1. **(7 pts)** Use simple induction to prove that $3^n > n^3 + n$ for all natural numbers $n > 3$. You may use the fact that $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

   sample solution: Define $P(n) : 3^n > n^3 + n$. Proof by simple induction that $\forall n \in \mathbb{N} \setminus \{0, 1, 2, 3\}, P(n)$.

   base case: $3^4 = 81 > 68 = 4^3 + 4$, so $P(4)$ holds.

   inductive step: Let n be an arbitrary natural number greater than 3. Assume $H(n)$, that is $3^n > n^3 + n$.

   derive inductive conclusion $C(n)$: that is, $3^{n+1} > (n+1)^3 + (n+1)$.

$$\begin{aligned}
3^{n+1} &= 3 \times 3^n > 3(n^3 + n) && \# \text{ by } H(n) \\
&= 3n^3 + 3n = n^3 + nn^2 + n^2n + 3n && \# \text{ expanding} \\
&> n^3 + 3n^2 + 4n + 2 && \# \, n > 3, n^2 > 9 > 4, n > 3 > 2/3 \\
&= (n+1)^3 + (n+1) && \# \text{ re-writing}
\end{aligned}$$

   So $C(n)$ ■

2. **(12 pts)** Read the definition of function f:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ (f(\lfloor n/3 \rfloor))^2 + f(\lfloor n/3 \rfloor) & \text{if } n > 2 \end{cases}$$

   Use complete induction to prove that for every natural number n larger than 1, $f(n)$ is even. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, where the inductive hypothesis is used, and why it is valid to use it there.

   sample solution: Define $P(n)$: $f(n)$ is a multiple of 2. Proof by complete induction that $\forall n \in \mathbb{N} \setminus \{0, 1\}, P(n)$.

   inductive step: Let $n \in \mathbb{N} \setminus \{0, 1\}$. Assume $H(n)$: $\forall i \in \mathbb{N}$ if $2 \leq i < n$ then $f(i)$ is a multiple of 2.

   must show inductive hypothesis $C(n)$: $f(n)$ is a multiple of 2.

   Base case, $n = 2$: Then $f(n) = 2$, from definition, and 2 is notoriously a multiple of 2.

Base case, $n \in \{3,4,5\}$: Then

$$\begin{aligned} f(n) &= (f(\lfloor n/3 \rfloor))^2 + f(\lfloor n/3 \rfloor) && \# \text{ from definition of } f(n), n > 2 \\ &= (f(1))^2 + f(1) && \# \lfloor n/3 \rfloor = 1 \text{ for } n \in \{3,4,5\} \\ &= 1 + 1 = 2 && \# \text{ from definition of } f(1),\ 2 \text{ is a multiple of } 2 \end{aligned}$$

Case $n > 5$: $f(\lfloor n/3 \rfloor)$ is a multiple of 2, by $H(n)$, since $2 \le \lfloor n/3 \rfloor < n$ when n is at least 6. Let $k \in \mathbb{N}$ such that $f(\lfloor n/3 \rfloor) = 2k$

$$\begin{aligned} f(n) &= (f(\lfloor n/3 \rfloor))^2 + f(\lfloor n/3 \rfloor) && \# \text{ from definition of } f(n), n > 2 \\ &= (2k)^2 + 2k = 2(2k^2 + k) \\ & \quad \# \text{ a multiple of 2 since } 2, k \in \mathbb{N} \text{ and } \mathbb{N} \text{ is closed under } +, \times \end{aligned}$$

In every possible case $C(n)$ follows ■

3. **(10 pts)** Define the set of non-empty binary trees $\mathcal{BT}^*$ (not **full** binary trees) as the smallest set such that:

   a) A solitary node with no edges is an element of $\mathcal{BT}^*$.

   b) If $t_1, t_2 \in \mathcal{BT}^*$ and n is a single node, then the following three trees are also elements of $\mathcal{BT}^*$:

      i. the tree formed with root n and an edge to left subtree t_1;

      ii. the tree formed with root n and an edge to right subtree t_2;

      iii. the tree formed with root n, an edge to left subtree t_1, and an edge to right subtree t_2.

   For $t \in \mathcal{BT}^*$ define $N(t)$ as the number of nodes in t, and $E(t)$ as the number of edges in t. Use structural induction to prove that for all $t \in \mathcal{BT}^*$, $N(t) = E(t) + 1$. For full marks your proof must clearly indicate any base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

sample solution: Define $P(t)$: $N(t) = E(t) + 1$. Proof by structural induction that $\forall t \in \mathcal{BT}^*, P(t)$

verify basis: A solitary node t with no edges has $N(t) = 1 = 0 + 1 = E(t) + 1$. So the claim holds for the basis.

inductive step: Let t_1, t_2 be arbitrary elements of $\mathcal{BT}^*$ and n be a single node. Assume $H(\{t_1, t_2\})$, that is $P(t_1)$ and $P(t_2)$.

must show $C(\{t_1, t_2\})$: that is, any tree t in $\mathcal{BT}^*$ formed from root n, with subtrees t_1, t_2 satisfies $P(t)$. There are 3 cases to consider:

Case i: If t is a tree formed from root n with left subtree t_1, then $N(t) = N(t_1) + 1$, since n provides one new node. Also $E(t) = E(t_1) + 1$, since there is a new edge from n to its subtree. Summing up:

$$\begin{aligned} N(t) &= N(t_1) + 1 = (E(t_1) + 1) + 1 \qquad \# \text{ by } H(\{t_1, t_2\} \\ &= E(t) + 1 \qquad \# \text{ since } n \text{ add one node.} \end{aligned}$$

$P(t)$ follows in this case.

Case ii: If t is a tree formed from root n with right subtree t_2, the argument is the same as Case ii with t_1 replace by t_2, left replaced by right, and $P(t)$ follows in this case.

Case iii: If t is a tree formed from room n with left subtree t_1 and right subtree t_2, then $N(t)$ is the sum of the nodes in the two subtrees, plus one for n itself, or $N(t) = N(t_1) + N(t_2) + 1$. $E(t)$ is the sum of the edges in the two subtrees, plus two edges connecting t to each subtree, or $E(t) = E(t_1) + E(t_2) + 2$. Summing up:

$$\begin{aligned} N(t) &= N(t_1) + N(t_2) + 1 = \\ &= (E(t_1) + 1) + (E(t_2) + 1) + 1 \qquad \# \text{ by } H(\{t_1, t_2\}) \\ &= (E(t_1) + E(t_2) + 2) + 1 = E(t) + 1 \end{aligned}$$

$P(t)$ follows in this case.

In every possible case $P(t)$ follows, that is $C(\{t_1, t_2\})$ ■