VE401 Probabilistic Methods in Eng.

RC#3 Multi-variate Random Variable

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if you want to edit this note, you can find it here https://github.com/joydddd/VE401-2020SP-notes

Discrete Multivariate random variable

joint density function

of the random variable $\boldsymbol{X}=(X_1,\ldots,X_n)$

$$\begin{aligned} \text{Discrete:} \quad & f_{\boldsymbol{X}}(x_1,\ldots,x_n)=P[X_1=x_1 \text{ and } X_2=x_2 \ldots \text{ and } X_n=x_n] \\ \text{Continuous:} \quad & P[X\in\Omega]=\int_\Omega f_X(x)dx \end{aligned}$$

has properties:

(i) $f_X(x)\geq 0$

(ii)discrete: $\sum_{x\in\Omega} f_X(x)=1$. continuous: $\int_{\mathbb{R}^n} f_X(x)dx=1$

marginal density

f_{X_k} for $X_k,\ k=1,\ldots,n$

$$\begin{aligned} \text{Discrete:} \quad & f_{X_k}(x_k)=\sum_{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n} f_X(x_1,\ldots,x_n) \\ \text{Continuous:} \quad & f_{X_k}(x_k)=\int_{\mathbb{R}^{n-1}} f_X(x)dx_1\ldots dx_{k-1}dx_{k+1}\ldots dx_n \end{aligned}$$

Bivariate Random Variable

Let $((X,Y),f_{XY})$ be a bivariate random variable with marginal densities f_X and f_Y.

Independent

$$\begin{gathered} \operatorname{dom} f_{XY}=(\operatorname{dom} f_X)\times(\operatorname{dom} f_Y) \\ f_{XY}(x,y)=f_X(x)f_Y(y) \quad \text{for all } (x,y)\in\operatorname{dom} f_{XY} \end{gathered}$$

conditional density

for X given $Y=y$ is

$$f_{X|y}(x)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

Expectations

Let $H:\Omega\to\mathbb{R}$ be some function. Then the expected value of $H\circ(X,Y)$ is

$$\begin{aligned} \text{Discrete:} \quad & \mathrm{E}[H\circ(X,Y)]=\sum_{(x,y)\in\Omega} H(x,y)\cdot f_{XY}(x,y) \\ \text{Continuous:} \quad & \mathrm{E}[H\circ(X,Y)]=\iint_{\mathbb{R}^2} H(x,y)\cdot f_{XY}(x,y)dxdy \end{aligned}$$

specially consider $H(x,y)=x$ and $H(x,y)=y$, giving

$$\text{Discrete:}$$
$$\mathrm{E}[X]=\sum_{(x,y)\in\Omega} x\cdot f_{XY}(x,y), \quad \mathrm{E}[Y]=\sum_{(x,y)\in\Omega} y\cdot f_{XY}(x,y)$$
$$\text{Continuous:}$$
$$\mathrm{E}[X]=\iint_{\mathbb{R}^2} x\cdot f_{XY}(x,y)dxdy, \quad \mathrm{E}[Y]=\iint_{\mathbb{R}^2} y\cdot f_{XY}(x,y)dxdy$$

Conditional Expectations

$$\text{Discrete:}$$
$$\mathrm{E}[Y\mid x]:=\sum_{y} y\cdot f_{Y|x}(y), \quad \mathrm{E}[X\mid y]:=\sum_{x} x\cdot f_{X|y}(x)$$
$$\text{Continuous:}$$
$$\mathrm{E}[Y\mid x]:=\int_{\mathbb{R}} y\cdot f_{Y|x}(y)dy, \quad \mathrm{E}[X\mid y]:=\int_{\mathbb{R}} x\cdot f_{X|y}(x)dxdsdf$$

Transformation of Variables

Let (X,f_X) be a continuous multivariate random variable and let $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ be a differentiable, bijective map with inverse φ^{-1}. Then $Y=\varphi\circ X$ is a continuous multivariate random variable with density

$$f_Y(y)=f_X\circ\varphi^{-1}(y)\cdot\left|\det D\varphi^{-1}(y)\right|$$

where $D\varphi^{-1}$ is the Jacobian of φ^{-1}.

Let $((X,Y),f_{XY})$ be a continuous bivariate random variable. Let $U=X/Y$. Then the density f_U of U is given by

$$f_U(u)=\int_{-\infty}^{\infty} f_{XY}(uv,v)\cdot|v|dv$$

> ## Sum of Two Continuous Random Variables
>
> Sum of Two Continuous Random Variables Let X and Y be continuous random variables with parameters with joint density f_{XY}. Let $U=X+Y$ and prove that the density of U is given by
>
> $$f_U(u)=\int_{-\infty}^{\infty} f_{XY}(u-v,v)dv$$
>
> Hint: Consider the transformation $(x,y)\mapsto(x+y,y)$.

