Calculus on Normed Vector Spaces

Joydeep Medhi
2013MT60599

Supervisor: Amit Priyadarshi

Dept. of Mathematics
Indian Institute of Technology

Final Presentation

February 21, 2018

Overview

1. Introduction
   - Aim
   - Quick Overview of Previous Work
2. Calculus on Normed Linear Spaces
   - Mean Value Theorem
   - Partial Differentials
   - Higher Differentials
   - Multilinear Mazpping
3. Taylor Formulas
   - Taylor's Theorem
   - Extrema

Aim

- To study the notion of derivatives on general normed vector spaces and do Calculus on them.
- To generalise the basic calculus of function of several variables to Normed Vector Spaces.'
- To explore the applications of these concepts.

Quick overview of previous work

Directional Derivatives

Let O be an open subset of a normed vector space E, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function $f_u : t \to f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

exists, then it is called the **directional derivative** of f at a in the direction of u, i.e. $\partial_u f(a)$.

Small "o" notation

Let E and F be normed vector spaces, O an open subset of E containing 0, and g a mapping from O into F such that $g(0) = 0$.
If there exists a mapping ϵ, defined on a neighbourhood of $0 \in E$ and with image in F , such that $\lim_{h \to 0} \epsilon(h) = 0$ and

$$g(h) = \|h\|_E \, \epsilon(h),$$

then we write $g(h) = o(h)$ and say that g is *"small o of h"*.
The condition $g(h) = o(h)$ is independent of the norms we choose for two spaces E and F.

Differentiability

- If $a \in O$ and there is a continuous linear mapping $\phi : E \to F$ such that
$$f(a+h) = f(a) + \phi(h) + o(h)$$
when h is close to 0, then we say that f is *differentiable* at a.

- If all the partial derivatives exist, then we know that the only possibility for $f'(a)$ is the linear function $\phi = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) dx_i$. We consider the expression,
$$\frac{f(a+h) - f(a) - \phi(h)}{\|h\|} = \epsilon(h)$$
If $\lim_{h \to 0} \epsilon(h) = 0$, then f is differentiable at a, otherwise it is not.

Differentiability of the Norm

If E is a normed vector space with norm $\|.\|$, then $\|.\|$ is itself a mapping from E into $\mathbb{R}$ and we can study its *differentiability*. We will write $Df(\|.\|)(x)$ for differentiability of the norm at x (if exists)

The equation becomes
$\|x+h\| = \|x\| + Df(\|.\|)(x)h + o(h)$

Mean Value Theorem

Let f be a real-valued function defined on a closed bounded interval $[a, b] \subset \mathbb{R}$. If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a point $c \in (a, b)$ such that

$$f(b) - f(a) = \dot{f}(c)(b - a)$$

Generalization of Mean Value Theorem

Let O be a open subset of a normed vector space E and $a, b \in E$ with $[a, b] \in O$. If $f : O \to \mathbb{R}$ is differentiable , then there is a element $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

If $E = \mathbb{R}^n$ then this result can be written as

$$f(b) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(c)(b_i - a_i)$$

Results

Let us consider the result $f : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (cos(2\pi t), sin(2\pi t))$
Here, $f(1) - f(0) = 0$. However, $f'(t) = (-2\pi sin(\pi t)dt, 2\pi cos(\pi t)dt)$, where dt is the identity on $\mathbb{R}$.

So we cannot find $t_0 \in (0, 1)$ such that $f(1) - f(0) = f'(t_0)(1 - 0)$. Therefore we cannot generalize Theorem to mappings whose image lies in a general normed vector space.

Results

Theorem

Let $[a,b]$ be an interval of $\mathbb{R}$, F a normed vector space and $f:[a,b]\to F$ and $g:[a,b]\to\mathbb{R}$ both continuous and differentiable on (a,b).
If $\left\|\dot{f}(t)\right\| \leq \dot{g}(t)$ for all $t\in(a,b)$, then

$$\|f(b)-f(a)\|_F \leq g(b)-g(a).$$

Corollary

Let E and F are normed vector spaces, O an open subset of E and $f:O\to F$ differentiable on O. If the segment $[a,b]\subset O$,then

$$\|f(b)-f(a)\|_F \leq sup_{x\in(a,b)}|f'(x)|_{L(E,F)}\,\|b-a\|_E$$

Partial Differentials

The continuous linear mappings between two normed vector spaces E and F form a vector space $L(E, F)$. If we set

$$|\phi|_{L(E,F)} = sup_{\|x\|\leq 1} \|\phi(x)\|_F$$

If $\phi \in L(E, F)$, then $|.|_L(E, F)$ is a norm on $L(E, F)$.

