Calculus On Normed Vector Spaces

MTP REPORT

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1 Introduction

The aim of the project is to study the notion of derivatives on general normed vector spaces and do Calculus on them. We will study generalizations of several well-known theorems of Calculus. We will also look at some applications of these concepts.

2 Definitions

In this section, some basic definations and elementary properties are discussed.

Norm: We will suppose that all vector spaces are real. Let E be a vector space. A mapping $\|.\| : E \to \mathbb{R}$, is said to be a *norm* if, for all x, $y \in E$ and $\lambda \in \mathbb{R}$

- $\|x\| \geq 0$;
- $\|x\| = 0 \Leftrightarrow x = 0$;
- $\|\lambda x\| = |\lambda| \, \|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|.\|)$ is called a *normed vector space* and we say that $\|x\|$ is the norm of x.

Continuity: Suppose now that we have two normed vector spaces, $(E, \|.\|_E)$ and $(F, \|.\|_F)$. Let A be a subset of E, f a mapping of A into F and $a \in A$. We say that f is *continuous* at a if the following condition is satisfied:

for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in A$ and $\|x - a\|_E < \delta$, then $\|f(x) - f(a)\|_F < \epsilon$

If f is *continuous* at every point $a \in A$, then we say that f is *continuous* on A.

Proposition 2.1. *The norm on a normed vector space is a continuous function.*

Proof. We have $\|x\| = \|x - y + y\| \leq \|x - y\| + \|x\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|$

In the same way, $\|y\| - \|x\| \leq \|y - x\|$. As $\|y - x\| = \|x - y\|$, We have

$$| \, \|x\| - \|y\| \, | \leq \|x - y\|$$

And hence the contunity.

Proposition 2.2. *Let E and F be normed vector spaces, $A \subseteq E$, $a \in A$, f and g are mappings from E into F and $\lambda \in \mathbb{R}$.*

- *If f and g are continuous at a, then so is $f + g$.*
- *If f is continuous at a, then so is λf.*
- *If α is a real-valued function defined on E and both f and α are continuous at a, then so is αf.*

Proposition 2.3. *Let $(E, \|.\|_E)$ be a normed vector space*

- *The mapping $f : E \times E \longrightarrow E, (x, y) \mapsto x + y$ is continuous.*
- *The mapping $f : \mathbb{R} \times E \longrightarrow E, (\lambda, x) \mapsto \lambda x$ is continuous.*

3 Differentiation

In this section we will be primarily concerned with extending the derivative defined for real-valued functions defined on an interval of $\mathbb{R}$. We will also consider minima and maxima of real-valued functions defined on a normed vector space.

3.1 Directional Derivatives

Let O be an open subset of a normed vector space E, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function $f_u : t \to f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

exists, then it is called the *directional derivative* of f at a in the direction of u, i.e. $\partial_u f(a)$.

If $E = \mathbb{R}_n$ and e_i is its standard basis, then the directional derivative $e_i f(a)$ is called the i th partial derivative of f at a, or the derivative of f with respect to x_i at a.

$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(a_1, .., a_i + t, ..., a_n) - f(a_1, ..., a_n)}{t}$$

If for every point $x \in O$, the partial derivative $\dfrac{\partial f}{\partial x_i}(x)$ is defined, then we obtain the function i th partial derivative defined on O. If these functions are defined and continuous for all i , then we say that the function f is of class C^1.

Example 3.1. *If f is the function defined on $\mathbb{R}^2$ by $f(x, y) = xe^{xy}$, then the partial derivatives with respect to x and y are defined at all points $(x, y) \in \mathbb{R}^2$ and*

$$\frac{\partial f}{\partial x}(x, y) = (1 + xy)e^{xy} \text{ and } \frac{\partial f}{\partial y}(x, y) = x^2 e^{xy}$$

As both are continuous, f is of class C^1.

However, a function of two or more variables may have all its partial derivatives defined at a given point without being *continuous* there. Here is an example.

