u)] e^{i\left(\frac{\alpha}{a}\right)u} \frac{du}{a}$$

$$= \frac{1}{a}\hat{f}\left(\frac{\alpha}{a}\right) \qquad (2)$$

Suppose a < 0. If we set again ax = u, then (1) becomes

$$F[f(ax)] = \int_{\infty}^{-\infty} [f(u)] e^{i\alpha\left(\frac{u}{a}\right)} \frac{du}{a}$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} [f(u)] e^{i\left(\frac{\alpha}{a}\right)u} du$$

$$= -\frac{1}{a}\hat{f}\left(\frac{\alpha}{a}\right) \qquad (3)$$

Expressions (2) and (3) may be combined as

$$F[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$$ This is the desired property

3. **Shifting Properties**

For any real constant 'a',

$$(i) \quad F[f(x-a)] = e^{i\alpha a}\hat{f}(\alpha)$$

$$(ii) \quad F[e^{iax} f(x)] = \hat{f}(\alpha + a)$$

Proof : (i) We have

$$F[f(x)] = \hat{f}(\alpha) = \int_{-\infty}^{\infty} [f(x)] e^{i\alpha x} dx$$

Hence, $F[f(x-a)] = \int_{-\infty}^{\infty} [f(x-a)] e^{i\alpha x} dx$

Set x-a = t. Then dx = dt.Then,

$$F[f(x-a)] = \int_{-\infty}^{\infty} [f(t)] e^{i\alpha(t+a)} dt$$

$$= e^{i\alpha a}\int_{-\infty}^{\infty} [f(t)] e^{i\alpha t} dt$$

$$= e^{i\alpha a}\hat{f}(\alpha)$$

ii) We have

$$\hat{f}(\alpha + a) = \int_{-\infty}^{\infty} f(x)\, e^{i(\alpha+a)x} dx$$

$$= \int_{-\infty}^{\infty} \left[f(x) e^{iax} \right] e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} g(x) e^{iax}\ dx,\ \ where\ \ g(x) = f(x) e^{iax}$$

$$= F[g(x)]$$

$$= F\left[e^{iax} f(x)\right]$$

This is the desired result.

4. **Modulation Property**

$$If\ \ F[f(x)] = \hat{f}(\alpha),$$

$$then,\ F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)\right]$$

where 'a' is a real constant.

Proof : We have

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

Hence

$$F[f(x)\cos ax] = F\left[f(x)\left(\frac{e^{iax} + e^{-iax}}{2} \right)\right]$$

$$= \frac{1}{2}\left[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)\right],$$ by using linearity and shift properties.

This is the desired property.

Note : Similarly

$$F[f(x)\sin ax] = \tfrac{1}{2}\left[\hat{f}(\alpha + a) - \hat{f}(\alpha - a)\right]$$

Problems:

1. Find the Fourier Transform of the function $f(x) = e^{-a|x|}$ where $a > 0$

For the given function, we have

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a|x|} e^{i\alpha x}\ dx$$

$$=\left[\int_{-\infty}^{0} e^{-a|x|} e^{i\alpha x}\ dx+\int_{0}^{\infty} e^{-a|x|} e^{i\alpha x}\ dx\right]$$

Using the fact that $|x| = x, 0 \le x < \infty$ and $|x| = -x, -\infty < x \le 0$, we get

$$F[f(x)]=\left[\int_{-\infty}^{0} e^{ax} e^{i\alpha x}\ dx+\int_{0}^{\infty} e^{-ax} e^{i\alpha x}\ dx\right]$$

$$=\left[\int_{-\infty}^{0} e^{(a+i\alpha)x}\ dx+\int_{0}^{\infty} e^{-(a-i\alpha)x}\ dx\right]$$

$$=\left[\left\{\frac{e^{(a+i\alpha)x}}{(a+i\alpha)}\right\}_{-\infty}^{0}+\left\{\frac{e^{-(a-i\alpha)x}}{-(a-i\alpha)}\right\}_{0}^{\infty}\right]$$

$$=\left[\frac{1}{(a+i\alpha)}+\frac{1}{(a-i\alpha)}\right]$$

$$=\left[\frac{2a}{(a^2+\alpha^2)}\right]$$

2. Find the Fourier Transform of the function

$$f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$$

where 'a' is a positive constant. Hence evaluate

$$(i)\quad \int_{-\infty}^{\infty}\frac{\sin\alpha a\ \cos\alpha x}{\alpha}\,d\alpha$$

$$(ii)\quad \int_{0}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha$$

For the given function, we have

$$F[f(x)]=\left[\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx\right]$$

$$=\left[\int_{-\infty}^{-a} f(x)e^{i\alpha x}dx+\int_{-a}^{a} f(x)e^{i\alpha x}dx+\int_{a}^{\infty} f(x)e^{i\alpha x}dx\right]$$

$$=\left[\int_{-a}^{a} e^{i\alpha x}dx\right]$$

$$=2\left[\frac{\sin\alpha a}{\alpha}\right]$$

$$Thus\quad F[f(x)]=\hat{f}(\alpha)=2\left(\frac{\sin\alpha\ a}{\alpha}\right)\qquad (1)$$

Inverting $\hat{f}(\alpha)$ by employing inversion formula, we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\left[\ \frac{\sin\alpha a}{\alpha}\ \right]e^{-i\alpha x}d\alpha$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha a(\cos\alpha x-i\sin\alpha x)}{\alpha}d\alpha$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin\alpha a(\cos\alpha\, x)}{\alpha} d\alpha - i\int_{-\infty}^{\infty} \frac{\sin\alpha\, a\sin\alpha x}{\alpha} d\alpha\right]$$

Here, the integrand in the first integral is even and the integrand in the second integral is odd. Hence using the relevant properties of integral here, we get

$$f(x) = \tfrac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha a\cos\alpha x}{\alpha} d\alpha$$

or

$$\int_{-\infty}^{\infty} \frac{\sin\alpha a\cos\alpha x}{\alpha} d\alpha = \pi f(x)$$

$$= \begin{cases} \pi, & |x| \le a \\ 0, & |x| > a \end{cases}$$

For $x = 0, a = 1$, this yields

$$\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} d\alpha = \pi$$

Since the integrand is even, we have

$$2\int_{0}^{\infty} \frac{\sin\alpha}{\alpha} d\alpha = \pi$$

or

$$\int_{0}^{\infty} \frac{\sin\alpha}{\alpha} d\alpha = \tfrac{\pi}{2}$$

3. Find the Fourier Transform of $f(x) = e^{-a^2x^2}$ where 'a' is a positive constant. Deduce that $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier Transform.

Here

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2 - i\alpha x)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{i\alpha}{2a}\right)^2 + \frac{\alpha^2}{4a^2}\right]} dx$$

$$= e^{-\left(\alpha^2/4a^2\right)} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2} dx$$

Setting $t = ax - \dfrac{i\alpha}{2a}$, we get

$$F[f(x)] = e^{-\left(\alpha^2/4a^2\right)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{1}{a} e^{-\left(\alpha^2/4a^2\right)} 2\int_0^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a} e^{-\left(\alpha^2/4a^2\right)} \sqrt{\pi}, \text{ using gamma function.}$$

$$\hat{f}(\alpha) = \frac{\sqrt{\pi}}{a} e^{-\left(\alpha^2/4a^2\right)}$$

This is the desired Fourier Transform of f(x).

For $a^2 = 1/2$ in $f(x) = e^{-a^2x^2}$

we get $f(x) = e^{-x^2/2}$ and hence,

$\hat{f}(\alpha) = \sqrt{2\pi}\, e^{-\alpha^2/2}$

Also putting $x = \alpha$ in $f(x) = e^{-x^2/2}$, we get $f(\alpha) = e^{-\alpha^2/2}$.

Hence, $f(\alpha)$ and $\hat{f}(\alpha)$ are same but for constant multiplication by $\sqrt{2\pi}$.

Thus $f(\alpha) = \hat{f}(\alpha)$

It follows that $f(x) = e^{-x^2/2}$ is self reciprocal

FOURIER SINE TRANSFORMS

Let f(x) be defined for all positive values of x.

The integral $\int_0^{\infty} f(x)\sin\alpha x dx$ is called the Fourier Sine Transform of f(x). This is denoted by $\hat{f}_s(\alpha)$ or $F_s[f(x)]$. Thus

$$\hat{f}_s(\alpha) = F_s[f(x)] = \int_0^{\infty} f(x)\sin\alpha x\, dx$$

The inverse Fourier sine Transform of $\hat{f}_s(\alpha)$ is defined

through the integral $\frac{2}{\pi}\int_0^{\infty} \hat{f}_s(\alpha)\sin\alpha x\, d\alpha$

This is denoted by f(x) or $F_s^{-1}[f_s(\alpha)]$. Thus

$$f(x) = F_s^{-1}[f_s(\alpha)] = \frac{2}{\pi}\int_0^{\infty} \hat{f}_s(\alpha)\sin\alpha x\, d\alpha$$

Properties

The following are the basic properties of Sine Transforms.

(1) **LINEARITY PROPERTY**

If 'a' and 'b' are two constants, then for two functions f(x) and φ(x), we have

$$F_s[af(x)+b\phi(x)] = aF_s[f(x)]+bF_s[g(x)]$$

Proof : By definition, we have

$$F_s[af(x)+b\phi(x)] = \int_0^\infty [af(x)+b\phi(x)]\sin\alpha x\, dx$$

$$= aF_s[f(x)]+bF_s[\phi(x)]$$

This is the desired result. In particular, we have

$$F_s[f(x)+\phi(x)] = F_s[f(x)]+F_s[\phi(x)]$$

and

$$F_s[f(x)-\phi(x)] = F_s[f(x)]-F_s[\phi(x)]$$

(2) **CHANGE OF SCALE PROPERTY**

If $\mathrm{F_s}[\mathrm{f}(\mathrm{x})] = \hat{\mathrm{f}}_s(\alpha)$, then for $a \neq 0$, we have

$$\mathrm{F_s}[\mathrm{f}(\mathrm{ax})] = \frac{1}{a}\hat{\mathrm{f}}_s\left(\frac{\alpha}{a}\right)$$

Proof : We have

$$\mathrm{F_s}[\mathrm{f}(\mathrm{ax})] = \int_0^\infty f(ax)\sin\alpha\ x\, dx$$

Setting ax = t , we get

$$\mathrm{F_s}[\mathrm{f}(\mathrm{ax})] = \int_0^\infty f(t)\sin\left(\frac{\alpha}{a}\right)t\left(\frac{dt}{a}\right)$$

$$= \frac{1}{a}\hat{\mathrm{f}}_s\left(\frac{\alpha}{a}\right)$$

(3) **MODULATION PROPERTY**

If $\mathrm{F_s}[\mathrm{f}(\mathrm{x})] = \hat{f}_s(\alpha)$, then for $a \neq 0$, we have

$$F_s[f(x)\cos ax] = \tfrac{1}{2}\left[\hat{f}_s(\alpha+a)+\hat{f}_s(\alpha-a)\right]$$

Proof : We have

$$F_s[f(x)\cos ax] = \int_0^\infty f(x)\cos ax\sin\alpha x\, dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\{\sin(\alpha+a)x+\sin(\alpha-a)x\}dx\right]$$

$$= \tfrac{1}{2}\left[\hat{f}_s(\alpha+a)+\hat{f}_s(\alpha-a)\right],$$ by using Linearity property.

Problems:

1. **Find the Fourier sine transform of**

$$f(x)=\begin{cases}1, & 0\le x\le a\\ 0, & x>a\end{cases}$$

For the given function, we have

$$\hat{f}_s(\alpha)=\left[\int_0^a \sin\alpha\ x\,dx+\int_a^\infty 0\sin\alpha\ x\,dx\right]$$

$$=\left[\frac{-\cos\alpha x}{\alpha}\right]_0^a$$

$$=\left[\frac{1-\cos\alpha a}{\alpha}\right]$$

2. Find the Fourier sine transform of $f(x)=\dfrac{e^{-ax}}{x}$

Here $\hat{f}_s(\alpha)=\left[\int_0^\infty \dfrac{e^{-ax}\sin\alpha x\,dx}{x}\right]$

Differentiating with respect to α, we get

$$\frac{d}{d\alpha}\hat{f}_s(\alpha)=\frac{d}{d\alpha}\left[\int_0^\infty \frac{e^{-ax}\sin\alpha x\,dx}{x}\right]$$

$$=\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial\alpha}(\sin\alpha x)\,dx$$

performing differentiation under the integral sign

$$=\int_0^\infty \frac{e^{-ax}}{x}x\cos\alpha x\,dx$$

$$=\left[\frac{e^{-ax}}{a^2+\alpha^2}\{-a\cos\alpha x+\alpha\sin\alpha x\}\right]_0^\infty$$

$$=\frac{a}{a^2+\alpha^2}$$

Integrating with respect to α, we get

$$\hat{f}_s(\alpha)=\tan^{-1}\frac{\alpha}{a}+c$$

But $\hat{f}_s(\alpha)=0$ *when* $\alpha=0$

$\therefore$ c=0

$$\hat{f}_s(\alpha)=\tan^{-1}\left(\frac{\alpha}{a}\right)$$

3. **Find f(x) from the integral equation**

$$\int_0^\infty f(x)\sin \alpha x dx = \begin{cases} 1, & 0 \le \alpha \le 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$$

Let $\phi(\alpha)$ be defined by

$$\phi(\alpha) = \begin{cases} 1, & 0 \le \alpha \le 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$$

Given

$$\phi(\alpha) = \int_0^\infty f(x)\sin \alpha x dx = \hat{f}_s(\alpha)$$

Using this in the inversion formula, we get

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \phi(\alpha)\sin \alpha x \, dx \\ &= \frac{2}{\pi}\left[\int_0^1 \phi(\alpha)\sin \alpha x \, d\alpha + \int_1^2 \phi(\alpha)\sin \alpha x \, d\alpha + \int_2^\infty \phi(\alpha)\sin \alpha x \, d\alpha\right] \\ &= \frac{2}{\pi}\left[\int_0^1 \sin \alpha x \, d\alpha \; + \int_1^2 2\sin \alpha x \, d\alpha + 0\right] \\ &= \frac{2}{\pi x}\left[1 + \cos x - 2\cos 2x\right] \end{aligned}$$

FOURIER COSINE TRANSFORMS

Let f(x) be defined for positive values of x. The integral $\int_0^\infty f(x)\cos \alpha x \, dx$ is called the Fourier Cosine Transform of f(x) and is denoted by $\hat{f}_c(\alpha)$ or $F_c[f(x)]$. Thus

$$\hat{f}_c(\alpha) = F_c[f(x)] = \frac{2}{\pi}\int_0^\infty f(x)\cos \alpha x dx$$

The inverse Fourier Cosine Transform of $\hat{f}_c(\alpha)$ is defined through the integral $\frac{2}{\pi}\int_0^\infty \hat{f}_c(\alpha)\cos \alpha \, x \, d\alpha$. This is denoted by f(x) or $F_c^{-1}\left[\hat{f}_c(\alpha)\right]$. Thus

$$f(x) = F_c^{-1}\left[\hat{f}(\alpha)\right] = \frac{2}{\pi}\int_0^\infty \hat{f}_c(\alpha)\cos \alpha \, x \, d\alpha$$

Basic Properties

The following are the basic properties of cosine transforms :

(1) **Linearity property**

If 'a' and 'b' are two constants, then for two functions f(x) and ϕ(x), we have
$F_c[af(x)+b\phi(x)] = aF_c(f(x))+bF_c(\phi(x))$

(2) **Change of scale property**

If $F_c\{f(x)\} = \hat{f}_c(\alpha)$, then for $a \neq 0$, we have
$F_c[f(ax)] = \frac{1}{a}\hat{f}_c\left(\frac{\alpha}{a}\right)$

(3) **Modulation property**

If $F_c\{f(x)\} = \hat{f}_c(\alpha)$, then for $a \neq 0$, we have
$F_c[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}_c(\alpha+a)+\hat{f}_c(\alpha-a)\right]$

The proofs of these properties are similar to the proofs of the corresponding properties of Fourier Sine Transforms.

Problems:

1) **Find the cosine transform of the function**

$$f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$$

We have

$$\hat{f}_c(\alpha) = \int_0^\infty f(x)\cos\alpha x dx$$

$$= \left[\int_0^1 x\cos\alpha x dx + \int_1^2 (2-x)\cos\alpha x dx + \int_2^\infty 0\cos\alpha x dx\right]$$

Integrating by parts, we get

$$\hat{f}_c(\alpha) = \left[\left\{x\left(\frac{\sin\alpha x}{\alpha}\right)-\left(\frac{-\cos\alpha x}{\alpha^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin\alpha x}{\alpha}\right)-(-1)\left(\frac{-\cos\alpha x}{\alpha^2}\right)\right\}_1^2\right]$$

$$= \left[\frac{2\cos\alpha-\cos 2\alpha-1}{\alpha^2}\right]$$

2) Find the cosine transform of $f(x) = e^{-ax}$, $a>0$. Hence evaluate $\int_0^\infty \frac{\cos kx}{x^2+a^2}dx$

Here

$$\hat{f}_c(\alpha) = \int_0^\infty e^{-ax} \cos \alpha x dx$$

$$= \left[\frac{e^{-ax}}{a^2 + \alpha^2} \{-a \cos \alpha x + \alpha \sin \alpha x\} \right]_0^\infty$$

Thus

$$\hat{f}_c(\alpha) = \left(\frac{a}{a^2 + \alpha^2} \right)$$

Using the definition of inverse cosine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{a^2 + \alpha^2} \right) \cos \alpha x d\alpha$$

or

$$\frac{\pi}{2a} e^{-ax} = \int_0^\infty \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha$$

Changing x to k, and α to x, we get

$$\int_0^\infty \frac{\cos kx}{x^2 + a^2} dx = \frac{\pi e^{-ax}}{2a}$$

3) **Solve the integral equation**

$$\int_0^\infty f(x) \cos \alpha x \, dx = e^{-a\alpha}$$

Let $\phi(\alpha)$ be defined by

$$\phi(\alpha) = e^{-a\alpha}$$

Given $\phi(\alpha) = \int_0^\infty f(x) \cos \alpha x \, dx = \hat{f}_c(\alpha)$

Using this in the inversion formula, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \phi(\alpha) \cos \alpha x d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty e^{-a\alpha} \cos \alpha x d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty \left[\frac{e^{-\alpha\alpha}}{a^2 + x^2} \{-a \cos \alpha x + \alpha \sin \alpha x\} \right]_0^\infty$$

$$= \frac{2a}{\pi (a^2 + x^2)}$$

UNIT III
Applications of Partial Differential Equations

CONTENTS:

- **Introduction**
- **Various possible solutions of the one dimensional wave equation**
- **Various possible solutions of the one dimensional heat equation**
- **Various possible solutions of the two dimensional Laplace's equation**

- **D'Alembert's solutions of the one dimensional wave equation**

APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Introduction

A number of problems in science and engineering will lead us to partial differential equations. In this unit we focus our attention on one dimensional wave equation ,one dimensional heat equation and two dimensional Laplace's equation.
Later we discuss the solution of these equations subject to a given set of boundary conditions referred to as boundary value problems.
Finally we discuss the D'Alembert's solution of one dimensional wave equation.

Various possible solutions of standard p.d.es by the method of separation of variables.

We need to obtain the solution of the ODEs by taking the constant k equal to
i) Zero ii)positive:$k=+p^2$ iii)negative:$k=-p^2$
Thus we obtain three possible solutions for the associated p.d.e

Various possible solutions of the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ by the method of separation of variables.