Let $E_1, E_2, ..., E_n$ and F be normed vector spaces. We set
$E = E_1 \times .... \times E_n$ and define a norm on E
$\|(x_1, ..., x_n)\|_E = \max_k \|x_k\|_{E_k}$.

Partial Differentials

Now let O be an open subset of E and f a mapping from O into F . If we take a point $a \in O$ and let the kth coordinate vary and fix the others, then we obtain a mapping $f_{a,k}$ from E_k into F , defined on an open subset of E_k containing a_k.
If $f_{a,k}$ is differentiable at a_k, then we call the differential $f'_{a,k}(a_k) \in L(E_k, F)$ the *kth partial differential of f at a* and write it as $\partial_k f(a)$ for $f'_{a,k}(a_k)$.

Theorem. Let $E_1, E_2, ..., E_n$ and F be normed vector spaces, O an open subset of $E = E_1 \times .... \times E_n$ and f is a mapping from O into F, then f is of class C^1 if and only if f has a continuous partial differentials defined on O.

Higher Derivatives and Differentials

Let $O \subset \mathbb{R}^n$ be open and f a real valued function defined on O. If the function $\frac{\partial f}{\partial x_i}$ is defined on O, then we can consider the existance of its partial derivatives. If $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_i})(a)$ exists, then we write for this derivative $\frac{\partial^2 x}{\partial x_j \partial x_i}(a)$ if $i \neq j$ and $\frac{\partial^2 x}{\partial^2 x_i}(a)$ if $i = j$.

If these functions are defined and continuous for all pairs (j, i), then we say that f is of class C^2.

We say that continuous functions are of class C^0 . If a function is of class C^K for all $K \in \mathbb{N}$, then we say that f is of class C^∞ , or smooth.

Higher Derivatives and Differentials

Schwarz's Theorem Let $O \subset \mathbb{R}^2$ be open and $f : O \to \mathbb{R}$ be such that the second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are defined on O. If these functions are continuous at $(a, b) \in O$, then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

S_k is the group of permutations of the set $1, ..., k$.

Theorem. Let $O \subset \mathbb{R}^n$ be open and $f : O \to \mathbb{R}$ of class C^K and $(i_1, ..., i_k) \in \mathbb{N}^K$ with $i_1 \leq ..... \leq i_n$. If $\sigma \in S_k$, then for $a \in O$ we have

$$\frac{\partial^k f}{\partial x_{i_1} .....\partial x_{i_k}}(a) = \frac{\partial^k f}{\partial x_{i_{\sigma(1)}} .....\partial x_{i_{\sigma(k)}}}(a).$$

Multilinear Mapping

- Second and higher differentials are more difficult to define than second and higher derivatives.
- The natural way of defining a second differential would be to take the differential of the mapping $x \mapsto f'(x)$.
- Unfortunately, if E and F are normed vector spaces and f a differentiable mapping from an open subset of E into F , then the image of f' lies not in F but in $L(E, F)$. This means that the differential of f' lies in $L(E, L(E, F))$.
- We get around this problem by identifying differentials with multilinear mappings.

Multilinear Mapping

Let E and F be normed vector spaces and O an open subset of E. If $f : O \to F$ is differentiable on an open neighbourhood V of $a \in O$, then the mapping

$$f' : V \mapsto L(E, F), x \mapsto f'(x) \text{ is defined.}$$

If f' is differentiable at a, then we would be tempted to define the second differential $f^{(2)}(a)$ of f at a as $f''(a) = (f')'(a)$. However, in this way $f^{(2)}(a) \in L_2(E, F)$ and it is difficult to work with these higher order spaces. Hence we proceed in a different way.

Multilinear Mapping

We will define linear continuous mappings Φ_k from $L_k(E,F) \to L(E^k,F)$. we define k-differentiability and the kth differential $f^{(k)}(a)$ for higher values of k. We will sometimes write $f^{(1)}$ for f'. To distinguish the differential in $L_k(E;F)$ corresponding to $f^{(k)}(a)$, we will write $f^{[k]}$ for it, i.e., $\Phi_k(f^{[k]}(a)) = f^{(k)}(a)$.

