CSC165
Mathematical Expression and Reasoning for Computer Science

Module 9

About Sequences

What is a Sequence?

- A sequence is an ordered collection of objects
- Order of the elements is important
- The number of elements is called the length of the sequence
- Examples:
  - Set of prime numbers: {2,3,5,7,11,13,17, ...}
  - Fibonacci numbers: $\{0,1,1,2,3,5,8,13,21,34, \dots\}$
  - Some repetitive sequence: {0,1,2,3,0,1,2,3,0,1,2,3, ...}
  - Generic sequence: $\{a_0, a_1, a_2, \dots\}$

For more, check Wikipedia page: https://en.wikipedia.org/wiki/Sequence

What is a Sequence?

- A sequence can be defined as a function whose domain is a countable totally ordered set, such as the natural numbers
- Example: Define sequence a_n as $\forall n \in \mathbb{N}: a_n = n + 2$
- a_n can also be represented as $a_n: \{2,3,4,5,6, \dots\}$
- Remember n is a dummy variable

n	0	1	2	3	4	5	...
a_n	2	3	4	5	6	7	...

- Define sequence b_n as $\forall n \in \mathbb{N}: b_n = -n^2$
- b_n can also be represented as $b_n: \{0, -1, -4, -9, -16, \dots\}$

n	0	1	2	3	4	5	...
b_n	0	-1	-4	-9	-16	-25	...

Example

- Consider the sequence $a_0, a_1, a_2,\ldots$ of integers
- Consider the following statement
  $S1: \exists j \in \mathbb{N}: [\forall i \in \mathbb{N}: (i \neq j) \rightarrow (a_i \geq a_j)]$
- Translate S1 into proper English
  There is a smallest element (or more than one) in the sequence
- Negate S1
  $\neg S1: \neg(\exists j \in \mathbb{N}: [\forall i \in \mathbb{N}: (i \neq j) \rightarrow (a_i \geq a_j)])$
  $\neg S1: \forall j \in \mathbb{N}: [\exists i \in \mathbb{N}: \neg((i \neq j) \rightarrow (a_i \geq a_j))]$
  $\neg S1: \forall j \in \mathbb{N}: [\exists i \in \mathbb{N}: (i \neq j) \wedge (a_i < a_j)]$

Example

- Consider the sequence $a_0, a_1, a_2,\ldots$ of integers
  $S2: \forall i \in \mathbb{N}: \forall j \in \mathbb{N}: [(j > i) \rightarrow (a_j \geq 2a_i)]$
- Translate S2 into proper English:
  Every element in the sequence is at least twice as large as every previous element in the sequence
- Negate S2
  $\neg S2: \neg(\forall i \in \mathbb{N}, \forall j \in \mathbb{N}: [(j > i) \rightarrow (a_j \geq 2a_i)])$
  $\neg S2: \exists i \in \mathbb{N}: \exists j \in \mathbb{N}: \neg[(j > i) \rightarrow (a_j \geq 2a_i)]$
  $\neg S2: \exists i \in \mathbb{N}: \exists j \in \mathbb{N}: [(j > i) \wedge (a_j < 2a_i)]$

Proof About Sequences

Example

- Define sequence a_n as
$\forall n \in \mathbb{N}: a_n = n^2$
$a_n: \{0,1,4,9,16,25,\ldots\}$

n	0	1	2	3	4	5	6	…
a_n	0	1	4	9	16	25	36	…

- Remember n is a dummy variable!
- Prove $\exists i \in \mathbb{N}: \left[\forall j \in \mathbb{N}: \left(a_j \leq i\right) \rightarrow (j < i)\right]$
- We need to pick some $i \in \mathbb{N}$
- For this i, we need to prove $\left[\forall j \in \mathbb{N}: \left(a_j \leq i\right) \rightarrow (j < i)\right]$ is true
- Need one "good" i

Proving a Statement About a Sequence

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Try $i = 0$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 0) \rightarrow (j < 0)$?
  - $a_j \leq 0$? is true for $j = 0$
  - $j < 0$? is false for $j = 0$
  - Counter example: $j = 0$

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

- Try $i = 1$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 1) \rightarrow (j < 1)$?
  - $a_j \leq 1$? is true for $j = 0,1$
  - $j < 1$? is true for $j = 0$ but false for $j = 1$
  - Counter example: $j = 1$

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

Proving a Statement About a Sequence

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Try $i = 2$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 2) \rightarrow (j < 2)$?
  - $a_j \leq 2$? is true for $j = 0,1$
  - $j < 2$? is true for both $j = 0,1$
  - Looks good