Consider $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Let u= XT where X=X(x),T=T(t) be the solution of the PDE

Hence the PDE becomes

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2} \quad or\ X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}$$

Dividing by c^2XT we have $\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$

Equating both sides to a common constant k we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad \text{and} \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = k$$

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} - c^2kT = 0$$

$$\left(D^2 - k\right)X = 0 \quad \text{and} \quad \left(D^2 - c^2k\right)T = 0$$

Where $D^2 = \frac{d^2}{dx^2}$ in the first equation and $D^2 = \frac{d^2}{dt^2}$ in the second equation

Case(i) : let k=0

AEs are m=0 amd m^2=0 amd m=0,0 are the roots

Solutions are given by

$T = c_1e^{0t} = c_1 \ and\ X = (c_2x + c_3)e^{0x} = (c_2x + c_3)$

Hence the solution of the PDE is given by

U= XT= $c_1(c_2x + c_3)$

Or u(x,t) =Ax+B where c_1c_2=A and c_1c_3=B

Case(ii) let k be positive say $k=+p^2$

AEs are $m - c^2p^2 = 0$ and $m^2 - p^2 = 0$

m= c^2p^2 and m=+p

solutions are given by

$T = c'_1e^{c^2p^2t} \ and X = c'_2e^{px} + c'_3e^{-px}$

Hence the solution of the PDE is given by

$u = XT = c'_1e^{c^2p^2t}.(c'_2e^{px} + c'_3e^{-px})$

Or u(x,t) = $c'_1e^{c^2p^2t}$(A' e^{px} +B e^{-px}) where $c_1'c_2'$=A' and $c_1'c_3'$=B'

Case(iii): let k be negative say $k=-p^2$

AEs are $m + c^2p^2 = 0$ and $m^2 + p^2 = 0$

m=- c^2p^2 and m=+ip

solutions are given by

$T = c''_1e^{-c^2p^2t} \ and \ X = c''_2 \cos px + c''_3 \sin px$

Hence the solution of the PDE is given by

$u = XT = c''_1 e^{-c^2p^2t} .(c''_2 \cos px + c''_3 \sin px)$

$u(x,t) = e^{-c^2p^2t}(A'' \cos px + B'' \sin px)$

Various possible solutions of the two dimensional Laplace's equation $u_{xx}+u_{yy}=0$ by the method of separation of variables

Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Let u=XY where X(x), Y=Y(y) e the solution of the PDE

Hence the PDE becomes

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y\frac{d^2(X)}{dx^2} + X\frac{d^2(Y)}{dy^2} = 0 \text{ and dividing by } XY \text{ we have}$$

$$\frac{1}{X}\frac{d^2(X)}{dx^2} = -\frac{1}{Y}\frac{d^2(Y)}{dy^2}$$

Equating both sides to a common constant k we have

$\frac{1}{X}\frac{d^2(X)}{dx^2}$=k and $-\frac{1}{Y}\frac{d^2(Y)}{dy^2}$=k

Or $(D^2-k)X=0$ and $(D^2+k)Y=0$

Where D $=\frac{d}{dx}$ in the first equation and D $=\frac{d}{dy}$ in the second equation

Case(i) Let k=0

AE are$m^2=0$ in respect of both the equations

M=0,0 and m=0,0

Solutions are given by

$X= c_1x+c_2$ and $Y= c_3y+c_4$

Hence the solution of the PDE is given by

$U=XY=(c_1x+c_2) (c_3y+c_4)$

Case(ii) : let k be positive ,say $k=+p^2$

$m^2-p^2=0$ and $m^2+p^2=0$

m=+p and m=+ip

solutions are given by

$X = c'_1 e^{px} + c'_2 e^{-px} \text{ and } Y = c'_3 \cos py + c'_4 \sin py$

Hence the solution of the PDE is given by

$u = XY = \left(c'_1 e^{px} + c'_2 e^{-px}\right)(c'3 \cos py + c'_4 \sin py)$

Case(iii) Let k be negative say $k= -p^2$

AEs are $m^2+p^2=0$ and $m^2-p^2=0$

M=+ip and m=+p

Solutions are given by

$X=(c''_1\cos px+c''_2\sin px)$ and $Y=(c''_3e^{py}+c''_4e^{-py})$

Hence the solution of the PDE is given by

$u=XY=(c''_1\cos px+c''_2\sin px)(c''_3e^{py}+c''_4e^{-py})$

EXAMPLES

1.Solve the wave equation $u_{tt}=c^2u_{xx}$ subject to the conditions

u(t,0)=0 ,u(l,t)=0, $\frac{\partial u}{\partial t}(x,0)=0$ and u(x,0) =$u_0\sin^3(\pi x/l)$

Soln: $u(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

Consider u(x,0) =$u_0\sin^3(\pi x/l)$

$u(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

$u_0\sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

$u_0\left[\frac{3}{4}\sin^3\frac{\pi x}{l}-\frac{1}{4}\sin\frac{3\pi x}{l}\right]=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

$\frac{3u_0}{4}\sin\ \frac{\pi x}{l}-\frac{u_0}{4}\sin\frac{3\pi x}{l}=b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}+b_3\sin\frac{3\pi x}{l}$

comparing both sides we get

$b_1=\frac{3u_0}{4}, b_2=0\ ,\ b_3=\frac{-u_0}{4},\ b_4=0\ b_5=0\ ,$

Thus by substituting these values in the expanded form we get

$u(x,t)=\frac{3u_0}{4}\sin\ \frac{\pi x}{l}\cos\frac{\pi ct}{l}-\frac{u_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$

2.Solve the wave equation $u_{tt}=c^2u_{tt}$ subject to the conditions

u(t,0)=0 ,u(l,t)=0, $\frac{\partial u}{\partial t}(x,0)=0$ when t=0and u(x,0) =f(x)

Soln: : $u(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

Consider u(x,0)=f(x) then we have

Consider u(x,0) = $\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

F(x) = $\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

The series in RHS is regarded as the sine half range Fourier series of f(x) in (0,l) and hence

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Thus we have the required solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

3. **Solve the Heat equation** $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ **given that u(0,t)=0,u(l,0)=0 and u(x,0)= 100x/l**

Soln: $b_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}\,dx == \frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$

$$b_n = \frac{200}{l^2}\left[\frac{x.-\cos\frac{n\pi x}{l}}{n\pi/l} - 1\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^l$$

$$b_n = \frac{200}{l^2}.\frac{-1}{n\pi}(l\cos n\pi) = -\frac{200(-1)^n}{n\pi}. = \frac{200(-1)^{n+1}}{n\pi}$$

The required solution is obtained by substituting this value of b_n

Thus $u(x,t) = \sum_{n=1}^{\infty}\frac{200(-1)^{n+1}}{n\pi} e^{\frac{-n^2\pi^2c^2t}{l^2}}\sin\frac{n\pi x}{l}$

4.**Obtain the solution of the heat equation** $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ **given that u(0,t)=0,u(l,t)and u(x,0) =f(x)where**

$$f(x) = \begin{cases} \frac{2Tx}{l} \; in\, 0 \le x \le \frac{l}{2} \\ \frac{2T}{l}(l-x) \; in\, \frac{l}{2} \le x \le l \end{cases}$$

Soln: $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}}\frac{2Tx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l}\frac{2Tx}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4T}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$b_{n=}\frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}$$

The required solution is obtained by substituting this value of b_n

Thus $u(x,t)=\dfrac{8T}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\,e\dfrac{-n^2\pi^2c^2t}{l^2}\sin\dfrac{n\pi x}{l}$

5. Solve the heat equation $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ with the boundary conditions u(0,t)=0,u(l,t)and u(x,0) =3sin π x

Soln: $u(x,t)=e^{-p^2t}(A\ \cos px+B\ \sin px)$............................(1)

Consider u(0,t)=0 now 1 becomes

0=e^{-p^2t} (A) thus A=0

Consider u(1,t)=0 using A=0 (1) becomes

0=e^{-p^2t} (Bsinp)

Since B≠0,sinp=0or p=nπ

$$u(x,t)=e^{-n^2\pi^2c^2t}(B\sin n\pi x)$$

In general $u(x,t)=\sum_{n=1}^{\infty}b_n e^{-n^2\pi^2c^2t}\ \sin n\pi x$

Consider u(x,0)= $3\sin n\pi x$ and we have

$$3\sin n\pi x=b_1\sin\pi x+b_2\sin 2\pi x+b_3\sin 3\pi x$$

Comparing both sides we get $b_1=3,b_2=0,b_3=0$

We substitute these values in the expanded form and then get

$$u(x,t)=3e^{-\pi^2t}(\sin\pi x)$$

6.Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ subject to the conditions u(0,y)=0,u(π ,y)=0 and u(x,∞) =0 and u(x,0)=ksin2x

Soln: The befitting solution to solve the given problem is given by

$$u(x,y)=(A\cos px+B\sin px)\left(Ce^{py}+De^{-py}\right)$$

$$u(0,y)=0\ gives(A)\left(Ce^{py}+De^{-py}\right)=0,A=0$$

$$u(\pi,y)=0\ gives(B\sin p\pi)\left(Ce^{py}+De^{-py}\right)=0,$$

Since A=0,B≠0 and we must have

$\sin p\pi$ =0,therefore p=n where n is a integer

$$u(x,y)=(B\sin nx)\left(Ce^{ny}+De^{-ny}\right)$$

The condition u(x,∞) =0 means that u→as y→∞

0=(Bsinnx)(Ce^{ny})since $e^{-ny} \to 0\ as y \to \infty$
Since B≠0 we must have C=0
We now have u(x,y) =BDsinnx e^{-ny}
Taking n= 1,2,3,……… and BD =b_1 ,b_2 ,b_3 ,b_4.....................
We obtain a set of independent solutions satisfying the first three conditions Their sum also satisfy these conditions hence

$$u(x,y)=\sum_{n=1}^{\infty} b_n \sin n\pi x e^{-ny}$$

Consider u(x,0)=ksin2x and we have

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin nx$$

Ksin2x=$b_1$sinx+$b_2$sin2x+$b_3$sin3x+…………….
Comparing both sides we get b_1=0, b_2=0, b_3=0
Thus by substituting these values in the expanded form we get

$$u(x,y)=k\sin 2xe^{-2y}$$

D'Alembert's solution of the one dimensional wave equation

We have one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

Let v=x+ct and w=x-ct
We treat u as a function of v and w which are functions of x and t
By chain rule we have,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}.\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}.\frac{\partial w}{\partial x}$$

Since v=x+ct and w=x-ct, $\frac{\partial v}{\partial x}=1\ and \frac{\partial w}{\partial x}=1$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}.(1)+\frac{\partial u}{\partial w}.(1)=\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}\right)$$

Again by applying chain rule we have,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}\right).\frac{\partial v}{\partial x}+\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial w}\right).\frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial^2 u}{\partial v^2}+\frac{\partial^2 u}{\partial v\partial w}\right).(1)+\left(\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial w^2}\right).(1)$$

But $\frac{\partial^2 u}{\partial v\partial w}=\frac{\partial^2 u}{\partial w\partial v}$

But $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}$

Similarly $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}.\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial t}$

Since v=x+ct and w=x-ct, $\frac{\partial v}{\partial t} = c \ and \frac{\partial w}{\partial t} = -c$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}.(c) + \frac{\partial u}{\partial w}.(-c)$ $,or$ $\frac{\partial u}{\partial t} = c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$

Again applying chain rule we have,

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial v} + \frac{\partial}{\partial w}\left[c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)\right].\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left[c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)\right].\frac{\partial w}{\partial t}$$

$$\frac{\partial^2 u}{\partial r^2} = c\left[\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w}\right].(c) + c\left[\frac{\partial^2 u}{\partial w \partial v} - \frac{\partial^2 u}{\partial w^2}\right].(-c)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w}\right] - c^2\left[\frac{\partial^2 u}{\partial w \partial v} - \frac{\partial^2 u}{\partial w^2}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}\right]$$

$$c^2\left[\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}\right] = c^2\left[\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}\right]$$

$-4\frac{\partial^2 u}{\partial w \partial v}=0$ or $\frac{\partial^2 u}{\partial w \partial v}=0$

We solve this PDE by direct integration, writing it in the form,

$$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right) = 0$$

Integrating w.r.t w treating v as constant we get $\frac{\partial u}{\partial v} = f(v)$

Now integrating w.r.t, we get $u = \int f(v)dv + G(w)$

U=F(v)+G(w) where F(v)= $\int f(v)dv$

But v=x+ct and w=x-ct

Thus u=u(x,t)=F(x+ct)+G(x-ct)

This is the D'Alembert's solution of the one dimensional wave equation

EXAMPLES

1. Obtain the D'Alembert's solution of the wave equation $u_{tt}=c^2u_{xx}$ subject to the conditions u(x,0)=f(x) and $\frac{\partial u}{\partial t}(x,0) = 0$

Soln: D'Alembert's solution of the wave equation is given by

u(x,t)=F(x+ct0=G(x-ct)………………………………………..(1)
Consider u(x,0)=f(x) (10 becomes
u(x,0)=F(x)+G(x)
f(x)=F(x)+G(x)………………………………………………….(2)
Differentiating partially wrt t we have

$\frac{\partial u}{\partial t}$ (x,t) =F'(x+ct).(c)+G'(x-ct).(-c)

$\frac{\partial u}{\partial t}$ (x,)=c[F'(x)-G'(x)]

0= c[F'(x)-G'(x)]
[F'(x)-G'(x)]=0 integrating wrt x we have
[F(x)-G(x)]=k
Where k is the constant of integration……………………………(3)
By solving simultaneously the equations,
[F(x)+G(x)]=f(x)
[F(x)-G(x)]=k

We obtain F(x)= $\frac{1}{2}\left[f(x)+k\right] \quad and G(x)=\frac{1}{2}\left[f(x)-k\right]$

Thus F(x+ct)= $\frac{1}{2}\left[f(x+ct)+k\right]$ and G(x-ct)= $\frac{1}{2}\left[f(x-ct)-k\right]$

Substituting these in(10 we have

U(x,t)= $\frac{1}{2}\left[f(x+ct)+k\right]+\frac{1}{2}\left[f(x-ct)-k\right]$

Thus the required solution is

u(x,t)= $\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$

2. **Obtain the D'Alembert's solution of the wave equation $u_{tt}=c^2u_{xx}$ given that $u(x,0)=f(x)=l^2-x^2$ and $u_t(x,0)=0$**

Soln. Assuming the D'Alembert's solution

u(x,t)= $\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$ and f(x)=l^2-x^2

u(x,t)= $\frac{1}{2}\left[\left\{l^2-(x+ct)^2\right\}+\left\{l^2-(x-ct)^2\right\}\right]$

$=\frac{1}{2}\left[2l^2-2x^2-2c^2t^2\right]$

u(x,t)= $\left[l^2-x^2-c^2t^2\right]$

3. . Obtain the D'Alembert's solution of the wave equation $u_{tt}=c^2u_{xx}$ given that $u(x,0)=a\sin^2\pi x$ and

$\frac{\partial u}{\partial t}=0 \ \ when\ t=0$

Soln: Assuming the D'Alembert's solution

u(x,t)= $\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$

By data f(x)= a$\sin^2\pi$x or f(x)= $\frac{a}{2}\left(1-\cos 2\pi x\right)$

u(x,t)= $\frac{1}{2}.\frac{a}{2}\left[1-\cos 2\pi(x+ct)+1-\cos 2\pi(x-ct)\right]$

u(x,t)= $\frac{a}{4}\left[2-\left\{\cos\left(2\pi x+2\pi ct\right)+\cos\left(2\pi x-2\pi ct\right)\right\}\right]$

$=\frac{a}{4}\left[2-2\cos 2\pi x\cos 2\pi ct\right]$

u(x,t)= $\frac{a}{2}\left[1-1\cos 2\pi x\cos 2\pi ct\right]$

UNIT IV

CURVE FITTING AND OPTIMIZATION

CONTENTS:

- **Curve fitting by the method of least squares**
- **Fitting of curves of the form**
    - ❖ $y=ax+b$
    - ❖ $y=ax^2+bx+c$
    - ❖ $y=ae^{bx}$

- $y = ax^b$

- **Optimization :**
  - **Linear programming**
  - **Graphical method**
  - **Simplex method**

CURVE FITTING AND OPTIMIZATION

CURVE FITTING [BY THE METHOD OF LEAST SQUARE]:

We can plot 'n' points (x_i, y_i) where i=0,1,2,3,………
At the XY plane. It is difficult to draw a graph y=f(x) which passes through all these points but we can draw a graph which passes through maximum number of point. This curve is called the curve of best fit. The method of finding the curve of best fit is called the curve fitting.

FITTING A STRAIGHT LINE Y = AX + B

We have straight line that sounds as best approximate to the actual curve y=f(x) passing through 'n' points (x_i, y_i), i=0,1.2………..n equation of a straight line is

$y = a + bx$ (1)

Then for 'n' points $Y_i = a + bx_i$ (2)

Where a and b are parameters to be determined; Y_i is called the estimated value. The given value Y_i corresponding to x_i.

$$Let \quad S = \sum (y_i - Y_i)^2 \qquad (3)$$

$$= \sum [y_i - (a + bx_i)]^2$$

$$S = \sum [y_i - a - bx_i]^2 \qquad (4)$$

We determined a and b so that S is minimum (least). Two necessary conditions for this $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$.

differentiate (4) w.r.t a and b partially

$$\frac{\partial S}{\partial a} = \sum [y_i - a - bx_i]$$

$$0 = \sum (y_i - a - bx_i)$$

$$0 = \sum y_i - \sum a - b\sum x_i$$

$$0 = \sum y_i - na - b\sum x_i$$

$$\boxed{\sum y_i = na + b\sum x_i}$$

$$\text{or } \boxed{\sum y = na + b\sum x}$$

$$\frac{\partial S}{\partial b} = 2\sum [y_i - a - bx_i](-x_i)$$

$$0 = -2\sum (x_i y_i - ax_i - bx_i^2)$$

$$0 = \sum x_i y_i - a\sum x_i - b\sum x_i^2$$

$$\sum x_i y_i = n\sum x_i + b\sum x_i^2$$

$$\text{or } \boxed{\sum xy = a\sum x + b\sum x^2}$$

where n = number of points or value.