> ## Sum of Two Exponential Distributions
>
> Let X and Y be independent exponentially distributed random variables with parameters $\beta_1=1/3$ and $\beta_2=1$ respectively. Let $U=X+Y$ and show that
>
> $$f_U(u)=\begin{cases}\left(e^{-u/3}-e^{-u}\right)/2 & u>0\\ 0 & u\leq 0\end{cases}$$

Covariance

$$\mathrm{Cov}[X,Y] = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$
$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$$

- $\mathrm{Cov}[X,X] = \mathrm{Var}[X]$.
- If X and Y are independent, then $\mathrm{Cov}[X,Y] = 0$.

covariance matrix

$$\mathrm{Var}[X] = \begin{pmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \dots & \mathrm{Cov}[X_1,X_n] \\ \mathrm{Cov}[X_1,X_2] & \mathrm{Var}[X_2] & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{Cov}[X_{n-1},X_n] \\ \mathrm{Cov}[X_1,X_n] & \dots & \mathrm{Cov}[X_{n-1},X_n] & \mathrm{Var}[X_n] \end{pmatrix}$$

for constant $n \times n$ matrix with real coefficients $C \in \mathrm{Mat}(n \times n; \mathbb{R})$

$$\mathrm{Var}[CX] = C\,\mathrm{Var}[X]C^\top$$

Standardized Random Variable

$$\tilde{X} := \frac{X-\mu_X}{\sigma_X}$$

$$\mathrm{E}[\tilde{X}] = 0, \quad \mathrm{Var}[\tilde{X}] = 1$$

Pearson coefficient

of correlation of (X, Y)

$$\rho_{XY} := \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \mathrm{Cov}\left[\tilde{X},\tilde{Y}\right]$$

- $-1 \leq \rho_{XY} \leq 1$
- $|\rho_{XY}| = 1$ if and only if there exist numbers $\beta_0, \beta_1 \in \mathbb{R}, \beta_1 \neq 0$, such that $Y = \beta_0 + \beta_1 X$ almost surely.

Linearity of X and Y

Fisher Transformation

$$\rho_{XY} = \tanh\left(\ln\left(\frac{\sigma_{\tilde{X}} + \tilde{Y}}{\sigma_{\tilde{X}-\tilde{Y}}}\right)\right)$$

- If $\rho_{XY} > 0,\ X$ and Y are positively correlated.
- If $\rho_{XY} < 0,\ X$ and Y are negatively correlated.

Bivariate normal distribution

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\varrho^2}} e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\varrho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$
$$\text{where} \quad -1 < \varrho < 1$$

Covariant and Bivariate normal distribution

Let $X=(X_1,X_2)$ be a random vector. Then we define the expectation vector and the variance-covariance matrix as follows:

$$\mathrm{E}[X]:=\begin{pmatrix}\mathrm{E}\left[X_1\right]\\ \mathrm{E}\left[X_2\right]\end{pmatrix},\quad \operatorname{Var} X:=\begin{pmatrix}\operatorname{Var} X_1 & \operatorname{Cov}(X_1,X_2)\\ \operatorname{Cov}(X_2,X_1) & \operatorname{Var} X_2\end{pmatrix}$$

Let A be a constant 2×2 matrix and $Y=(Y_1,Y_2)=AX$.

1. Show that $\mathrm{E}[AX]=A\mathrm{E}[X]$.
2. Show that $\operatorname{Var}(AX)=A(\operatorname{Var} X)A^T$.
3. Suppose that X_1 and X_2 follow independent normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the joint density is given by

$$f_X(x)=f_X\left(x_1,x_2\right)=\frac{1}{2\pi\sqrt{\det\Sigma_X}}e^{-\frac{1}{2}\left\langle x-\mu_X,\Sigma_X^{-1}(x-\mu_X)\right\rangle}$$

where $\mu_X=(\mu_1,\mu_2)$ and $\Sigma_X=\operatorname{diag}\left(\sigma_1^2,\sigma_2^2\right)$ is the 2×2 matrix with the variances on the diagonal and all other entries vanishing. (1 Mark)
iv) Suppose that X_1 and X_2 follow independent normal distributions with means $\mu_1,\mu_2\in\mathbb{R}$ and variances $\sigma_1^2,\sigma_2^2>0$, respectively. Let $Y=AX$ where A is an invertible $n\times n$ matrix. Show that

$$f_Y(y)=\frac{1}{2\pi\sqrt{|\det\Sigma_Y|}}e^{-\frac{1}{2}\left\langle y-\mu_Y,\Sigma_Y^{-1}(y-\mu_Y)\right\rangle}$$

where $\mu_Y=\mathrm{E}[Y],\Sigma_Y=\operatorname{Var} Y$ and $\langle\cdot,\cdot\rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$.
v) Show that $(*)$ can be written as

$$f_Y\left(y_1,y_2\right)=\frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\varrho^2}}e^{-\frac{1}{2\left(1-\varrho^2\right)}}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2-2\varrho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)+\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]$$

where μ_{Y_i} is the mean and $\sigma_{Y_i}^2$ the variance of $Y_i, i=1,2,$ and ϱ is the correllation of Y_1 and Y_2.