CSC165
Mathematical Expression and Reasoning for Computer Science

Module 10

Proof by Cases

Proof by Cases

- Prove $\forall x \in D: \big(P(x) \lor Q(x)\big) \rightarrow R(x)$
- Split your argument into differences cases
- Prove the conclusion for each case
- Sometimes the different cases are implicit
- What makes a valid proof?
  - If the union of the cases is covering all possibilities

Proof Structure

- Prove $\forall x \in D: \big(P(x) \lor Q(x)\big) \rightarrow R(x)$
- Generic proof:

Let $x \in D$. # x is a typical element of domain D
    Assume $P(x) \lor Q(x)$. # assume antecedent true
        Case 1: Assume $P(x)$. # case 1: assume $P(x)$ to be true
            ⋮ # find the chain
            Then $R(x)$. # conclude $R(x)$ is true
        Case 2: Assume Q(x). # case 2: assume Q(x) to be true
            ⋮ # find the chain
            Then $R(x)$. # conclude $R(x)$ is true
        Then $R(x)$. # conclude $R(x)$ is true for both (all) cases
    Then $\big(P(x) \lor Q(x)\big) \rightarrow R(x)$. # since $\big(P(x) \lor Q(x)\big)$ and in both (all) cases we concluded $R(x)$
Therefore, $\forall x \in D: \big(P(x) \lor Q(x)\big) \rightarrow R(x)$. # introduce universal quantifier

Example

- Prove $\forall x \in \mathbb{R}: \big((x \le 0) \lor (x \ge 1)\big) \rightarrow (x^2 \ge x)$
- Consider two cases for x:
  - $x \le 0$
  - $x \ge 1$
- For each case, prove that $x^2 \ge x$

Proof: $\forall x \in \mathbb{R}: \big((x \le 0) \lor (x \ge 1)\big) \rightarrow (x^2 \ge x)$

Let $x \in \mathbb{R}$. # x is a typical element of domain $\mathbb{R}$
 Assume $(x \le 0) \lor (x \ge 1)$. # two cases that cover all possibilities
  Case 1: Assume $x \le 0$. # case 1: $x \le 0$
   Then $x^2 \ge 0$. # find the chain
   Then $x^2 \ge x$. # conclusion is true for this case
  Case 2: Assume $x \ge 1$. # case 2: $x \ge 1$
   Then $x.x \ge x$. # find the chain
   Then $x^2 \ge x$. # conclusion is true for this case
  Then $x^2 \ge x$. # conclusion is true for both (all) cases
 Then $(x \le 0) \lor (x \ge 1) \rightarrow x^2 \ge x$. # implication is satisfied
Therefore, $\forall x \in \mathbb{R}: \big((x \le 0) \lor (x \ge 1)\big) \rightarrow (x^2 \ge x)$. # introduce universal quantifier

Example

- Prove $\forall n \in \mathbb{N}: n^2 + n$ is even
- Thoughts:
  - $n^2 + n = n(n+1)$
  - n and $(n+1)$ are consecutive numbers
  - If one of them is an odd number, then the other is an even number
- Consider two cases for n:
  - n is even, then $(n+1)$ is odd. Prove $n(n+1)$ is even
  - n is odd, then $(n+1)$ is even. Prove $n(n+1)$ is even

Proof: $\forall n \in \mathbb{N}: n^2 + n$ is even

Let $n \in \mathbb{N}$. # n is a typical element of domain D

Then (n is even) $\vee$(n is odd). # two cases that cover all possibilities

Case 1: Assume n is even. # case 1: even number

Then $\exists j \in \mathbb{N}: n = 2j$. # definition of even number

Let $j_0 \in \mathbb{N}$ such that $n = 2j_0$.

Then $n^2 + n = (2j_0)^2 + 2j_0 = 2(2j_0^2 + j_0)$. # find the chain

Let $k_0 = 2j_0^2 + j_0$.

