

**MST Construction Theorem**

The following theorem shows how one can extend any spanning forest of a graph  $G$  that is contained in some MST of  $G$ , into a *larger* forest that is also contained in some MST of  $G$ . Many algorithms use this theorem to construct an MST efficiently: Roughly speaking, they start from the trivial forest of  $G$  with no edges (i.e., this initial forest of  $G$  consists of  $n$  trivial trees, each tree with just one node of  $G$ ), and then they “apply” this theorem  $n - 1$  times to grow this initial forest of  $G$  into a MST of  $G$ . Each application of the theorem extends the spanning forest by one edge, and the resulting MST has  $n - 1$  edges.

**Theorem:**

Let  $G = (V, E)$  be a connected undirected graph.

Let  $T_1 = (V_1, E_1), T_2 = (V_2, E_2), \dots, T_k = (V_k, E_k)$  be a spanning forest of  $G$ .

Suppose some MST  $T$  of  $G$  contains this spanning forest, i.e.,  $T$  contains all the edges in  $E_1 \cup E_2 \cup \dots \cup E_k$ .

Let  $(u, v)$  be an edge of minimum-weight among all edges such that  $u \in V_i$  and  $v \in V - V_i$  (i.e.,  $(u, v)$  is an edge with minimum weight among all the edges “out of  $T_i$ ”).

Then there is a MST  $T^*$  of  $G$  that contains all the edges in  $E_1 \cup E_2 \cup \dots \cup E_k \cup \{(u, v)\}$ .

**Proof (sketch):**

By assumption, there is an MST  $T = (V, E')$  that contains all the edges in  $E_1 \cup E_2 \cup \dots \cup E_k$ . There are 2 possible cases:

- (a)  $T$  also contains edge  $(u, v)$ . In this case,  $T^* = T$  and we are done.
- (b)  $T$  does not contain  $(u, v)$ . In this case, add edge  $(u, v)$  to  $T$ . We get graph  $T' = (V, E' \cup \{(u, v)\})$ . By Fact 2 (seen in class),  $T'$  has a unique cycle, and this cycle includes edge  $(u, v)$ . Since  $u \in V_i$  and  $v \in V - V_i$ , this cycle must also contain another edge  $(u', v')$  such that  $u' \in V_i$  and  $v' \in V - V_i$  (intuitively, this is because a cycle that has an edge “out of  $T_i$ ” must also have an edge “into  $T_i$ ”). Note that by the definition of  $(u, v)$ , we have  $w(u, v) \leq w(u', v')$ .

Now remove  $(u', v')$  from  $T'$ . By Fact 2, this cuts the unique cycle of  $T'$ , and we get back a spanning tree of  $G$ , denoted  $T^*$ . By construction,  $T^* = (V, E' \cup \{(u, v)\} - \{(u', v')\})$ .

Note that:

1.  $T^*$  contains all the edges in  $E_1 \cup E_2 \cup \dots \cup E_k$ . This is because  $T$ , and therefore  $T'$ , contain all these edges, and the only edge that we removed from  $T'$  to obtain  $T^*$ , namely,  $(u', v')$ , is not in  $E_1 \cup E_2 \cup \dots \cup E_k$ .
2.  $T^*$  contains  $(u, v)$ .
3. The weight of spanning tree  $T^*$  is  $w(T^*) = w(T) - w(u', v') + w(u, v)$ . Since  $w(u, v) \leq w(u', v')$ , we have  $w(T^*) \leq w(T)$ . In other words, the weight of  $T^*$  is less or equal to the weight of  $T$ . Since  $T$  is a *minimum* spanning tree of  $G$ , we conclude that  $T^*$  is also a *minimum* spanning tree of  $G$ .

By (3) above,  $T^*$  is an MST of  $G$ . Furthermore, by (1) and (2),  $T^*$  contains all the edges in  $E_1 \cup E_2 \cup \dots \cup E_k \cup \{(u, v)\}$ .

Q.E.D.