Calculus on Normed Vector Spaces

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Overview

Aim

- To study the notion of derivatives on general normed vector spaces and do Calculus on them.
- To generalise the basic calculus of function of several variables to Normed Vector Spaces.'
- To explore the applications of these concepts.

Basic Definations 1

Norm : A mapping $\|.\| : E \to \mathbb{R}$, is said to be a *norm* if, for all $x, y \in E$ and $\lambda \in \mathbb{R}$ if the given properties are true.
The pair $(E, \|.\|)$ is called a *normed vector space* and we say that $\|x\|$ is the norm of x.

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Properties

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|\lambda x\| = |\lambda| \, \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

Basic Definations 2

Continuity

Suppose now that we have two normed vector spaces, $(E, \|.\|_E)$ and $(F, \|.\|_F)$.
Let A be a subset of E, f a mapping of A into F and $a \in A$. We say that f is *continuous* at a if the following condition is satisfied:

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for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in A$ and $\|x - a\|_E < \delta$, then $\|f(x) - f(a)\|_F < \epsilon$

If f is *continuous* at every point $a \in A$, then we say that f is *continuous* on A.

Basic Definations 3

The norm on a normed vector space is a continuous function

We have

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|x\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

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In the same way, $\|y\| - \|x\| \leq \|y - x\|$.

As $\|y - x\| = \|x - y\|$, We have

$$|\,\|x\| - \|y\|\,| \leq \|x - y\|$$

And hence the contunity.

Basic Definations 4

Let E and F be normed vector spaces, $A \subseteq E$, $a \in A$, f and g are mappings from E into F and $\lambda \in \mathbb{R}$:

- If f and g are continuous at a, then so is $f + g$.
- If f is continuous at a, then so is λf.
- If α is a real-valued function defined on E and both f and α are continuous at a, then so is αf.

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Let $(E, \|.\|_E)$ be a normed vector space

- The mapping $f : E \times E \longrightarrow E, (x, y) \mapsto x + y$ is continuous.
- The mapping $f : \mathbb{R} \times E \longrightarrow E, (\lambda, x) \mapsto \lambda x$ is continuous.

Directional Derivatives 1

Let O be an open subset of a normed vector space E, f a real-valued function defined on O, $a \in O$ and u a nonzero element of E. The function $f_u : t \to f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\lim_{t\to 0} \frac{f(a+tu) - f(a)}{t}$$

exists, then it is called the **directional derivative** of f at a in the direction of u, i.e. $\partial_u f(a)$.

Directional Derivatives 2

Example (1)

If f is the function defined on $\mathbb{R}^2$ by $f(x, y) = xe^{xy}$, then the partial derivatives with respect to x and y are defined at all points $(x, y) \in \mathbb{R}^2$ and

$$\frac{\partial f}{\partial x}(x, y) = (1 + xy)e^{xy} \text{ and } \frac{\partial f}{\partial y}(x, y) = x^2 e^{xy}$$

As both are continuous, f is of class C^1.

Directional Derivative 3

If $E = \mathbb{R}_n$ and e_i is its standard basis, then the directional derivative $\partial e_i f(a)$ is called the i th partial derivative of f at a, or the derivative of f with respect to x_i at a.

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$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(a_1, .., a_i + t, ..., a_n) - f(a_1, ..., a_n)}{t}$$

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If for every point $x \in O$, the partial derivative $\dfrac{\partial f}{\partial x_i}(x)$ is defined, then we obtain the function i th partial derivative defined on O. If these functions are defined and continuous for all i , then we say that the function f is of class C^1.

Directional Derivatives 4

Example (2)

Consider the function f defined on $\mathbb{R}^2$ by

$$f(x,y) = \frac{x^6}{x^8 + (y - x^2)^2} \text{ if } (x,y) \neq (0,0)$$

$$0 \text{ otherwise}$$

Directional Derivatives 5

Example (2)

We have (for x and y)

$$\lim_{t\to 0} \frac{t^6}{t^8+t^4}/t = 0 \text{ and } \lim_{t\to 0} \frac{0}{t^2}/t = 0$$

and so,

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

However, $\lim_{t\to 0} f(x, x^2) = \infty$, which indicates f is not continuous at 0.

