UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER Fall 2015 Final Examination

CSC 263H1F

Duration - 3 hours

No Aids Allowed

PLEASE COMPLETE THE SECTION BELOW.

First (Given) Name:

Last (Family) Name:

Student Number:

Exam Instructions and Course Policy

- Check that your exam has 21 pages (including this cover page and 5 blank pages at the end of this book). Bring any discrepancy to the attention of an invigilator.
- There are ten questions worth a total of 112 points.
- For rough work, use the backs of the pages or the last 5 pages; these will not be marked.
- The exam will be graded for correctness, completeness, clarity and conciseness.
- Course policy reminder: a mark of at least 40% on the final exam is necessary to pass the course.
- Unless stated otherwise, you can use standard data structures or algorithms discussed in class or in tutorial and you do not need to describe their implementation. Simply state which algorithm or data structure you are using. You may also state its complexity without proof. If you modify a data structure or an algorithm from class, you must describe the modification and its effects.
- DFS stands for depth-first search, BFS stands for depth-first search, MST stands for minimum spanning tree.

Good Luck!

This page is for marker's use only. Do NOT use it for answering or for scratch paper.

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1. (14 points, 2 points each) Circle the correct answer; no justification is necessary.

   (a) [✓ True] or False: Let X and Y be two random variables. Then $E[X+Y] = E[X] + E[Y]$.

   (b) [✓ True] of False: The worst case complexity is $O(\log n)$ for performing a search in an AVL tree.

   (c) True or [✓ False]: The worst case complexity is $O(\log n)$ for search in a binary search tree.

   (d) True or [✓ False]: The operations of delete and insert applied to a binary search tree are commutative. That is, given a binary search tree T, deleting x and then delete y leaves the same tree as deleting y and then x.

   (e) Suppose we have n keys and they are searched for with equal probability. They are stored in a binary search tree that minimizes the expected number of nodes examined to find a key. The *worst-case* number of nodes examined to find a key in that tree is:

      i. $\Theta(1)$
      ii. $\Theta(n)$
      [✓ iii.] $\Theta(\log n)$
      iv. None of the above.

   (f) Suppose we insert n keys into a binary search tree. The first inserted key is 0, the second one is 1, and from then on, each inserted key is the average of the previous two. Thus the sequence of inserted keys begins as follows: $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots$. The height of the resulting tree is:

      i. $\Theta(1)$
      ii. $\Theta(\log n)$
      [✓ iii.] $\Theta(n)$
      iv. $\Theta(n \log n)$
      v. None of the above.

   (g) Let G be a connected, undirected graph with n nodes and m edges. Suppose its edges have distinct weights $1, 2, \ldots, m$. Then the cost of a minimum cost spanning tree of G must be exactly:

      i. $n-1$
      ii. n
      iii. $n(n-1)/2$
      iv. $n(n+1)/2$
      [✓ v.] None of the above.

2. (6 points, 2 points each) Consider the Randomized Quicksort algorithm where we pick the pivot element at random at each iteration. Answer each question below. No justification is necessary.

   (a) What is the worst-case complexity of Randomized Quicksort?

   O(n^2) : worst case

   (b) What is the best-case complexity of Randomized Quicksort?

   O(n log n) : equal split

   (c) What is the average-case complexity of Randomized Quicksort?

   O(n log n)

3. (10 points) Consider an algorithm that determines whether two n-bit numbers $x = x_{n-1}, \ldots, x_0$ and $y = y_{n-1}, \ldots, y_0$ are equal, by comparing them one bit at a time, from the most significant to the least significant bit, until the bits differ or until all bits have been compared. Suppose that x and y are each equally likely to be any n-bit binary number, and they are chosen independently.

   (a) Define your probability space.

   All pairs of n-bit number with equal probability

   (b) Define all necessary random variables.

   E1 ... En iid {0,1}

   (c) What is the expected number of bit comparisons for $n = 3$? Explain your answer.

   1.75

   1st bit must be compared
   2nd bit will be compared at 1/2 chance

   (d) What is the expected number of bit comparisons for arbitrary n? (Simplification is not necessary.) Explain your answer.

