CSC165 Mathematical Expression and Reasoning for Computer Science

Module 5

Universally Quantified Implication

Mathematical Statements

- A universal statement says that a certain property is true for all elements in a set
  - For example: All positive numbers are greater than zero
- A conditional/implication statement says that if one thing is true then some other thing also has to be true
  - For example: If 378 is divisible by 18 then 378 is divisible by 6

Universally Quantified Implication

- It is a frequent logical form for claims (Boolean statements: true or false)
- Here is such a claim, which we'll name C_0:
- (C_0): Every (real) number larger than seven has the property such that its square is larger than sixteen
- Notice: C_0 is a complete English sentence
- Is it a claim?
- Is C_0 true? What precisely does it claim?
  - C_0 claims something about every number larger than seven

About C_0

- Since one-hundred-sixty-five is larger than seven:
  - C_0 claims that the square of one-hundred-sixty-five is larger than sixteen
- Since ten is larger than seven:
  - C_0 claims that the square of ten is larger than sixteen
- Since eight is larger than seven:
  - C_0 claims that the square of eight is larger than sixteen
- Since seven-and-a-half is larger than seven:
  - C_0 claims that the square of seven-and-a-half is larger than sixteen
- ... and many other such claims

About C_0

- But the number six is NOT larger than seven:
  - C_0 does not DIRECTLY make a claim about six
  - Although the square of six IS larger than sixteen, that is IRRELEVANT for determining whether C_0 is true
- The number three is NOT larger than seven:
  - C_0 does not directly make a claim about three
  - Although the square of three is NOT larger than sixteen, that is IRRELEVANT for determining whether C_0 is true
- The number negative-one-hundred-sixty-five is NOT larger than seven:
  - C_0 doesn't directly make a claim about -165
  - Although the square of -165 IS larger than sixteen, that is IRRELEVANT for determining whether C_0 is true

About C_o

- Let us put some of that information into a table:

Number	larger than seven?	square	square larger than 16?	
10	$10 > 7?$ True	$10^2 = 100$	$10^2 > 16?$ True	Okay
6	$6 > 7?$ False	$6^2 = 36$	$6^2 > 16?$ True	Irrelevant
3	$3 > 7?$ False	$3^2 = 9$	$3^2 > 16?$ False	Irrelevant
-165	$-165 > 7?$ False	$(-165)^2 = 27225$	$(-165)^2 > 16?$ True	Irrelevant

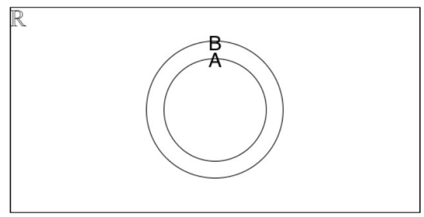
About C_o

- Consider two "sets" (bunches) of numbers:
  - A: The set of all numbers larger than seven
  - B: The set of all numbers that when squared are larger than sixteen
- Look at these sets on a number line:

```
A:                                  o-----+-------->
B:     <----------o              o----+---------+-------->
       <----------|---------|------+--|----+--|-----+-------->
                 -4         0      3  4    6  7     10
```

About C_0: Venn Diagram

- What does C_0 claim about A vs B?
- To be able to illustrate relationships between sets of things that are not numbers there is a common type of general diagram: a "Venn Diagram"
- The relationship between A and B is represented by this Venn Diagram

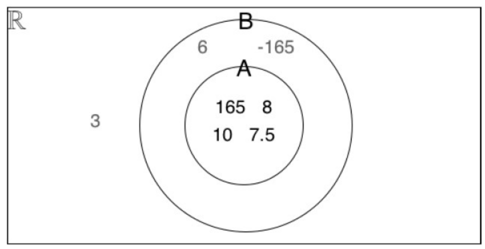


About C_0: Venn Diagram

- The enclosing rectangle in a Venn Diagram represents the "Universe":
  - The type of the things being discussed
- For C_0 it is the set of real numbers, called "$\mathbb{R}$", and also called "the Reals"
- The things in a set are called its "elements“ or its "members"
  - What are some numbers inside the A circle (i.e., elements of A)?
  - What are some numbers inside the B circle (i.e., elements of B) that are not inside the A circle (i.e., not elements of A)?
  - What are some numbers outside the B circle (i.e., not elements of B nor A)?

