……转动惯量，细圆环、细棒、圆柱、圆筒和圆球的回转半径分别为 r、$\sqrt{\frac{1}{12}}L$、$\sqrt{\frac{1}{2}}r$、$\sqrt{\frac{1}{2}(r_1^2+r_2^2)}$ 和 $\sqrt{\frac{2}{5}}r$。在相同质量条件下，回转半径 k 越大，转动惯量越大。 (6.33)

(2) 平行轴定理和垂直轴定理

为了简化转动惯量的计算。这里介绍两个相关定理。

定理 1　平行轴定理

刚体相对某一条轴线的转动惯量 I_0 必等于相对通过质心并与之平行的轴的转动惯量 I_C 加上总质量 M 与两轴距离 d 平方的乘积，即

$$I = I_C + Md^2 \tag{6.34}$$

下面利用图 6-4，给出证明。

由转动惯量定义，有

$$I_0 = \sum m_i r_i^2 = \sum m_i (r_i' + d)^2$$
$$= \sum m_i r_i'^2 + \sum m_i d^2 + 2\sum m_i r_i' d = I_C + Md^2$$

其中 $M = \sum m_i$，$\sum m_i r_i' = Mr_C'$，即等于质心系中质心位矢与总质量乘积，它等于零。证毕。

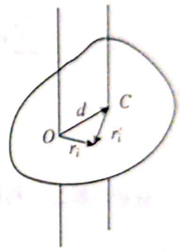


图 6-4

定理 2　垂直轴定理

在所考察的刚体上，任取直角坐标系 $Oxyz$，记刚体对 x、y、z 轴的转动惯量分别为 I_x、I_y、I_z，则有结论

$$I_x + I_y + I_z = 2\sum_i m_i r_i^2 \tag{6.35}$$

其中 r_i 是刚体上第 i 个质量元到坐标原点的距离。

下面利用图 6-5 给出证明。

利用 I_x、I_y、I_z 的定义，可以得到

$$I_x + I_y + I_z = \sum m_i (y_i^2 + z_i^2) + \sum m_i (z_i^2 + x_i^2) + \sum m_i (x_i^2 + y_i^2)$$
$$= 2\sum m_i (x_i^2 + y_i^2 + z_i^2) = 2\sum m_i r_i^2$$

证毕。

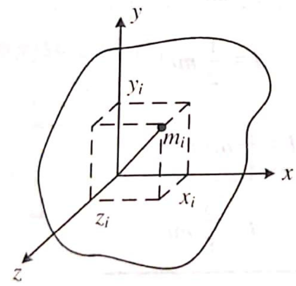


图 6-5

对于现阶段，如下推论可能更常用。

推论：一个平面分布的质点组（质量连续分布或分立离散分布均可），取 z 轴垂直于此平面，x、y 轴取在平面内，那么此质点组对三根轴的转动惯量有关系

$$I_z = I_x + I_y \tag{6.36}$$

证明：这是因为

$$I_z = \sum m_i (x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2 = I_y + I_x$$

显然，在式(6.35)中，取 $z_i = 0$，立即得到相同的结论。

例如，一块质量为 M、长为 l_1、宽为 l_2 的匀质矩形薄板，对于穿过平板中心且垂直于平板的轴，转动惯量为

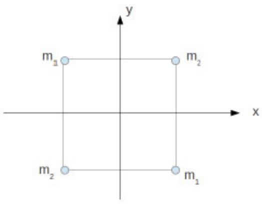
$$I = \frac{1}{12}Ml_1^2 + \frac{1}{12}Ml_2^2 = \frac{1}{12}M(l_1^2 + l_2^2)$$

2. A square with side length $2a$, lies in plane $z = 0$ and has masses m_1 and m_2 in its vertices (figure).

   (a) Find the components of the tensor of inertia with respect to axes x, y, z.

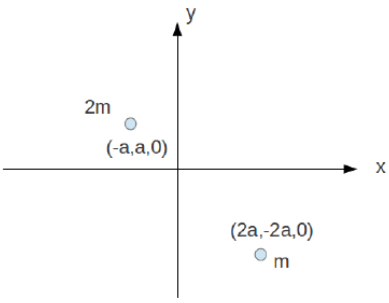
   (b) Diagonalize this tensor (find principal moments of inertia and directions of the principal axes).

You can easily get the result here from symmetry, but go through the diagonalization.



3. Do the same for the system of masses in the figure below. Solve the problem using symmetry first, and then check by doing calculations.

   [You will have a degenerate eigenvalue here, i.e. two vectors belonging to one eigenvalue. Physically, it means two principal axes have the same moment of inertia]



4. Using symmetry, find the principal axes and corresponding principal moments of inertia for: (a) thin disk (two axes contained in the disk $MR^2/4$; perpendicular axis $MR^2/2$), (b) thin-walled hollow sphere (all three $2MR^2/3$), (c) torus with mean radius R and the radius of cross-section r (two axes crossing the torus $M(4R^2 + 5r^2)/8$, perpendicular axis $M(R^2 + 3r^2/4)$).

   Which of these objects are symmetrical tops? spherical tops?

7. A wedge with mass M and angle α rests on a frictionless horizontal surface. A cylinder with mass m rolls down the wedge without slipping. Find the acceleration of the wedge.

   [It's more convenient to solve this problem in the frame of reference associated with the wedge. Answer: $a = \frac{g \sin 2\alpha}{[3(M-m)/m] - 2\cos^2\alpha}$]

8. A ball with mass m, moving with in the horizontal direction with speed v, hits the upper edge of a rectangular box with dimensions $l \times l \times 2l$. Assuming that the box can rotate about a fixed axis containing the edge AA', and the collision of the ball with the box is elastic (and the ball moves back in the horizontal direction after the collision), find

   (a) angular velocity the box starts moving with at the moment of collision [answer: $\omega_0 = \frac{2v}{I_{AA'}/ml + l}$],

   (b) equation of motion of the box after the collision [$I_{AA'}\ddot{\alpha} + Mgl\frac{\sqrt{5}}{2}\cos\alpha = 0$],

   (c) the minimum speed of the ball needed to put the box in the upright position [$v = (\frac{I_{AA'}}{ml} + l)\frac{1}{2}\sqrt{\frac{mg}{l}(\sqrt{5} - 1)}$].