FITTING A SECOND DEGREE PARABOLA Y=AX^2 + BX + C

Let us take equation of parabola called parabola of best fit in the form

$$y = a + bx + cx^2 \qquad (1)$$

Where a, b, c are parameters to be determined. Let be the value of corresponding to the v

$Y_i = a + bx_i + cx_i^2$ (2)

Also

$S=\sum\left(y_i - Y_i\right)^2$ (3)

$=\sum\left[y_i - \left(a + bx_i + cx_i^2\right)\right]^2$

$\boxed{S = \sum\left(y_i - a - bx_i - cx_i^2\right)}$ (4)

We determine a, b, c so that S is least (minimum).
The necessary condition for this are

$\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0 \ \& \ \frac{\partial S}{\partial c} = 0$

diff (4) w.r.t 'a' partially

$\frac{\partial S}{\partial a} = -2\sum\left(y_i - a - bx_i - cx_i^2\right)$

$0 = -2\sum\left(y_i - a - bx_i - cx_i^2\right)$

$0 = \sum\left(y_i - a - bx_i - cx_i^2\right)$

$0 = \sum y_i - \sum a - b\sum x_i - c\sum x_i^2$

$\sum y_i = \sum a + b\sum x_i + c\sum x_i^2$

$\boxed{\sum y_i = na + b\sum x_i + c\sum x_i^2}$

$or \ \boxed{\sum y = na + b\sum x + c\sum x^2}$

diff (4) w.r.t 'b' partially

$\frac{\partial S}{\partial b} = 2\sum\left(y_i - a - bx_i - cx_i^2\right)\left(-x_i\right)$

$0 = -2\sum\left(x_i y_i - ax_i - bx_i^2 - cx_i^3\right)$

$0 = \sum\left(x_i y_i - ax_i - bx_i^2 - cx_i^3\right)$

$0 = \sum x_i y_i - a\sum x_i - b\sum x_i^2 - c\sum x_i^3$

$\boxed{\sum x_i y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3}$

$or \ \boxed{\sum xy = a\sum x + b\sum x^2 + c\sum x^3}$

diff (4) w.r.t 'c' partially

$$\frac{\partial S}{\partial c} = 2\sum\left(y_i - a - bx_i - cx_i^2\right)\left(-x_i^2\right)$$

$$0 = -2\sum\left(x_i^2 y_i - ax_i^2 - bx_i^3 - cx_i^4\right)$$

$$0 = \sum\left(x_i^2 y_i - ax_i^2 - bx_i^3 - cx_i^4\right)$$

$$0 = \sum x_i^2 y_i - a\sum x_i^2 - b\sum x_i^3 - c\sum x_i^4$$

$$\boxed{\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4}$$

$$or\ \boxed{\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4}$$

Hence the normal equation for second degree parabola are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

PROBLEMS:

1) **Fit a straight line** $y = a + bx$ **to the following data**

$x:$	5	10	15	20	25
$y:$	16	19	23	26	30

Solution: Let $y = a + bx$ (1)

Normal equation

$$\sum y = na + b\sum x \quad (2)$$

$$\sum xy = a\sum x + b\sum x^2 \quad (3)$$

x	y	x^2	xy
5	16	25	80
10	19	100	190
15	23	225	345
20	26	400	520
25	30	625	750
$\sum x$ $= 75$	$\sum y$ $=114$	$\sum x^2$ $=1375$	$\sum xy$ $=1885$

(1) & (2) $\Rightarrow$

$114 = 15a + b \times 75$

$1885 = 75a + b \times 1375$

$$\boxed{\begin{aligned} a &= 12.3 \\ b &= 0.7 \end{aligned}}$$

(1) $\Rightarrow$ $\boxed{y = 12.3 + 0.7x}$

2) Fit equation of straight line of best fit to the following data

x:	1	2	3	4	5
y:	14	13	9	5	2

Solution:

Let $y = a + bx$ (1)

Normal equation

$$\sum y = na + b\sum x \quad (2)$$

$$\sum xy = a\sum x + b\sum x^2 \quad (3)$$

x	y	x^2	xy
1	14	1	14
2	13	4	26
3	9	9	27
4	5	16	20
5	2	25	10
$\sum x$ $=15$	$\sum y$ $=43$	$\sum x^2$ $=55$	$\sum xy$ $=97$

(2) & (3) $\Rightarrow$

$43 = 15a + 15b$

$97 = 15a + 56b$

Solving above equations we get

$$\boxed{\begin{aligned} a &= 18.2 \\ b &= -3.2 \end{aligned}}$$

(1) $\Rightarrow$ $\boxed{y = 18.2 - 3.2x}$

3) The equation of straight line of best fit find the equation of best fit

x:	0	1	2	3	4
y:	1	1.8	3.3	4.5	6.3

Solution: let $y = a + bx$ (1)

Normal equation

$\sum y = na + b\sum x$ (2)

$\sum xy = a\sum x + b\sum x^2$ (3)

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x$ =10	$\sum y$ =16.9	$\sum x^2$ =30	$\sum xy$ =47.1

(2) & (3) $\Rightarrow$

$16.9 = 5a + 10b$

$47.1 = 10a + 30b$

$\boxed{\begin{aligned} a &= 0.72 \\ b &= 1.33 \end{aligned}}$

(1) $\Rightarrow$ $\boxed{y = 0.72 + 1.33x}$

4) If p is the pull required to lift a load by means of pulley block. Find a linear block of the form p=MW+C Connected p &w using following data

ω:	50	70	100	120
p:	12	15	21	25

Compute p when W=150.

Solution: Given p=y & W=x

$\therefore$ equation of straight line is

:

let $y = a + bx$ (1)

Normal equations

$\sum y = na + b\sum x$ (2)

$\sum xy = a\sum x + b\sum x^2$ (3)

$x=\omega$	$p=y$	x^2	xy
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	144000	3000

$\sum x = 10$ $\sum y$=16.9 $\sum x^2$=30 $\sum xy$=47.1

(2) & (3) $\Rightarrow$

$73 = 4a + 340b$

$6750 = 340a + 31800b$

$\boxed{\begin{array}{l} a = 2.27 \\ b = 0.187 \end{array}}$

(1) $\Rightarrow$ $\boxed{y = 2.27 + 0.187x}$

put $\omega = 150$

$y = 30.32$

5) Fit a curve of the form $y = ab^x$

Solution: Consider

$y = ab^x$ (1)

Take log on both side

logy=log$\left(ab^x\right)$

=loga+logb^x

logy=loga+logb^x

$y = A + Bx$ (2) ; logy=Y $\Rightarrow$ y=e^y

Corresponding normal equation loga=A $\Rightarrow$ a=e^A

$\sum Y = nA + B\sum x$ (3) logb=B $\Rightarrow$ b=e^B

$\sum xY = A\sum x + B\sum x^2$ (4)

Solving the normal equation (3) & (4) for a & b . Substitute these values in (1) we get curve of best fit of the form $y = ab^x$

6) Fit a curve of the curve $y = ab^x$ **for the data**

x:	1	2	3	4	5	6	7	8
y:	1.0	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution:

$let\ y = ab^x$ (1)

Normal equations are

$\sum Y = nA + B\sum x$ (2); A=loga

$\sum xY = A\sum x + B\sum x^2$ (3) Y=logy, B=logb

x	y	Y=logy	x^2	xY
1	1.0	0	1	0
2	1.2	0.182	4	0.364
3	1.8	0.587	9	1.761
4	2.5	0.916	16	3.664
5	3.6	1.280	25	6.4
6	4.7	1.547	36	9.282
7	6.6	1.887	49	13.209
8	9.1	2.208	64	17.664
$\sum x = 36$		$\sum Y = 8.607$	$\sum x^2 = 204$	$\sum xY = 52.34$

(1) & (2) $\Rightarrow$

$8.607 = 8A + 36B$

$52.34 = 36A + 203B$

$$\boxed{\begin{aligned} A &= -0.382 \\ B &= 0.324 \end{aligned}}$$

then $e^A = a = 0.682$

$e^B = b = 1.382$

(1) $\Rightarrow$ $\boxed{y = 0.682(1.382)^x}$

7) Fit a curve of the form $y = ax^b$ for the following data

x:	1	1.5	2	2.5
y:	2.5	5.61	10.0	15.6

Solution:: Consider

$let\ y = ax^b$ (1)

Normal equations are

$\sum Y = nA + b\sum x$ (2); Y=logy

$\sum XY = A\sum X + B\sum X^2$ (3) A=loga, X=logx

x	y	X= $\log x$	X^2	$Y=\log y$	XY
1	2.5	0	0	0.916	0
1.5	5.62	0.405	0.164	1.726	0.699
2	10.0	0.693	0.480	2.302	1.595
2.5	15.6	0.916	0.839	2.747	2.516
		$\sum$X=2.014	$\sum X^2$=1.483	$\sum$Y=7.691	$\sum$XY=4.81

(1) & (2) $\Rightarrow$

$7.691 = 4A + 2.014b$

$4.81 = 2.014A + 1.483b$

A=0.916, $e^A = a = 2.499 \approx 2.5$

$b = 1.999 \approx 2$

(1) $\Rightarrow$ $\boxed{y = 2.5(x)^2}$

8) Fit a parabola $y = a + bx + cx^2$ for the following data

x: 1 2 3 4

y: 1.7 1.8 2.3 3.2

Sol:

$y = a + bx + cx^2$ (1)

Normal equation

$\sum y = na + b\sum x + c\sum x^2$ (2)

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ (3)

$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$ (4)

x	y	x^2	x^3	x^4	xy	x^2y
1	1.7	1	1	1	1.7	1.7
2	1.8	4	8	16	3.6	7.2
3	2.3	9	27	81	6.9	20.7
4	3.2	16	64	256	12.8	51.2

$\sum x \quad \sum y = 9 \quad \sum x^2 = 30 \quad \sum x^3 = 100 \quad \sum x^4 = 354 \quad \sum xy = 25 \quad \sum x^2 y = 80.8$

(1),(3) & (4)

$9 = 4a + 10b + 30c$

$25 = 10a + 30b + 100c$

$80.8 = 30a + 100b + 354c$

$a = 2$

$b = -0.5$

$c = 0.2$

(1) $\Rightarrow$ $y = 2 - 0.5x + 0.2x^2$

9) Fit a curve of the form $y = ae^{bx}$ **for the following**

x: 0 2 4

y: 8.12 10 31.82

Sol:

$y = ae^{bx}$ (1)

Normal equation

$\sum$ Y=nA+b$\sum x$ (2); Y=logy

$\sum xy = A\sum x + b\sum x^2$ (3) A=loga

x	y	Y=logy	x^2	xY
0	8.12	2.094	0	0
2	10	2.302	4	4.604
4	31.82	3..46	16	13.84

$\sum x = 6$ $\sum$ Y=7.86 $\sum x^2 = 20$ $\sum xY = 18.444$

(2) & (3) $\Rightarrow$

7.856=3A+6b

18.444=6A+20b

A=1.935, $a = e^A = 6.924$

b=0.341

(1) $\Rightarrow$ $y = 6.924e^{0..341x}$

10) Fit a II degree parabola ax^2+bx+c to the least square method & hence find y when x=6

$x: \; 1 \quad 2 \quad 3 \quad 4 \quad 5$

$y: \; 10 \quad 12 \quad 13 \quad 16 \quad 19$

Sol:

$$y = ax^2 + bx + c \qquad (1)$$

$$y = c + bx + ax^2 \qquad (1)$$

Normal equation

$$\sum y = nc + b\sum x + a\sum x^2 \qquad (2)$$

$$\sum xy = c\sum x + b\sum x^2 + a\sum x^3 \qquad (3)$$

$$\sum x^2 y = c\sum x^2 + b\sum x^3 + a\sum x^4 \qquad (4)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	14
3	13	9	27	81	39	117
4	16	16	64	256	64	256
5	19	25	125	625	95	475
$=15$	$=70$	$=55$	$=225$	$=979$	$=232$	$=906$

$$70 = 5c + 15b + 55a$$

$$232 = 15c + 55b + 225a$$

$$906 = 55c + 225b + 979a$$

$$a = 0.285$$

$$b = 0.485$$

$$c = 9.4$$

$$(1) \Rightarrow y = 0.285x^2 + 0.485x + 9.4$$

at $x = 6, \quad y = 22.6$

OPTIMIZATION:

Optimization is a technique of obtaining the best result under the given conditions. Optimization means maximization or minimization

LINEAR PROGAMMING:

Linear Programming is a decision making technique under the given constraints on the condition that the relationship among the variables involved is linear. A general relationship among the variables involved is called objective function. The variables involved are called decision variables.

The optimization of the objective function Z subject to the constraints is the mathematical formulation of a LPP.

A set real values $X = (x_1, x_2, ...x_n)$ which satisfies the constraint $AX \le (\ge) B$ is called solution.

A set of real values xi which satisfies the constraints and also satisfies non negativity constraints $x_i \ge 0$ is called feasible solution.

A set of real values xi which satisfies the constraints along with non negativity restrictions and optimizes the objective function is called optimal solution.

An LPP can have many solutions.

If the optimal value of the objective function is infinity then the LPP is said to have unbounded solutions. Also an LPP may not possess any feasible solution.

GRAPHICAL METHOD OF SOLVING AN LPP

LPP involved with only two decision variables can be solved in this method. The method is illustrated step wise when the problem is mathematically formulated.

The constraints are considered in the form of $\frac{x}{a} + \frac{y}{b} = 1$ which graphically represents straight lines passing through the points (a,0) and (0,b) since there are only two decision variables.

These lines along with the co-ordinate axes forms the boundary of the region known as the feasible region and the figure so formed by the vertices is called the convex polygon.

The value of the objective function $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ is found at all these vertices.

The extreme values of Z among these values corresponding the values of the decision variables is required optimal solution of the LPP.

PROBLEMS:

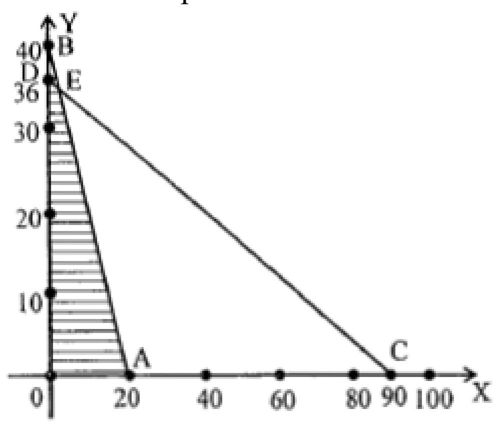
1) **Use the graphical method to maximize Z = 3x + 4y subject to the constraints** $2x + y \le 40$, $2x + 5y \le 180$, $x \ge 0$, $y \ge 0$.

Solution: Let us consider the equations

$$2x + y = 40 \qquad\qquad 2x + 5y = 180$$

$$\Rightarrow \qquad \frac{x}{20} + \frac{y}{40} = 1 ....(1) \qquad\qquad \frac{x}{90} + \frac{y}{36} = 1 ....(2)$$

Let (1) and (2) represent the straight lines AB and CD respectively where we have
A = (20, 0), B = (0, 40) ; C = (90 , 0), D = (0,36)
We draw these lines in XOY plane.



Shaded portion is the feasible region and OAED is the convex polygon. The point E being the point of intersection of lines AB and CD is obtained by solving the equation: $2x + y = 40$, $2x + 5y = 180$. $E(x, y) = (2.5,35)$

The value of the objective function at the corners of the convex polygin OAED are tabulated.

Corner	Value of Z = 3x + 4y
O(0,0)	0
A(20, 0)	60
E(2.5, 35)	147.5
D(0,36)	144

$$Thus \ \ (Z)_{Max} = 147.5 \ \ when \ \ x = 2.5, \ y = 35$$

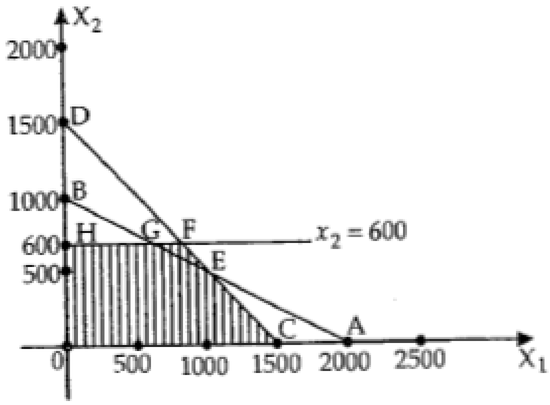
2) Use the graphical method to maximize $Z = 3x_1 + 5x_2$ **subject to the constraints** $x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 1500$, $x_2 \le 600$, $x_1, x_2 \ge 0$.

Solution: Let us consider the equations

$x_1 + x_2 = 2000$, $\quad x_1 + x_2 = 1500$, $\quad x_2 = 600$

$$\Rightarrow \frac{x_1}{2000} + \frac{x_2}{1000} = 1.....(1) \qquad \frac{x_1}{1500} + \frac{x_2}{1500} = 1...........(2) \qquad x_2 = 600.........(3)$$

Let (1) and (2) represent the straight lines AB and CD respectively where we have
A = (2000, 0), B = (0, 1000) ; C = (1500 , 0), D = (0,1500)
$x_2 = 600$. is a line parallel to the x_1 axis.
We draw these lines in XOY plane.



On solving

$x_1 + x_2 = 2000, \quad x_1 + x_2 = 1500, \quad we\ get \quad E = (1000,500)$

$x_1 + x_2 = 1500, \quad x_2 = 600, \quad we\ get \quad F = (900,600)$

$x_1 + 2x_2 = 2000, \ x_2 = 600, \quad we\ get \quad G = (800,600)$

Also we have C = (1500 , 0), H= (0, 600)

The value of the objective function at the corners of the convex polygin OAED are tabulated.

Corner	Value of $Z = 3x_1 + 5x_2$
O(0,0)	0
C(1500, 0)	4500
E(1000, 500)	5500
F(900, 600)	5700
G(800,600)	5400

$$Thus \quad (Z)_{Max} = 5700 \quad when \quad x_1 = 900, \ x_2 = 600$$

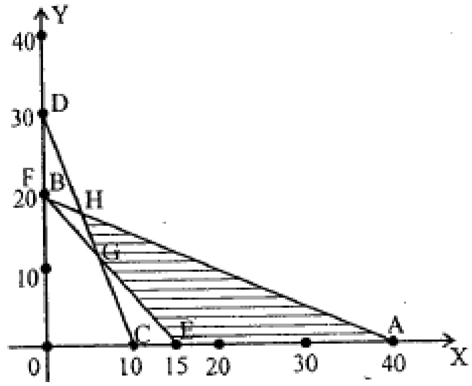
3**) Use the graphical method to minimize** $Z = 20x_1 + 10x_2$ **subject to the constraints** $x_1 + 2x_2 \le 40, \quad 3x_1 + x_2 \ge 30, \ \ 4x_1 + 3x_2 \ge 60 \ , \quad x_1, x_2 \ge 0.$

Solution: Let us consider the equations

$x_1 + 2x_2 = 40, \qquad 3x_1 + x_2 = 30, \qquad 34x_1 + 3x_2 = 60$

$$\Rightarrow \ \frac{x_1}{40} + \frac{x_2}{20} = 1.....(1) \qquad \frac{x_1}{10} + \frac{x_2}{30} = 1...........(2) \qquad \frac{x_1}{15} + \frac{x_2}{20} = 1.........(3)$$

Let (1) ,(2) and (3) represent the straight lines AB ,CD and EF respectively where we have A = (40, 0), B = (0,20) ; C = (10 , 0), D = (0,30) ; E = (15,0), F = (0, 20)
We draw these lines in XOY plane.



Shaded portion is the feasible region and EAHG is the convex polygon. The point G being the point of intersection of lines EF and CD. The point H being the point of intersection of lines AB and CDis obtained by solving the equation:

The value of the objective function at the corners of the convex polygin OAED are tabulated.

Corner	Value of $Z = 3x_1 + 5x_2$
A(15 , 0)	300
A(40, 0)	800
H(4 , 18)	260
G (6, 12)	240

$Thus \quad (Z)_{MIN} = 240 \quad when \quad x_1 = 6, \quad x_2 = 12$

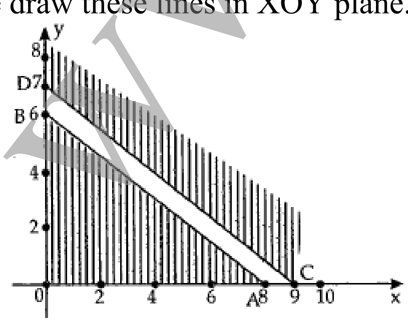
4) Show that the following LPP does not have any feasible solution.
Objective function for maximization:Z = 20x+30y.
Constraints: $3x + 4y \le 24$, $7x + 9y \ge 63$, $x \ge 0$, $y \ge 0$.

