CSC236 tutorial exercises, Week #11
best before Thursday evening

These exercises are intended to give you some practice devising deterministic finite state automata (DFAs).

1. Let $L_1 = \{x \in \{a,b\}^* \mid \text{ the number of } a\text{s in } x \text{ is even}\}$, and let $L_2 = \{z \in \{a,b\}^* \mid |z| \equiv 0 \bmod 3\}$. Build DFAs that accept L_1, L_2, and use the product procedure to build a DFA that accepts $L_1 \cap L_2$.

   sample solution: Here is my specification for $M_1 = \{Q, \Sigma, \delta, q_0, F\}$ that accepts L_1:

$$
\begin{aligned}
&\{Q = \{E, O\},\\
&\Sigma = \{a, b\},\\
&\delta = \begin{array}{|c|c|c|}\hline \delta & E & O \\ \hline a & O & E \\ \hline b & E & O \\ \hline \end{array},\\
&q_0 = E,\\
&F = \{E\}\}
\end{aligned}
$$

   Here is my specification for $M_2 = \{Q, \Sigma, \delta, q_0, F\}$ that accepts L_2:

$$
\begin{aligned}
&\{Q = \{0, 1, 2\},\\
&\Sigma = \{a, b\},\\
&\delta = \begin{array}{|c|c|c|c|}\hline \delta & 0 & 1 & 2 \\ \hline a & 1 & 2 & 0 \\ \hline b & 1 & 2 & 0 \\ \hline \end{array},\\
&s = q_0 = 0,\\
&F = \{0\}\}
\end{aligned}
$$

Here is my specification for the product machine $M_{1\wedge 2}$ that accepts $L_1 \cap L_2$:

$$
\begin{aligned}
&\{Q = \{(E,0),(E,1),(E,2),(O,0),(O,1),(O,2)\},\\
&\Sigma = \{a,b\},\\
&\delta = \begin{array}{||c|c|c|c|c|c|c||}
\delta & (E,0) & (E,1) & (E,2) & (O,0) & (O,1) & (O,2) \\ \hline
a & (O,1) & (O,2) & (O,0) & (E,1) & (E,2) & (E,0) \\ \hline
b & (E,1) & (E,2) & (E,0) & (O,1) & (O,2) & (O,0)
\end{array},\\
&s = q_0 = (E,0),\\
&F = \{(E,0)\}\}
\end{aligned}
$$

2. Use structural induction to prove that the DFAs you propose accept L_1 and L_2. Without any further induction, prove that your product machine accepts $L_1 \cap L_2$ by constructing a state invariant consisting of conjunctions of the state invariants of the other two machines, and then using your earlier proofs to show that this new state invariant is correct.

sample solution: First, define Σ^* as the smallest set such that:

(a) $\varepsilon \in \Sigma^*$

(b) $s \in \Sigma^* \Rightarrow sa \in \Sigma^* \wedge sb \in \Sigma^*$

prove that M_1 accepts L_1: Define $P(s)$ as:

$$P(s) : \delta^*(E,s) = \begin{cases} E & \text{if } s \text{ has an even number of } a\text{s} \\ O & \text{if } s \text{ has an odd number of } a\text{s} \end{cases}$$

I prove $\forall s \in \Sigma^*, P(s)$ by structural induction.

basis case: $|\varepsilon| = 0$, an even number, and $\delta^*(E,\varepsilon) = E$ so the implication in the first line of the invariant is true in this case. Also, since $|\varepsilon|$ is not odd, the implication in the second line of the invariant is vacuously true. So $P(\varepsilon)$ holds.

inductive step: Let $s \in \Sigma^*$ and assume $P(s)$. I will show that $P(sa)$ and $P(sb)$ follow. There are two cases to consider:

case sa: Then

$$
\begin{aligned}
\delta^*(E,sa) = \delta(\delta^*(E,s),a) &= \begin{cases} \delta(E,a) & \text{if } s \text{ has even number of } a\text{s} \\ \delta(O,a) & \text{if } s \text{ has odd number of } a\text{s} \end{cases} && \# \text{ by } P(s) \\
&= \begin{cases} O & \text{if } sa \text{ has odd number of } a\text{s} \\ E & \text{if } sa \text{ has even number of } a\text{s} \end{cases} && \# \text{ one more } a
\end{aligned}
$$

case sb: Then

$$
\begin{aligned}
\delta^*(E,sb) = \delta(\delta^*(E,s),b) &= \begin{cases} \delta(E,b) & \text{if } s \text{ has even number of } a\text{s} \\ \delta(O,b) & \text{if } s \text{ has odd number of } a\text{s} \end{cases} && \# \text{ by } P(s) \\
&= \begin{cases} E & \text{if } sb \text{ has even number of } a\text{s} \\ O & \text{if } sb \text{ has odd number of } a\text{s} \end{cases} && \# \text{ same number of } a\text{s}
\end{aligned}
$$

So $P(sa)$ and $P(sb)$ follow.