Example 3.2. *Consider the function f defined on $\mathbb{R}_2$ by*

$$f(x,y) = \begin{cases} \dfrac{x^6}{x^8 + (y-x^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

We have (for x and y)

$$\lim_{t\to 0} \frac{t^6}{t^8+t^4}/t = 0 \qquad \text{and} \qquad \lim_{t\to 0} \frac{0}{t^2}/t = 0$$

and so,

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

However, $\lim_{t\to 0} f(x,x^2) = \infty$*, which indicates f is not continuous at 0.*

Suppose now that O is an open subset of $\mathbb{R}^n$ and f a mapping defined on O with image in $\mathbb{R}^m$. f has m coordinate mappings $f_1, ..., f_m$. If $a \in O$ and the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ of a, for $1 \leq i \leq m$ and $1 \leq j \leq n$, are all defined, then the $m \times n$ matrix

$$J_f(a) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ . & . & . \\ . & . & . \\ . & . & . \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

is called the *Jacobian Matrix* of f at a.

3.2 The Differential

Let E and F be normed vector spaces, O an open subset of E containing 0, and g a mapping from O into F such that $g(0) = 0$. If there exists a mapping $\epsilon, defined on a neighbourhood of 0 \in E$ and with image in F , such that $\lim_{h\to 0} \epsilon(h) = 0$ and

$$g(h) = \|h\|_E \, \epsilon(h),$$

then we write $g(h) = o(h)$ and say that g is *"small o of h"*.

The condition $g(h) = o(h)$ is independent of the norms we choose for two spaces E and F.

Let O be an open subset of a normed vector space E and f a mapping from O into a normed vector space F . If $a \in O$ and there is a continuous linear mapping $\phi : E \to F$ such that

$$f(a+h) = f(a) + \phi(h) + o(h)$$

when h is close to 0, then we say that f is *differentiable* at a.

Proposition 3.1. *If f is differentable at a, then*
(a) f is continuous at a;
(b) ϕ is unique.

Remark 3.1. *If E and F are normed vector spaces and $f : E \to F$ is constant, then $f'(a)$ is the zero mapping at any point $a \in E$. If $f : E \to F$ is linear and continuous, then $f'(a) = f$ at any point $a \in E$.*

Proposition 3.2. *Let f be a mapping defined on an open subset O of a normed vector space E with image in the cartesian product $F = F_1 \times ... \times F_p$. Then f is differentiable at $a \in O$ if and only if the coordinate mappings f_i , for $i = 1, ..., p$, are differentiable at a.*

$$f'(a) = (f_1'(a), ..., f_p'(a))$$

Suppose that dim $E = n < \infty$ and that e_i is a basis of E. If $x = \sum_{i=1}^{n} x_i e_i$, then

$$f'(a)x = \sum_{i=1}^{n} x_i f'(a)e_i = \sum_{i=1}^{n} \partial_{e_i} f(a)e_i^*(x),$$

where (e_i^*) is the dual basis of (e_i). We thus obtain the expression. If $E = \mathbb{R}^n$ and (e_i) is its standard basis, then we usually write dx_i for e_i^*. This gives us the expression

$$f'(a)x = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i.$$

Differentiability at a given point: If we wish to determine whether a real-valued function f defined on an open subset of $\mathbb{R}^n$ is differentiable at a given point a, then first we determine whether all its partial derivatives at a exist. If this is not the case, then f is not differentiable at a.
If all the partial derivatives exist, then we know that the only possibility for $f'(a)$ is the linear function $\phi = \sum_{i=1}^{n} frac\partial f \partial x_i(a)dx_i$. We consider the expression,

$$\frac{f(a+h) - f(a) - \phi(h)}{\|h\|} = \epsilon(h)$$

If $\lim_{h \to 0} \epsilon(h) = 0$, then f is differentiable at a, otherwise it is not.

3.3 Differentials of Compositions

Let E, F and G be normed vector spaces, O an open subset of E, U an open subset of F and $f : O \to F$, $g : U \to G$ be such that $f(O) \subset U$. Then the mapping $g \circ f$ is defined on O.

Theorem 3.1. *If f is differentiable at a and g is differentiable at f(a), then $g \circ f$ is differentiable at a and*
$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$
This expression is referred to as Chain Rule.