Solution: Let us consider the equations

$$3x + 4y = 24 \qquad\qquad 7x + 9y = 63$$

$$\Rightarrow \quad \frac{x}{8} + \frac{y}{6} = 1 \text{.............(1)} \qquad\qquad \frac{x}{9} + \frac{y}{7} = 1 \text{...........(2)}$$

Let (1) and (2) represent the straight lines AB and CD respectively where we have
A = (8, 0), B = (0,6) ; C = (9 , 0), D = (0,7)
We draw these lines in XOY plane.



It is evident that there is no feasible region.Thus we conclude that the LPP does not have any feasible solution.

SIMPLEX METHOD

Simplex method is an efficient algebraic method to solve a LPP by systematic procedure and hence an algorithm can be evolved called the simplex algorithm.

In this method it is necessary that all the constraints in the inequality formn is converted into equality form thus arriving at a system of algebraic equations.

If the constraint is involved with $\leq$ we add a non zero ariables $s_1(say) \geq 0$ to the LHS to make it an equality and the same variable is called slack variable.

LPP with all constraints being equalities is called a standard form of LPP.

A minimizing LPP is converted into an equivalent maximization problem. Minimizing the given objective function P is equivalent to maximizing –P under the same constraints and Min.P = -(Max value of –P)

PROBLEMS:

1) Use Simplex method to maximize z= 2x + 4y subject to the constraints
$3x + y \leq 22,\ 2x + 3y \leq 24,\ x \geq 0,\ y \geq 0$

Solution: Let us introduce slack variables s_1 and s_2 to the inequalities to write them in the following form.

$3x + y + 1.s_1 + 0.s_2 = 22$

$2x + 3y + 0.s_1 + 1.s_2 = 24$

$2x + 4y + 0.s_1 + 0.s_2 = z$ is the objective function

Solution by the simplex method is presented in the following table.

NZV	x	y	s_1	s_2	Qty	Ratio	
s_1	3	1	1	0	22	22/1=22	8 is least, 3 is pivot,
s_2	2	3	0	1	24	24/3=8	s_2 is replaced by y. Also 1/3.R2
Indicators (Δ)	-2	-4	0	0	0		
s_1	3	1	1	0	22		$R_1 \rightarrow -R_2 + R_1$
y	2/3	1	0	1/3	8		
Δ	-2	-4	0	0	0		$R_3 \rightarrow 4R_2 + R_3$
s_1	7/3	0	1	-1/3	14		
y	2/3	0	0	1/3	8		
Δ	2/3	0	0	4/3	32		No negative indicators

Thus the maximum value of Z is 32 at x=0 and y=8

2) Use Simplex method to maximize Z =2x + 3y + z subject to the constraints
$x+3y+2z \le 11,\ x+2y+5z \le 19, 3x+y+4z \le 25\ \ x \ge 0,\ y \ge 0, z \ge 0$

Solution: Let us introduce slack variables s_1, s_2, s_3 to the inequalities to write them in the following form.

$$x+3y+2z+1.s_1+0.s_2+0.s_3 = 11$$

$$x+2y+5z+0.s_1+1.s_2+0.s_3 = 19$$

$$3x+y+4z+0.s_1+0.s_2+1.s_3 = 25$$

$$2x+3y+z+0.s_1+0.s_2+0.s_3 = P \text{ is the objective function}$$

Solution by the simplex method is presented in the following table.

NZV	x	y	z	s_1	s_2	s_3	Qty	Ratio	
s_1	1	3	2	1	0	0	11	11/3=3.33	3.33 is least, 3 is pivot s_1 is replaced by y. Also 1/3.R1
s_2	1	2	5	0	1	0	19	19/3=9.5	
s_3	3	1	4	0	0	1	35	25/1=25	
Indicators (Δ)	-2	-3	-1	0	0	0	0		
y	1/3	1	2/3	1/3	0	0	11/3		
s_2	1	2	5	0	1	0	19		$R_2 \to -2R_1 + R_2$
s_3	3	1	4	0	0	1	25		$R_3 \to -R_1 + R_3$
Δ	-2	-3	-1	0	0	0	0		$R_4 \to 3R_1 + R_4$
Y	1/3	1	2/3	1/3	0	0	11/3	11/3÷1/3=11	8 is least, 8/3 is pivot s_3 is replaced by x. Also 3/8.R3
s_2	1/3	0	11/3	-2/3	1	0	35/3	35/3÷1/3=35	
s3	8/3	0	10/3	-1/3	0	1	64/3	64/3÷8/3=8	
Δ	-1	0	1	1	0	0	11		
y	1/3	1	2/3	1/3	0	0	11/3		$R_1 \to -1/3R_3 + R_1$
s_2	1/3	0	11/3	-2/3	1	0	35/3		$R_2 \to -1/3R_3 + R_2$
x	1	0	10/8	-1/8	0	3/8	8		
Δ	-1	0	1	1	0	0	1		$R_4 \to R_3 + R_4$
y	0	1	1/4	3/8	0	-1/8	1		
s_2	0	0	13/4	-5/8	1	-1/8	9		
x	1	0	10/8	-1/8	0	3/8	9		
Δ	0	0	9/4	7/8	0	3/8	19		No negative indicators

Thus the maximum value of P is 19 at x = 8 and y = 1, z = 0

3) Use Simplex method to minimize P =x - 3y + 2z subject to the constraints
$3x - y + 2z \le 7, -2x + 4y \le 12,\ -4x + 3y + 8z \le 10\ \ x \ge 0,\ y \ge 0, z \ge 0$

Solution: The given LPP is equivalent to maximizing the objective function –P subject to the same constraints

That is –P=P'=-x + 3y - 2z is to be maximized

Let us introduce slack variables s_1, s_2, s_3 to the inequalities to write them in the following form.

$3x - y + 2z + 1.s_1 + 0.s_2 + 0.s_3 = 7$

$-2x + 4y + 0z + 0.s_1 + 1.s_2 + 0.s_3 = 12$

$-4x + 3y + 8z + 0.s_1 + 0.s_2 + 1.s_3 = 10$

$-x + 3y - 2z + 0.s_1 + 0.s_2 + 0.s_3 = P'$ is the objective function

Solution by the simplex method is presented in the following table.

NZV	x	y	z	s_1	s_2	s_3	Qty	Ratio	
s_1	3	-1	2	1	0	0	7	7/-1=-7	3 is least, 4 is pivot s_2 is replaced by y. Also 1/4.R2
s_2	-2	4	0	0	1	0	12	12/4=3	
s_3	-4	3	8	0	0	1	10	10/3=3.3	
Indicators (Δ)	1	-3	2	0	0	0	0		
s1	3	-1	2	1	0	0	7		$R_1 \to R_1 + R_2$
y	-1/2	1	0	0	1/4	0	3		
s_3	-4	3	8	0	0	1	10		$R_3 \to -3R_2 + R_3$
Δ	1	-3	2	0	0	0	0		$R_4 \to 3R_2 + R_4$
s1	5/2	0	2	1	1/4	0	10	10÷5/2=4	4 is least, 5/2 is pivot, s_1 is replaced by x. Also 2/5.R1
y	-1/2	1	0	0	1/4	0	3	3÷-1/2=-6	
s_3	-5/2	0	8	0	-3/4	1	1	1÷-5/2=-2/5	
Δ	-1/2	0	2	0	3/4	0	9		
x	1	0	4/5	2/5	1/10	0	4		
y	-1/2	1	0	0	1/4	0	3		$R_2 \to 1/2R_1 + R_2$
s_3	-5/2	0	8	0	-3/4	1	1		$R_3 \to 5/2R_1 + R_3$
Δ	-1/2	0	2	0	3/4	0	9		$R_4 \to 1/2R_1 + R_4$
x	1	0	4/5	2/5	1/10	0	4		
y	0	1	2/5	1/5	3/10	0	5		
s_3	0	0	10	1	-1/2	1	11		
Δ	0	0	12/5	1/5	4/5	0	11		No negative indicators

Thus the maximum value of P is 19 at x = 8 and y = 1, z = 0

UNIT V
NUMERICAL METHODS - 1

CONTENTS:

- **Introduction**
- **Numerical solution of algebraic and transcendental equations**
    - **Regula –falsi method**
    - **Newton –Raphson method**
- **Iterative methods of solution of a system of equation**
    - **Gauss –Seidel method**
    - **Relaxation method**
- **Largest eigen value and the corresponding eigen vector by Rayeigh's power method**

NUMERICAL METHODS-1

Introduction:

Limitations of analytical methods led to the evolution of Numerical methods. Numerical Methods often are repetitive in nature i.e., these consist of the repeated execution of the same procedure where at each step the result of the proceeding step is used. This process known as iterative process is continued until a desired degree of accuracy of the result is obtained.

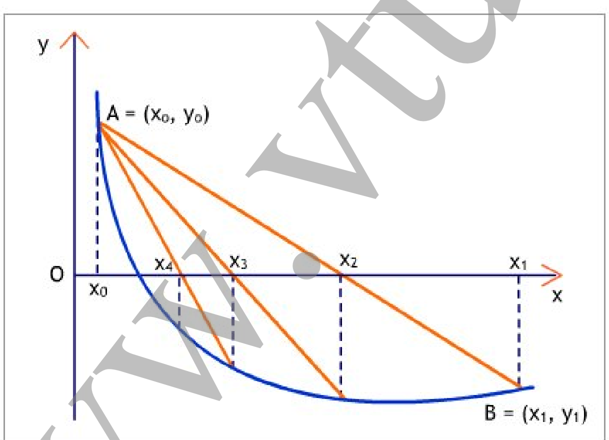
Solution of Algebraic and Transcendental Equations:

The equation f(x) = 0 said to be purely algebraic if f(x) is purely a polynomial in x.
If f(x) contains some other functions like Trigonometric, Logarithmic, exponential etc. then f(x) = 0 is called a Transcendental equation.

Ex: (1) $x^4 - 7x^3 + 3x + 5 = 0$ is algebraic
(2) $e^x - x \tan x = 0$ is transcendental

Method of false position or Regula-Falsi Method:

This is a method of finding a real root of an equation f(x) = 0 and is slightly an improvisation of the bisection method.
Let x_0 and x_1 be two points such that $f(x_0)$ and $f(x_1)$ are opposite in sign.



Let $f(x_0) > 0$ and $f(x_1) < 0$
The graph of $y = f(x)$ crosses the x-axis between x_0 and x_1

$\therefore$ **Root of f(x) = 0 lies between x_0 and x_1**

Now equation of the Chord AB is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad ...(1)$$

When y =0 we get $x = x_2$

$$\text{i.e. } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad ...(2)$$

Which is the first approximation

If $f(x_0)$ and $f(x_2)$ are opposite in sign then second approximation

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

This procedure is continued till the root is found with desired accuracy.

Poblems:

1. Find a real root of x^3 - 2x -5 = 0 by method of false position correct to three decimal places between 2 and 3.

Answer:

Let $f(x) = x^3 - 2x - 5 = 0$

$f(2) = -1$

$f(3) = 16$

$\therefore$ a root lies between 2 and 3

Take $x_0 = 2$, $x_1 = 3$

$\therefore$ $x_0 = 2$, $x_1 = 3$

$$\text{Now } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3 - 2}{16 + 1}(-1)$$

$= 2.0588$

$f(x_2) = f(2.0588) = -0.3908$

$\therefore$ Root lies between 2.0588 and 3

Taking $x_0 = 2.0588$ and $x_1 = 3$

$f(x_0) = -0.3908$, $f(x_1) = 16$

$$\text{We get } x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} .f(x_0)$$

$$= 2.0588 - \frac{0.9412}{16.3908}(-0.3908)$$

$= 2.0813$

$f(x_3) = f(2.0813) = -0.14680$

$\therefore$ Root lies between 2.0813 and 3

Taking $x_0 = 2.0813$ and $x_1 = 3$

$f(x_0) = 0.14680$, $f(x_1) = 16$

$$x_4 = 2.0813 - \frac{0.9187}{16.1468}(-0.14680) = 2.0897$$

Repeating the process the successive approximations are

$x_5 = 2.0915$, $x_6 = 2.0934$, $x_7 = 2.0941$, $x_8 = 2.0943$

Hence the root is 2.094 correct to 3 decimal places.

2. **Find the root of the equation $xe^x = \cos x$ using Regula falsi method correct to three decimal places**.

Solution:

Let $f(x) = \cos x - xe^x$

Observe

$f(0) = 1$

$f(1) = \cos 1 - e = -2.17798$

$\therefore$ root lies between 0 and 1

Taking $x_0 = 0$, $x_1 = 1$

$f(x_0) = 1$, $f(x_1) = -2.17798$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}.f(x_0)$$

$$= 0 - \frac{1}{-3.17798}(1) = 0.31467$$

$f(x_2) = f(0.31467) = 0.51987$ +ve

$\therefore$ Root lies between 0.31467 and 1

$x_0 = 0.31467$, $x_1 = 1$

$f(x_0) = 0.51987$, $f(x_1) = -2.17798$

$$x_3 = 0.31467 - \frac{1 - 0.31467}{-2.17798 - 0.51987}(0.51987) = 0.44673$$

$f(x_3) = f(0.44673) = 0.20356$ +ve

$\therefore$ Root lies between 0.44673 and 1

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process

$x_5 = 0.50995$, $x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$

$x_9 = 0.51767$, etc

Hence the root is 0.518 correct to 4 decimal places

Newton Raphson Method

This method is used to find the isolated roots of an equation f(x) = 0, when the derivative of f(x) is a simple expression.

Let m be a root of f(x) = 0 near a.

$\therefore \quad f(m) = 0$

We have by Taylor's series

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!}\, f''(a) + \ldots..$$

$\therefore \quad f(m) = f(a) + (m-a)\, f'(a) + \ldots..$

Ignoring higher order terms

$f(m) = f(a) + (m - a)\, f'(a) = 0$

$$\text{or } m - a = -\frac{f(a)}{f'(a)}$$

$$\text{or } m = a - \frac{f(a)}{f'(a)}$$

Let $a = x_0$, $m = x_1$

then $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is the first approximation

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ is the second approximation

.

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.

$x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ is the iterative formula for Newton Raphson Method

1. **Using Newton's Raphson Method find the real root of x $\log_{10}$ x = 1.2 correct to four decimal places**.

Answer:

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2$, $f(2) = -0.59794$, $f(3) = 0.23136$

$$\text{We have } f(x) = \frac{x \log_e x}{\log_e 10} - 1.2 \Rightarrow f'(x) = \frac{1 + \log_e x}{\log_e 10}$$

$$= \log_{10} e + \log_{10} x$$

$$\therefore \quad x_{k+1} = x_k - \frac{x_k \log_{10} x_k - 1.2}{\log_{10} e + \log_{10} x_K}$$

Let $x_0 = 2.5$ (you may choose 2 or 3 also)

$$x_1 = 2.5 - \frac{2.5 \log_{10} 2.5 - 1.2}{\log_{10} e + \log_{10} 2.5} = 2.7465$$

$$x_2 = 2.7465 - \frac{2.7465 \log 2.7465 - 1.2}{\log_{10} e + \log_{10} 2.7465} = 2.7406$$

Repeating the procedure

$$x_3 = 2.7406$$

$$\therefore \quad x \approx 2.7406 \text{ is the root of the given equation}$$

2.**Using Newton's Method, find the real root of $xe^x = 2$. Correct to 3 decimal places.**
Answer:
Let $f(x) = xe^x - 2$
$f(0) = -2$
$f(1) = e - 2 = 0.7182$
Let $x_0 = 1$
$f'(x) = (x + 1)e^x$
We have

$$x_{k+1} = x_k - \frac{x_k e^{x_k} - 2}{(x_k + 1)\, e^{x_k}}$$

$$x_1 = 1 - \frac{e - 2}{2e} = 0.8678$$

$$x_2 = 0.8678 - \frac{(0.8678)\, e^{0.8678} - 2}{(1.8678)\, e^{0.8678}} = 0.8527$$

$$x_3 = 0.8527 - \frac{(0.8527)\, e^{0.8527} - 2}{(1.8527)\, e^{0.8527}} = 0.8526$$

$$\therefore \quad x \approx 0.8526, \text{ is the required root. Correct to 3 decimal places}$$

3. Find by Newton's Method the real root of $3x = \cos x + 1$ near 0.6, x is in radians. Correct for four decimal places.
Answer:
Let $f(x) = 3x - \cos x - 1$
$f'(x) = 3 + \sin x$

$$x_{k+1} = x_k - \frac{3x_k - \cos x_{k-1}}{3 + \sin x_k}$$

When $x_0 = 0.6$ $\quad x_1 = 0.6 - \dfrac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6071$

$$x_2 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)} = 0.6071$$

Since $x_1 = x_2$
The desired root is 0.6071

4. Obtain the iterative formula for finding the square root of N and find $\sqrt{41}$

Answer:

Let $x = \sqrt{N}$

or $x^2 - N = 0$

$\therefore f(x) = x^2 - N$

$f'(x) = 2x$

Now

$$x_{k+1} = x_k - \frac{x_k^2 - N}{2x_k}$$

$$= x_k - \frac{x_k}{2} + \frac{N}{2x_k}$$

i.e. $x_{k+1} = \frac{1}{2}\left\{x_k + \frac{N}{x_k}\right\}$

To find $\sqrt{41}$

Observe that $\sqrt{36} < \sqrt{41}$

$\therefore$ Choose $x_0 = 6$

$$x_1 = \frac{1}{2}\left\{6 + \frac{41}{6}\right\} = 6.4166$$

$$x_2 = \frac{1}{2}\left\{6.4166 + \frac{41}{6.4166}\right\} = 6.4031$$

$$x_3 = \frac{1}{2}\left\{6.4031 + \frac{41}{6.4031}\right\} = 6.4031$$

Since $x_2 = x_3 = 6.4031$

The value of $\sqrt{41} \approx 6.4031$

5. Obtain an iterative formula for finding the p-th root of N and hence find $(10)^{1/3}$ correct to 3 decimal places.

Answer:

Let $x^p = N$

or $x^p - N = 0$

Let $f(x) = x^p - N$

$f'(x) = px^{p-1}$

Now $x_{k+1} = x_k - \dfrac{x_k^p - N}{px_k^{p-1}}$

Observe that 8 < 10

$\Rightarrow \quad 8^{1/3} < 10^{1/3}$

i.e. $\quad 2 < (10)^{1/3}$

$\therefore$ Use $x_0 = 2$, $p = 3$, $N=10$

$$x_1 = 2 - \frac{2^3 - 10}{3(2^2)} = 2.1666$$

$$x_2 = 2.1666 - \frac{(2.1666)^3 - 10}{3(2.1666)^2} = 2.1545$$

$$x_3 = 2.1545 - \frac{(2.1545)^3 - 10}{3(2.1545)^2} = 2.1544$$

$\therefore \quad (10)^{1/3} \approx 2.1544$

6. Obtain an iterative formula for finding the reciprocal of p-th root of N. Find $(30)^{-1/5}$ correct to 3 decimal places.

Answer:

Let $x^{-p} = N$

or $x^{-p} - N = 0$

$\therefore$ $f(x) = x^{-p} - N$

$f'(x) = -px^{-p-1}$

Now

$$x_{k+1} = x_k + \frac{x_k^{-p} - N}{p\, x_k^{-p-1}}$$

$$\text{since } (32)^{-1/5} = \frac{1}{2} = 0.5$$

We use $x_0 = 0.5$, $p = 5$, $N = 30$

$$x_1 = 0.5 + \frac{(0.5)^{-5} - 30}{5(0.5)^{-6}} = 0.50625, \text{ Repeating the process}$$

$$x_2 = 0.506495,\ x_3 = 0.506495$$

$\therefore \quad (30)^{-1/5} \approx 0.5065$

GAUSS-SEIDAL ITERATION METHOD (to solving the system linear simultaneous equations.)