About C_0: Venn Diagram

- Is every element of A an element of B?
- Is every element of B an element of A?
- Is C_0 true?
  - A has to be "completely contained" in B
  - All elements of A are elements of B
  - Yes, C_0 is true



Converse of C_0

- (C_1): The numbers whose squares are larger than sixteen have the property of being larger than seven
- Is that a claim?
  - Is it a "complete English Sentence"?
  - Is it Boolean?
- Is C_1 true?
  - Let us check

About C_1

- Since the square of one-hundred-sixty-five IS larger than sixteen:
  - C_1 claims that one-hundred-sixty-five IS larger than seven
- Since the square of eight IS larger than sixteen:
  - C_1 claims that eight IS larger than seven
- Since the square of three IS NOT larger than sixteen:
  - C_1 DOES NOT directly make a claim about three
- Since the square of six IS larger than sixteen:
  - C_1 claims that six IS larger than seven!
  - That is false, which violates the "universal" aspect of C_1, so C_1 is false
- The number six is called a "counter-example" to the Universally Quantified Implication C_1
- One counter-example is enough to disprove C_1

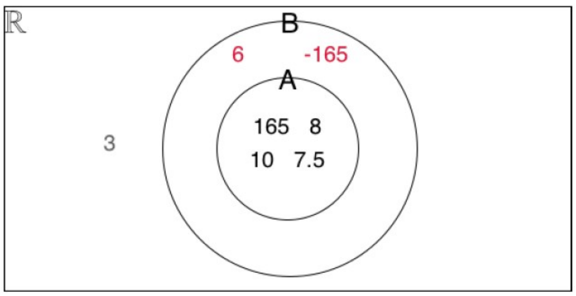
About C_1

- How is this reflected in:
  - The table?
  - The number line diagram?
  - The Venn Diagram?

number	square	square larger than 16?		number larger than 7?		
10	$10^2 = 100$	$10^2 > 16?$	True	$10 > 7?$	True	Still okay
6	$6^2 = 36$	$6^2 > 16?$	True	$6 > 7?$	False	Counter example
3	$3^2 = 9$	$3^2 > 16?$	False	$3 > 7?$	False	Irrelevant
-165	$(-165)^2 = 27225$	$(-165)^2 > 16?$	True	$-165 > 7?$	False	Counter example

About C_1

```
A:                                          o------+-------->
B:     <----------o                  o----+---------+-------->
       <----------|---------|------+--|----+--|------+-------->
                 -4         0      3  4    6  7     10
```



Example

Example

- Determine explicitly all the numbers x such that $x + \sqrt{x} = 2$
  - Note: "numbers" here mean "real numbers“
  - Later we might use "natural numbers", "integers", "rational numbers", etc.
- Discussing the "manipulating equations" approach to solving an equation will continue to illuminate Universally Quantified Implications
- OK, from your pre-UofT experience, how to solve it?

Pre-UofT Solution?

$x + \sqrt{x} = 2$
$\sqrt{x} = 2 - x$
$x = (2 - x)^2$
$x = 4 - 4x + x^2$
$0 = 4 - 5x + x^2$
$0 = (x - 1)(x - 4)$

$x = 1$ or $x = 4$

- Alright!
  - Are these complete Boolean English sentences? Read out loud!
  - Are these “connected” statements?
  - So what is your solution?