Then $k_0 \in \mathbb{N}$.

Then $n^2 + n = 2k_0$. # definition of even number

Then $\exists k \in \mathbb{N}: n^2 + n = 2k$. # definition of even number

Then $n^2 + n$ is even. # conclusion is true for this case

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Proof: $\forall n \in \mathbb{N}: n^2 + n$ is even

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Case 2: Assume n is odd. # case 2: odd number

Then $\exists j \in \mathbb{N}: n = 2j + 1$. # definition of odd number

Let $j_1 \in \mathbb{N}$ such that $n = 2j_1 + 1$.

Then $n^2 + n = (2j_1 + 1)^2 + 2j_1 + 1 = 2(2j_1 + 1)(j_1 + 1)$. # find the chain

Let $k_1 = (2j_1 + 1)(j_1 + 1)$.

Then $k_1 \in \mathbb{N}$.

Then $n^2 + n = 2k_1$.

Then $\exists k \in \mathbb{N}: n^2 + n = 2k$. # definition of even number

Then $n^2 + n$ is even. # conclusion is true for this case

Then $n^2 + n$ is even. # conclusion is true for both (all) cases

Therefore, $\forall n \in \mathbb{N}: n^2 + n$ is even. # introduce universal quantifier

Examples

Remainder

- If a and d are integers, with d nonzero, it can be proven that there exist unique integers q and r, such that $a = qd + r$ and $0 \leq r < |d|$
- q is called the quotient
- r is called the remainder: $r = a - qd$
- Let $r(a, d)$ denote the remainder after division of a by d
- Examples:

a	d	$a = qd + r$	$r(a, d)$
43	5	$43 = 8(5) + 3$	3
43	-5	$43 = (-8)(-5) + 3$	3
-43	5	$-43 = (-9)(5) + 2$	2
-43	-5	$-43 = 9(-5) + 2$	2

Example

- Prove $\forall n \in \mathbb{N}$: $\left[\text{remainder of } \frac{n^2}{4} \text{ is either 0 or 1}\right]$
- Prove $\forall n \in \mathbb{N}$: $[r(n^2, 4)$ is either 0 or 1$]$

n	n^2	$n^2 = 4q + r$	$r(n^2, 4)$	$r(n^2, 4) = 0?$	$r(n^2, 4) = 1?$	$r(n^2, 4) = 0$ or $r(n^2, 4) = 1$	**Observation**
0	0	$0 = 4(0) + 0$	0	T	F	T	n is even
1	1	$1 = 4(0) + 1$	1	F	T	T	n is odd
2	4	$4 = 4(1) + 0$	0	T	F	T	n is even
3	9	$9 = 4(2) + 1$	1	F	T	T	n is odd
4	16	$16 = 4(4) + 0$	0	T	F	T	n is even
5	25	$25 = 4(6) + 1$	1	F	T	T	n is odd

Prove: $\forall n \in \mathbb{N}$: $[r(n^2, 4)$ is 0 or 1$]$

- Observations:
  - When n is even, $r(n^2, 4) = 0$
  - When n is odd, $r(n^2, 4) = 1$
- What are the cases here?
  - n is even
  - n is odd
  - $\forall n \in \mathbb{N}$: (n is even) $\vee$ (n is odd)

Proof: $\forall n \in \mathbb{N}$: $[r(n^2, 4)$ is 0 or 1$]$

Let $n \in \mathbb{N}$. # n is a typical element of domain D

Then (n is even) $\vee$(n is odd). # two cases that cover all possibilities

Case 1: Assume n is even. # case 1: even number

Then $\exists j \in \mathbb{N}: n = 2j$. # definition of even number

Let $j_0 \in \mathbb{N}$ such that $n = 2j_0$.