   1 + 1/2 + 1/4 + ... + 1/2^(n-1)
   1 - for the first bit
   1/2 for the second bit
   1/2^(i-1) for the i-th bit

4. (10 marks) Suppose we implement min-heap as an array. To perform an INSERT operation, if the array is not full we proceed as explained in class. If the array is full, we first move the elements of the heap from the current (full) array into a new array of twice the size and then we perform the insertion in the new (half-full) array. To perform an EXTRACTMIN operation, if the array is more than a quarter full we proceed as explained in class. If the array is only a quarter full, we first move the elements of the heap from the current (quarter-full) array into a new array of half the size, and then we perform the EXTRACTMIN in the new (half-full) array.

   Suppose that the heap is initially empty and we perform an arbitrary sequence of n INSERT and EXTRACTMIN operations.

   (a) What is the *worst-case* cost for an individual operation in such a sequence? Explain your answer.

   O(n)

(b) Use the accounting method to determine the *amortized* cost per operation in a sequence of n operations. State the cost of each operation, and the credit invariant, and explain why the credit invariant can be maintained.

Amortized cost: 2 + (log n for heap ops)

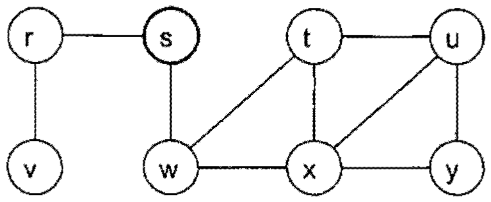
Let m be the array size:
Credit invariant: credit + 2(m - n) >= m AND credit + 2(n-m/4) >= m/4

We can give the array some credit in the beginning so it still hold.

We we insert or remove element we add 2 credit to the pool,
so the credit invariant still holds. YOU SHOULD EXPLAIN MORE!!!

When we are up-sizing, we have m element, which means we have m credit
and after we (potentially) exhaust all the credit, the invariant still hold since now
we have m' = 2m, which means 2(m'-m) = m' AND 2(m-m'/4) = m >= m'/4
When we are down-sizing, we have m/4 element, which means we have enough
credit to down size, after we (potentially) exhaust all the credit, we have m' = m/2
and now we have 2(m'-m/4)=m' AND 2(m/4 - m'/4) = m/4 >= m'/4

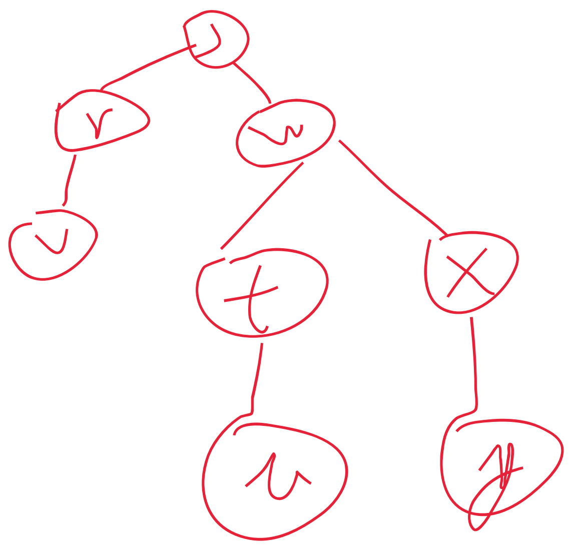
5. BFS/DFS problem (12 points)

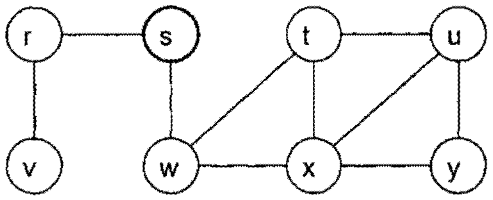


Assume that the graph above is given as an array of adjacency lists where the vertices are listed in alphabetical order, and for each vertex, the vertices in its adjacency list are listed in alphabetical order. For example, the adjacency list for vertex t contains the vertices u, w, x, in this order.

(a) Execute the BFS algorithm on the following undirected graph starting from vertex s. Draw the resulting BFS tree. What is the distance label $d[u]$ of vertex u in this BFS?

