1. **Sample Solution**

   Let $u, v, w, z \in \mathbb{R}^+$.

   #Proof by contrapositive.

   Assume $\frac{u+z}{v+w} \geqslant \frac{z}{w}$.

   Then $uw + zw \geqslant vz + wz$.

   Then $uw \geqslant vz$.

   Then $\frac{u}{v} \geqslant \frac{z}{w}$.

   Then $\frac{u+z}{v+w} \geqslant \frac{z}{w} \Rightarrow \frac{u}{v} \geqslant \frac{z}{w}$.

   Then $\frac{u}{v} < \frac{z}{w} \Rightarrow \frac{u+z}{v+w} < \frac{z}{w}$.

   Therefore $\forall u, v, w, z \in \mathbb{R}^+ : \frac{u}{v} < \frac{z}{w} \Rightarrow \frac{u+z}{v+w} < \frac{z}{w}$.

2. **Sample Solution**

   Let $x, y \in \mathbb{R}$.

   #Proof by contrapositive.

   Assume $y > x$.

   Then $y - x > 0$.

   Then $x^2 + y^2 > 0$. #if $x = 0$ then $y \neq 0$ and vice versa

   Then $(y - x)(x^2 + y^2) > 0(x^2 + y^2)$.

   Then $yx^2 + y^3 - x^3 - xy^2 > 0$.

   Then $y^3 + yx^2 > x^3 + xy^2$.

   Then $x > y \Rightarrow y^3 + yx^2 > x^3 + xy^2$.

   Then $(y^3 + yx^2 \leqslant x^3 + xy^2) \Rightarrow (y \leqslant x)$.

   Therefore $\forall x, y \in R : (y^3 + yx^2 \leqslant x^3 + xy^2) \Rightarrow (y \leqslant x)$.

3. **Sample Solution**

   Let $n \in \mathbb{N}$.

   #Proof by contrapositive.

   Assume $E(n)$.

   Then $\exists k \in \mathbb{N} : n = 2k$.

   Then let $k_0 \in \mathbb{N}$ be s.t. $n = 2k_0$.

   Then $n^2 + 3 = 4k_0^2 + 3$.

$$= 2(2k_0^2 + 1) + 1.$$

Let $k_1 = 2k_0^2 + 1$.

Then $k_1 \in \mathbb{N}$.

Then $n^2 + 3 = 2k_1 + 1$.

Then $\exists k \in \mathbb{N} : n^2 + 3 = 2k + 1$.

Then $O(n^2 + 3)$.

Then $E(n) \Rightarrow O(n^2 + 3)$.

Then $E(n^2 + 3) \Rightarrow O(n)$.

Therefore $\forall n \in \mathbb{N} : E(n^2 + 3) \Rightarrow O(n)$

4. **Sample Solution**

Let $n \in \mathbb{N}$.
Assume $\exists a, b \in \mathbb{Z} : n = 5a + 7b$.
Let $a_0, b_0 \in \mathbb{Z}$ such that $n = 5a_0 + 7b_0$.
Then, $n + 1 = 5a_0 + 7b_0 + 1$.
Then, $n + 1 = 5a_0 + 7b_0 - 20 + 20 + 1$.
Then, $n + 1 = 5a_0 + 7b_0 - 5(4) + 21$.
Then, $n + 1 = 5a_0 + 7b_0 - 5(4) + 7(3)$.
Then, $n + 1 = 5(a_0 - 4) + 7(b_0 + 3)$.
Let $a_1 = a_0 - 4$.
Then, $a_1 \in \mathbb{Z}$.
Let $b_1 = b_0 + 3$.
Then, $b_1 \in \mathbb{Z}$.
Then, $n + 1 = 5a_1 + 7b_1$.
Then, $\exists a, b \in \mathbb{Z} : n + 1 = 5a + 7b$.
Then, $[\exists a, b \in \mathbb{Z} : n = 5a + 7b] \Rightarrow [\exists a, b \in \mathbb{Z} : n + 1 = 5a + 7b]$.
Then, $\forall n \in \mathbb{N} : [\exists a, b \in \mathbb{Z} : n = 5a + 7b] \Rightarrow [\exists a, b \in \mathbb{Z} : n + 1 = 5a + 7b]$.