Then $n^2 = (2j_0)^2 = 4j_0^2$. # find the chain

Then $\frac{n^2}{4} = j_0^2$. # find the chain

Then $r(n^2, 4) = 0$. # conclusion is true for this case

Case 2: Assume n is odd. # case 2: odd number

Then $\exists j \in \mathbb{N}: n = 2j + 1$. # definition of odd number

Let $j_0 \in \mathbb{N}$ such that $n = 2j_0 + 1$.

Then $n^2 = (2j_0 + 1)^2 = 4j_0^2 + 4j_0 + 1$. # find the chain

Then $\frac{n^2}{4} = j_0^2 + j_0 + \frac{1}{4}$. # find the chain

Then $r(n^2, 4) = 1$. # conclusion is true for this case

Then $r(n^2, 4)$ is 0 or 1. # conclusion is true for both (all) cases

Therefore, $\forall n \in \mathbb{N}: r(n^2, 4)$ is 0 or 1. # introduce universal quantifier

Example

- Prove $\forall n \in \mathbb{N}$: if n is not divisible by 5, the remainder of $\frac{n^2}{5}$ is 1 or 4
- Thoughts:
  - n is not divisible by 5: n is not a multiple of 5
  - n is a multiple of 5: $\exists j \in \mathbb{N}: n = 5j$
  - n is not a multiple of 5: $\forall j \in \mathbb{N}: n \neq 5j$
  - $\forall n \in \mathbb{N}: [(\forall j \in \mathbb{N}: n \neq 5j) \rightarrow (r(n^2, 5)$ is 1 or 4$)]$
  - n is divisible by 5: remainder of $\frac{n}{5} = 0$
  - n is not divisible by 5: remainder of $\frac{n}{5} \neq 0$
  - $\forall n \in \mathbb{N}: [(r(n, 5) \neq 0) \rightarrow (r(n^2, 5)$ is 1 or 4$)]$
  - $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5) = 1) \vee (r(n^2,5) = 4)\right)\right]$

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Prove: $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5) = 1) \vee (r(n^2,5) = 4)\right)\right]$

n	$r(n,5)$	$r(n,5) \neq 0?$	n^2	$r(n^2,5)$	$r(n^2,5) = 1?$	$r(n^2,5) = 4?$	$r(n^2,5) = 1 \text{ or } 4?$	**Significance**
0	0	F	0	0	F	F	F	Irrelevant
1	1	T	1	1	T	F	T	Okay
2	2	T	4	4	F	T	T	Okay
3	3	T	9	4	F	T	T	Okay
4	4	T	16	1	T	F	T	Okay
5	0	F	25	0	F	F	F	Irrelevant
6	1	T	36	1	T	F	T	Okay
7	2	T	49	4	F	T	T	Okay
8	3	T	64	4	F	T	T	Okay
9	4	T	81	1	T	F	T	Okay
10	0	F	100	0	F	F	F	Irrelevant

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Prove: $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5) = 1) \vee (r(n^2,5) = 4)\right)\right]$

- Observations:
  $(r(n,5) \neq 0) \rightarrow \left((r(n,5) = 1) \vee (r(n,5) = 2) \vee (r(n,5) = 3) \vee (r(n,5) = 4)\right)$
- Four cases to consider:
  - $r(n,5) = 1: \exists j \in \mathbb{N}: n = 5j + 1$
  - $r(n,5) = 2: \exists j \in \mathbb{N}: n = 5j + 2$
  - $r(n,5) = 3: \exists j \in \mathbb{N}: n = 5j + 3$
  - $r(n,5) = 4: \exists j \in \mathbb{N}: n = 5j + 4$

Proof: $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5) = 1) \vee (r(n^2,5) = 4)\right)\right]$

Let $n \in \mathbb{N}$. # n is a typical element of domain D

Assume $r(n,5) \neq 0$. # assume antecedent

Then $\left((r(n,5) = 1) \vee (r(n,5) = 2) \vee (r(n,5) = 3) \vee (r(n,5) = 4)\right)$. # four cases

Case 1: Assume $r(n,5) = 1$. # case 1

Then $\exists j \in \mathbb{N}: n = 5j + 1$. # definition

Let $j_0 \in \mathbb{N}$ such that $n = 5j_0 + 1$.

