1. **Sample Solution - using strong induction**

Base step: Prove $P(1), P(2), P(3)$

Let $k = 0, c_0 = 1$

Then $1 = 1 \cdot 2^0$

Then $P(1)$.

Let $k = 1, c_0 = 0, c_1 = 1$.

Then $2 = 1 \cdot 2^1 + 0 \cdot 2^0$.

Then $P(2)$.

Let $k = 1, c_0 = 1, c_1 = 1$.

Then $3 = 1 \cdot 2^1 + 1 \cdot 2^0$

Then $P(3)$.

Inductive step: $\forall k \in \mathbb{Z}^+ : [(k \geqslant 3) \to ((\forall i \in \{1, \cdots, k\} : P(i)) \to P(k+1))]$

Let $k \in \mathbb{Z}^+$

Assume $k \geqslant 3$.

Assume $\forall i \in \{1, \cdots, k\} : P(i)$.

Case 1: k odd.

Then $k+1$ is even.

Let $q = \frac{n+1}{2} \in \mathbb{Z}^+$.

Then $q < k$.

Then $\exists j \in \mathbb{Z}^+, c_0, \ldots, c_j \in \{0,1\} : q = \sum_{l=0}^{j} c_l 2^l$

Then $k+1 = 2q = \sum_{l=0}^{j} c_l 2^{l+1}$

Let $j' = j+1$

Let $c'_0 = 0, c'_1 = c_0, \ldots, c'_{j+1} = c_j$.

Then $j' \in \mathbb{Z}^+, c'_l \in \{0,1\} \forall 0 \leqslant l \leqslant j'$

Then $k+1 = \sum_{l=0}^{j'} c'_l 2^l$.

Then $P(k+1)$.

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Case 2: k even.

Then $k+1$ is odd.

Let $q = \frac{n}{2} \in \mathbb{Z}^+$.

Then $q < k$.

Then $\exists j \in \mathbb{Z}^+, c_0, \ldots, c_j \in \{0,1\} : q = \sum_{l=0}^{j} c_l 2^l$

Then $k+1 = 2q = \sum_{l=0}^{j} c_l 2^{l+1} + 1$

Let $j' = j+1$.

Let $c'_0 = 1, c'_1 = c_0, \ldots, c'_{j+1} = c_j$.

Then $j' \in \mathbb{N}, c'_l \in \{0,1\} \forall 0 \leqslant l \leqslant j'$

Then $k+1 = \sum_{l=0}^{j'} c'_l 2^l$.

Then $P(k+1)$.

Then $P(k+1)$.

Then $(\forall i \in \{1, \cdots, k\} : P(i)) \rightarrow P(k+1)$.

Then $(k \geqslant 3) \rightarrow ((\forall i \in \{1, \cdots, k\} : P(i)) \rightarrow P(k+1))$

Then $\forall k \in \mathbb{Z}^+ : [(k \geqslant 3) \rightarrow ((\forall i \in \{1, \cdots, k\} : P(i)) \rightarrow P(k+1))]$

Then $\forall n \in \mathbb{N} \in \mathbb{Z}^+ : P(n)$

2. **Sample Solution**

This claim is true, since the highest degrees of both polynomials are the same. By definition of Ω, we need to prove the following statement:

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geqslant c \cdot (5n^3 - n^2 + n + 1)$$

Use the chain of overestimate/underestimate to find proper c and B.

Proof:

Let $c_0 = 2/7, B_0 = 1$.
Then $c_0 \in \mathbb{R}^+, B_0 \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ and $n \geqslant B_0$ # generic natural number and the antecedent

Then $6n^3 - 4n^2 + 3n + 2 > 6n^3 - 4n^2$ # remove positive term $3n+2$

$\geqslant 6n^3 - 4n^2 \times n = 2n^3$# multiply a negative term by $n \geqslant B_0 = 1$

$= (2/7) \cdot (7n^3) = c \cdot (7n^3)$ # we picked $c_0 = 2/7$

$= c_0 \cdot (5n^3 + n^3 + n^3)$ # $7 = 5+1+1$

$\geqslant c_0 \cdot (5n^3 + n + 1)$ # $n^3 \geqslant n, n^3 \geqslant 1$ since $n \geqslant B_0 = 1$

$> c_0 \cdot (5n^3 - n^2 + n + 1)$ # add a negative term $-n^2$

Then $6n^3 - 4n^2 + 3n + 2 \geqslant c_0 \cdot (5n^3 - n^2 + n + 1)$ # transitivity

Then $\forall n \in \mathbb{N}, n \geqslant B_0 \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geqslant c_0 \cdot (5n^3 - n^2 + n + 1)$ # introduce $\forall, \Rightarrow$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 6n^3 - 4n^2 + 3n + 2 \geqslant c \cdot (5n^3 - n^2 + n + 1)$ # intro $\exists$

Therefore $6n^3 - 4n^2 + 3n + 2 \in \Omega(5n^3 - n^2 + n + 1)$ # by definition of Ω

3. **Sample Solution**

We provide only the estimation. Use the slides as a template to write a formal proof.

$$\begin{aligned}\frac{n^3+1}{2n+1} &\leqslant \frac{8n^3+1}{2n+1} = \frac{(2n+1)(4n^2-2n+1)}{2n+1} \\ &= 4n^2 - 2n + 1 \leqslant 4n^2 + 1 \leqslant 4n^2 + n^2 = 5n^2\end{aligned}$$

4. **Sample Solution**

We provide only the estimation. Use the slides as a template to write a formal proof.

Observe that we want some $M \in \mathbb{N}$ and some $c \in \mathbb{R}^+$ such that $\frac{1}{5}n^2 - 42n - 8 > cn^2$ for all $n \geqslant B$. Considering $\frac{1}{10} = \frac{1}{5} - \frac{1}{10}$ and $10(42+8) = 500$, we pick $c = \frac{1}{10}$ and $B = 501$. Then we have the following:

$$n > 500 = 10(42+8) > 10(42 + \frac{8}{n})$$

$$\frac{n^2}{10} - 42n - 8 > 0$$

$$\frac{n^2}{5} - \frac{n^2}{10} - 42n - 8 > 0$$

$$\frac{n^2}{5} - 42n - 8 > \frac{n^2}{10}$$

5. **We disprove it, that is:**

$$\forall c \in \mathbb{R}^+ : \forall B \in \mathbb{N} : \exists n \in \mathbb{N} : (n \geqslant B) \wedge (9n^2 + 3n - 1 < cn^3).$$

Let $c \in \mathbb{R}^+$. Let $B \in \mathbb{N}$.
Let $n_0 = \max(B, \lceil \frac{12}{c} \rceil) + 1$.
Then $n_0 \geqslant B$.
Then $n_0 > 12/c$.
Then $cn_0 > 12$.
Then $12 < cn_0$.
Then $12n_0^2 < cn_0^3$.
Then $9n_0^2 + 3n_0^2 < cn_0^3$.
Then $9n_0^2 + 3n_0 < cn_0^3$.
Then $9n_0^2 + 3n_0 - 1 < cn_0^3$.
Then $(n_0 \geqslant B) \wedge (9n_0^2 + 3n_0 - 1 < cn_0^3)$.
Then $\exists n \in \mathbb{N} : (n \geqslant B) \wedge (9n^2 + 3n - 1 < cn^3)$.
Then $\forall c \in \mathbb{R}^+ : \forall B \in \mathbb{N} : \exists n \in \mathbb{N} : (n \geqslant B) \wedge (9n^2 + 3n - 1 < cn^3)$.
Therefore, $9n^2 + 3n - 1 \notin \Omega(n^3)$.