5. **Sample Solution.**

(a) False, so disproof. Write the negation symbolically:

$$\exists r, s \in \mathbb{R} : (r > 0 \wedge s > 0) \wedge \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$$

Let $r_0 = 1, s_0 = 1$ # the first, and easiest, reals to work with to introduce $\exists$
Then $r_0, s_0 \in \mathbb{R}$ # $r_0 = s_0 = 1 \in \mathbb{R}$
Then $r_0 > 0 \wedge s_0 > 0$ # $1 > 0$
Then $\sqrt{r_0} + \sqrt{s_0} = \sqrt{1} + \sqrt{1} = 1 + 1 = 2 \neq \sqrt{2} = \sqrt{1 + 1} = \sqrt{r_0 + s_0}$ # substitute $r_0 = s_0 = 1$
Then $\exists r, s \in \mathbb{R} : (r > 0 \wedge s > 0) \wedge \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$ # introduced $\exists$

(b) True, write the statement symbolically:

$$\forall r \in \mathbb{R} : \forall s \in \mathbb{R} : r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$$

Let $r, s \in \mathbb{R}$.

proof by contraposition (indirect proof)
Assume $\sqrt{r}+\sqrt{s}=\sqrt{r+s}$.
Then, $(\sqrt{r}+\sqrt{s})^2=(\sqrt{r+s})^2$. # square both sides
Then, $(\sqrt{r})^2+2\sqrt{r}\sqrt{s}+(\sqrt{s})^2=r+s$. # expand both sides
Then, $2\sqrt{rs}=0$. # subtract $r+s$ from both sides
Then, $rs=0$. # divide by 2 and square both sides
Then, $r=0 \vee s=0$.
Now, do a sub-proof by cases.
Case 1: Assume $r=0$.
Then, $r \not> 0$.
Then, $r \not> 0 \vee s \not> 0$.
Then, $\neg(r>0 \wedge s>0)$.
Case 2: Assume $s=0$.
Then, $s \not> 0$.
Then, $r \not> 0 \vee s \not> 0$.
Then, $\neg(r>0 \wedge s>0)$.
Then, $\neg(r>0 \wedge s>0)$. # for both cases
Then, $\sqrt{r}+\sqrt{s}=\sqrt{r+s} \Rightarrow \neg(r>0 \wedge s>0)$. # introduced contrapositive
Then, $r>0 \wedge s>0 \Rightarrow \sqrt{r}+\sqrt{s} \neq \sqrt{r+s}$.
Then, $\forall r \in \mathbb{R} : \forall s \in \mathbb{R} : r>0 \wedge s>0 \Rightarrow \sqrt{r}+\sqrt{s} \neq \sqrt{r+s}$.

6. **Sample Solution.**

Let $x \in \mathbb{Q}$.
Assume $x \neq 0$.
Then $\exists p, q \in \mathbb{Z} : q \neq 0 \wedge x = p/q$.
Let $p_0, q_0 \in \mathbb{Z} : x = p_0/q_0$.
Then $p_0 = xq_0 \neq 0$.
Let $a_0 = \sqrt{2}$.
Then $a_0 \in \mathbb{I}$. #proved in class.
Let $b_0 = x/\sqrt{2}$.
Will prove that b is irrational by contradiction.
Suppose $b_0 \notin \mathbb{I}$.
Then $b_0 \in \mathbb{Q}$. # b is real number so if it is not irrational, has to be rational.
Then $\exists s_0, t_0 \in \mathbb{Z} : b_0 = s_0/t_0$.
Let $r_0 = xt_0/s_0$.
Then $r_0 \in \mathbb{Q}$.
Also $\sqrt{2} = r_0$.
Then $\sqrt{2} \in \mathbb{Q}$. #contradicts the known fact $\sqrt{2} \in \mathbb{I}$.
Then $b_0 \in \mathbb{I}$.
Then $x = a_0 b_0$.
Then $\exists a, b \in \mathbb{I} : x = ab$.
Then $x \neq 0 \Rightarrow \exists a, b \in \mathbb{I} : x = ab$.
Therefore $\forall x \in \mathbb{Q} : x \neq 0 \Rightarrow [\exists a, b \in \mathbb{I} : x = a \cdot b]$.