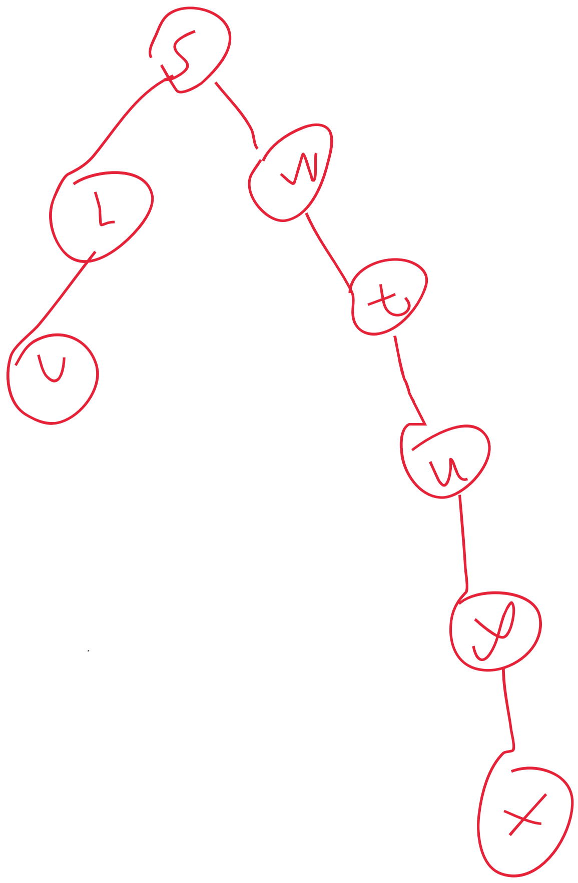
d[u] = 3



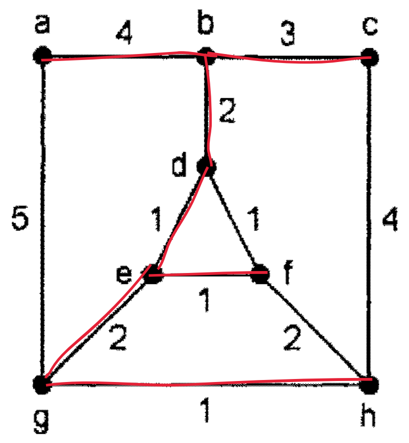


(b) Now execute the DFS algorithm on the same graph starting from vertex s. Draw the resulting DFS forest. What are the discovery and finish times of vertex u in this DFS? Draw the back, forward and cross edges (if any) with dotted lines and label them as back, forward, cross accordingly.

NOTE Back cross edges are not drawn



6. MST problem (10 points)



Execute Kruskal's MST algorithm on this graph and list the MST edges in the order they are included in the MST by the algorithm. Let the sorted order of the edges be:

$$(d,e), (g,h), (e,f), \cancel{(d,f)}, (b,d), (e,g), \cancel{(f,h)}, (b,c), \cancel{(a,b)}, (c,h), (a,g).$$

7. (16 points) Consider the following ADT that consists of a subset $S \subseteq \{1, \ldots, n\}$ and supports the following operations:

- DELETE(S, i): Delete integer $i \in \{1, \ldots, n\}$ from the subset S. If $i \notin S$, this operation has no effect.
- PRED(S, i): Return the predecessor in S of $i \in \{1, \ldots, n\}$, i.e. $\max\{j \in S \mid j < i\}$. If i has no predecessor in S (i.e. if $i \leq \min(S)$), then return 0.

Initially, $S = \{1, \ldots, n\}$.

In this question, you will explain how to use a disjoint set data structure to implement this ADT so that the amortized cost of each operation is in $O(\log^* n)$.

(a) (1 point) Let $n = 12$ and $S = \{3, 5, 6, 8, 12\} \subseteq \{1, \ldots, 12\}$.

What is the set $\{i \in \{1, \ldots, 12\} \mid \text{PRED}(S, i) = 0\}$? 1, 2, 3