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

- Try $i = 3$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 3) \rightarrow (j < 3)$?
  - $a_j \leq 3$? is true for $j = 0,1$
  - $j < 3$? is true for both $j = 0,1$
  - Looks good

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

Proving a Statement About a Sequence

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Try $i = 4$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 4) \rightarrow (j < 4)$?
  - $a_j \leq 4$? is true for $j = 0,1,2$
  - $j < 4$? is true for $j = 0,1,2$
  - Looks good

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

- Try $i = 5$
  - Can we prove that $\forall j \in \mathbb{N}: (a_j \leq 5) \rightarrow (j < 5)$?
  - $a_j \leq 5$? is true for $j = 0,1,2$
  - $j < 5$? is true for $j = 0,1,2$
  - Looks good

j	0	1	2	3	4	5	6	…
a_j	0	1	4	9	16	25	36	…

Nested Tracing of the Claim

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Need one example… Check for values of i

$i = 0$, check for j	j	$(a_j \leq i = 0)$?	$(j < i = 0)$?	$(a_j \leq i) \rightarrow (j < i)$	Significance
	0	$0 \leq 0$? True	$0 < 0$? False	False	Counter example
	1	$1 \leq 0$? False	$1 < 0$? False	True	Irrelevant
	2	$4 \leq 0$? False	$2 < 0$? False	True	Irrelevant
$i = 1$, check for j	j	$(a_j \leq i = 1)$?	$(j < i = 1)$?	$(a_j \leq i) \rightarrow (j < i)$	Significance
	0	$0 \leq 1$? True	$0 < 1$? True	True	Okay
	1	$1 \leq 1$? True	$1 < 1$? False	False	Counter example
	2	$4 \leq 1$? False	$2 < 1$? False	True	Irrelevant

Nested Tracing of the Claim

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Need one example… Check for values of i

$i = 2$, check for j	j	$(a_j \leq i = 2)$?	$(j < i = 2)$?	$(a_j \leq i) \rightarrow (j < i)$	Significance
	0	$0 \leq 2$? True	$0 < 2$? True	True	Okay
	1	$1 \leq 2$? True	$1 < 2$? True	True	Okay
	2	$4 \leq 2$? False	$2 < 2$? False	True	Irrelevant
i =4, check for j	j	$(a_j \leq i = 4)$?	$(j < i = 4)$?	$(a_j \leq i) \rightarrow (j < i)$	Significance
	0	$0 \leq 4$? True	$0 < 4$? True	True	Okay
	1	$1 \leq 4$? True	$1 < 4$? True	True	Okay
	2	$4 \leq 4$? True	$2 < 4$? True	True	Okay

Nested Tracing of the Claim

- Prove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)]$ for $\forall n \in \mathbb{N}: a_n = n^2$
- Need one example… Check for values of i

i	$\forall j \in \mathbb{N}: (a_j \leq i) \rightarrow (j < i)$	True? False?	Significance
0	$\forall j \in \mathbb{N}: (a_j \leq 0) \rightarrow (j < 0)$	F: Counter Example: $j = 0$	Not example
1	$\forall j \in \mathbb{N}: (a_j \leq 1) \rightarrow (j < 1)$	F: Counter example: $j = 1$	Not example
2	$\forall j \in \mathbb{N}: (a_j \leq 2) \rightarrow (j < 2)$	T	Okay, example
3	$\forall j \in \mathbb{N}: (a_j \leq 3) \rightarrow (j < 3)$	T	Okay, example
4	$\forall j \in \mathbb{N}: (a_j \leq 4) \rightarrow (j < 4)$	T	Okay, example

- You can find some example where the claim is true
- The claim is true

Proof

- $\forall n \in \mathbb{N}: a_n = n^2$ Prove $\exists i \in \mathbb{N}: \left[\forall j \in \mathbb{N}: \left(a_j \leq i\right) \rightarrow (j < i)\right]$

Let $i' = 4$. # choose an example that works
Then $i' \in \mathbb{N}$. # 4 is a natural number
Let $j \in \mathbb{N}$. # assume a generic natural number
  Assume $a_j \leq i'$. # assume $P(j)$ to be true
    Then $j^2 \leq 4$. # $a_j = j^2, i' = 4$
    Then $j \leq 2$. # if $j^2 \leq 4$ then $j \leq 2$ (j is natural number)
    Then $j < 4$. # if $j \leq 2$, then $j < 4$, prove $Q(j)$
  Then $(a_j \leq 4) \rightarrow (j < 4)$. # introduce implication
Then $\forall j \in \mathbb{N}: \left(a_j \leq i'\right) \rightarrow (j < i')$. # introduce universal quantifier
Therefore, $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: \left(a_j \leq i\right) \rightarrow (j < i)]$. # introduce existential quantifier