Then $n^2 = 25j_0^2 + 10j_0 + 1$. # find the chain

Then $\frac{n^2}{5} = 5j_0^2 + 2j_0 + \frac{1}{5}$. # find the chain

Then $r(n^2,5) = 1$. # conclusion is true for this case

Case 2: Assume $r(n,5) = 2$. # case 2

Then $\exists j \in \mathbb{N}: n = 5j + 2$. # definition

Let $j_0 \in \mathbb{N}$ such that $n = 5j_0 + 2$.

Then $n^2 = 25j_0^2 + 20j_0 + 4$. # find the chain

Then $\frac{n^2}{5} = 5j_0^2 + 4j_0 + \frac{4}{5}$. # find the chain

Then $r(n^2,5) = 4$. # conclusion is true for this case

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Proof: $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5)=1) \vee (r(n^2,5)=4)\right)\right]$

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Case 3: Assume $r(n,5) = 3$. # case 3

Then $\exists j \in \mathbb{N}: n = 5j + 3$. # definition

Let $j_0 \in \mathbb{N}$ such that $n = 5j_0 + 3$.

Then $n^2 = 25j_0^2 + 30j_0 + 9$. # find the chain

Then $\frac{n^2}{5} = 5j_0^2 + 6j_0 + 1 + \frac{4}{5}$. # find the chain

Then $r(n^2,5) = 4$. # conclusion is true for this case

Case 4: Assume $r(n,5) = 4$. # case 4

Then $\exists j \in \mathbb{N}: n = 5j + 4$. # definition

Let $j_0 \in \mathbb{N}$ such that $n = 5j_0 + 4$.

Then $n^2 = 25j_0^2 + 40j_0 + 16$. # find the chain

Then $\frac{n^2}{5} = 5j_0^2 + 8j_0 + 3 + \frac{1}{5}$. # find the chain

Then $r(n^2,5) = 1$. # conclusion is true for this case

Then $(r(n^2,5) = 1) \vee (r(n^2,5) = 4)$. # conclusion is true for all cases

Then $(r(n,5) \neq 0) \rightarrow \left((r(n^2,5)=1) \vee (r(n^2,5)=4)\right)$. # implication is satisfied

Therefore, $\forall n \in \mathbb{N}: \left[(r(n,5) \neq 0) \rightarrow \left((r(n^2,5)=1) \vee (r(n^2,5)=4)\right)\right]$. # introduce universal quantifier

Absolute Value

- The absolute value of a real number x is the non-negative value of x without regard to its sign
- $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$
- Examples:
  - $|2| = 2$
  - $|-2| = 2$
  - $|0| = 0$
- $|x-y| = \begin{cases} (x-y), & \text{if } (x-y) \geq 0 \\ -(x-y), & \text{if } (x-y) < 0 \end{cases}$
- $|x-y| = \begin{cases} x-y, & \text{if } x \geq y \\ y-x, & \text{if } x < y \end{cases}$

Example

- Prove $\forall x \in \mathbb{R}: [x \leq |x|]$
- Thoughts:
  - $|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$
  - $(x < 0) \rightarrow (|x| = -x > x) \rightarrow (x < |x|) \rightarrow (x \leq |x|)$
  - $(x \geq 0) \rightarrow (|x| = x) \rightarrow (x \leq |x|)$
- What are the cases?
  - $\forall x \in \mathbb{R}: (x < 0) \vee (0 \leq x)$
  - Consider two cases that cover all the possibilities of $x \in \mathbb{R}$
  - For each case, we need to prove that $x \leq |x|$