Trace the Solution

	If $x = 1$		If $x = 4$	
$x + \sqrt{x} = 2$	$1+\sqrt{1} = 2?$	True	$4+\sqrt{4} = 2?$	False!!!
$\sqrt{x} = 2 - x$	$\sqrt{1} = 2 - 1?$	True	$\sqrt{4} = 2 - 4?$	False!!!
$x = (2 - x)^2$	$1 = (2 - 1)^2?$	True	$4 = (2 - 4)^2?$	True
$x = 4 - 4x + x^2$	$1 = 4 - 4(1) + (1)^2?$	True	$4 = 4 - 4(4) + (4)^2?$	True
$0 = 4 - 5x + x^2$	$0 = 4 - 5(1) + (1)^2?$	True	$0 = 4 - 5(4) + (4)^2?$	True
$0 = (x - 1)(x - 4)$	$0 = (1 - 1)(1 - 4)?$	True	$0 = (4 - 1)(4 - 4)?$	True
$x = 1$ or $x = 4$	$1 = 1$ or $1 = 4?$	True	$4 = 1$ or $4 = 4?$	True

Comments

- Did we make an algebraic error or typo?
  - No, but tracing our proofs helps catch those
- Solving an equation is creating a proof… of what?
- Is squaring both sides of an equality 'wrong'?
  - Cannot we 'do' the same 'thing' to both sides of an equation?
  - Example: For all numbers x : if $x < 2$ then $x^2 < 4$. (this is False)

Comments

- Do we have to check every result of a 'solution', not just to catch typos and algebraic errors?
- If we are worried that 'solving' an equation can produce extra results, should we be worried that it can miss some results?
- The main source of the problem is much more general:
  - The 'solution' was not a paragraph. It was an unconnected sequence of claims
- Let us fix that

Solution

Let x be a real number.
Assume $x + \sqrt{x} = 2$.
Then $\sqrt{x} = 2 - x$.
Then $(\sqrt{x})^2 = (2 - x)^2$.
Then $x = (2 - x)^2$.
Then $x = 4 - 4x + x^2$.
Then $0 = 4 - 5x + x^2$.
Then $0 = (x - 1)(x - 4)$.
Then $x = 1$ or $x = 4$.
Therefore, for any real number x: if $x + \sqrt{x} = 2$ then [$x = 1$ or $x = 4$].

Solution

- This conclusion is a Universally Quantified Implication
  - Can we take the course's standard approach to exploring it?
  - Is it true?
  - Can it miss any solutions to the equation?
  - Can it generate extra solutions to the equation?

Trace the Solution

	If $x = 1$		If $x = 4$		Solution Set
$x + \sqrt{x} = 2$	$1+\sqrt{1} = 2?$	True	$4+\sqrt{4} = 2?$	False	1
$\sqrt{x} = 2 - x$	$\sqrt{1} = 2 - 1?$	True	$\sqrt{4} = 2 - 4?$	False	1
$x = (2 - x)^2$	$1 = (2 - 1)^2?$	True	$4 = (2 - 4)^2?$	True	1 or 4
$x = 4 - 4x + x^2$	$1 = 4 - 4(1) + (1)^2?$	True	$4 = 4 - 4(4) + (4)^2?$	True	1 or 4
$0 = 4 - 5x + x^2$	$0 = 4 - 5(1) + (1)^2?$	True	$0 = 4 - 5(4) + (4)^2?$	True	1 or 4
$0 = (x - 1)(x - 4)$	$0 = (1 - 1)(1 - 4)?$	True	$0 = (4 - 1)(4 - 4)?$	True	1 or 4
$x = 1$ or $x = 4$	$1 = 1$ or $1 = 4?$	True	$4 = 1$ or $4 = 4?$	True	1 or 4

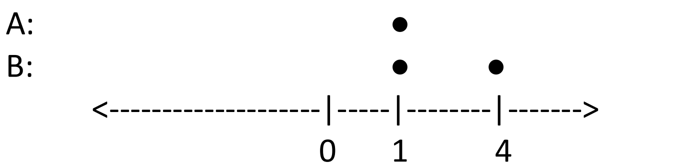
- Solution set stays the same or gets bigger
- This is consistent with consecutive pairs of lines representing TRUE Universally Quantified Implications!
- That is consistent with the use of "Then" to connect them!