Disproving a Statement About a Sequence

Example

- Consider the following sequence:
  a_n: {0,0,1,1,2,2,3,3,4,4,5,5,6,…}
- Consider the following statement:
  $S1: \exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)]$
- Disprove S1
- This is equivalent to proving the negation of S1
  $\neg S1: \neg(\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)])$
  $\neg S1: \forall i \in \mathbb{N}: [\exists j \in \mathbb{N}: (j > i) \wedge (a_j \neq a_i)]$

Nested Tracing of the Claim

- Disprove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)]$ for a_n:{0,0,1,1,2,2,3,3,4,4,5,5,…}
- Need to show there are no "good" examples… Check values of i

$i = 0$, check for j	j	$(j > i = 0)$?	$(a_j = a_i = 0)$?	$(j > i) \rightarrow (a_j = a_i)$	Significance
	0	$0 > 0$? False	$0 = 0$? True	True	Irrelevant
	1	$1 > 0$? True	$0 = 0$? True	True	Okay
	2	$2 > 0$? True	$1 = 0$? False	False	Counter example
$i = 1$, check for j	j	$(j > i = 1)$?	$(a_j = a_i = 0)$?	$(j > i) \rightarrow (a_j = a_i)$	Significance
	0	$0 > 1$? False	$0 = 0$? True	True	Irrelevant
	1	$1 > 1$? False	$0 = 0$? True	True	Irrelevant
	2	$2 > 1$? True	$1 = 0$? False	False	Counter example

Nested Tracing of the Claim

- Disprove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)]$ for a_n:{0,0,1,1,2,2,3,3,4,4,5,5,…}
- Need to show there are no "good" examples… Check values of i

$i = 2$, check for j	j	$(j > i = 2)$?	$(a_j = a_i = 1)$?	$(j > i) \rightarrow (a_j = a_i)$	Significance
	2	$2 > 2$? False	$1 = 1$? True	True	Irrelevant
	3	$3 > 2$? True	$1 = 1$? True	True	Okay
	4	$4 > 2$? True	$2 = 1$? False	False	Counter example
$i = 3$, check for j	j	$(j > i = 3)$?	$(a_j = a_i = 1)$?	$(j > i) \rightarrow (a_j = a_i)$	Significance
	2	$2 > 3$? False	$1 = 1$? True	True	Irrelevant
	3	$3 > 3$? False	$1 = 1$? True	True	Irrelevant
	4	$4 > 3$? True	$2 = 1$? False	False	Counter example

Nested Tracing of the Claim

- Disprove $\exists i \in \mathbb{N}: [\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)]$ for a_n:{0,0,1,1,2,2,3,3,4,4,5,5,…}
- Need to show there are no "good" examples… Check values of i

i	$\forall j \in \mathbb{N}: (j > i) \rightarrow (a_j = a_i)$	True? False?	Significance
0	$\forall j \in \mathbb{N}: (j > 0) \rightarrow (a_j = a_0)$	F: Counter Example: $j = 2$	Not example
1	$\forall j \in \mathbb{N}: (j > 1) \rightarrow (a_j = a_1)$	F: Counter example: $j = 2$	Not example
2	$\forall j \in \mathbb{N}: (j > 2) \rightarrow (a_j = a_2)$	F: Counter Example: $j = 4$	Not example
3	$\forall j \in \mathbb{N}: (j > 3) \rightarrow (a_j = a_3)$	F: Counter example: $j = 4$	Not example
4	$\forall j \in \mathbb{N}: (j > 4) \rightarrow (a_j = a_4)$	F: Counter Example: $j = 6$	Not example

- You cannot find any single example where the claim is true
- The claim is false…. Negate statement and proved the negated form

Disproving a Statement About a Sequence

- For a_n:{0,0,1,1,2,2,3,3,4,4,5,5,6,…}, prove
  $\forall i \in \mathbb{N}: \left[\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)\right]$
- For every natural number i there is a natural number j such that $(j > i)$ and $\left(a_j \neq a_i\right)$
- Are we allowed to make j dependent on i?
- Notice that:
  - i is in scope when we pick j
  - i has been declared and can be seen from where we declare j
  - j is not in scope when we declare i
  - When we pick i, we are not allowed to use j