Jacobian

Suppose now that O is an open subset of $\mathbb{R}^n$ and f a mapping defined on O with image in $\mathbb{R}^m$. f has m coordinate mappings $f_1, ..., f_m$. If $a \in O$ and the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ of a, for $1 \leq i \leq m$ and $1 \leq j \leq n$, are all defined, then the $m \times n$ matrix

$$J_f(a) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

is called the *Jacobian Matrix* of f at a.

Small "o" notation

Let E and F be normed vector spaces, O an open subset of E containing 0, and g a mapping from O into F such that $g(0) = 0$.
If there exists a mapping ϵ, defined on a neighbourhood of $0 \in E$ and with image in F , such that $\lim_{h\to 0} \epsilon(h) = 0$ and

$$g(h) = \|h\|_E \, \epsilon(h),$$

then we write $g(h) = o(h)$ and say that g is *"small o of h"*.
The condition $g(h) = o(h)$ is independent of the norms we choose for two spaces E and F.

Differentiability

- If $a \in O$ and there is a continuous linear mapping $\phi : E \to F$ such that
$$f(a+h) = f(a) + \phi(h) + o(h)$$
when h is close to 0, then we say that f is *differentiable* at a.

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- If all the partial derivatives exist, then we know that the only possibility for $f'(a)$ is the linear function $\phi = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i$. We consider the expression,
  $$\frac{f(a+h) - f(a) - \phi(h)}{\|h\|} = \epsilon(h)$$
  If $\lim_{h \to 0} \epsilon(h) = 0$, then f is differentiable at a, otherwise it is not.

Composition

Let E, F and G be normed vector spaces, O an open subset of E, U an open subset of F and $f : O \to F$, $g : U \to G$ be such that $f(O) \subset U$. Then the mapping $g \circ f$ is defined on O.

If f is differentiable at a and g is differentiable at f(a), then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a).$$

This expression is referred to as Chain Rule.

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If in the above theorem the normed vector spaces are euclidian spaces, then

$$J_{g\circ f}(a) = J_g(f(a)) \circ J_f(a).$$

Differentiability of the Norm

If E is a normed vector space with norm $\|.\|$, then $\|.\|$ is itself a mapping from E into $\mathbb{R}$ and we can study its *differentiability*. We will write $Df(\|.\|)(x)$ for differentiability of the norm at x (if exists)

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The equation becomes
$\|x+h\| = \|x\| + Df(\|.\|)(x)h + o(h)$

Norm is not differentiable at the origin.

Suppose $Df(\|.\|)$ exists. Then for small non-zero values of h, we have

$$\|h\| = Df(\|.\|)(0)h + o(h) \Rightarrow \lim_{h\to 0}\left(1 - Df(\|.\|)\frac{h}{\|h\|}\right) = 0$$

And

$$\|h\| = \|-h\| = -Df(\|.\|)(0)h + o(h) \Rightarrow \lim_{h\to 0}\left(1 + Df(\|.\|)\frac{h}{\|h\|}\right) = 0$$

Summing the two limits we obtain $2 = 0$, which is a contradiction. Hence $Df(\|.\|)(0)$ does not exist.

Differentiation of Norm

Let E be a normed vector space and $\|.\|$ its norm. If $\|.\|$ is differentiable at $a \neq 0$ and $\lambda > 0$, then $\|.\|$ is differentiable at λa and $Df(\|.\|)(\lambda a) = Df(\|.\|)(a)$.

If $\|.\|$ is differentiable at $a, \lambda > 0$ and $h \in E \setminus \{0\}$, then we have

$$\|\lambda a + h\| = \lambda \left\|a + \frac{h}{\lambda}\right\| = \lambda \left(\|a\| + Df(\|.\|)(\tfrac{h}{\lambda}) + o(\tfrac{h}{\lambda})\right)$$

$$= \|\lambda a\| + Df(\|.\|)(a)h + o(h)$$

It follows that $Df(\|.\|)(\lambda a)$ exists and $Df(\|.\|)(\lambda a) = Df(\|.\|)(a)$.

Example (Basic Example)

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \|x\|_1 = |x|$$

Then f is continuous at x = 0,
But f is not differentiable at $x = 0$

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Then f is continuous at x = 0,
But f is not differentiable at $x = 0$
Again,

$$f'(x) = \frac{x}{|x|}$$

$$\lambda > 0$$

$$f'(\lambda x) = f'(x)$$

References

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Thank You!

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