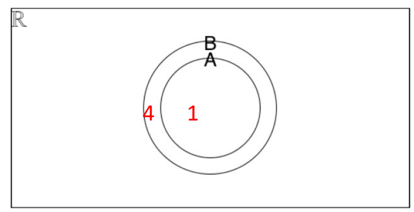
Proof Process

x	$x + \sqrt{x} = 2$?	$[x = 1$ **or** $x = 4]$?	
1	True	True	Okay
4	False	True	Irrelevant
0	False	False	Irrelevant

A: The set of all numbers such that $x + \sqrt{x} = 2$

B: The set of all numbers such that $x = 1$ or $x = 4$





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Understanding the Solution

- The first "Then" is part of the following Universally Quantified Implication:

  For all numbers x: if $x + \sqrt{x} = 2$ then $\sqrt{x} = 2 - x$
- It uses a known true Universally Quantified Implication:

  For all numbers a and b and c: if $a = b$ then $a - c = b - c$
- From that known one, we can deduce:

  For all numbers x: if $x + \sqrt{x} = 2$ then $x + \sqrt{x} - x = 2 - x$
- There is some basic arithmetic on the LHS, which could be justified by using other known Universally Quantified Implications

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Introducing "for all"

- Now that we are writing lots of them, we will allow the abbreviations:
  - "$\forall$" to mean "for all"
  - "$\forall x \in \mathbb{R}$" to mean "for all real numbers x"
  - "$\forall x, y \in \mathbb{R}$" to mean "for all numbers x and y"
  - And so on…

If… Then

Solution	If… Then representation
Let x be a real number.	
Assume $x + \sqrt{x} = 2$.	
Then $\sqrt{x} = 2 - x$.	$\forall x \in \mathbb{R}$: if $x + \sqrt{x} = 2$ then $\sqrt{x} = 2 - x$
Then $(\sqrt{x})^2 = (2 - x)^2$.	$\forall x \in \mathbb{R}$: if $\sqrt{x} = 2 - x$ then $(\sqrt{x})^2 = (2 - x)^2$
Then $x = (2 - x)^2$.	$\forall x \in \mathbb{R}$: if $(\sqrt{x})^2 = (2 - x)^2$ Then $x = (2 - x)^2$.
Then $x = 4 - 4x + x^2$.	$\forall x \in \mathbb{R}$: if $x = (2 - x)^2$ then $x = 4 - 4x + x^2$.
Then $0 = 4 - 5x + x^2$.	$\forall x \in \mathbb{R}$: if $x = 4 - 4x + x^2$ then $0 = 4 - 5x + x^2$.
Then $0 = (x - 1)(x - 4)$.	$\forall x \in \mathbb{R}$: if $0 = 4 - 5x + x^2$ then $0 = (x - 1)(x - 4)$.
Then $x = 1$ or $x = 4$.	$\forall x \in \mathbb{R}$: if $0 = (x - 1)(x - 4)$ then $[x = 1$ or $x = 4]$.
Therefore, for any real number x: if $x + \sqrt{x} = 2$ then $[x = 1$ or $x = 4]$.	