Example 1.Use Gauss-Seidal iteration method to solve the following system of equations.

$3x + 20y - 2z = -18$

$20x + y - 2z = 17$

$2x - 3y + 20z = 25$

Solution:. Rearranging thegiven system of equations

$20x + y - 2z = 17$

$3x + 20y - 2z = -18$ ………………….(1)

$2x - 3y + 20z = 25$

The above system equations is arranged such that, diagonally dominant.

System *(1)*

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z] \quad \text{………… (2)}$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

Let the initial approximations to the solution of the system (A) be

$$x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$$

First Iteration:Using (2)

$$x^{(1)} = \frac{1}{20}[17 - y^{(0)} + 2z^{(0)}]$$

$$x^{(1)} = \frac{1}{20}[17 - 0 + 2(0)] = 0.8500$$

$$y^{(1)} = \frac{1}{20}[-18 - 3x^{(1)} + z^{(0)}]$$

$$y^{(1)} = \frac{1}{20}[-18 - 3(0.8500) + 0] = -1.0275$$

$$z^{(1)} = \frac{1}{20}[25 - 2x^{(1)} + 3y^{(1)}]$$

$$z^{(1)} = \frac{1}{20}[25 - 2(0.8500) + 3(-1.0275)]$$

$= 1.0109$

Second Iteration: Usining (2)

$$x^{(2)} = \frac{1}{20}[17 - y^{(1)} + 2z^{(1)}]$$

$$x^{(2)} = \frac{1}{20}[\,17-(-10275) + 2(1.0109)\,]$$
$= 1.0025$

$$y^{(2)} = \frac{1}{20}[-18-3x^{(2)} + z^{(1)}]$$

$$y^{(2)} = \frac{1}{20}[\,-18-3(1.0025) + 1.0109\,]$$
$= -0.99928$

$$z^{(2)} = \frac{1}{20}[25-2x^{(2)} + 3y^{(2)}]$$

$$z^{(2)} = \frac{1}{20}[25-2(1.0025) + 3(-0.99928)]$$

$= 0.9998$

Third Iteration: Usining (2)

$$x^{(3)} = \frac{1}{20}[\,17-y^{(2)} + 2z^{(2)}]$$

$$x^{(3)} = \frac{1}{20}[\,17-(-0.99928)+2(0.9998)\,]$$
$= 1.0000$

$$y^{(3)} = \frac{1}{20}[-18-3x^{(3)} + z^{(2)}]$$

$$y^{(3)} = \frac{1}{20}[\,-18-3(1.0000) + 0.9998\,]$$
$= -1.0000$

•

$$z^{(3)} = \frac{1}{20}[25-2x^{(3)} + 3y^{(3)}]$$

$$z^{(3)} = \frac{1}{20}[25 - 2(1.00) + 3(-1.0000)]$$

$= 1.0000$

Answer after three iterations
$x = 1.000$, $y = -1.0000$ and $z = 1.000$

2) Use Gauss-Seidal iteration method to solve the following system of equations.

x + y +54z=110
27x +6 y - z = 85 …………(A)
6x+15y + 2z =72

Solution :. Rearranging the system of equations (*A*)

$$\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned} \quad \rightarrow (B)$$

The above system equations is arranged such that, the system is diagonally dominant.

System *(B)* $\Longrightarrow$

$$\begin{aligned} x &= \frac{1}{27}[85 - 6y + z] \\ y &= \frac{1}{15}[72 - 6x - 2z] \\ z &= \frac{1}{54}[110 - x - y] \end{aligned} \qquad (C)$$

Let the initial approximations to the solution of the system (A) be

$$x^{(0)} = 1, y^{(0)} = 0, z^{(0)} = 0$$

First Iteration:Using (C)

$$x^{(1)} = \frac{1}{27}[85 - 6y^{(0)} + z^{(0)}]$$

$$x^{(1)} = \frac{1}{27}[85 - 6(0) + 0] = 3.148148$$

$$y^{(1)} = \frac{1}{15}[72-6x^{(1)} -2z^{(0)}] \qquad y^{(1)} = \frac{1}{15}[72-6(3.148148) -2(0)]$$

=3.54074

$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}]$$

$$z^{(1)} = \frac{1}{54}[110 - 3.148148 - 3.54074] = 1.913168$$

Second Iteration: Usining (C)

$$x^{(2)} = \frac{1}{27}[85-6y^{(1)} + z^{(1)}]$$

$$x^{(2)} = \frac{1}{27}[85-6(3.54074) + 1.913168]$$

=2.432175

$$y^{(2)} = \frac{1}{15}[72-6x^{(2)} -2z^{(1)}]$$

$$y^{(2)} = \frac{1}{15}[72-6(2.432175) -2(1.913168)]$$

= 3.57204

$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}]$$

$$z^{(2)} = \frac{1}{54}[110 - 2.432175 - 3.57204]$$

=1.925837

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Third Iteration: Usining (C)

$$x^{(3)} = \frac{1}{27}[85-6y^{(2)} + z^{(2)}]$$

$$x^{(3)} = \frac{1}{27}[85 - 6(3.57204) + 2(1.925837)]$$
$=2.425689$

$$y^{(3)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(2)}]$$
$$y^{(3)} = \frac{1}{15}[72 - 6(2.425689) - 2(1.913168)]$$
$=3.57313$

$$z^{(3)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}]$$
$$z^{(3)} = \frac{1}{54}[110 - 2.432175 - 3.57204]$$
$=1.925947$

Answer after three iterations
x =2.425689, y = 3.57313 and z =

RELAXATION METHOD:

The method is illustrated for the following diagonally dominant system of three independent equations in three unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

From these equations the residuals R_1, R_2, R_3 are defined as follows

$$R_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1$$
$$R_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2$$
$$R_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - b_3$$

The values of x_1, x_2, x_3 that make the residuals R_1, R_2, R_3 simultaneously zero constitutes the exact solution of the system of equations. If this is not possible we need to make the residuals as close to zero as possible.

The values of x_1, x_2, x_3 at that stage constitutes an approximate solution. The relaxation method aims in reducing the values of residuals as close to zero as possible by modifying the value of the variables at every stage.

At every step the numerically largest residual is reduced to zero by choosing an appropriate integral value for the corresponding variable.

These integral values are called the increments in x_1, x_2, x_3 denoted by $\Delta x_1, \Delta x_2, \Delta x_3$ respectively. If the system possess exact solution then R_1, R_2, R_3 becomes zero simultaneously and the required solution will be the sum of all these increments.
i.e., $x_1 = \sum \Delta x_1, \quad x_2 = \sum \Delta x_2, \quad x_3 = \sum \Delta x_3$

When the system does not possess exact solution, we need to continue the process b y magnifying the prevailing residuals on multiplication with 10. Later we divide by 10 to obtain the solution correct to one decimal place.

The process will be contined to meet the requirement of the solution to the desired accuracy.

PROBLEMS:

1) Solve the following system of equations by relaxation method.

$$12x_1 + x_2 + x_3 = 31$$
$$2x_1 + 8x_2 - x_3 = 24$$
$$3x_1 + 4x_2 + 10x_3 = 58$$

Solution:

The given system of equations are diagonally dominant.

Let $R_1 = 12x_1 + x_2 + x_3 - 31$

$R_2 = 2x_1 + 8x_2 - x_3 - 24$ be the residuals.

$R_3 = 3x_1 + 4x_2 + 10x_3 - 58,$

Let $\Delta x_1, \Delta x_2, \Delta x_3$ respectively represent the increments for x_1, x_2, x_3 in the following relaxation table.

Δx_1	Δx_2	Δx_3	R_1	R_2	R_3
0	0	0	-31	-24	-58
0	0	6	-25	-30	2
0	4	0	-21	2	18
2	0	0	3	6	24
0	0	-2	1	8	4
0	-1	0	0	0	0

R_1, R_2, R_3 are all zero and the exact solution of the given system is

$x_1 = \sum \Delta x_1 = 2, \quad x_2 = \sum \Delta x_2 = 4 - 1 = 3, \quad x_3 = \sum \Delta x_3 = 6 - 2 = 4$

Thus (x_1, x_2, x_3) = (2, 3, 4) is the solution of the given system of equations.

2) Solve the following system of equations by relaxation method.

$$10x - 2y - 3z = 205$$
$$-2x + 10y - 2z = 154$$
$$-2x - y + 10z = 120$$

Solution:

The given system of equations are diagonally dominant.

Let $R_1 = 10x - 2y - 3z - 205$

$R_2 = -2x + 10y - 2z - 154$ be the residuals.

$R_3 = -2x - y + 10z - 120,$

Let $\Delta x, \Delta y, \Delta z$ respectively represent the increments for x, y, z in the following relaxation table.

Δx	Δy	Δz	R_1	R_2	R_3
0	0	0	-205	-154	-120
21	0	0	5	-196	-162
0	20	0	-35	4	-182
0	0	18	-89	-32	-2
9	0	0	1	-50	-20
0	5	0	-9	0	25
0	0	3	-18	-6	5
2	0	0	2	-10	1
0	1	0	0	0	0

R_1, R_2, R_3 are all zero and the exact solution of the given system is

$x = \sum \Delta x = 32, \quad y = \sum \Delta y = 26, \quad z = \sum \Delta z = 21$

Thus (x, y, z) = (32 , 26, 21) is the solution of the given system of equations.

Rayleigh's power method:

Given square matrix A, if there exist a scalar λ and a non zero column matrix X, such that AX = λ X, then λ is called an eigen value of A and X is called an eigen vector of A corresponding to an eigen value λ.

Rayleigh's power method is an iterative method to determine the numerically largest eigen value and the corresponding eigen vector of a square matrix.

Working procedure:

- Suppose A is the given square matrix, we assume initially an eigen vector X_0 in a simple for like $[1,0,0]'$ *or* $[0,1,0]'$ *or* $[0,0,1]'$ *or* $[1,1,1]'$ and find the matrix product AX_0 which will also be a column matrix.
- We take out the largest element as the common factor to obtain $AX_0 = \lambda^{(1)} X^{(1)}$.
- We then find $AX^{(1)}$ and again put in the form $AX^{(1)} = \lambda^{(2)} X^{(2)}$ by normalization.
- The iterative process is continued till two consecutive iterative values of λ and X are same upto a desired degree of accuracy.
- The values so obtained are respectively the largest eigen value and the corresponding eigen vector of the given square matrix A.

Problems:

1) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \textit{given} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

$$\textit{Solution}: \quad AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.994 \end{bmatrix} = 2.997 \begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Thus the **largest eigen value** is approximately **3** and the corresponding **eigen vector** is [1 ,0, 1]'

2) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \textit{given} \begin{bmatrix} 1 & 0.8 & -0.8 \end{bmatrix}^T$$

$$\textit{Solution}: \quad AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6 \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86 \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96 \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98 \begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \begin{bmatrix} 5.994 \\ 5.988 \\ -5.988 \end{bmatrix} = 5.994 \begin{bmatrix} 1 \\ 0.999 \\ -0.999 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

Thus after five iterations the numerically largest eigen value is **5.994** and corresponding eigen vector is **[1, 0.999, -0,999]'**

UNIT VI

NUMERICAL METHOD -2

CONTENTS:

- **Introduction**
- **Finite difference:**

  Forward and Backward difference
- **Newtons forward and backward interpolation formula**
- **Newtons divided difference formula**
- **Lagranges interpolation formula**
- **Numerical integration**

NUMERICAL METHODS-2

Finite Differences

Let y = f(x) be represented by a table

x :	x_0	x_1	x_2	x_3	….	x_n
y :	y_0	y_1	y_2	y_3	…	y_n

where $x_0, x_1, x_2 \ldots x_n$ are equidistant. ($x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h$)

We now define the following operators called the difference operators.

Forward difference operator (Δ)

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta y_r = y_{r+1} - y_r, \ \ r = 0,1,2,\ldots,n-1$$

$$\left.\begin{array}{l} \Delta y_0 = y_1 - y_0 \\ \Delta y_1 = y_2 - y_1 \\ . \\ . \\ \Delta y_{n-1} = y_n - y_{n-1} \end{array}\right\} \text{first forward differences}$$

$\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \ldots,$ are called the second differences

Now $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$

$$= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$|||^{ly}$ $\quad \Delta^2 y_1 = y_3 - 2y_2 + y_1$

$$\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r$$

Note : $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

$\therefore \quad \Delta^k y_r = y_{r+k} - {}^k C_1 y_{r+k-1} + {}^k C_2 \; y_{r+k-2} - + ... + (-1)^{\;k} C_r$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_0$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, ...$are called the leading differences.

Ex: The following table gives a set of values of x and the corresponding values ofy = f(x)

x :	**10**	**15**	**20**	**25**	**30**	**35**
y :	**19.97**	**21.51**	**22.47**	**23.52**	**24.65**	**25.89**

Form the difference table and find $\Delta f(10), \Delta^2 f(10), \Delta^3 f(20), \Delta^4 f(15)$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
10	19.97					
		1.54				

15	21.51		-0.58			
		0.96		0.67		
20	22.47		0.09		-0.68	
		1.05		-0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

$\Delta f(10) = 1.54,\ \Delta^2 f(10) = -0.58,\ \Delta^3 f(20) = 0.03,\ \Delta^4 f(15) = 0.04$

Note: The nth differences of a polynomial of n the degree are constant.

Backward difference operator (∇)

Let y = f(x)

We define $\nabla f(x) = f(x) - f(x - h)$

i.e. $\nabla y_1 = y_1 - y_0 = \Delta y_0$

$\nabla y_2 = y_2 - y_1 = \Delta y_1$

$\nabla y_3 = y_3 - y_2 = \Delta y_2$

'

'

$\nabla y_n = y_n - y_{n-1} = \Delta y_{n-1}$

$\therefore \quad \nabla y_r = y_r - y_{r-1} = \Delta y_{r-1}$

Note:

1. $\nabla f(x + h) = f(x + h) - f(x) = \Delta f(x)$
2. $\nabla^2 f(x + 2h) = \nabla(\nabla f(x + 2h))$

$= \nabla \{f(x + 2h) - f(x + h)\}$

$= \nabla f(x + 2h) - \nabla f(x + h)$

$= f(x + 2h) - f(x + h) - f(x + h) + f(x)$

$= f(x + 2h) - 2f(x + h) + f(x)$

$= \Delta^2 f(x)$

|||ly $\nabla^n f(x + nh) = \Delta^n f(x)$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
X_0	y_0			
		∇y_1		
X_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
X_2	y_2		$\nabla^2 y_3$	
		∇y_3		
X_3	y_3			

1. **Form the difference table for**

x	**40**	**50**	**60**	**70**	**80**	**90**
y	**184**	**204**	**226**	**250**	**276**	**304**

and find ∇y (30), $\nabla^2 y$ (70), $\nabla^5 y$ (90)

Soln:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

∇y (80) = 26, $\nabla^2 y$ (70) = 2, $\nabla^5 y$ (90) = 0

2. **Given**

x	**0**	**1**	**2**	**3**	**4**
f(x)	**4**	**12**	**32**	**76**	**156**

Construct the difference table and write the values of $\nabla f(4)$, $\nabla^2 f(4)$, $\nabla^3 f(3)$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	4			
		8		
1	12		12	
		20		**12**
2	32		24	
		44		12
3	76		**36**	
		80		
4	156			

3) Find the missing term from the table:

x	**0**	**1**	**2**	**3**	**4**
y	**1**	**3**	**9**	**-**	**81**

Explain why the value obtained is different by putting x = 3 in 3^x.

Denoting the missing value as a, b, c ..etc. Construct a difference table and solve.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	2			
1	3	6	4		
2	9	a - 9	a - 15	a - 19	-4a + 124
3	a	81 - a	81 - a	-3a +105	
4	81				

Put $\Delta^4 y = 0$ (assuming f(x) its be a polynomial of degree 3)

i.e., $-4a + 124 = 0$

$a = 31$

Since we have assumed f(x) to be a polynomial of degree 3 which is not 3^x we obtained a different value.

4) **Given $u_1 = 8, u_3 = 64, u_5 = 216$ find u_2 and u_4**

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
x_1	8			
x_2	a	a - 8	-2a + 72	b + 3a - 200
x_3	64	64 - a	b + a - 128	-3b - a + 408
x_4	b	b - 64	-2b + 280	
x_5	216	216 -b		

We carryout upto the stage where we get two entries ($\because$ 2 unknowns) and equate each of those entries to zero. (Assuming) to be a polynomial of degree 2.

b + 3a - 200 = 0

-3b - a + 408 = 0 We get a = 24 b = 128

Interpolation:

The word interpolation denotes the method of computing the value of the function y = f(x) for any given value of x when a set $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ are given.

Note:

Since in most of the cases the exact form of the function is not known. In such cases the function f(x) is replaced by a simpler function φ(x) which has the same values as f(x) for $x_0, x_1, x_2 \ldots, x_n$.

$$\varphi(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$+\frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

is called the Newton Gregory forward difference formula

Note :

1. Newton forward interpolation is used to interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table but the formula contains those values of y which come after the value chosen as y_0.

Problems:

1) The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earths surface.

x = height	**100**	**150**	**200**	**250**	**300**	**350**	**400**
y = distance	**10.63**	**13.03**	**15.04**	**16.81**	**18.42**	**19.90**	**21.27**

Find the values of y when i) x = 120, ii) y =218

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

Choose $x_0 = 100$

i) $x = 120,\ u = \dfrac{120-100}{50} = 0.4$

$$f(120) = 10.63 + \frac{0.4}{1!}(2.40) + \frac{(0.4)(0.4-1)}{2!}(-0.39)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0.15)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(-0.07)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!}(0.02)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)(0.4-5)}{6!}(0.02) = 11.649$$

ii) Let $x = 218$, $x_0 = 200$, $u = \dfrac{218-200}{50} = \dfrac{18}{50} = 0.36$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36\,(-0.64)}{2}(-0.16)$$

$$+ \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.7$$

2) **Find the value of f(1.85).**

x	**y**	**Δy**	**$\Delta^2 y$**	**$\Delta^3 y$**	**$\Delta^4 y$**	**$\Delta^5 y$**	**$\Delta^6 y$**
1.7	5.474						
		0.575					
1.8	6.049		0.062				
		0.637		0.004			
1.9	6.686		0.066		0.004		
		0.703		0.008		-0.004	
2.0	7.389		0.074		0		0.004
		0.777		0.008		0	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.090				
		0.949					
23	9.974						

Choose $x_0 = 1.8,\ x = 1.85\ u = \frac{x - x_0}{h} = \frac{1.85 - 1.8}{0.1} = 0.5$

$$f(1.85) = 6.049 + (0.5)\,(0.637) + \frac{(0.5)(-0.5)}{2}(0.066)$$

$$+ \frac{(0.5)(-0.5)\,(-1.5)}{6}\,(0.008)$$

$$= 6.049 + 0.3185 - 0.0008 + 0.0005$$

$$= 6.359$$

3) Given sin 45^o = 0.7071, sin 50^o = 0.7660, sin 55^o =0.8192, sin 60^o = 0.8660. Find sin 48^o.

x	**y**	**Δ**	**Δ²**	**Δ³**
45	0.7071			
		0.589		
50	0.7660		-0.0057	
		0.0532		0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$x = 48,\ x_0 = 45;\ h = 5\ u = \frac{x - x_0}{h} = 0.6$

$$\sin 48^o = 0.7071 + (0.6)\ (0.0589)$$

$$+ \frac{(0.6)(-0.4)}{2}\ (-0.0057)\ \ + \frac{(0.6)\,(-0.4)\ (-1.4)}{6}\ (0.0007) = 0.7431$$

4) From the following data find the number of students who have obtained ≤ 45 marks. Also find the number of students who have scored between 41 and 45 marks.

