CSC165
Mathematical Expression and Reasoning for Computer Science

Module 4

Equivalence

Equivalence

- If $P \rightarrow Q$ is true, and if $Q \rightarrow P$ is true:
  - P if and only if Q
  - P iff Q
- Notation:
  - P only if Q: $P \rightarrow Q$
  - P if Q: $P \leftarrow Q$
  - P iff Q: $P \leftrightarrow Q$
- Note: $P \leftrightarrow Q: (P \rightarrow Q) \wedge (Q \rightarrow P)$

Equivalence

- Other sayings of equivalence:
  - P implies Q, and conversely
  - P is true exactly when Q is true
  - P is necessary and sufficient for Q
- Notation:
  - P is sufficient for Q: $P \rightarrow Q$
  - P is necessary for Q: $P \leftarrow Q$
  - P is necessary and sufficient for Q: $P \leftrightarrow Q$

Example

- $\forall n \in \mathbb{N}: n$ is even $\rightarrow n^2$ is even: True
- $\forall n \in \mathbb{N}: n^2$ is even$\rightarrow n$ is even: True
- Statement and its converse are both true: P and Q are equivalent
- $\forall n \in \mathbb{N}: n$ is even $\leftrightarrow n^2$ is even
- $\forall n \in \mathbb{N}: n^2$ is even $\leftrightarrow n$ is even
- Similarly:
- $\forall n \in \mathbb{N}: n$ is odd $\leftrightarrow n^2$ is odd
- $\forall n \in \mathbb{N}: n$ is even $\leftrightarrow \exists k \in \mathbb{N}: n = 2k$
- $\forall n \in \mathbb{N}: n$ is odd $\leftrightarrow \exists k \in \mathbb{N}: n = 2k + 1$

Conjunctions and Disjunctions

Conjunction (AND,∧)

- Conjunction: "the action or an instance of two or more events or things occurring at the same point in time or space"
- Synonyms: co-occurrence, coexistence, simultaneity

- Combine two statements by claiming they are both true
- Conjunction is true if both statements are true; otherwise, it is false

Example

- Define the following predicates for some domain C:
  - $R(x)$: car x is red
  - $F(x)$: car x is Ferrari
- Car x is red and a Ferrari:
  - $R(x) \wedge F(x)$
  - x is both red and Ferrari
  - Both $R(x)$ and $F(x)$ are true

Be Careful!

- There is a pen, and a telephone
- Define:
  - O: set of all objects
  - $P(x)$: x is a pen
  - $T(x)$: x is a telephone
- $\exists x \in O: P(x) \wedge T(x)$
  - There is an object that is both pen and telephone
  - There is a pen-phone!
- $\left(\exists x \in O: P(x)\right) \wedge \left(\exists x \in O: T(x)\right)$
  - There is an object that is a pen and there is an object that is a telephone
  - There is a pen and a telephone

Disjunction (OR, V)

- Combine two statements by claiming that at least one of them is true
- Disjunction is false if both statements are false; otherwise, it is true
- $R(x)$: car x is red
- $F(x)$: car x is a Ferrari
- Car x is red or a Ferrari:
  - $R(x) \vee F(x)$
  - x is either red, Ferrari, or both
  - $R(x)$ or $F(x)$ is true

Examples

- $\forall n \in \mathbb{N}: (n \text{ is even}) \vee (n \text{ is odd})$
- $\forall n \in \mathbb{N}: (n^2 \text{ is even}) \vee (n^2 \text{ is odd})$
- $\forall z \in \mathbb{Z}: (z \geq 0) \vee (z < 0)$
- $\forall x \in \mathbb{R}: (x \geq 77.452) \vee (x < 77.452)$
- $\forall x \in \mathbb{R}^+: \left(x^2 \geq \frac{\pi^3}{15}\right) \vee \left(x^2 < \frac{\pi^3}{15}\right)$

De Morgan's Law

De Morgan's Law

- S1: Jon is tall and Alex is fast
- Negate S1:
- Conjunction is true if both statements are true; otherwise, it is false
- Negated S1: Jon is not tall OR Alex is not fast
- $S1: P \wedge Q$
- $\neg S1: \neg P \vee \neg Q$

- $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$

De Morgan's Law

- S2: Casey is absent or Morgan is happy
- Negate S2:
- Disjunction is false if both statements are false; otherwise, it is true
- Negated S2: Casey is not absent AND Morgan is not happy
- $S2: P \vee Q$
- $\neg S2: \neg P \wedge \neg Q$

- $\neg(P \vee Q) \leftrightarrow (\neg \mathrm{P} \wedge \neg Q)$

Examples

- $S1: \neg P \vee Q$
- $\neg S1: \neg(\neg P \vee Q)$
- $\neg S1: \neg\neg P \wedge \neg Q$
- $\neg S1: P \wedge \neg Q$

- $S2: \neg P \wedge \neg Q$
- $\neg S2: \neg(\neg P \wedge \neg Q)$
- $\neg S2: \neg\neg P \vee \neg\neg Q$
- $\neg S2: P \vee Q$

- $S3: P \wedge \neg Q$
- $\neg S3: \neg(P \wedge \neg Q)$
- $\neg S3: \neg P \vee \neg\neg Q$
- $\neg S3: \neg P \vee Q$

- $S4: \neg P \vee \neg Q$
- $\neg S4: \neg(\neg P \vee \neg Q)$
- $\neg S4: \neg\neg P \wedge \neg\neg Q$
- $\neg S4: P \wedge Q$

Example

- For a real number x: negate $-1 < x \leq 4$
- This is equivalent to negating $-1 < x$ and $x \leq 4$
- Use De Morgan's Law, the negation is: $-1 \nless x$ or $x \nleq 4$
- This is equivalent to $-1 \geq x$ or $x > 4$
- Try it on the numbers line!

