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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(6 Marks)
Define the sequence $\{a_k\}$ as follows: $a_0 = 0, a_1 = 0, a_2 = 2, a_k = 3a_{\lfloor \frac{k}{2} \rfloor} + 2$ for all $k \geq 3$.

1. (2 Marks) Find the first 8 terms of the sequence.
2. (4 Marks) Prove that
$$\forall n \in \mathbb{N} : E(a_n)$$
where $E(n)$ is the usual predicate "n is even".

Solution

1. $a_0 = 0, a_1 = 0, a_2 = 2, a_3 = 3a_1 + 2 = 2, a_4 = 3a_2 + 2 = 8, a_5 = 3a_2 + 2 = 8, a_6 = 3a_3 + 2 = 8, a_7 = 3a_3 + 2 = 8.$
2. # Proof by induction.
   # Base Step.
   $a_0 = 0.$
   Then $E(a_0)$.
   #Inductive Step.
   Let $n \in \mathbb{N}$.
   Assume $\forall k \in \mathbb{N} : k \leq n \implies E(a_k)$. # Strong induction
   Let $k = \lfloor \frac{n+1}{2} \rfloor$.
   Then $k \in \mathbb{N}$.
   Also $k \leq \frac{n+1}{2} \leq n$.
   Then $E(a_k)$.
   Then $\exists q \in \mathbb{N} : a_k = 2q$.
   Let $q_0 \in \mathbb{N}$ such that $a_k = 2q_0$.
   Then $a_{n+1} = 3a_{\lfloor \frac{n+1}{2} \rfloor} + 2 = 6q_0 + 2 = 2(3q_0 + 1)$..
   Let $q_1 = 3q_0 + 1$.
   Then $q_1 \in \mathbb{N}$ and $a_{n+1} = 2q_1$.
   Then $\exists q \in \mathbb{N} : a_{n+1} = 2q$.
   Then $E(a_{n+1})$.
   Then $[\forall k \in \mathbb{N} : k \leq n \implies E(a_k)] \implies E(a_{n+1})$.
   Then $\forall n \in \mathbb{N} : E(a_n)$.

Problem 2.

(6 Marks)
Prove or disprove:

1. (3 Marks) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$

2. (3 Marks) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor nx \rfloor = n\lfloor x \rfloor$

Solution

1. Let $x \in \mathbb{R}, n \in \mathbb{N}$.
   Then $\lfloor x + n \rfloor \leq x + n$. # By definition of floor(x+n).
   Then $\lfloor x + n \rfloor - n \leq x$.
   Then $\lfloor x + n \rfloor - n \leq \lfloor x \rfloor$. #By definition of floor(x).
   Then $\lfloor x + n \rfloor \leq \lfloor x \rfloor + n$. (*)
   Also $\lfloor x \rfloor + n \leq x + n$. # By definition of floor(x).
   Then $\lfloor x \rfloor + n \leq \lfloor x + n \rfloor$. # By definition of floor(x+n).
   Then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$. # Combine the above statement with (*).
   Then $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$.

2. # We disprove it.
   Let $n = 4, x = 0.5$.
   Then $\lfloor nx \rfloor = \lfloor 2 \rfloor = 2 \neq 0 = 4 \cdot 0 = 4\lfloor 0.5 \rfloor$.
   Then $\exists x \in \mathbb{R}, \exists n \in \mathbb{N} : \lfloor nx \rfloor \neq n\lfloor x \rfloor$.

Problem 3.

(5 MARKS)
Prove the following claims:

1. (3 MARKS) $\forall x \in [0, \pi/2] : \sin x + \cos x \geq 1$.

2. (2 MARKS) Prove that $\log_2 3$ is irrational.

Solution

1. # Proof by contradiction.
   Assume $\exists x \in [0, \pi/2] : \sin x + \cos x < 1$.
   Then $\sin x \geq 0$. # For $x \in [0, \pi/2]$.
   Also $\cos x \geq 0$. # For $x \in [0, \pi/2]$.
   Then $0 \leq \sin x + \cos x < 1$.
   Then $(\sin x + \cos s)^2 < 1$.
   Then $\sin^2 x + \cos^2 x + 2 \sin x \cos x < 1$.
   Then $1 + 2 \sin x \cos x < 1$. # By fundamental indetity of trigonometry.
   Then $\sin x \cos x < 0$.
   Also $\sin x \cos x \geq 0$ # See above - so we have contradiction.
   Then $\forall x \in [0, \pi/2] : \sin x + \cos x \geq 1$.