Marks	**0 - 40**	**41 - 50**	**51 - 60**	**61 -70**	**71 - 80**
No. of students	**31**	**42**	**51**	**35**	**31**

x	y	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$f(45) = 31 + (0.5)\ (42) + \frac{(0.5)\ (-0.5)\ 9}{2} + \frac{(0.5)\ (-0.5)\ (-1.5)(-25)}{3!}$$

$$+ \frac{(0.5)\ (-0.5)\ (-1.5)\ (-2.5)}{4!}\ (37) = 47.8672 \approx 48$$

f(45) - f(40) = 70 = Number of students who have scored between 41 and 45.

5) Find the interpolating polynomial for the following data:

f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 10. Hence evaluate f(0.5)

x	y	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$$u = \frac{x-0}{1} = x$$

$$f(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)\ (x-2)}{3!}\ 6 = x^3 - 2x^2 + 1$$

6) Find the interpolating polynomial for the following data:

x:	**0**	**1**	**2**	**3**	**4**
f(x) :	**3**	**6**	**11**	**18**	**27**

x	y	Δ	Δ^2	Δ^3	Δ^4
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$$u = \frac{x-0}{1} = x$$

$$f(x) = 3 + x(3) + \frac{x(x-1)}{2}(2) + \frac{x(x-1)}{x!}(0) \qquad = 3 + 2x + x^2$$

Newton Gregory Backward Interpolation formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots$$

$$\text{where} \quad u = \frac{x - x_n}{h}$$

1) **The values of tan x are given for values of x in the following table. Estimate tan (0.26)**

x	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**
y	**0.1003**	**0.1511**	**0.2027**	**0.2553**	**0.3093**

x	y	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		

		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

$$u = \frac{0.26 - 0.3}{0.05} = -0.8$$

$$f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)}{2}(0.2)\ (0.0014) + \frac{(-0.8)\ (0.2)\ (1.2)}{6}(0.0004) = 0.2659$$

2) **The deflection d measured at various distances x from one end of a cantilever is given by the following table. Find d when x = 0.95**

$$u = \frac{0.95 - 1}{0.2} = -0.25 \qquad d = 0.3308 \text{ when } x = 0.95$$

x	**d**	∇	∇^2	∇^3	∇^4	∇^5
0	0					
		0.0347				
0.2	0.0347		0.0479			
		0.0826		-0.0318		
0.4	0.1173		0.0161		0.0003	
		0.0987		-0.0321		-0.0003
0.6	0.2160		-0.016		0	
		0.0827		-0.032		
0.8	0.2987		-0.0481			
		0.0346				
1.0	0.3333					

3) **The area y of circles for different diameters x are given below:**

x :	**80**	**85**	**90**	**95**	**100**
y :	**5026**	**5674**	**6362**	**7088**	**7854**

Calculate area when x = 98

x	**y**	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			

85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Answer:

$$u = \frac{x - x_n}{n} = -0.4$$

y = 7542

4) Find the interpolating polynomial which approximates the following data.

x	0	1	2	3	4
y	-5	-10	-9	4	35

x	**y**	∇	∇^2	∇^3	∇^4
0	-5				
		-5			
1	-10		6		
		1		6	
2	-9		12		0
		13		6	
3	4		18		
		31			
4	35				

$$u = \frac{x - 4}{1}$$

$$f(x) = 35 + (x-4)\ (31) + (x-4)\ (x-3)\frac{18}{2!} \ + \frac{(x-4)\ (x-3)\ (x-2)\ (6)}{3!}$$

$$\boxed{f(x) = x^3 + 2x^2 + 6x - 5}$$

Interpolation with unequal intervals

Newton backward and forward interpolation is applicable only when x_0, x_1,…,x_{n-1} are equally spaced. Now we use two interpolation formulae for unequally spaced values of x.

i) Lagranges formula for unequal intervals:

If y = f(x) takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2) + \ldots.$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} f(x_n)$$ is known as the lagrange's interpolation formula

ii) Divided differences (Δ)

$$\Delta f(x_0) = \Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0} = [x_0, x_1]$$

$$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1} = [x_2, x_1]$$

$$\Delta y_{n-1} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = [x_{n-1}, x_n]$$

second divided difference

$$\Delta^2 f(x_0) = \Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}$$

$$= \frac{[x_2, x_1] - [x_1, x_0]}{x_2 - x_0} = [x_0, x_1, x_2]$$

$$\|^{ly} \ \Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \frac{[x_3, x_2] - [x_2, x_1]}{x_3 - x_1} = [x_1, x_2, x_3]$$

similarly $\Delta^3 y_0, \ldots.$ can be defined

Newton's divided difference interpolation formula

$$y = f(x) = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)\,(x - x_1)\,\Delta^2 y_0 + (x - x_0)\,(x - x_1)(x - x_2)\,\Delta^3 y_0$$

$$+ \ldots + (x - x_0)\,(x - x_1)\,\ldots(x - x_n)\,\Delta^n y_0$$

is called the Newton's divided difference formula.

Note: Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients need to be recalculated.

Inverse interpolation: Finding the value of y given the value of x is called interpolation where as finding the value of x for a given y is called inverse interpolation.

Since Lagrange's formula is only a relation between x and y we can obtain the inverse interpolation formula just by interchanging x and y.

$$\therefore x = \frac{(y - y_1)\,(y - y_2)\ldots.(y - y_n)}{(y_0 - y_1)\,(y_0 - y_2)\ldots(y_0 = y_n)}.x_0$$

$$+ \frac{(y - y_0)\,(y - y_0)\,(y - y_2)\,(y - y_3)\ldots(y - y_n)}{(y_1 - y_0)\,(y_1 - y_2)\,(y_1 - y_3)\,\ldots(y_1 - y_n)}\,x_1 + \ldots$$

$$+ \ldots + \frac{(y - y_0)\,(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)\,(y_n - y_1)\ldots(y_n - y_{n-1})}.x_n$$

is the Lagranges formula for inverse interpolation

1) **The following table gives the values of x and y**

x :	**1.2**	**2.1**	**2.8**	**4.1**	**4.9**	**6.2**
y :	**4.2**	**6.8**	**9.8**	**13.4**	**15.5**	**19.6**

Find x when y = 12 using Lagranges inverse interpolation formula.

Using Langrages formula

$$x = \frac{(y - y_1)\,(y - y_2)\,(y - y_3)\,(y - y_4)\,(y - y_5)}{(y_0 - y_1)\,(y_0 - y_2)\,(y_0 - y_3)\,(y_0 - y_4)\,(y_0 - y_5)}\,x_0$$

$$+\dots+\frac{(y-y_0)\,(y-y_1)\,(y-y_2)\,(y-y_3)\,(y-y_4)}{(y_5-y_0)\,(y_5-y_1)\,(y_5-y_2)\dots(y_5-y_4)}\,x_4$$

$$=0.022-0.234+1.252+3.419-0.964+0.055$$

$$=3.55$$

2) **Given the values**

x :	**5**	**7**	**11**	**13**	**17**
f(x) :	**150**	**392**	**1452**	**2366**	**5202**

Evaluate f(9) using (i) Lagrange's formula (ii) Newton's divided difference formula.

i) **Lagranges formula**

$$f(9)=\frac{(9-7)\,(9-11)\,(9-13)\,(9-17)}{(5-7)\,(5-11)\,(5-13)\,(5-17)}(150)+\frac{(9-5)\,(9-11)\,(9-13)\,(9-17)}{(7-5)\,(7-11)\,(7-13)\,(7-17)}.392$$

$$+\frac{(9-5)\,(9-7)\,(9-13)\,(9-17)}{(11-5)\,(11-7)\,(11-13)\,(11-17)}(1452)+\frac{(9-5)\,(9-7)\,(9-11)\,(9-17)}{(13-5)\,(13-7)\,(13-11)\,(13-17)}(2366)$$

$$+\frac{(9-5)\,(9-7)\,(9-11)\,(9-13)}{(17-5)\,(17-7)\,(17-11)\,(17-13)}(5202)=810$$

f(9) = 810

ii)

5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0

		457		1	
13	2366		42		
		709			
17	5202				

f(9) = 150 + 121 (9 - 5) + 24 (9 - 5) (9 - 7) + 1(9 - 5) (9 - 7) (9 - 11) = 810

3) **Using i) Langranges interpolation and ii) divided difference formula. Find the value of y when x = 10.**

x :	**5**	**6**	**9**	**11**
y :	**12**	**13**	**14**	**16**

i) Lagranges formula

$$y = f(10) = \frac{(10-6)\ (10-9)\ (10-11)}{(5-6)\ (5-9)\ (5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{44}{3}$$

ii)Divided difference

x	y	Δ	Δ^2	Δ^3
5	12			
		1		
6	13		$-\frac{2/3}{4} = \frac{-1}{6}$	
		$\frac{1}{3}$		$\frac{\frac{2}{15}+\frac{1}{6}}{11-5} = \frac{\frac{27}{90}}{6} = \frac{3/10}{6} = \frac{1}{20}$
9	14		$\frac{2/3}{5} = \frac{2}{15}$	
		$\frac{2}{2} = 1$		
11	16			

$$f(10) = 12 + (10-5) + (10-5)(10-6).\left(-\frac{1}{6}\right) + (10-5)(10-6)(10-9)\left(\frac{1}{20}\right)$$

$$= \frac{44}{3}$$

4) If y(1) = -3, y(3) = 9, y(4) =30, y(6) = 132 find the lagranges interpolating polynomial that takes the same values as y at the given points.

Given:

x	1	3	4	6
y	-3	9	30	132

$$f(x) = \frac{(x-3)\ (x-4)\ (x-6)}{(1-3)(1-4)(1-6)}\ .\ (-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}.9$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}.30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}.132$$

$$= x^3 - 3x^2 + 5x - 6$$

5) Find the interpolating polynomial using Newton divided difference formula for the following data:

x	**0**	**1**	**2**	**5**
y	**2**	**3**	**12**	**147**

x	y	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1
2	12		9	

		45		
5	147			

F(x) = 2 + (x - 0)(1) + (x - 0) (x - 1) (4) + (x - 0) (x - 1) (x - 2) 1

$= x^3 + x^2 - x + 2$

Numerical Integration:-

To find the value of $I = \int_a^b y\,dx$ numerically given the set of values (x_i, y_i), $i = 0,1,2,...,n$ at regular intervals.

(i) **Simpson's one third rule**:-

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

when n is even.

(ii) **Simpson's three-eighth rule**:-

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1}) + 2(y_3 + y_6 + ... + y_{n-3})\right]$$

when n is a multiple of 3.

(iii) **Weddle's rule**:-

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + ...\right]$$

when n is a multiple of 6.

Problems:

1) Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval $(0, 1)$ into 4 equal sub intervals and hence find the value of π correct to four decimal places.

Solution: Let us divide [0,1] into 4 equal strips (n = 4)

$\therefore$ length of each strip: $h = \frac{1-0}{4} = \frac{1}{4}$

The points of division are $x = 0, \frac{1}{4}, \frac{2}{4} = \frac{1}{2}, \frac{3}{4}, \frac{4}{4} = 1$

By data $y=\frac{1}{1+x^2}$

Now we have the following table

x	0	1/4	1/2	3/4	1
$y=\frac{1}{1+x^2}$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}^{rd}$ rule for n = 4 is given by

$$\int_a^b y\,dx=\frac{h}{3}\left[\left(y_0+y_4\right)+4\left(y_1+y_3\right)+2\left(y_2\right)\right]$$

$$\therefore\int_0^1\frac{1}{1+x^2}dx=\frac{1/4}{3}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{16}{17}+\frac{16}{25}\right)+2.\frac{4}{5}\right]=0.7854$$

$$Thus\quad\int_0^1\frac{1}{1+x^2}dx=0.7854$$

To deduce the value of π: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore\int_0^1\frac{1}{1+x^2}dx=\left[\tan^{-1}x\right]_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}$$

$$We\,must\,have,\ \frac{\pi}{4}=0.7854\Rightarrow\pi=4(0.7854)=3.1416$$

$$Thus\quad\boxed{\pi=3.1416}$$

2) **Given that**

x	**4**	**4.2**	**4.4**	**4.6**	**4.8**	**5**	**5.2**
$\log x$	**1.3863**	**1.4351**	**1.4816**	**1.5261**	**1.5686**	**1.6094**	**1.6487**

Evaluate $\int_4^{5.2}\log x\;dx$ **using Simpson's** $\frac{3}{8}^{th}$ **rule**

Solution: Simpson's $\frac{3}{8}^{th}$ rule for n = 6 is given by

$$\int_a^b ydx = \frac{3h}{8}\left[\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2\left(y_3\right)\right]$$

$$\int_4^{5.2} \log_e xdx = \frac{3(0.2)}{8}\left[\left(1.3863+1.6487\right)+3\left(1.4351+1.4816+1.5686+1.6094\right)+2\left(1.5261\right)\right]$$

$$\int_4^{5.2} \log_e xdx = 1.8279$$

3) Using Weddle's rule evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by taking seven ordinates and hence find $\log_e 2$

Solution: Let us divide [0,1] into 6 equal strips (since seven ordinates)

$\therefore$ length of each strip: $h = \frac{1-0}{6} = \frac{1}{6}$

The points of division are $x = 0, \frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3}, \frac{5}{6}, \frac{6}{6} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/6	1/3	1/2	2/3	5/6	1
$y = \frac{x}{1+x^2}$	0	6/37	3/10	2/5	6/13	30/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Weddle's rule for n = 6 is given by

$$\int_a^b ydx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{3(1/6)}{10}\left[0+5(6/37)+3/10+6(2/5)+6/13+5(30/61)+1/2\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = 0.3466$$

To deduce the value of $\log_e 2$: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log_e(1+x^2)\Big]_0^1 = \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$

$$Hence \ \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log_e 2$$

$$We\,must\,have,\ \frac{1}{2}\log_e 2 = 0.3466 \Rightarrow \log_e 2 = 2(0.3466) = 0.6932$$

$$Thus \quad \boxed{\log_e 2 = 0.6932}$$

UNIT VII

NUMERICAL METHODS -3

CONTENTS:

- **Numerical solution of PDE**
- **Finite difference approximation to derivatives**
- **Numerical solution of**
    - **One dimensional wave equation**

- **One dimensional heat equation**

- **Two dimensional Laplace's equation**

UNIT-VII
NUMERICAL METHODS-3

Classification of PDE s of second order

The general second order linear PDE in two independent variables x,yis of the form
$Au_{xx}+ Bu_{xy}+ Cu_{yy}+ Du_{x}+ Eu_{y}+ Fu=0$
Where A,B,C,D,E,F are in general functions of x,y
This equation is said to be

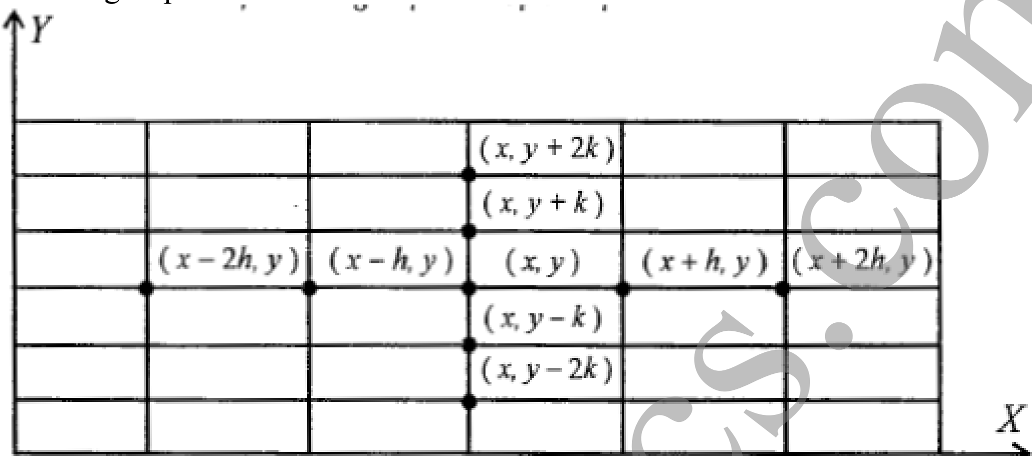
i) Parabolic if $B^2-4AC=0$
ii) Elliptic if $B^2-4AC<0$
iii) Hyperbolic if $B^2-4AC>0$

Now let us examine the nature of three PDE which are under our discussion

	P.D.E	A	B	C	$B^2 4AC=0$	Nature of PDE
1	One dimensional wave equation$c^2u_{xx}-u_{tt}=0$	C^2	0	-1	$4C^2>0$	Hyperbolic
2	One dimensional heat equation$c^2u_{xx}-u_{t}=0$	C^2	0	0	0	parabolic
3	two dimensional Laplace'sequationuxx+$u_{yy}=0$	1	0	C^2	-4<0	elliptic

Numerical solution of a PDE

Consider a rectangular region in the x—y plane. Let us divide this region into a network of rectangular of sides h and k. In other words , we draw lines x=ih,y=jk:I,j=1,2,3…….being parallel to the Y-axis and X-axis respectively resulting into a network of rectangles.the points of intersection of these lines are called mesh points or grid points.



We write u(x,y) =u(ih,jk) and the finite difference approximation for the the partial derivatives are put in the modified form

$$u_x = \frac{1}{h}\left[u_{i+1,j} - u_{i,j}\right]$$

$$u_y = \frac{1}{h}\left[u_{i,j} - u_{i-1,j}\right]$$

$$u_x = \frac{1}{2h}\left[u_{i+1,j} - u_{i-1,j}\right]$$

$$u_y = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right]$$

$$u_y = \frac{1}{k}\left[u_{i,j} - u_{i,j-1}\right]$$

$$u_y = \frac{1}{2k}\left[u_{i,j+1} - u_{i,j-1}\right]$$

$$u_{xx} = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$u_{yy} = \frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right]$$

The substitution of these finite diference approximation into the given PDE converts the PDE into a finite difference equation.

Now we discuss numerical solution of the three important PDE namely

1) One dimensional wave equation
2) One dimensional heat equation

3) Two dimensional laplaceequation

Numerical solution of the one dimensional wave equation

We seek the numerical solution of the wave equation

$c^2 u_{xx} = u_{tt}$................................(1)

Subject to the boundary conditions

u(0,t)=0………………………(2)

u(l,t)=0……………………….(3)

and the initial conditions

u(x,0)=f(x)……………………(4)

u_t(x,0)=0……………………(5)

We shall substitute the finite difference approximation for the partial derivatives present in (1)

$$c^2.\frac{1}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]=\frac{1}{k^2}\left[u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right]$$

$$c^2\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]=\frac{h^2}{k^2}\left[u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right]$$

Taking k/h= λ we have

$$c^2\lambda^2\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]=\left[u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right]$$

$$u_{i,j+1}+2\left(1-c^2\lambda^2\right)u_{i,j}+c^2\lambda^2\left[u_{i-1,j}+u_{i+1,j}\right]-u_{i,j-1}\ \text{………………..(6)}$$

For convenience let us choose λ such that 1- $c^2\lambda^2$ =0

$$1-c^2\left(\frac{k^2}{h^2}\right)=0 \quad or \quad k^2=\frac{k^2}{h^2}\Rightarrow k=\frac{h}{c}$$

Thus (6) reduces to the form

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1}$$

This is called the explicit formula for the solution of wave equation.