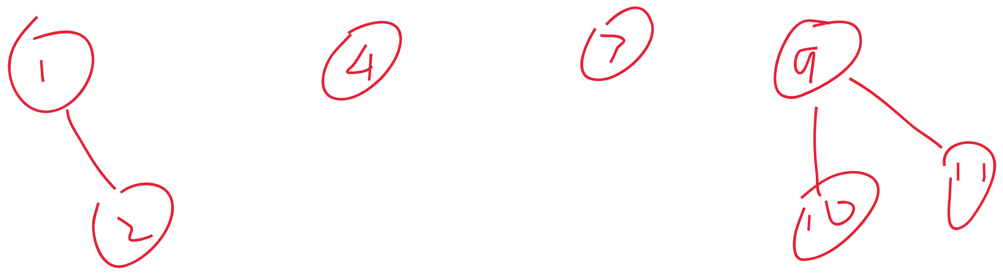
What is the set $\{i \in \{1, \ldots, 12\} \mid \text{PRED}(S, i) = 5\}$? 6

What is the set $\{i \in \{1, \ldots, 12\} \mid \text{PRED}(S, i) = 8\}$? 6, 7

What is the set $\{i \in \{1, \ldots, 12\} \mid \text{PRED}(S, i) = 12\}$? {}

(b) (4 points) Explain how the disjoint set abstract data type can be used to represent a subset $S \subseteq \{1, \ldots, n\}$. Include a diagram of your data structure for the subset $S = \{3, 5, 6, 8, 12\} \subseteq \{1, \ldots, 12\}$. (Hint: Look at part (a).)

The elements deleted are connected if they are 1 apart. We add the smallest element of the component to the root (nil if it is not a part of the disjoint forest yet).



(c) (1 point) Which disjoint data set structure should you use?

The typical union-find data structure (w/ path compression & union by rank)

(d) (3 points) Explain how to implement DELETE(S, i).

We union it with i-1 and i+1 if they are within 1 ... n, update the representative if the to the smallest element (EXPLAIN MORE)

(e) (3 points) Explain how to implement PRED(S, i).

We check if i-1 still in the set, it not then we find the smallest element of the disjoint component of i-1 and subtract 1 from that number

(f) (2 points) Briefly explain why the amortized cost of each operation is in $O(\log^* n)$.

Each operation is a union find operations (which has amortized cost of log* n).

(g) (2 points) Explain how to initialize your data structure. How much time does it take?

O(n) We make set on each of the element (and do the corresponding initialization).

8. (12 marks) Recall that a *simple cycle* in an undirected graph $G = (V, E)$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_{k-1}$, where $k \geq 3$ and $\{v_0, v_1\} \in E$, $\{v_1, v_2\} \in E$, $\ldots$, $\{v_{k-1}, v_0\} \in E$. An undirected graph G is *acyclic* (i.e., it contains no simple cycle) if and only if G is a *forest* (i.e., G consists of one or more tree(s)).

Explain how to use a graph algorithm discussed in class to determine, in $O(|V|)$ worst case time, whether a given undirected graph $G = (V, E)$ contains a cycle. State which algorithm from class you will use, and then how to use it to solve the cycle problem. Briefly explain why your solution runs in $O(|V|)$ time.

We either BFS or DFS, and if we find any cross edge or back edge, then there is a cycle. Note that we will examine at most |v|-1 edges since more if we examine |v| edges, there must be a cycle.

9. (12 points, 3 each) Explain how to augment an AVL tree representing a set of at least two integers so that, given a pointer p to any node in the tree, it is possible to determine in $O(1)$ time, the integer c in the set that is closest, but not equal, to the integer stored in the node to which p points.

(a) What additional field(s) do you add to each node?

A previous pointer and a next pointer.

(b) Explain how to modify INSERT and explain why it still takes $O(\log n)$ time, where n is the number of integers in the tree when the operation is performed.

After insertion, we find the largest element smaller than the new element and insert the new element after that element.

Note that we can do this since it is a double linked list.

(c) Explain how to modify DELETE and explain why it still takes $O(\log n)$ time, where n is the number of integers in the tree when the operation is performed.

We we find the element we remove it from the double linked list (O(1)).
And then we do the normal deletion

(d) Explain how to to determine, in $O(1)$ time, the integer c in the set that is closest, but not equal, to the integer stored in the node to which a given pointer p points.

We find the previous integer and next integer.

10. Universal Hashing (10 points) Let $U = \{1, \ldots, m\}$ and let $R = \{0, \ldots, m-1\}$.

(a) (5 points) Give the definition of a universal family of hash functions from U to R.

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(b) (5 points) Let $f : U \to R$ denote the function $f(x) = x - 1$. Is $\{f\}$ a universal family of hash functions from U to R? Justify your answer.

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