Disproving a Statement About a Sequence

- For a_n:{0,0,1,1,2,2,3,3,4,4,5,5,6,…}, prove
  $\forall i \in \mathbb{N}: \left[\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)\right]$
- For every i we need to find a j such that $(j > i) \wedge \left(a_j \neq a_i\right)$ is true
- Need both $(j > i)$ and $\left(a_j \neq a_i\right)$ to be true

index	n	0	1	2	3	4	5	6	7	8	9	10	11	12	…
value	a_n	0	0	1	1	2	2	3	3	4	4	5	5	6	…

- For $i = 0$:
  - Let $j = 0$: $(j > i)$ is false, and $\left(a_j \neq a_i\right)$ is false
  - Let $j = 1$: $(j > i)$ is true, but $\left(a_j \neq a_i\right)$ is false
  - Let $j = 2$: $(j > i)$ is true, and $\left(a_j \neq a_i\right)$ is true… good
  - Let $j = 3$: $(j > i)$ is true, and $\left(a_j \neq a_i\right)$ is true… good

Disproving a Statement About a Sequence

- For a_n:{0,0,1,1,2,2,3,3,4,4,5,5,6,…}, prove
  $\forall i \in \mathbb{N}: [\exists j \in \mathbb{N}: (j > i) \wedge (a_j \neq a_i)]$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	…
a_n	0	0	1	1	2	2	3	3	4	4	5	5	6	…

- For $i = 1$:
  - Let $j = 0$: $(j > i)$ is false, and $(a_j \neq a_i)$ is false
  - Let $j = 1$: $(j > i)$ is false, and $(a_j \neq a_i)$ is false
  - Let $j = 2$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good
  - Let $j = 3$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good
  - ….

Disproving a Statement About a Sequence

- For a_n:{0,0,1,1,2,2,3,3,4,4,5,5,6,…}, prove
  $\forall i \in \mathbb{N}: [\exists j \in \mathbb{N}: (j > i) \wedge (a_j \neq a_i)]$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	…
a_n	0	0	1	1	2	2	3	3	4	4	5	5	6	…

- For $i = 2$:
  - Let $j = 3$: $(j > i)$ is true, but $(a_j \neq a_i)$ is false
  - Let $j = 4$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good
  - Let $j = 5$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good

n	0	1	2	3	4	5	6	7	8	9	10	11	12	…
a_n	0	0	1	1	2	2	3	3	4	4	5	5	6	…

- For $i = 3$:
  - Let $j = 4$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good
  - Let $j = 5$: $(j > i)$ is true, and $(a_j \neq a_i)$ is true… good

Disproving a Statement About a Sequence

- For a_n:{0,0,1,1,2,2,3,3,4,4,5,5,6,…}, prove
  $\forall i \in \mathbb{N}: \left[\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)\right]$
- For any specific i, we need $j > i$ so that $(j > i)$ is true. Also:
- If we choose $j = i + 1$: $(j > i)$ is true
  - $\left(a_j \neq a_i\right)$ is true (for odd values of i)
  - $\left(a_j \neq a_i\right)$ is false (for even values of i)
- If we choose $j = i + 2$: $(j > i)$ is true
  - $\left(a_j \neq a_i\right)$ is true (for both odd and even values of i)
- If we choose $j = i + 3$: $(j > i)$ is true
  - $\left(a_j \neq a_i\right)$ is true
- For any generic value of i, $j = i + 2$ will work

Proof

- For a_n: {0,0,1,1,2,2,3,3,4,…}, prove $\forall i \in \mathbb{N}: \left[\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)\right]$

Let $i \in \mathbb{N}$. # assume a generic natural number

Let $j' = i + 2$. # choose an example that works

Then $j' \in \mathbb{N}$. # if $i \in \mathbb{N}$ then $i + 2 \in \mathbb{N}$

Then $j' > i$. # if $j' = i + 2$ then $j' > i$

Then $a_{j'} \neq a_i$. # if $j' = i + 2$ then $a_{j'} \neq a_i$ (by inspection)

Then $(j' > i) \wedge \left(a_{j'} \neq a_i\right)$. # if A is true and B is true, then $A \wedge B$ is true

Then $\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)$. # introduce existential quantifier

Therefore, $\forall i \in \mathbb{N}: \left[\exists j \in \mathbb{N}: (j > i) \wedge \left(a_j \neq a_i\right)\right]$. # introduce universal quantifier