The first line of the invariant ensures that all strings with an even number of as are accepted. The contrapositive of the second line of the invariant ensures that any string that does not drive the machine to state O does not have an odd number of as, in other words all strings that drive the machine to state E have an even number of as. So M_1 accepts L_1.

prove that M_2 accepts L_2: Define $P(s)$ as:

$$P(s) : \delta^*(0, s) = \begin{cases} 0 & \text{if } |s| \equiv 0 \bmod 3 \\ 1 & \text{if } |s| \equiv 1 \bmod 3 \\ 2 & \text{if } |s| \equiv 2 \bmod 3 \end{cases}$$

I prove $\forall s \in \Sigma^*, P(s)$ by structural induction.

basis case: $|\varepsilon| = 0$, a multiple of 3, and $\delta^*(0, \varepsilon) = 0$, so the implication in the first line of the invariant is true in this case. Since $|\varepsilon|$ leaves a remainder of neither 1 nor 2 when divided by 3, the implications on the second and third lines of the invariant are vacuously true. So $P(\varepsilon)$ holds.

induction step: Let $s \in \Sigma^*$ and assume $P(s)$. Let $c \in \{a, b\}$. I will prove that $P(sc)$ follows.

$$\begin{aligned} \delta^*(0, sc) = \delta(\delta^*(0, s), c) &= \begin{cases} \delta(0, c) & \text{if } |s| \equiv 0 \bmod (3) \\ \delta(1, c) & \text{if } |s| \equiv 1 \bmod (3) \\ \delta(2, c) & \text{if } |s| \equiv 2 \bmod (3) \end{cases} && \# \text{ by } P(s) \\ &= \begin{cases} 1 & \text{if } |sc| \equiv 1 \bmod (3) \\ 2 & \text{if } |sc| \equiv 2 \bmod (3) \\ 0 & \text{if } |sc| \equiv 0 \bmod (3) \end{cases} && \# \text{ one more character} \end{aligned}$$

So $P(sc)$ follows ■

The invariant ensures that all strings with a multiple of 3 characters drive the machine to state 0. The contrapositives of the second and third lines ensure that any string that does not drive the machine to state 1 does not have a length equivalent to 1 mod 3, and any string that does not drive the machine to state 2 does not have a length equivalent to 2 mod 3, so any strings that drive the machine to state 0 have lengths equivalent to 0 mod 3. Hence M_2 accepts L_2.

prove $M_{1\wedge 2}$ accepts $L_1 \cap L_2$: Denote the states for M_1 as Q_1, the states for M_2 as Q_2, their respective transition functions as δ_1 and δ_2, and the transition function for $M_{1\wedge 2}$ as $\delta_{1\wedge 2}$. Inspection of $\delta_{1\wedge 2}$ shows that if $(q_1, q_2, c) \in Q_1 \times Q_2 \times \Sigma$, then $\delta_{1\wedge 2}((q_1, q_2), c) = (\delta_1(q_1, c), \delta_2(q_2, c))$. Thus the following invariant follows by simply taking conjunctions of the invariants of the component machines, for any $s \in \Sigma^*$

$$P(s) : \delta^*((E, 0), s) = \begin{cases} (E, 0) & \text{if } s \text{ has an even number of } a\text{s} \wedge |s| \equiv 0 \bmod 3 \\ (E, 1) & \text{if } s \text{ has an even number of } a\text{s} \wedge |s| \equiv 1 \bmod 3 \\ (E, 2) & \text{if } s \text{ has an even number of } a\text{s} \wedge |s| \equiv 2 \bmod 3 \\ (O, 0) & \text{if } s \text{ has an odd number of } a\text{s} \wedge |s| \equiv 0 \bmod 3 \\ (O, 1) & \text{if } s \text{ has an odd number of } a\text{s} \wedge |s| \equiv 1 \bmod 3 \\ (O, 2) & \text{if } s \text{ has an odd number of } a\text{s} \wedge |s| \equiv 2 \bmod 3 \end{cases}$$

The implication on the first line ensures that all strings with an even number of as and a length that is a multiple of 3 end up in state $(E, 0)$. The contrapositive of the implications on the other lines ensure that any string that does not drive the machines to one of those 5 states must have an even number of as and a length that is a multiple of 3. Hence $M_{1\wedge 2}$ accepts $L_1 \cap L_2$