# CSC236 tutorial exercises, Week #7
## sample solution

These exercises are intended to give you practice proving facts about a recurrence.

1. Examine the recurrence $R(n)$ below.

$$R(n) = \begin{cases} 0 & \text{if } n = 1 \\ n + 3R(\lceil n/3 \rceil) & \text{if } n > 1 \end{cases}$$

Last week we conjectured that if $n = 3^k$ for some natural number $k$, then $R(n) = n \log_3 n$.

(a) Use induction to prove this conjecture.

**sample solution:** Define $P(k) : R(3^k) = k3^k$. Note that when $n = 3^k$ this is equivalent to $R(n) = n \log_3 n$. I will prove this by simple induction on $k$:

**base case:** $R(3^0) = R(1) = 0 = 0 \times 3^0$, so $P(0)$ holds.

**induction step:** Let $k \in \mathbb{N}$. Assume $P(k)$, that is $R(3^k) = k3^k$. I will show that $P(k+1)$ follows, that is $R(3^{k+1}) = (k+1)3^{k+1}$. Since $k + 1 > 0$, $3^{k+1} > 1$, so by definition:

$$\begin{aligned} R(3^{k+1}) &= 3^{k+1} + 3R(3^k) && \text{\# by function definition} \\ &= 3^{k+1} + 3(k3^k) && \text{\# by IH } P(k) \\ &= 3^{k+1} + k3^{k+1} = (k+1)3^{k+1} && \blacksquare \end{aligned}$$

(b) Emulate the lemma 3.6 on page 84 of the CSC236 notes, to prove that $R$ is nondecreasing on natural numbers greater than, or equal to, 1.

**sample solution:** Define:

$$P(n) : \bigwedge_{m=1}^{m=n} R(m) \leq R(n)$$

I will prove $\forall n \in \mathbb{N}^+, P(n)$ using complete induction. I will assume, without proof, that if $n$ is a natural number greater than 1, then $n > \lceil n/3 \rceil \geq 1$.

**inductive step:** Let $n \in \mathbb{N}^+$. Assume $\wedge_{i=1}^{i=n-1} P(i)$. I will show that $P(n)$ follows.

**base case** $n < 3$: $R(1) = 0$, which is no smaller than the value of $R$ on any smaller positive natural number — since there are no smaller natural numbers — so $P(1)$ holds. $R(1) = 0 \leq 2 = R(2)$, so $P(2)$ holds since 1 is the only positive natural number less than 2.

**case** $n \geq 3$: Since $n > 2$ we know that $n > n - 1 > 1$ so by our IH we know that $P(n-1)$ holds and (by transitivity of $\leq$) we need only show that $R(n-1) \leq R(n)$. Since $n - 1 > 1$ we have

$$\begin{aligned} R(n-1) &= (n-1) + 3R(\lceil (n-1)/3 \rceil) && \text{\# by definition of } R\text{, since } n-1 > 1 \\ &\leq n + 3R(\lceil n/3 \rceil) && \text{\# by } P(\lceil n/3 \rceil), 1 \leq \lceil (n-1)/3 \rceil \leq \lceil n/3 \rceil < n \\ &= R(n) && \blacksquare \end{aligned}$$