Let E and F be normed vector spaces, O an open subset of E and f a mapping from O into F . Then f is $k+1$-differentiable at $a \in O$ if and only if f^k is differentiable at a and in this case

$$f^{(k)'}(a)h(h_1,....,h_k) = f^{(k+1)}(a)(h,h_1,.....,h_k)$$

for $h, h_1, ...,h_k \in E$.

Notations

- Let E and F be normed vector spaces, O an open subset of E containing 0 and g a mapping from O into F such that $g(0) = 0$. If there exists a mapping ϵ, defined on a neighbourhood of $0 \in E$ and with image in F, such that $lim_{h \to 0} \epsilon(h)$ and
$$g(h) = \|h\|_E^k \, \epsilon(h),$$
then we will write $g(h) = o(\|h\|_E^k)$ or $g(h) = o(\|h\|^k)$ when the norm is understood. If k =1, then $o(\|h\|) = o(h)$
- If E is a normed vector space and h is a vector in E, then we will write h^k for the vector $(h, ..., h) \in E^k$.

Taylor's Formula

Let E and F be normed vector spaces, O an open subset of E and $a \in O$. If $f : O \to F$ is $(k-1)$-differentiable and $f^{(k)}(a)$ exists, then for x is sufficiently small

$$f(a+x) = f(a) + f^{(1)}(a)(x) + \frac{1}{2} f^{(2)}(a)(x^2) + \dots + \frac{1}{k!} f^{(k)}(a)(x^k) + o(\|h\|^k).$$

Asymptotic Development

Let E and F be normed vector spaces, O an open subset of E and f a mapping from O into F. We say that f has an asymptotic development of order k at a point $a \in O$ if there are symmetric continuous i-linear mappings A_i , for $i = 1, \ldots\ldots, k$, such that for small values of x we have

$$f(a+x) = f(a) + A_1 x + \frac{1}{1} A_2(x^2) + \ldots\ldots\ldots + \frac{1}{k!} A_k(x^k) + o(\|x\|_k).$$

If f is k-differentiable at a, then f has an asymptotic development of order k at a. By definition, if f has an asymptotic development of order 1 at a, then f is differentiable at a; however, f may have an asymptotic development of order $k > 1$ without being k-differentiable.

Example

Let $f : \mathbb{R} \to R$ be defined by

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For x close to 0 we can write

$$f(x) = x^2(x \sin \tfrac{1}{x}) = x^2 \epsilon(x),$$

where $lim_{x \to 0} \epsilon(x) = 0$. Hence, f has an asymptotic development of order 2 at 0. Also,

$$f'(x) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and so,

$$\frac{f'(x) - f'(0)}{x} = 3x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}$$

which has no limit at 0 and it follows that $f^{(2)}(0)$ does not exist.

Asymptotic Development

Theorem. Let E and F be normed vector spaces, O an open subset of E and f a mapping from O into F . If f has an asymptotic development at $a \in E$ of order k, then this development is unique.

corollary Let E and F be normed vector spaces, O an open subset of E and f a mapping from O into F . If f has a kth differential at $a \in O$ and

$$f(a+x) = f(a) + \sum_{i=1}^{k} \frac{1}{i!} A_i(x^i) + o(\|x\|_k),$$

then $A_i = f^{(i)}(a)$ for all i.

Extrema: 2nd order

A local extremum of a differentiable function is always a critical point. We now suppose that the function is 2-differentiable at the critical point.

proposition Let O be an open subset of a normed vector space E, $a \in O$ and f a real-valued function defined on O having a second differential at a. If a is a local minimum, then for $h \in E$

$$f^{(2)}(a)(h, h) \geq 0$$

Inverse of this proposition is not true. Example, $x \to x^3$. This is a necessary condition for a point to be minimum.

Extrema: 2nd order

Let O be an open subset of a normed vector space E, $a \in O$ and f a 2-differentiable real-valued function defined on O and $a \in O$. If a is a critical point of f and there is an open ball B centered on a such that

$$f^{(2)}(a)(h, h) \geq 0$$

for $x \in B$ and $h \in E$, then a is a local minimum.

Future work

- Extend these study to Hilbert space, Convex functions.
- Calculus of Variations
- Practical Applications of these concepts.

References

Rodney Colman(2012)
Calculus on Normed Linear Spaces

Avez, A.(1986)
Differential Calculus. J. Wiley and Sons Ltd, New York (1986)

Thank You!

The End