Implication and De Morgan's Law

- The implication $P \rightarrow Q$ is "mathematically" equivalent to $\neg P \vee Q$
- How?
- From the negation of an implication we know that: $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$
- What is the negation of $(\neg P \vee Q)$?
- Use De Morgan's law: $\neg(\neg P \vee Q) \leftrightarrow (P \wedge \neg Q)$
- So, $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

Tautology and Contradiction

Tautology

- Tautology is a statement that is always true
- Let $\boldsymbol{t}$ denote a tautology statement (i.e., always true)
- Let P denote some predicate (could be either true or false depending on its argument)
- Example:
  - $P \vee \neg P \leftrightarrow \boldsymbol{t}$
- Consequences:
  - $P \vee \boldsymbol{t} \leftrightarrow \boldsymbol{t}$
  - $P \wedge \boldsymbol{t} \leftrightarrow P$

Contradiction

- Contradiction is a statement that is always false
- Let $\boldsymbol{c}$ denote a contradiction statement (i.e., always false)
- Let P denote some predicate (could be either true or false depending on its argument)
- Example:
  - $P \wedge \neg P \leftrightarrow \boldsymbol{c}$
- Consequences:
  - $P \vee \boldsymbol{c} \leftrightarrow P$
  - $P \wedge \boldsymbol{c} \leftrightarrow \boldsymbol{c}$

Identity and Idempotent Laws

- Let P, Q denote some predicates

Identity laws:

- $P \wedge (Q \vee \neg Q) \leftrightarrow P$
  - Remember $(Q \vee \neg Q)$ is tautology
- $P \vee (Q \wedge \neg Q) \leftrightarrow P$
  - Remember $(Q \wedge \neg Q)$ is contradiction

Idempotent laws:

- $P \wedge P \leftrightarrow P$
- $P \vee P \leftrightarrow P$

Vacuous Truth

Vacuous Truth

- An implication $\forall x \in D: P(x) \rightarrow Q(x)$ is false if you can find an x such that $P(x)$ is true and $Q(x)$ is false. Otherwise, the implication is true
- Specifically, if $P(x)$ is false, the implication is true
- A vacuous truth is a statement that asserts that all members of the empty set have a certain property
- An implication that is true only because $P(x)$ is false is called vacuous truth
- Examples:
  - $\forall x \in \mathbb{R}: (x^2 < 0) \rightarrow (x > x + 5)$
  - If a prime number is even and bigger than two, then it must be divisible by three

Equality

Equality

- Consider these true claims:
  - $\forall a, b, c \in \mathbb{R}$: if $a < b$ then $a - c < b - c$
  - $\forall a, b, c \in \mathbb{R}$: if $a = b$ then $a - c = b - c$
  - Why true?
- $\forall a, b, c \in \mathbb{R}$: if $a < b$ then $a - c < b - c$:
  - True because of the meaning of numbers, subtraction and order
- $\forall a, b, c \in \mathbb{R}$: if $a = b$ then $a - c = b - c$:
  - True because if $a = b$, then "a" and "b" are names for the same *one* thing

Equality Claims

1. $\forall a, b \in \mathbb{R}$: if $a = b$ then $a^2 = b^2$:
   - True: if "a" and "b" refer to the same [one] thing, then "a^2" and "b^2“ refer to its square
2. $\forall a, b \in \mathbb{R}$: if $a < b$ then $a^2 < b^2$:
   - False: one counter-example is $a = -3$ and $b = 1$
3. $\forall a, b \in \mathbb{R}$: if $a^2 = b^2$ then $a = b$:
   - False: one counter-example is $a = -3$ and $b = 3$
4. $\forall a, b \in \mathbb{R}$: if $a^2 < b^2$ then $a < b$:
   - False: one counter-example is $a = -1$ and $b = -2$

Scope

Parentheses

- Consider this statement:
$\forall x \in \mathbb{R}: [P(x) \vee Q(x) \rightarrow R(x)]$
- Do you mean
$\forall x \in \mathbb{R}: \left[\left(P(x) \vee Q(x)\right) \rightarrow R(x)\right]$
or
$\forall x \in \mathbb{R}: \left[P(x) \vee \left(Q(x) \rightarrow R(x)\right)\right]$
- Parentheses are important!

Scope inside Parentheses

- The following statement
$(\forall x \in \mathbb{R}: [\exists y \in \mathbb{R}: x < y]) \rightarrow (\forall x \in \mathbb{R}: [\exists y \in \mathbb{R}: x > y])$
is the same as:
$(\forall x \in \mathbb{R}: [\exists y \in \mathbb{R}: x < y]) \rightarrow (\forall z \in \mathbb{R}: [\exists w \in \mathbb{R}: z > w])$

- Everything happens in parentheses stays in parentheses

Examples

- Also:
$\forall x \in \mathbb{N}: [(\exists y \in \mathbb{N}: x = 2y) \rightarrow (\exists z \in \mathbb{N}: x^2 = 2z)]$
is the same as:
$\forall x \in \mathbb{N}: [(\exists y \in \mathbb{N}: x = 2y) \rightarrow (\exists y \in \mathbb{N}: x^2 = 2y)]$
- Similarly:
$\forall x \in \mathbb{N}: [(\exists y \in \mathbb{N}: x = 2y + 1) \rightarrow (\exists z \in \mathbb{N}: x^2 = 2z + 1)]$
is the same as:
$\forall x \in \mathbb{N}: [(\exists y \in \mathbb{N}: x = 2y + 1) \rightarrow (\exists y \in \mathbb{N}: x^2 = 2y + 1)]$