CSC165 Mathematical Expression and Reasoning for Computer Science

Module 13

Proofs About Non-Boolean Functions

Non-Boolean Functions

- Boolean function: a function whose return value is True or False
- Non-Boolean function: a function whose return value is not True or False
- Boolean or non-Boolean?
  - $x > 5$
  - x^2
  - $x^2 \neq x$
  - $|x|$
  - $\sin x$
- Remember that quantifiers are only applied to variables, not to functions
  - You cannot say: $\forall |x| \in \mathbb{R}$

Floor Function

- Floor of real number x: the largest integer that is $\leq x$
- x: real number $\in \mathbb{R}$
- $\lfloor x \rfloor$: integer number $\in \mathbb{Z}$
- $\lfloor x \rfloor \leq x$
- For all integers less than or equal x, $\lfloor x \rfloor$ is the largest

x	$\lfloor x \rfloor$
2	2
2.4	2
2.99	2
-2	-2
-2.4	-3
-2.99	-3

Definition of Floor Function

- $\forall x \in \mathbb{R}: [(y = \lfloor x \rfloor) \leftrightarrow \big((y \in \mathbb{Z}) \land (y \leq x) \land (\forall z \in \mathbb{Z}: [(z \leq x) \rightarrow (z \leq y)])\big)]$

- $y = \lfloor x \rfloor$ is an integer
- y is less than or equal to x
- Among all integers that are $\leq x$, y is the largest one

Example

- Prove $\forall x \in \mathbb{R}: [\lfloor x \rfloor < x + 1]$
- Thoughts:
  - By definition, "$\lfloor x \rfloor$ is the largest integer that is $\leq x$"
  - $(\lfloor x \rfloor \leq x) \rightarrow \big((\lfloor x \rfloor < x) \lor (\lfloor x \rfloor = x)\big)$
  - Two cases:
  - $(\lfloor x \rfloor < x) \rightarrow (\lfloor x \rfloor < x + 1)$
  - $(\lfloor x \rfloor = x) \rightarrow (\lfloor x \rfloor < x + 1)$

Proof: $\forall x \in \mathbb{R}: [\lfloor x \rfloor < x + 1]$

Let $x \in \mathbb{R}$. # x is a typical element of domain D
Then $\lfloor x \rfloor \leq x$. # by definition of floor function
Then $(\lfloor x \rfloor < x) \vee (\lfloor x \rfloor = x)$. # two (all) cases to consider
Case 1: Assume $\lfloor x \rfloor < x$. # case 1
Then $\lfloor x \rfloor < x + 1$. # conclusion is true for this case
Case 2: Assume $\lfloor x \rfloor = x$. # case 2
Then $\lfloor x \rfloor < x + 1$. # conclusion is true for this case
Then $\lfloor x \rfloor < x + 1$. # conclusion is true for all cases
Therefore, $\forall x \in \mathbb{R}: [\lfloor x \rfloor < x + 1]$. # introduce universal quantifier

Example

- Prove $\forall x \in \mathbb{R}: [\lfloor x \rfloor > x - 1]$
- Thoughts:
  - $(\lfloor x \rfloor \in \mathbb{Z}) \wedge (\lfloor x \rfloor \leq x) \wedge (\forall z \in \mathbb{Z}: [(z \leq x) \rightarrow (z \leq \lfloor x \rfloor)])$
  - Contrapositive of $\forall z \in \mathbb{Z}: [(z \leq x) \rightarrow (z \leq \lfloor x \rfloor)]$ is $\forall z \in \mathbb{Z}: [(z > \lfloor x \rfloor) \rightarrow (z > x)]$
  - $\forall z \in \mathbb{Z}: [(z > \lfloor x \rfloor) \rightarrow (z > x)]$ is true for all $z \in \mathbb{Z}$
  - Especially, if $z = \lfloor x \rfloor + 1$, then $z > \lfloor x \rfloor$ and $z \in \mathbb{Z}$
  - Consequently, $(\lfloor x \rfloor + 1 > \lfloor x \rfloor) \rightarrow (\lfloor x \rfloor + 1 > x)$
  - Then, $(\lfloor x \rfloor + 1 > \lfloor x \rfloor) \rightarrow (\lfloor x \rfloor > x - 1)$

Proof: $\forall x \in \mathbb{R}: [\lfloor x \rfloor > x - 1]$

Let $x \in \mathbb{R}$. # x is a typical element of domain D

Then $\lfloor x \rfloor \in \mathbb{Z}$. # by definition of floor function

Then $\lfloor x \rfloor + 1 \in \mathbb{Z}$. # $\mathbb{Z}$ is closed under summation

Then $\lfloor x \rfloor + 1 > \lfloor x \rfloor$. # add $\lfloor x \rfloor$ to $1 > 0$

Then $\lfloor x \rfloor + 1 > x$. # contrapositive of $\forall z \in \mathbb{Z}: [(z \leq x) \rightarrow (z \leq \lfloor x \rfloor)]$

Then $\lfloor x \rfloor > x - 1$. # subtract 1

Therefore, $\forall x \in \mathbb{R}: [\lfloor x \rfloor > x - 1]$. # introduce universal quantifier