2. # Proof by contradiction.
   Assume $\log_2 3$ is rational.
   Then $\exists p, q \in \mathbb{Z} : q \neq 0 \land \log_2 3 = \frac{p}{q}$.
   Let $p', q' \in \mathbb{Z} : q' \neq 0 \land \log_2 3 = \frac{p'}{q'}$.
   Then $\frac{p'}{q'} > 0$ #Log of a number greater than the base is positive. Then $q' \log_2 3 = p'$.
   Then $\log_2 3^{q'} = p'$.
   Then $3^{q'} = 2^{p'}$. # By def. of logarithms. Also, if $p' < 0$, then also $q' < 0$
   . # so by coss multiplying the fraction, we get same result.
   Then $p' > 0$.
   Then $3^{q'}$ is odd # Product of odd numbers
   Then $2^{p'}$ is even # Product of even numbers
   Then $\exists k \in \mathbb{N} : 3^{q'} = 2k + 1$.
   Let $k_0 \in \mathbb{N} : 3^{q'} = 2k_0 + 1$.
   Then $2^{p'} = 2 \cdot 2^{p'-1}$.
   Let $l_0 = 2^{p'-1}$.

Then $l_0 \in \mathbb{N}$.
Then $2l_0 = 2k_0 + 1$.
Then $l_0 - k_0 = \frac{1}{2}$.
Then $l_0 - k_0 \in \mathbb{Z}$. # Difference of two naturals is integer.
Also $l_0 - k_0 = \frac{1}{2} \notin \mathbb{Z}$ # 1/2 is not integer - Contradiction.
Then $\log_2 3$ is irrational.
Then $\log_2 3$ is irrational.

Problem 4.

(6 Marks) Prove the following:

1. (3 Marks) $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \Leftrightarrow x = 0 \vee y = 0$.
2. (3 Marks) $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \vee y = -x$.

Solution

1. # Prove $\Rightarrow$.
   Let $x, y \in \mathbb{R}$.
   Assume $x^2 + y^2 = (x + y)^2$.
   Then $x^2 + y^2 = x^2 + 2xy + y^2$.
   Then $2xy = 0$.
   Then $x = 0 \vee y = 0$.
   Then $x^2 + y^2 = (x + y)^2 \implies x = 0 \vee y = 0$.
   Then $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \implies x = 0 \vee y = 0$.

   #Prove $\Leftarrow$.
   Let $x, y \in \mathbb{R}$.
   Assume $x = 0 \vee y = 0$.
   #Case 1: $x = 0$.
   . Then $x^2 + y^2 = y^2$.
   . Also $(x + y)^2 = y^2$.
   . Then $x^2 + y^2 = (x + y)^2$.
   #Case 2: $y = 0$.
   . Then $x^2 + y^2 = x^2$.
   . Also $(x + y)^2 = x^2$.
   . Then $x^2 + y^2 = (x + y)^2$.
   Then $x^2 + y^2 = (x + y)^2$.
   Then $x = 0 \vee y = 0 \implies x^2 + y^2 = (x + y)^2$.
   Then $\forall x, y \in \mathbb{R} : x = 0 \vee y = 0 \implies x^2 + y^2 = (x + y)^2$.

   Then $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \Leftrightarrow x = 0 \vee y = 0$.

2. # Prove $\Rightarrow$.
   Let $x, y \in \mathbb{R}$.
   Assume $x^3 + x^2y = y^2 + xy$.

Then $x^3 - xy + x^2y - y^2 = 0$.
Then $x(x^2 - y) + y(x^2 - y) = 0$.
Then $(x^2 - y)(x + y) = 0$.
Then $x^2 - y = 0 \vee x + y = 0$.
Then $y = x^2 \vee y = -x$.
Then $x^3 + x^2y = y^2 + xy \implies y = x^2 \vee y = -x$.
Then $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \implies y = x^2 \vee y = -x$.

#Prove $\Leftarrow$.
Let $x, y \in \mathbb{R}$.
Assume $y = x^2 \vee y = -x$.
#Case 1: $y = x^2$.
. Then $y - x^2 = 0$.
. Then $0 = 0 \cdot (x + y) = (y - x^2)(x + y)$.
. Then $x^3 + x^2y = y^2 + xy$. # After some algebra
#Case 2: $y = -x$.
. Then $x + y = 0$.
. Then $0 = 0 \cdot (y - x^2) = (x + y)(y - x^2)$.
. Then $x^3 + x^2y = y^2 + xy$. # After some algebra
Then $x^3 + x^2y = y^2 + xy$.
Then $y = x^2 \vee y = -x \implies x^3 + x^2y = y^2 + xy$.
Then $\forall x, y \in \mathbb{R} : y = x^2 \vee y = -x \implies x^3 + x^2y = y^2 + xy$.

Then $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \vee y = -x$.