CSC236 tutorial exercises, Week #6
best before Thursday evening

These exercises are intended to give you practice with recurrences and using the technique of repeated substitution.

1. Consider the recurrence:

$$T(n) = \begin{cases} 2 & \text{if } n = 1 \\ 2n + T(\lceil n/3 \rceil) & \text{if } n > 1 \end{cases}$$

Find a closed form for T in the special case where n is a power of 3, that is $\exists k \in \mathbb{N}, n = 3^k$. Use the technique of repeated substitution (aka unwinding) from week 5 slides, or the CSC236 course notes, page 82.

sample solution: Let $n \in \mathbb{N}$ and assume that $\exists k \in \mathbb{N}^+, n = 3^k$, so $k = \log_3 n$. In this case $\lceil n/3 \rceil = n/3$, and:

$$\begin{aligned} T(n) &= T(3^k) = 2 \times 3^k + T(3^{k-1}) \\ &= 2 \times 3^k + 2 \times 3^{k-1} + T(3^{k-2}) \\ &= \vdots \\ &= \vdots \text{ (some intuition going on here)} \\ &= \vdots \\ &= \left(2 \sum_{i=1}^{i=k} 3^i\right) + T(3^{k-k}) = 2 \sum_{i=0}^{i=k} 3^i \\ &= 3^{k+1} - 1 = 3^{\log_3(n)+1} - 1 \end{aligned}$$

Because of the $\vdots$ portion, this is not a proof, but simple induction on k yields this result. Also note that in the special case where $k = 0$ the proposed closed form is correct.

2. Consider the recurrence:

$$R(n) = \begin{cases} 0 & \text{if } n = 1 \\ n + 3R(\lceil n/3 \rceil) & \text{if } n > 1 \end{cases}$$

Find a closed form for T in the special case where n is a power of 3, that is $\exists k \in \mathbb{N}, n = 3^k$. Use the technique of repeated substitution (aka unwinding) from week 5 slides, or the CSC236 course notes, page 82.

sample solution: Let $n \in \mathbb{N}$, and assume $\exists k \in \mathbb{N}^{+}, n = 3^k, sok = \log_3 n$. In this case $\lceil n/3 \rceil = n/3$, and:

$$
\begin{aligned}
R(n) &= R(3^k) = 3^k + 3R(3^{k-1}) = 3^k + 3(3^{k-1} + 3R(3^{k-2}) \\
&= 2 \times 3^k + 3^2 R(3^{k-2}) = 3 \times 3^k + 3^3 R(3^{k-3}) \\
&= \vdots \\
&= \vdots \text{ (intuition going on here)} \\
&= \vdots \\
&= k3^k + 3^k R(3^{k-k}) = k3^k + 3^k \times 0 = n \log_3(n)
\end{aligned}
$$

Since we make a leap in the $\vdots$ section, this is not a proof, but simple induction on k establishes the result. Also notice that in the special case $k = 0$ the proposed closed form matches $R(1)$.