Proof: $\forall x \in \mathbb{R}: [x \leq |x|]$

Let $x \in \mathbb{R}$. # x is a typical element of domain D

  Then $(x < 0) \vee (0 \leq x)$. # two cases that cover all possibilities

  Case 1: Assume $x < 0$. # case 1

    Then $x < |x|$. # find the chain

    Then $x \leq |x|$. # conclusion is true for this case

  Case 2: Assume $0 \leq x$. # case 2:

    Then $x = |x|$. # find the chain

    Then $x \leq |x|$. # conclusion is true for this case

  Then $x \leq |x|$. # conclusion is true for all cases

Therefore, $\forall x \in \mathbb{R}: [x \leq |x|]$. # introduce universal quantifier

Example

- Prove $\forall x \in \mathbb{R}: [-5 \leq |x+2| - |x-3| \leq 5]$
- Thoughts:
  - $|x+2| = \begin{cases} -(x+2), & \text{if } (x+2) < 0 \\ x+2, & \text{if } (x+2) \geq 0 \end{cases}$
  - $|x+2| = \begin{cases} -(x+2), & \text{if } x < -2 \\ x+2, & \text{if } x \geq -2 \end{cases}$
  - $|x-3| = \begin{cases} -(x-3), & \text{if } (x-3) < 0 \\ x-3, & \text{if } (x-3) \geq 0 \end{cases}$
  - $|x-3| = \begin{cases} -(x-3), & \text{if } x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases}$
  - $|x+2| - |x-3| = \begin{cases} -(x+2) - (-(x-3)), & \text{if } x < -2 \\ (x+2) - (-(x-3)), & \text{if } -2 \leq x < 3 \\ (x+2) - (x-3), & \text{if } x \geq 3 \end{cases}$
  - $|x+2| - |x-3| = \begin{cases} -5, & \text{if } x < -2 \\ 2x-1, & \text{if } -2 \leq x < 3 \\ 5, & \text{if } x \geq 3 \end{cases}$

Prove: $\forall x \in \mathbb{R}: [-5 \leq |x+2| - |x-3| \leq 5]$

- $|x+2| - |x-3| = \begin{cases} -5, & \text{if } x < -2 \\ 2x-1, & \text{if } -2 \leq x < 3 \\ 5, & \text{if } x \geq 3 \end{cases}$
- What are the cases?
  - $\forall x \in \mathbb{R}: (x < -2) \vee (-2 \leq x < 3) \vee (x \geq 3)$
  - Consider three cases that cover all the possibilities of $x \in \mathbb{R}$
  - For each case, we need to prove that $-5 \leq |x+2| - |x-3| \leq 5$
  - That is: prove $(-5 \leq |x+2| - |x-3|) \wedge (|x+2| - |x-3| \leq 5)$

Proof: $\forall x \in \mathbb{R}: [-5 \leq |x+2| - |x-3| \leq 5]$

Let $x \in \mathbb{R}$.	# x is a typical element of domain D
Then $(x < -2) \vee (-2 \leq x < 3) \vee (x \geq 3)$.	# three cases that cover all possibilities
Case 1: Assume $x < -2$.	# case 1
Then $|x+2| - |x-3| = -5$.	# find the chain
Then $-5 \leq |x+2| - |x-3|$.	# find the chain
Then $-5 \leq |x+2| - |x-3| \leq 5$.	# conclusion is true for this case
Case 2: Assume $-2 \leq x < 3$.	# case 2
Then $|x+2| - |x-3| = 2x - 1$.	# find the chain
Also $-4 \leq 2x < 6$.	# find the chain
Then $-5 \leq 2x - 1 < 5$.	# find the chain
Then $-5 \leq |x+2| - |x-3| < 5$.	# find the chain
Then $-5 \leq |x+2| - |x-3| \leq 5$.	# conclusion is true for this case
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Proof: $\forall x \in \mathbb{R}: [-5 \leq |x+2| - |x-3| \leq 5]$