Examples

1.Solve the wave equation $\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$ subject to u=(0,t) =0,u(4,t)=0,u_t(x,0)=0 and u(x,0)=x(4-x) by taking h=1,k=0.5 upto four steps

Soln: Below is the initial table wherein the first and last column are zero since (0,t)=0=u(4,t)

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0}=0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}=0$
t_1	0.5	$u_{0,1}=0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}=0$
t_2	1	$u_{0,2}=0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}=0$
t_3	1.5	$u_{0,3}=0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}=0$
t_4	2	$u_{0,4}=0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}=0$

Now consider u(x,0)=x(4-x)

$u_{1,0}=u(1,0)=3; u_{2,0}=u(2,0)=4;\ u_{3,0}=3$

$$u_{i,1}=\frac{1}{2}\left[u_{i-1,0}+u_{i+1,0}\right]$$

$$u_{1,1}=\frac{1}{2}\left[u_{0,0}+u_{2,0}\right]=\frac{1}{2}(0+4)=2$$

Next consider
$$u_{2,1}=\frac{1}{2}\left[u_{1,0}+u_{3,0}\right]=\frac{1}{2}(3+3)=3$$

$$u_{1,1}=\frac{1}{2}\left[u_{0,0}+u_{4,0}\right]=\frac{1}{2}(4+0)=2$$

We now consider the explicit formula to find the remaining values in the table

$\left[u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1}\right]$

$u_{1,2}=u_{0,1}+u_{2,1}-u_{1,0}=0+3-3=0$

$u_{2,2}=u_{1,1}+u_{3,1}-u_{2,0}=2+2-4=0$

$u_{3,2}=u_{2,1}+u_{4,1}-u_{3,0}=3+0-3=0$

The third row is completed

$u_{1,3}=u_{0,2}+u_{2,2}-u_{1,1}=0+0-2=-2$

$u_{2,3}=u_{1,2}+u_{3,2}-u_{2,1}=0+0-3=-3$

$u_{3,3}=u_{2,2}+u_{4,2}-u_{3,1}=0+0-2=-2$

The fourth row is completed

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 3 - 0 = -3$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -2 - 2 - 0 = -4$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -3 + 0 - 0 = -3$$

The last row is completed
Thus the required values 0f $u_{i,j}$ are tabulated

t/x	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	-2	-3	-2	0
2	0	-3	-4	-3	0

2. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ given that u(x,0) =0; u(0,t)=0 u_t(x,0)=0 and u(l,t)=100sin(πt) in the range 0≤t≤1 by taking h=1/4

Soln: Comparing the wave equation $c^2u_{xx} = u_{tt}$ with the equation $u_{xx}= u_{tt}$ we have $c^2=1$ or c=1
By data h=1/4,k=h/c=1/4

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0$	$u_{2,0} = 0$	$u_{3,0} = 0$	$u_{4,0} = 0$
t_1	1/4	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$
t_2	1/2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$
t_3	3/4	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$
t_4	1	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$

We have seen that the condition u_t(x,0)=0 will lead us to the formula

$$u_{i,j} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$$

$$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+0) = 0$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(0+0) = 0$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(0+0) = 0$$

$u_{4,1} = \frac{1}{2}\left[u_{3,0} + u_{5,0}\right]$

Hence

$u_{4,1} = u\left(x_4, t_1\right) = u(1,1/4) = 100\sin(\pi/4) = 0.707$

(Second row of the table is completed)

We shall now conider the explicit formula to find the remaining values in the table

$\left[u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}\right]$

$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0$

$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0$

$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 70.7$

$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0}$ is inapplicable

$u_{4,2} = u\left(x_4, t_2\right) = u(1,1/2) = 100\sin(\pi/2) = 100$

(Third row of the table is completed)

$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0$

$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 70.7$

$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 100$

$u_{4,3} = u\left(x_4, t_3\right) = u(1,3/4) = 100\sin(3\pi/4) = 70.7$

(Fourth row of the table is completed)

$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 70.7$

$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = 100$

$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = 70.7$

$u_{4,4} = u\left(x_4, t_4\right) = u(1,1) = 100\sin(\pi) = 0$

(Last row is completed)

Thus the required values 0f $u_{i,j}$ are tabulated

t/x	0	1/4	1/2	3/4	1
0	0	0	0	0	0
0.25	0	0	0	0	70.7
0.5	0	0	0	70.7	100
.75	0	0	70.7	100	70.7
1	0	70.7	100	70.7	0

Numerical solution of the one dimensional heat equation

We seek the numerical solution of heat equation

$u_t=c^2u_{xx}$...(1)

subject to the boundary conditions

$u(0,t)=0$……………………………………………(2)

$u(l,t)=0$……………………………………………(3)

and the initial condition

$u(x,0)=f(x)$………………………………………(4)

we shall substitute the finite difference approximation for the partial derivatives present in (1)

$$\frac{1}{k}\left[u_{i,j+1}-u_{i,j}\right]=c^2\frac{1}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]$$

$$\left[u_{i,j+1}-u_{i,j}\right]=\frac{kc^2}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]$$

Taking $\frac{kc^2}{h^2}$ =a the above equation becomes

$$\left[u_{i,j+1}\right]=\left[u_{i,j}+au_{i-1,j}-2au_{i,j}+au_{i+1,j}\right]$$

$$\left[u_{i,j+1}\right]=\left[au_{i-1,j}+(1-2a)u_{i,j}+au_{i+1,j}\right]\text{……………………(5)}$$

This is called Schmidt explicit formula valid for $0<a<1/2$

For convenience let us set $1-2a=0$ or $a=1/2$

That is $\frac{kc^2}{h^2}=1/2$

Thus (5) becomes $\left[u_{i,j+1}\right]=\frac{1}{2}\left[u_{i-1,j}+u_{i+1,j}\right]$

This is called Bendre Schmidt formula .

EXAMPLES

1) Find the numerical solution of the parabolic equation $\frac{\partial^2 u}{\partial x^2}=2\frac{\partial u}{\partial t}$

When u(0,t)=0=u(4,t) and u(x,00=x94-x0 by taking h=1

Soln: the standard form of the one dimensional heat equation is $u_t=c^2u_{xx}$ and the given equation can be put in the form $u_t=(1/2)u_{xx}$

$c^2=1/2$ since h=1,k=$\frac{h^2}{2c^2}=1$

the values of x=0 in 0<x<4 with h=1 are 0,1,2,3,4 and the values of t with k=1 are 0,1,2,3,4,5

The initial table is given by

Consider the initial condition u(x,0)=x(4-x)

$u_{1,0}=u(1,0)=3$; $u_{2,0}=4$; $u_{3,0}=3$

(First row in the table is completed)

Now we consider the relation

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0}=0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}=0$
t_1	1	$u_{0,1}=0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}=0$
t_2	2	$u_{0,2}=0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}=0$
t_3	3	$u_{0,3}=0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}=0$
t_4	4	$u_{0,4}=0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}=0$
t_5	5	$u_{0,5}=0$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5}=0$

In particular

$$u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$$

$$u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}(0+4) = 2$$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}(3+3) = 3$$

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}(4+0) = 2$$

(The second row in the table is completed)

Again $u_{i,2} = \frac{1}{2}\left[u_{i-1,1} + u_{i+1,1}\right]$

$$u_{1,2} = \frac{1}{2}\left[u_{0,1} + u_{2,1}\right] = \frac{1}{2}(0+3) = 1.5$$

$$u_{2,2} = \frac{1}{2}\left[u_{1,1} + u_{3,1}\right] = \frac{1}{2}(2+2) = 2$$

$$u_{3,2} = \frac{1}{2}\left[u_{2,1} + u_{4,1}\right] = \frac{1}{2}(3+0) = 1.5$$

(The third row in the table is completed)

Again $u_{i,3} = \frac{1}{2}\left[u_{i-1,3} + u_{i+1,2}\right]$

$$u_{1,3} = \frac{1}{2}\left[u_{0,2} + u_{2,2}\right] = \frac{1}{2}(0+2) = 1$$

$$u_{2,3} = \frac{1}{2}\left[u_{1,2} + u_{3,2}\right] = \frac{1}{2}(1.5+1.5) = 1.5$$

$$u_{3,3} = \frac{1}{2}\left[u_{2,2} + u_{4,2}\right] = \frac{1}{2}(2+0) = 1$$

(Fourth row in the table is completed)

Again $u_{i,4} = \frac{1}{2}\left[u_{i-1,3} + u_{i+1,3}\right]$

$$u_{1,4} = \frac{1}{2}\left[u_{0,3} + u_{2,3}\right] = \frac{1}{2}(0+1.5) = 0.75$$

$$u_{2,4} = \frac{1}{2}\left[u_{1,3} + u_{3,3}\right] = \frac{1}{2}(1+1) = 1$$

$$u_{3,4} = \frac{1}{2}\left[u_{2,3} + u_{4,3}\right] = \frac{1}{2}(1.5+0) = 0.75$$

(Fifth row in the table is completed)

Again $u_{i,5} = \frac{1}{2}\left[u_{i-1,4} + u_{i+1,4}\right]$

$$u_{1,5} = \frac{1}{2}\left[u_{0,4} + u_{2,4}\right] = \frac{1}{2}(0+1) = 0.5$$

$$u_{2,5} = \frac{1}{2}\left[u_{1,4} + u_{3,4}\right] = \frac{1}{2}(0.75+0.75) = 0.75$$

$$u_{3,5} = \frac{1}{2}\left[u_{2,4} + u_{4,4}\right] = \frac{1}{2}(1+0) = 0.5$$

He last row in the table is completed)

Thus the required values 0f $u_{i,j}$ are tabulated

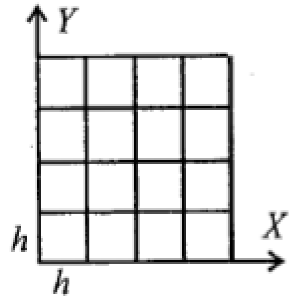
t/x	0	1	2	3	4
0	0	3	4	3	0
1	0	2	03	2	0
2	0	1.5	02	1.5	0
3	0	1	701.5.7	1	0
4	0	0.75	1001	0.75	0
5	0	0.5	0.75	0.5	0

Numerical solution of the Laplace's equation in two dimensions

Laplace's equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Here we consider a rectangular region R for which u(x,y) is know at the boundary. Let us suppose that the region is such that it can divided into a network of square mesh of side h.



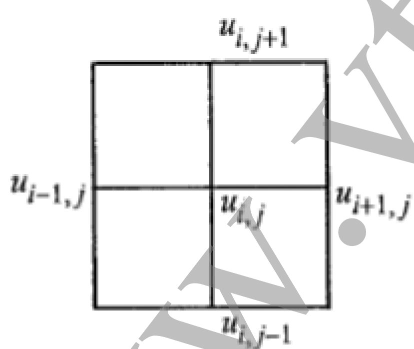
We shall substitute the finite difference approximation for the partial derivatives ,hence we have

$$\frac{1}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]+\frac{1}{h^2}\left[u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right]=0$$

$$\left[u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}\right]=4u_{i,j}$$

$$u_{i,j}=\frac{1}{4}\left[\left[u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}\right]\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This is called the standard five point formula. It may be observed that the value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighboring points which is given in the figure below

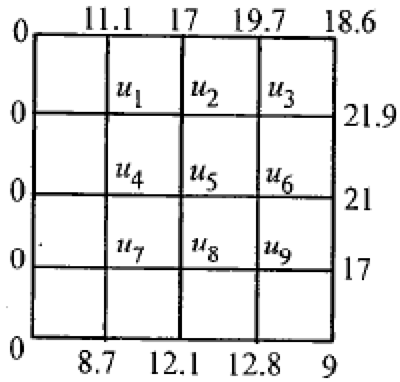


$$u_{i,j}=\frac{1}{4}\left[\left[u_{i-1,j\_+1}+u_{i+1,j-1}+u_{i+1,j+1}+u_{i-1,j-1}\right]\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (30 is called diagonal five point formula.

Examples

1.solve Laplace's $u_{xx}+u_{yy}=0$ for the following square mesh with boundary values as shown below



u_5 is located at the centre of the region.

$$u_5 = \frac{1}{4}(0+21+17+12.1) = 12.525 \, by \, S.F$$

Next we shall compute u_7, u_9, u_1, $u_3$by applying D.F

$$u_7 = \frac{1}{4}(0+12.525+0+12.1) = 6.15625$$

$$u_9 = \frac{1}{4}(12.1+21+12.525+9) = 13.65625$$

$$u_1 = \frac{1}{4}(0+17+0+12.525) = 7.38125$$

$$u_3 = \frac{1}{4}(12.525+18.6+17+21) = 17.28125$$

Finally we shall compute u_2, u_4, u_6, u_8 by applying S.F

$$u_2 = \frac{1}{4}(7.38125+17.28125+17+12.525) = 13.546875$$

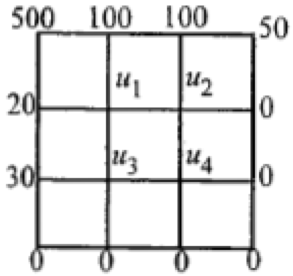
$$u_4 = \frac{1}{4}(0+12.525+7.38125+6.15625) = 6.515625$$

$$u_6 = \frac{1}{4}(12.525+21+17.28125+13.65625) = 16.115625$$

$$u_8 = \frac{1}{4}(6.15625+13.65625+12.525+12.1) = 11.109375$$

Thus, u_1=7.38, u_2=13.55 , u_3=17.28, u_4=6.25, u_5=12.53, u_6=16.12, u_7=6.16, u_8=11.11, u_9=13.66,

2.Solve $u_{xx}+u_{yy}=0$ in the following square region with the boundary conditions as indicated in the figure below



Soln: we shall apply standard five point formula for u_1, u_2, u_3, u_4, to obtain a system of equations

$$u_1 = \frac{1}{4}\left(20 + u_2 + 100 + u_3\right)$$

$$u_2 = \frac{1}{4}\left(u_1 + 0 + 100 + u_4\right)$$

$$u_3 = \frac{1}{4}\left(30 + u_4 + u_1 + 0\right)$$

$$u_4 = \frac{1}{4}\left(u_3 + 0 + u_2 + 0\right)$$

Now we have a system of equations to be solved

$4u_1 - u_2 - u_3 = 120$ ……………………………(1)

$-u_1 + 4u_2 - u_4 = 100$ ………………………….(2)

$-u_1 + 4u_3 - u_4 = 30$ ……………………………..(3)

$-u_2 - u_3 + 4u_4 = 0$ ……………………………..(4)

Let us eliminate u_1 from (1) and (2);(20 and (3)

That is 15u_2-u_3-4u_4=520………………………..(5)

4u_2-4u_3=70 …………..(6)

we shall now eliminate u_4 from (4) and (5)

That is 14u_2-2u_3=520

Let us solve (6) and (7) : 2u_2-2u_3=35

14u_2-2u_3=520

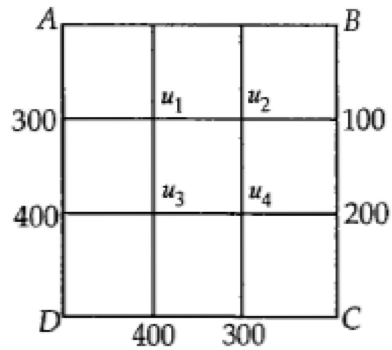
Thus u_2=40.4, and u_3=22.9 hence from (1)

u_1=45.825 and from(4) u_4=15.825

Thus the required values at th2 interior mesh points are

u_1=45.825 , u_2=40.4, u_3=22.9 u_4=15.825

3. solve the elliptic equation u_{xx}+u_{yy}=0 at the pivotal points for the following square mesh



Soln: it is evident that the values at the boundary points on AB are respectively 200 and 100

Thus $u_1 = \frac{1}{4}(300 + u_2 + 200 + u_3)$

$u_2 = \frac{1}{4}(u_1 + 100 + 100 + u_4)$

$u_3 = \frac{1}{4}(400 + u_4 + u_1 + 400)$

$u_4 = \frac{1}{4}(u_3 + 200 + u_2 + 300)$

$4u_1 - u_2 - u_3 = 500$ ……………………………(1)

$-u_1 + 4u_2 - u_4 = 200$ …………………………(2)

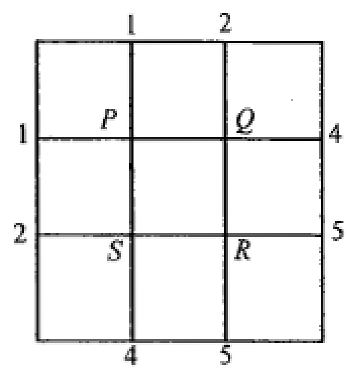
$-u_1 + 4u_3 - u_4 = 800$ ……………………………..(3)

$-u_2 - u_3 + 4u_4 = 500$ ……………………………..(4)

Thus by solving simultaneously we obtain the values at the pivotal points

u_1=250 , u_2=175, u_3=325 u_4=250

4. Solve the elliptic partial differential equation for the following square mesh using the 5 point difference formula by setting up the linear equations at the unknown points P,Q,R,S



The linear equations for the unknownfor the 5 point difference formula are

Thus $P = \frac{1}{4}(1 + Q + 1 + S)$

$$Q=\frac{1}{4}(P+4+R+2)$$

$$R=\frac{1}{4}(S+5+5+Q)$$

$$S=\frac{1}{4}(2+R+4+P)$$

4P-Q-S=2……………………………(1)

-P+4Q-R=6…………………………(2)

-Q+4R-S=10………………………(3)

-P-R+4S=6…………………………(4)

Thus by solving we get P=2,Q=3,S=3,R=4

UNIT VIII

DIFFERENCE EQUATIONS AND Z- TRANSFORMS

CONTENTS:

- **Introduction**
- **Z-Transforms –Definition**

- **Damping Rule**
- **Shifting RuleTheorem**
- **Inverse Z-Transforms**
- **Power Series Method**
- **Applications Of Z-Transforms To Solve Difference Equations**

DIFFERENCE EQUATIONS AND Z- TRANSFORMS

Introduction

The Z-transform plays an important role plays in the study of communications, sample data control systems, discrete signal processing , solutions of difference equations etc.

Definition

Let $u_n = f(n)$ be a real-valued function defined for n=0,1,2,3,….. and $u_n = 0$ for n<0. Then the Z-transform of u_n denoted by $Z(u_n)$ is defined by

$$\bar{u}(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} \qquad (1)$$

The transform also is referred to as the one sided Z-transform or unilateral Z-transform. Next, we define $u_n = f(n)$ for n=0, ±1, ±2, …… ±∞.

The two-sided Z-transform is defined by

$$Z(u_n) = \sum_{-\infty}^{\infty} u_n z^{-n} \qquad (2)$$

The region of the Z-plane in which the series (1) or (2) converges is called the region of convergence of the transform.