Justify Solution

Solution	Justification
Let x be a real number.	
Assume $x + \sqrt{x} = 2$.	it implicitly assumes x is not negative.
Then $\sqrt{x} = 2 - x$.	[1]since $\forall a, b, c \in \mathbb{R}$: if $a = b$ then $a - c = b - c$. And by some other basic arithmetic for the LHS.
Then $(\sqrt{x})^2 = (2 - x)^2$.	since $\forall a, b \in \mathbb{R}$: if $a = b$ then $a^2 = b^2$.
Then $x = (2 - x)^2$.	since $\forall a \in \mathbb{R}$: if $a \geq 0$ then $(\sqrt{a})^2 = a$.
Then $x = 4 - 4x + x^2$.	since $\forall a, b \in \mathbb{R}$: $(a + b) \cdot (a + b) = a^2 + 2ab + b^2$.
Then $0 = 4 - 5x + x^2$.	for the same reason as [1], but with some basic arithmetic for the RHS.
Then $0 = (x - 1)(x - 4)$.	from some basic arithmetic for the RHS.
Then $x = 1$ or $x = 4$.	since $\forall a, b \in \mathbb{R}$: if $a \cdot b = 0$ then $[a = 0$ or $b = 0]$.
Therefore, for any real number x: if $x + \sqrt{x} = 2$ then $[x = 1$ or $x = 4]$.	

Solution

- So now we know that at there are at MOST two solutions to the equation: $x = 1$ or $x = 4$
- We checked and found that $x = 4$ is NOT a solution, so the result of the proof can be strengthened:

$$\forall x \in \mathbb{R}: \text{if } x + \sqrt{x} = 2 \text{ then } x = 1$$

- We checked and found that $x = 1$ IS a solution, which can also be summarized as a "converse" Universally Quantified Implication:

$$\forall x \in \mathbb{R}: \text{if } x = 1 \text{ then } x + \sqrt{x} = 2$$

- Together, those two Universally Quantified Implications say that the equation has exactly one solution: when $x = 1$
- Solving an equation means proving a Universally Quantified Implication and its converse

Proof:
Let x be a real number.
Assume $x = 1$.
Then $x + \sqrt{x} = 2$.
Then if $x = 1$ then $x + \sqrt{x} = 2$.
Therefore, for any number x: if $x = 1$ then $x + \sqrt{x} = 2$.

Solution of $x + \sqrt{x} = 2$

- We found that

  $\forall x \in \mathbb{R}$: if $x + \sqrt{x} = 2$ then $[x = 1$ or $x = 4]$ is TRUE
- We also found that

  $\forall x \in \mathbb{R}$: if $[x = 1$ or $x = 4]$ then $x + \sqrt{x} = 2$ is False

A: The set of all numbers such that $x + \sqrt{x} = 2$

B: The set of all numbers such that $x = 1$ or $x = 4$

A: ●

B: ● ●

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<--------------------|-----|--------|------->
                     0     1        4
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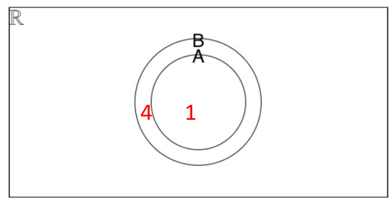

Notes

Alternative Phrasing of UQI

- An alternative phrasing of Universally Quantified Implication (about a number):
  - "If a number … then it …."
- (C_0’): If a number is larger than seven then its square is larger than sixteen
- (C_1’): If a number's square is larger than sixteen then the number is larger than seven
  - We avoided an "it" ambiguity in C_1’ by saying "the number" again

Rephrasing of C_0

- For all numbers x: if $x > 7$ then $x^2 > 16$
- Note: it is still readable as a complete English sentence “For all numbers x: if x is larger than seven then the square of x is larger than sixteen"
- Every line you write must be readable as a complete sentence. Read every line out loud to yourself
- You are writing sentences, paragraphs, and essays, but with the convenience of some symbolic abbreviations
- So the usual standards apply: complete grammatical sentences