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Case 3: Assume $x \geq 3$.	# case 3
Then $|x+2| - |x-3| = 5$.	# find the chain
Then $|x+2| - |x-3| \leq 5$.	# find the chain
Then $-5 \leq |x+2| - |x-3| \leq 5$.	# conclusion is true for this case
Then $-5 \leq |x+2| - |x-3| \leq 5$.	# conclusion is true for all cases
Therefore, $\forall x \in \mathbb{R}: [-5 \leq |x+2| - |x-3| \leq 5]$.	# introduce universal quantifier

Practice

- $\forall x \in \mathbb{R}: \left[|x| < 4 \rightarrow \left((-4 < x) \wedge (4 > x)\right)\right]$

another way to write it is $\forall x \in \mathbb{R}: [|x| < 4 \rightarrow (-4 < x < 4)]$

- $\forall x \in \mathbb{R}: \left[|x| > 4 \rightarrow \left((-4 > x) \vee (4 < x)\right)\right]$

- $\forall x \in \mathbb{R}: \left[|x+1| < 4 \rightarrow \left((-5 < x) \wedge (3 > x)\right)\right]$

another way to write it is $\forall x \in \mathbb{R}: [|x+1| < 4 \rightarrow (-5 < x < 3)]$

- $\forall x \in \mathbb{R}: \left[|x+1| > 4 \rightarrow \left((-5 > x) \vee (3 < x)\right)\right]$

Maximum Value

- $\forall x, y \in \mathbb{R}: \max(x, y) = \begin{cases} x, & \text{if } x \geq y \\ y, & \text{if } x < y \end{cases}$
- Examples:
  - $\max(2,3) = 3$
  - $\max(2, -3) = 2$
  - $\max(-2,3) = 3$
  - $\max(-2, -3) = -2$
  - $\max(-2, -2) = -2$

Example

- Prove $\forall x, y \in \mathbb{R}: \max(x, y) = \frac{x+y+|x-y|}{2}$
- Thoughts:
- $x \geq y$:
  - $\max(x, y) = x$
  - $|x - y| = x - y$
  - $\frac{x+y+|x-y|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x = \max(x, y)$
- $x < y$:
  - $\max(x, y) = y$
  - $|x - y| = y - x$
  - $\frac{x+y+|x-y|}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y = \max(x, y)$

- What are the cases?
  - $\forall x, y \in \mathbb{R}: (x \geq y) \lor (x < y)$
  - Consider two cases that cover all the possibilities of $x, y \in \mathbb{R}$
  - For each case, we need to prove that $\max(x, y) = \frac{x+y+|x-y|}{2}$

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Proof: $\forall x, y \in \mathbb{R}: \max(x, y) = \frac{x+y+|x-y|}{2}$

Let $x, y \in \mathbb{R}$.	# x, y are typical elements of domain D
Then $(x \geq y) \lor (x < y)$.	# two cases that cover all possibilities
Case 1: Assume $x \geq y$.	# case 1
Then $\max(x, y) = x$.	# find the chain
Then $\|x - y\| = x - y$.	# find the chain
Then $\frac{x+y+\|x-y\|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$.	# find the chain
Then $\max(x, y) = \frac{x+y+\|x-y\|}{2}$.	# conclusion is true for this case
Case 2: Assume $x < y$.	# case 2
Then $\max(x, y) = y$.	# find the chain
Then $\|x - y\| = y - x$.	# find the chain
Then $\frac{x+y+\|x-y\|}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y$.	# find the chain
Then $\max(x, y) = \frac{x+y+\|x-y\|}{2}$.	# conclusion is true for this case
Then $\max(x, y) = \frac{x+y+\|x-y\|}{2}$.	# conclusion is true for all cases
Therefore, $\forall x, y \in \mathbb{R}: \max(x, y) = \frac{x+y+\|x-y\|}{2}$.	# introduce universal quantifier

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