Properties of Z-transform

1. Linearity property

Consider the sequences $\{u_n\}$ and $\{v_n\}$ and constants a and b. Then

$$Z[au_n + bv_n] = aZ(u_n) + bZ(v_n)$$

Proof : By definition, we have

$$Z[au_n + bv_n] = \sum_{n=0}^{\infty} [au_n + bv_n] z^{-n}$$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n)$$

In particular, for a=b=1, we get

$$Z[u_n+v_n] = Z(u_n) + Z(v_n)$$

and for a=-b=1, we get

$$Z[u_n - v_n] = Z(u_n) - Z(v_n)$$

2. Damping property

Let $Z(u_n) = \bar{u}(z)$. Then (i) $Z(a^n u_n) = \bar{u}\left(z/a\right)$ (ii) $Z(a^{-n} u_n) = \bar{u}(az)$

Proof : By definition, we have

$$Z(a^n u_n) = \sum_{n=0}^{\infty}(a^n u_n)z^{-n} = \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n}$$

$$= \bar{u}\left(\frac{z}{a}\right)$$

Thus

$$Z(a^n u_n) = \bar{u}\left(\frac{z}{a}\right)$$

This is the result as desired. Here, we note that that if Z(u_n) = $\bar{u}(z)$, then

$$Z(a^n u_n) = [\bar{u}(z)]_{Z \to Z/a} = \bar{u}\left(\frac{z}{a}\right)$$

Next,

$$Z(a^{-n} u_n) = \sum_{n=0}^{\infty}(a^{-n} u_n)z^{-n} = \sum_{n=0}^{\infty} u_n (az)^{-n}$$

$$= \bar{u}(az)$$

Thus

$$Z(a^{-n} u_n) = \bar{u}(az)$$

This is the result as desired.

3. Shifting property

(a) Right shifting rule :

If Z(u_n) = $\bar{u}(z)$, then Z(u_{n-k}) = $z^{-k}\,\bar{u}(z)$ where k>0

Proof : By definition, we have

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$$

Since u_n = 0 for n<0, we have u_{n-k} = 0 for n=0,1,……(k-1)

Hence

$$Z(u_{n-k}) = \sum_{n=k}^{\infty} u_{n-k} z^{-n}$$

$$= u_0 z^{-k} + u_1 z^{-(k+1)} + ...\infty$$

$$= z^{-k}[u_0 + u_1 z^{-1} + ... + \infty]$$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= z^{-k}\bar{u}(z)$$

Thus

$$Z(u_{n-k}) = z^{-k}\,\overline{u}(z)$$

(b)Left shifting rule :

$$Z(u_{n+k}) = z^k[\overline{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - ...... - u_{k-1} z^{-(k-1)}]$$

Proof :

$$Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$$

$$= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} = z^k \left\{ \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} \right\}, where \quad m = n+k$$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

$$= z^k[\overline{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - ...... - u_{k-1} z^{-(k-1)}]$$

Particular cases :

In particular, we have the following standard results :

1. $Z(u_{n+1}) = z[\overline{u}(z) - u_0]$
2. $Z(u_{n+2}) = z^2[\overline{u}(z) - u_0 - u_1 z^{-1}]$
3. $Z(u_{n+3}) = z^3[\overline{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$ etc.

Some Standard Z-Transforms :

1.Transform of a^n

By definition, we have

$$Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + ... + \infty$$

The series on the RHS is a Geometric series. Sum to infinity of the series is $\frac{1}{1-a/z}$ or $\frac{z}{z-a}$ Thus, $Z(a^n) = \frac{z}{z-a}$

In particular, when a=1, we get Z(1) = $\frac{z}{z-1}$

2. Transform of e^{an}

Here

$$Z(e^{an}) = Z(k^n) \qquad \text{where } k = e^a$$
$$= \frac{z}{z-k} = \frac{z}{z-e^a}$$

Thus

$$Z(e^{an}) = \frac{z}{z-e^a}$$

3. Transform of n^p , p being a positive integer

We have,

$$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n}$$
$$= z\sum_{n=0}^{\infty} n^{p-1} z^{-(n+1)} n \qquad (1)$$

Also, we have by defintion

$$Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

Differentiating with respect to z, we get

$$\frac{d}{dz}\left[Z(n^{p-1})\right] = \frac{d}{dz}\sum_{n=0}^{\infty} n^{p-1} z^{-n}$$
$$= \sum_{n=0}^{\infty} n^{p-1}(-n) z^{-(n+1)}$$

Using this in (1), we get

$$Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$$

Particular cases of $Z(n^p)$:-

1. For p = 1, we get

$$Z(n) = -z\frac{d}{dz}\left[Z(1)\right] = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

Thus,

$$Z(n) = \frac{z}{(z-1)^2}$$

2. For p = 2, we get

$$Z(n^2) = -z\frac{d}{dz}\left[Z(n)\right] = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z^2+z}{(z-1)^3}$$

Thus,

$$Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

3. For p = 3, we get

$$Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

4. Transform of na^n

By damping property, we have

$$Z(na^n) = \left[Z(n)\right]_{Z\to Z/a} = \left[\frac{z}{(z-1)^2}\right]_{Z\to Z/a}$$

$$= \frac{z/a}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

, in view of damping property

Thus,

$$Z(na^n) = \frac{az}{(z-a)^2}$$

5. Transform of n^2a^n

We have,

$$Z(n^2a^n) = \left[Z(n^2)\right]_{Z\to \frac{Z}{a}} = \left[\frac{z^2+z}{(z-1)^3}\right]_{Z\to \frac{Z}{a}}$$

Thus,

$$Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$$

6. Transforms of coshnθ and sinhnθ

We have

$$\cosh n\theta = \frac{e^{n\theta} + e^{-n\theta}}{2}$$

$$Z(\cosh n\theta) = \frac{1}{2} Z(e^{n\theta} + e^{-n\theta})$$

$$= \frac{1}{2}\left[Z(e^{n\theta}) + Z(e^{-n\theta})\right], \text{ by using the linearity property}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}+z-e^{\theta}}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right] = z\left[\frac{z - \dfrac{(e^{\theta}+e^{-\theta})}{2}}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right]$$

$$= \frac{z[z-\cosh\theta]}{z^2 - 2z\cosh\theta + 1}$$

Next,

$$\sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2}$$

$$Z(\sinh n\theta) = \frac{z}{2}\left[\frac{1}{z-e^{\theta}} - \frac{1}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^2 - 2z\cosh\theta + 1}\right]$$

$$= \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

7. Transforms of cosnθ and sinnθ

We have

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

$$Z(\cos n\theta) = \frac{1}{2} Z(e^{in\theta} + e^{-in\theta})$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\theta}} + \frac{z}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{in\theta}+e^{-in\theta})}{z^2 - z(e^{in\theta}+e^{-in\theta})+1}\right]$$

$$= \frac{z[z-\cos\theta]}{z^2-2z\cos\theta+1}$$

Next,

$$\sin n\theta = \frac{e^{in\theta}-e^{-in\theta}}{2i}$$

$$Z(\sin n\theta) = \frac{1}{2i}\left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2i}\left[\frac{e^{i\theta}-e^{-i\theta}}{z^2-2z\cos\theta+1}\right]$$

$$= \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Examples :

Find the Z-transforms of the following :

1. $$u_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n$$

We have,

$$Z(u_n) = Z\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right]$$

$$= Z\left(\frac{1}{2}\right)^n + Z\left(\frac{1}{4}\right)^n$$

$$= \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{4}} = \frac{2z}{2z-1} + \frac{4z}{4z-1}$$

$$= \frac{2z(8z-3)}{(2z-1)(4z-1)}$$

2. $$u_n = \frac{1}{n!}$$

Here

$$Z(u_n) = Z\frac{1}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!},$$

$$= 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + ... = e^{\frac{1}{z}}$$, by exponential theorem

3. $u_n = a^n \cos n\theta$

We have

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

By using the damping rule, we get

$$Z(a^n \cos n\theta) = \left[\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{Z \to Z/a}$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{z(z - a\cos\theta)}{z^2 - 2az + a^2}$$

4. $u_n = \frac{1}{(n+2)!}$

Let us denote $v_n = \frac{1}{n!}$, so that $v_{n+2} = \frac{1}{(n+2)!} = u_n$

Here

$$Z(v_n) = e^{\frac{1}{z}}$$

Hence

$$Z(u_n) = Z(v_{n+2}) = z^2\left[Z(v_n) - v_0 - \frac{v_1}{z}\right],$$ by left shifting rule

$$= z^2\left[e^{\frac{1}{z}} - \frac{1}{0!} - \frac{1}{z}.\frac{1}{1!}\right]$$

$$= z^2\left[e^{\frac{1}{z}} - 1 - \frac{1}{z}\right]$$

List of standard inverse Z-transforms

1. $Z_T^{-1}\left[\dfrac{z}{z-1}\right]=1$

2. $Z_T^{-1}\left[\dfrac{z}{z-k}\right]=k^n$

3. $Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]=n$

4. $Z_T^{-1}\left[\dfrac{kz}{(z-k)^2}\right]=k^n n$

5. $Z_T^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right]=n^2$

6. $Z_T^{-1}\left[\dfrac{kz^2+k^2z}{(z-k)^3}\right]=k^n n^2$

7. $Z_T^{-1}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right]=n^3$

8. $Z_T^{-1}\left[\dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right]=k^n n^3$

9. $Z_T^{-1}\left[\dfrac{z}{z^2+1}\right]=\sin(n\pi/2)$

10. $Z_T^{-1}\left[\dfrac{z^2}{z^2+1}\right]=\cos(n\pi/2)$

Evaluation of inverse Z-transforms:

We obtain the inverse Z-transforms using any of the following three methods:

(I) Power series Method:- This is the simplest of all the method of finding the inverse Z-transform. If U (z) is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Problems:

(1)Find the inverse transform of $\log\left[z/_{z+1}\right]$ **by power series method.**

Putting

$$z = 1/t\,,\quad U(z)=\log\left(\frac{1/y}{1/y+1}\right)=-\log(1+y)=-y+\frac{1}{2}y^2-\frac{1}{3}y^3+\ldots$$

$$=-z^{-1}+\frac{1}{2}z^{-2}-\frac{1}{3}z^{-3}+\ldots$$

$$i.e, U(z)=\sum_{n=1}^{\infty}u_n z^{-n}$$

$$Thus\quad u_n=\begin{cases}0 & for\ n=0\\ (-1)^n/n & otherwise\end{cases}$$

(2) Find the inverse Z-transform of $z/(z+1)^2$ **by division method**.

$$U(z) = \frac{z}{z^2 + 2z + 1}$$
$$= \frac{z}{z^2(1 + 2/z + 1/z^2)}$$
$$= \frac{z}{z^2(1 + 1/z)^2}$$
$$= \frac{1}{z}(1 + 1/z)^{-2}$$
$$= \frac{1}{z}\left[1 - 2(1/z) + 3(1/z^2) - \tfrac{1}{4}(1/z^3) + \cdots\right]$$
$$\therefore U(z) = \frac{1}{z} - 2\frac{1}{z^2} + 3\frac{1}{z^3} - 4\frac{1}{z^4} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} nZ^{-n}$$

Thus $u_n = (-1)^{n-1} n$

<u>(II) Partial fractions Method:-</u> This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

<u>Problems:</u>

(1) $\dfrac{2z^2 + 3z}{(z+2)(z-4)}$

We write $U(z) = \dfrac{2z^2 + 3z}{(z+2)(z-4)}$ as

$$\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4} \quad where\ A = 1/6 \quad and \quad B = 11/6$$

Therefore

$$U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^{-n} + \frac{11}{6}(4)^n \qquad \left[\because z^{-1}\left(\tfrac{z}{z-k}\right) = k^n\right]$$

(2) $\dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

We write $U(z) = \dfrac{z^3 - 20z}{(z-2)^3(z-4)}$ as

$$\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A + Bz + Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Multiplying throughout by $(z-2)^3(z-4)$

we get $z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3$.

Comparing the coefficients of z in the above equation
we get A = 6, B=0, C = ½ and D = -½ .
Thus

$$\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{6 + 0z + \frac{1}{2}z^2}{(z-2)^3} - \frac{\frac{1}{2}}{z-4}$$

$$\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{6 + 0z + \frac{1}{2}z^2}{(z-2)^3} - \frac{\frac{1}{2}}{z-4}$$

$$U(z) = \frac{1}{2}.\left\{\frac{12z + z^3}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right) = \frac{1}{2}\left\{\frac{12z + 4z^2 - 4z^2 + z^3}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right)$$

$$= \frac{1}{2}\left\{\frac{(8z + 4z) + 4z^2 - 4z^2 + z^3}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right) = \frac{1}{2}\left\{\frac{(z^3 - 4z^2 + 4z) + 8z + 4z^2}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right)$$

$$= \frac{1}{2}\left\{\frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right) \quad = \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2 + 4z}{(z-2)^3}\right\} - \frac{1}{2}\left(\frac{z}{z-4}\right)$$

On inversion, we get $u_n = \dfrac{1}{2}\left[(2^n + 2.n^2 2^n) - 4^n\right]$

$$= 2^{n-1} + n^2 2^n - 2^{2n-1}$$

(3)Find the inverse Z-transform of $\dfrac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2}$, *for* $2 < |z| < 3$

Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$=\frac{1}{2}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{1}{9}\left(1-\frac{z}{3}\right)^{-2} \quad \text{so that } 2/z<1 \text{ and } z/3<1$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)+\frac{1}{9}\left(1+\frac{2z}{3}+\frac{3z^2}{9}+\frac{4z^3}{27}+\cdots\right) \quad \text{where } 2<|z|<3$$

$$=\left(\frac{1}{2}+\frac{2}{z^2}+\frac{2^2}{z^3}+\frac{2^3}{z^4}+\cdots\right)-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\frac{z^3}{3^4}+\cdots\right)+\left(\frac{1}{3^2}+\frac{2z}{3^3}+\frac{3z^2}{3^4}+\frac{4z^3}{3^5}+\cdots\right)$$

$$=\sum_{n=1}^{\infty}2^{n-1}z^{-n}-\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^{n-1}z^n+\sum_{n=0}^{\infty}(n+1)\left(\tfrac{1}{3}\right)^{n+2}z^n$$

on inversion, we get $u_n = 2^{n-1},\ n\ge 1$ *and* $u_n = -(n+2)3^{n-2},\ n\le 0.$

Difference Equations

INTRODUCTION

Difference equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such difference equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of difference equations grew out of differential calculus to become one of the must powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

Definition

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Eg: 1) $\Delta y_{n+1} + y_n = 2$

2) $\Delta^2 y_n + 5\Delta y_n + 6y_n = 0$

3) $\Delta^3 y_n - 3\Delta^2 y_n + 2\Delta y_n + y_n = x^2$ are difference equations.

An alternative way of writing a difference equation is as follows:

Putting $\Delta = E - 1$, we get

(1) may be written as,

$$(E-1)y_{n+1} + y_n = 2$$

$$Ey_{n+1} - y_{n+1} + y_n = 2 \quad [\text{since } E^r y_n = y_{n+r}]$$

$$y_{n+2} - y_{n+1} + y_n = 2 \quad ------------- (4)$$

(2) may be written as,

$$(E-1)^2 y_n + 5(E-1)y_n + 6y_n = 0$$
$$(E^2-2E+1)y_n + (5E-5)y_n + 6y_n = 0$$
$$y_{n+2} - 2y_{n+1} + y_n + 5y_{n+1} + y_n = 0$$
$$y_{n+2} + 3y_{n+1} + 2y_n = 0 \quad ----------(5)$$

(3) may be written as

$$(E-1)^3 y_n - 3(E-1)^2 y_n + 2(E-1)y_n + y_n = x^2$$
$$(E^3-3E^2+3E-1)y_n - 3(E^2-2E+1)y_n + 2(E-1)y_n + y_n = x^2$$
$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n - 3y_{n+2} + 6y_{n+1} - 3y_n + 2y_{n+1} - 2y_n + y_n = x^2$$
$$y_{n+3} - 6y_{n+2} + 11y_{n+1} - 5y_n = x^2 \quad --------- \quad (6)$$

The equations (4), (5) and (6) can also be written in terms of the operator E. i.e,

$$(E^2 - E + 1)y_n = 2$$
$$(E^2 + 3E + 2)y_n = 0$$
$$(E^3 - 6E^2 + 11E - 5)y_n = x^2$$

Order of a difference equation:

The order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

$$i.e \quad \text{order of a difference.eq} = \frac{\text{largest argument - smallest argument}}{\text{unit of increment (interval)}}$$

Note: *To find the order, the equation must be expressed in a form free of $\Delta's$. [because the highest power of Δ does not give the order of difference equation]*

therefore the order of the difference equation

(4) is = $\dfrac{(n+2)-(n)}{1} = 2$

(5) is = $\dfrac{(n+2)-(n)}{1} = 2$

(6) is = $\dfrac{(n+3)-(n)}{1} = 3$

The order of a difference equation can also be obtained by considering the highest power of the operator E involved in the equation.

Solution of a difference Equation is an expression for $y_{(n)}$ which satisfies the given difference equation.

The **general solution** of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A **particular solution** (or particular integral) is that solution which is obtained from the general solution by giving particular values to the constants.

APPLICATION OF Z-TRANSFORM TO SOLVE DIFFERENCE EQUATIONS

Working Procedure to solve a linear difference equation with constant coefficient by Z-transforms:

1) Take the Z-transform of both sides of the difference equations using the Z-transforms formulae and the given conditions.
2) Transpose all terms without $U(z)$ to the right.
3) Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z.
4) Express this function in terms of Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.

Problems:

(1) Using the Z-transform, solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ *with* $u_0 = 0, u_1 = 1$

We note that If $Z(u_n) = U(z)$, *then* $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - u_1 z^{-1}\right]$$

Also $Z(2^n) = z/(z-2)$

$\therefore$ Taking the Z-transforms of both sides of the above difference equation, we get

$$z^2\left[U(z) - u_0 - u_1 z^{-1}\right] + 4z\left[U(z) - u_0\right] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$= \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions, then

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right)$$

$$= \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n$$

(2) Solve $y_{n+2}+6y_{n+1}+9y_n=2^n \quad with\ y_0=y_1=0,$ **using Z-transform.**

If $Z(y_n)=Y(z),\ then\ Z(y_{n+1})=z[Y(z)-y_0],\ Z(y_{n+2})=z^2\left[Y(z)-y_0-y_1z^{-1}\right]$

Also $Z(2^n)=z/(z-2)$

Taking Z-transforms of both sides, we get

$$z^2\left[Y(z)-y_0-y_1z^{-1}\right]+6z[Y(z)-y_0]+9Y(z)=z/(z-2)$$

Since $y_0=0,\quad and \quad y_1=0,\ we\ have\ Y(z)(z^2+6z+9)=z/(z-2)$

Or $\dfrac{Y(z)}{z}=\dfrac{1}{(z-2)(z+3)^2}=\dfrac{1}{25}\left[\dfrac{1}{z-2}-\dfrac{1}{z+3}-\dfrac{5}{(z+3)^2}\right]$, on splitting into partial fractions.

Or $Y(z)=\dfrac{1}{25}\left[\dfrac{z}{z-2}-\dfrac{z}{z+3}-5\dfrac{z}{(z+3)^2}\right]$

On taking inverse Z-transform of both sides, we obtain

$$y_n=\frac{1}{25}\left[Z^{-1}\left(\tfrac{z}{z-2}\right)-Z^{-1}\left(\tfrac{z}{z+3}\right)+\tfrac{5}{3}Z^{-1}\left(-\tfrac{3z}{(z+3)^2}\right)\right]$$

$$=\frac{1}{25}\left\{2^n-(-3)^n+\tfrac{5}{3}n(-3)^n\right\} \qquad \left[\because\ Z^{-1}\left\{\frac{az}{(z-a)^2}\right\}=na^n\right]$$