Ceiling Function

- Ceiling of real number x: the smallest integer that is $\geq x$

- x: real number $\in \mathbb{R}$
- $\lceil x \rceil$: integer number $\in \mathbb{Z}$
- $\lceil x \rceil \geq x$
- For all integers greater than or equal x, $\lceil x \rceil$ is the smallest

x	$\lceil x \rceil$
2	2
2.4	3
2.99	3
-2	-2
-2.4	-2
-2.99	-2

Definition of Ceiling Function

- $\forall x \in \mathbb{R}: [(y = \lceil x \rceil) \leftrightarrow \big((y \in \mathbb{Z}) \wedge (y \geq x) \wedge (\forall z \in \mathbb{Z}: [(z \geq x) \rightarrow (z \geq y)])\big)]$

- $y = \lceil x \rceil$ is an integer
- y is greater than or equal to x
- Among all integers that are $\geq x$, y is the smallest one

Example

- Prove $\forall x \in \mathbb{R}: [\lceil x \rceil > x - 1]$
- Thoughts:
  - By definition, "$\lceil x \rceil$ is the smallest integer that is $\geq x$"
  - $(\lceil x \rceil \geq x) \rightarrow \big((\lceil x \rceil > x) \vee (\lceil x \rceil = x)\big)$
  - Two cases:
  - $(\lceil x \rceil > x) \rightarrow (\lceil x \rceil > x - 1)$
  - $(\lceil x \rceil = x) \rightarrow (\lceil x \rceil > x - 1)$

Proof: $\forall x \in \mathbb{R}: [\lceil x \rceil > x - 1]$

Let $x \in \mathbb{R}$. # x is a typical element of domain D

Then $\lceil x \rceil \geq x$. # by definition of ceiling function

Then $(\lceil x \rceil > x) \vee (\lceil x \rceil = x)$. # two (all) cases to consider

Case 1: Assume $\lceil x \rceil > x$. # case 1

Then $\lceil x \rceil > x - 1$. # conclusion is true for this case

Case 2: Assume $\lceil x \rceil = x$. # case 2

Then $\lceil x \rceil > x - 1$. # conclusion is true for this case

Then $\lceil x \rceil > x - 1$. # conclusion is true for all cases

Therefore, $\forall x \in \mathbb{R}: [\lceil x \rceil > x - 1]$. # introduce universal quantifier

Example

- Prove $\forall x \in \mathbb{R}: [\lceil x \rceil < x + 1]$
- Thoughts:
  - $(\lceil x \rceil \in \mathbb{Z}) \wedge (\lceil x \rceil \geq x) \wedge (\forall z \in \mathbb{Z}: [(z \geq x) \rightarrow (z \geq \lceil x \rceil)])$
  - Contrapositive of $\forall z \in \mathbb{Z}: [(z \geq x) \rightarrow (z \geq \lceil x \rceil)]$ is $\forall z \in \mathbb{Z}: [(z < \lceil x \rceil) \rightarrow (z < x)]$
  - $\forall z \in \mathbb{Z}: [(z < \lceil x \rceil) \rightarrow (z < x)]$ is true for all $z \in \mathbb{Z}$
  - Especially, if $z = \lceil x \rceil - 1$, then $z < \lceil x \rceil$ and $z \in \mathbb{Z}$
  - Consequently, $(\lceil x \rceil - 1 < \lceil x \rceil) \rightarrow (\lceil x \rceil - 1 < x)$
  - Then, $(\lceil x \rceil - 1 < \lceil x \rceil) \rightarrow (\lceil x \rceil < x + 1)$

Proof: $\forall x \in \mathbb{R}: [\lceil x \rceil < x + 1]$

Let $x \in \mathbb{R}$. # x is a typical element of domain D

Then $\lceil x \rceil \in \mathbb{Z}$. # by definition of floor function

Then $\lceil x \rceil - 1 \in \mathbb{Z}$. # $\mathbb{Z}$ is closed under summation

Then $\lceil x \rceil - 1 < \lceil x \rceil$. # add $\lceil x \rceil$ to $-1 < 0$

Then $\lceil x \rceil - 1 < x$. # contrapositive of $\forall z \in \mathbb{Z}: [(z \geq x) \rightarrow (z \geq \lceil x \rceil)]$

Then $\lceil x \rceil < x + 1$. # add 1

Therefore, $\forall x \in \mathbb{R}: [\lceil x \rceil < x + 1]$. # introduce universal quantifier

Summary

- We know that for real number x:

$x - 1 < x < x + 1$

- We found that:
  - $\lfloor x \rfloor \leq x$
  - $\lfloor x \rfloor < x + 1$
  - $\lfloor x \rfloor > x - 1$
  - Then $x - 1 < \lfloor x \rfloor \leq x < x + 1$

- We also found that:
  - $\lceil x \rceil \geq x$
  - $\lceil x \rceil > x - 1$
  - $\lceil x \rceil < x + 1$
  - Then $x - 1 < x \leq \lceil x \rceil < x + 1$

- We can tell that

$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

Summary