Corollary 3.1. *If in the above theorem the normed vector spaces are euclidian spaces, then*
$$J_{g \circ f}(a) = J_g(f(a)) \circ J_f(a).$$

Example 3.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be definrd by*
$$f(x, y, z) = (xy, e^x z) \qquad\qquad g(u, v) = u^2 v. \text{ Then,}$$

$$J_f(x, y, z) = \begin{bmatrix} y & x & 0 \\ ze^{xz} & 0 & xe^{xz} \end{bmatrix}$$

and

$$J_g(u, v) = \begin{bmatrix} 2uvu^2 \end{bmatrix}$$

Multiplying the matrices $J_g(f(x, y, z))$ and $J_f(x, y, z)$, we obtain

$$J_{g \circ f}(x, y, z) = (2xy^2 + x^2y^2z)e^{xz}2x^2ye^{xz}x^3y^3e^{xz} \ .$$

3.4 Differentiability of the Norm

If E is a normed vector space with norm $\|.\|$, then $\|.\|$ is itself a mapping from E into $\mathbb{R}$ and we can study its *differentiability*. We will write $Df(\|.\|)(x)$ for differentiability of the norm at x (if exists).

Proposition 3.3. *Norm is not differentiable at the origin.*

Proof. Suppose $Df(\|.\|)$ exists. Then for small non-zero values of h, we have

$$\|h\| = Df(\|.\|)(0)h + o(h) \Rightarrow \lim_{h\to 0}\left(1 - Df(\|.\|)\frac{h}{\|h\|}\right) = 0$$

And

$$\|h\| = \|-h\| = -Df(\|.\|)(0)h + o(h) \Rightarrow \lim_{h\to 0}\left(1 + Df(\|.\|)\frac{h}{\|h\|}\right) = 0$$

Summing the two limits we obtain $2 = 0$, which is a contradiction. Hence $Df(\|.\|)(0)$ does not exist.

At points where the differential exists, we have the following interesting result:

Proposition 3.4. *Let E be a normed vector space and $\|.\|$ its norm. If $\|.\|$ is differentiable at $a \neq 0$ and $\lambda > 0$, then $\|.\|$ is differentiable at λa and $Df(\|.\|)(\lambda a) = Df(\|.\|)(a)$. In addition, $|Df(\|.\|)(a)|_{E^*} = 1$*

Proof. If $\|.\|$ is differentiable at $a, \lambda >=$ and $h \in E \setminus \{0\}$, then we have

$$\|\lambda a + h\| \ = \lambda \left\|a + \tfrac{h}{\lambda}\right\| = \lambda \left(\|a\| + Df(\|.\|)(\tfrac{h}{\lambda}) + o(\tfrac{h}{\lambda})\right) = \|\lambda a\| + Df(\|.\|)(a)h + o(h)$$

It follows that $Df(\|.\|)(\lambda a)$ exists and $Df(\|.\|)(\lambda a) = Df(\|.\|)(a)$.

For 2nd part, Consider the function, $f : \mathbb{R}_+^* \to \mathbb{R}, \lambda \mapsto \|\lambda a\|$

For a given $\lambda \in \mathbb{R}_+^*$ and $h \in \mathbb{R}$ sufficiently small, we have

$$\|(\lambda + h)a\| = (\lambda + h)\|a\|$$

and so

$$\lim_{h\to 0}\frac{\|(\lambda + h)a - \|\lambda a\|\|}{h} = \lim_{h\to 0}\frac{h\|a\|}{h} = \|a\|$$

Therefore $\dot{f}(\lambda) = \|a\|$ for all values of λ. On the other hand, $f = \|.\|\circ\phi$, where $\phi(\lambda) = \lambda a$, and so

$$(f'(\lambda))s = Df(\|.\|)(\lambda a)sa = a(Df(\|.\|)(a))a$$

This implies, $\dot{f}(\lambda) = Df(\|.\|)(a)a$ and hence $Df(\|.\|)(a)a = \|a\|$. It follows that $|Df(\|.\|)(a)|_{E^*} = 1$.

4 References

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