Proving a claim

- Consider a claim "for all numbers x: if $A(x)$ then $B(x)$"
- Read out as: for all real numbers x: if $A(x)$ is True then $B(x)$ is True
- The claim is False if there is an $A(x)$ that is True while $B(x)$ is False
- In the table and examples:
  - We were trying to see if there are counter examples to invalidate the claim
  - A counter example: $A(x)$ is True and $B(x)$ is False
  - If a counter example is found, then a claim in false
- The irrelevant cases occur when $A(x)$ is False:
  - Remember the claim is about if $A(x)$ being true, then $B(x)$ is true
  - The claim does not directly "promise" anything about $B(x)$ if $A(x)$ is not True
  - Consequently, to disprove a claim, we cannot use the results of an irrelevant case because the claim does not (directly) promise anything about this case

Proving a claim

- In Venn Diagram:
  - The claim is True if A is completely contained in B
  - That means all members of A are members of B
  - That does not mean that all members of B have to be in A
  - The irrelevant cases occur for elements that are not contained in A

Writing precisely

- In this course: DO NOT USE COMMAS IN PLACE OF "and", "then", "such that", etc.
- Commas replacing words are a frequent source of ambiguity and confusion in student writing
- DO NOT WRITE: If $x > 7, x^2 > 16$
- DO NOT WRITE: If x is a number, $x > 7$, then $x^2 > 16$
- DO NOT WRITE: If $x > 165, x + y > 148$, then $x + y > 200$
  - Which commas are replacing an "and"?
  - Which commas are replacing a "then"?
  - Which commas are replacing "such that"?
- Remember "$\forall x, y \in \mathbb{R}$" means "for all numbers x and y"
  - This is a violation of our "no commas replacing words" rule (replaces an "and"), but we are explicitly allowing that violation for this course

"for all" Symbol

- "$\forall$" can be read in various synonymous ways:
  - "for all" "for any", "for each", "for every"
  - "$\forall x \in \mathbb{R}$" means "for all real numbers x"
  - "$\forall x, y \in \mathbb{R}$" means "for all numbers x and y"
- "$\forall$" is used with variables
- DO NOT use "$\forall$" on NON-VARIABLE EXPRESSIONS
- DO NOT write things likes:
  - " $\forall x^2 \in \mathbb{R}$: ...."
  - "For all numbers x^2 : ...."
- Also DO NOT write things like:
  - " $\forall x > 0 \in \mathbb{R}$: ...."
  - " $\forall$ positive $x \in \mathbb{R}$: ...."
  - "For all $x > y$: ...."

Existential Statement

Existential Statement

- Given a property that may or may not be true, an existential statement says that there is at least one thing for which the property is true
- Example:
  - There is a prime number that is even
  - At least one of prime numbers is even
- "$\exists$": There is at least one
- Claim: $\exists x \in \mathbb{R}: x > 10$ [True or False?]
- Claim: $\exists x \in \mathbb{R}: x^2 < 0$ [True or False?]

Even and Odd Numbers

- Even numbers: $\{\ldots, -4, -2, 0, 2, \ldots\}$
- Odd numbers: $\{\ldots, -3, -1, 1, 3, \ldots\}$
- x is even: $\exists j \in \mathbb{Z}: x = 2(j)$
- x is odd: $\exists j \in \mathbb{Z}: x = 2(j) + 1$

Even Number	Representation
-4	$-2(2)$
-2	$-1(2)$
0	$0(2)$
2	$1(2)$
4	$2(2)$
x	$\exists j \in \mathbb{Z}: x = 2(j)$

Odd Number	Representation
-3	$-2(2) + 1$
-1	$-1(2) + 1$
1	$0(2) + 1$
3	$1(2) + 1$
5	$2(2) + 1$
x	$\exists j \in \mathbb{Z}: x = 2(j) + 1$

Even and Odd Numbers

- Even numbers: $\{\ldots, -4, -2, 0, 2, \ldots\}$
- Odd numbers: $\{\ldots, -3, -1, 1, 3, \ldots\}$
- $\forall x \in \mathbb{Z}$: if x is even then $[\exists j \in \mathbb{Z}: x = 2(j)]$
- $\forall x \in \mathbb{Z}$: if x is odd then $[\exists j \in \mathbb{Z}: x = 2(j) + 1]$