CSC165
Mathematical Expression and Reasoning for Computer Science

Module 7

Negation of Universal & Existential

Claims (i.e., Statements)

- Verifying a claim is proving the claim true
- Falsifying a claim is disproving the claim; i.e., proving the claim false

Universal Statement

- Claim: Every P (in the domain) is a Q
- To prove a universal claim true:
  - Verify that every element of the domain is an example that satisfies the quantification
  - There is no counter example
- To disprove a universal claim:
  - Find one counter example
  - What does a counter example look like?

  There is some P that is not Q

Universal Statement

S: $\forall x \in \mathbb{R}: x > 10$

- S is a universal claim… $\forall x \in \mathbb{R}: Q(x)$
- S claims that every element in the domain has the property of being Q
- S is false ($x = 5$ is a counter example)
- Counter example: There is some element in the domain that is not Q

$\exists x \in \mathbb{R}: x \ngtr 10$… or
$\exists x \in \mathbb{R}: x \leq 10$

- The negation of a universal statement is an existential statement
- The negation of $\forall x \in \mathbb{R}: Q(x)$ is $\exists x \in \mathbb{R}: \neg Q(x)$

Existential Statement

- Claim: Some P (in the domain) is a Q
- To prove an existential claim true:
  - Find one element of the domain as an example that satisfies the quantification
  - There is one example
- To disprove an existential claim:
  - Verify that every element of the domain is a counter example that does not satisfy the quantification
  - All elements of the domain are counter examples
  - All P are not Q

Existential Statement

S: $\exists x \in \mathbb{R}: x^2 < 0$

- S is an existential claim… $\exists x \in \mathbb{R}: Q(x)$
- S claims that some element in the domain has the property of being Q
- S is false (we cannot find any example)
- Every element of the domain is a counter example
- All elements of the domain have the property of being not Q

$\forall x \in \mathbb{R}: x^2 \nless 0$… or
$\forall x \in \mathbb{R}: x^2 \geq 0$

- The negation of an existential statement is a universal statement
- The negation of $\exists x \in \mathbb{R}: Q(x)$ is $\forall x \in \mathbb{R}: \neg Q(x)$

Example

- S: All cars are red
- Negation of S in “English”:
  - Not all care are red
  - There is some car that is not red
- Define the domain C: set of cars
- Define predicate $R(x)$: x has the property of being red
- S: $\forall x \in C: R(x)$
- Negation of S in “logical expression”:
  - ¬S: $\neg(\forall x \in C: R(x))$
  - ¬S: $\exists x \in C: \neg R(x)$

Example

- S: There is some student from the south pole
- Negation of S in "English":
  - There is not any student from the south pole
  - There is no student from the south pole
  - All the students are not from the south pole
- Define domain T: set of students
- Define $P(x)$: x from the south pole
- S: $\exists x \in T: P(x)$
- Negation of S in "logical expression":
  - $\neg S$: $\neg(\exists x \in T: P(x))$
  - $\neg S$: $\forall x \in T: \neg P(x)$

Negation and Implication

About Negation

- Negation of a true statement (or predicate) is false
- Negation of a false statement (or predicate) is true
- The negation of a universally quantified statement is an existentially quantified statement
  - "not all..." means "there is one that is not..."
- The negation of an existentially quantified statement is a universally quantified statement
  - "there does not exist..." means "all...are not..."
- Push the negation sign inside layer by layer (like peeling an onion)

Example

- S1: All red cars are Honda cars
- Also S1: For all x in cars: if x is a red car, then x is a Honda car
- Define:
  C: set of cars
  $R(x)$: x is red
  $H(x)$: x is Honda
- Write S1 in logical expression
  $S1:\ \forall x \in C: R(x) \rightarrow H(x)$
- Negate S1
  $\neg S1$: Not all red cars are Honda
  $\neg S1: \neg(\forall x \in C: R(x) \rightarrow H(x))$
  $\neg S1$: There exists a car that is red and not Honda
  $\neg S1: \exists x \in C: R(x) \wedge \neg H(x)$
- $\wedge$ means "logical and"

Example

- S2: There exists a car that is red and Honda
- Write S2 in logical expression
  $S2: \exists x \in C: R(x) \land H(x)$
- Negate S2
  $\neg S2$: There does not exists a car that is red and Honda
  $\neg S2: \neg(\exists x \in C: R(x) \land H(x))$
  $\neg S2$: For all cars, if car is red, then it is not Honda
  $\neg S2: \forall x \in C: R(x) \rightarrow \neg H(x)$
  Also $\neg S2$: For all cars, if car is Honda, then it is not red
  $\neg S2: \forall x \in C: H(x) \rightarrow \neg R(x)$

Mixed Quantifiers

Example

$\forall n \in \mathbb{N}: [(\exists j \in \mathbb{N}: n = 2j) \rightarrow (\exists k \in \mathbb{N}: n^2 = 2k)]$

- What is that?

$\forall n \in \mathbb{N}$: n is even, then n^2 is even

$\forall n \in \mathbb{N}: [(\exists j \in \mathbb{N}: n = 4j) \rightarrow (\exists k \in \mathbb{N}: n^2 = 4k)]$

- What is that?

$\forall n \in \mathbb{N}$: n is a multiple of 4, then n^2 is a multiple of 4

Examples

- What does the following mean?

$\exists m \in \mathbb{N}: [\forall n \in \mathbb{N}: m \geq n]$	There exists a BIGGEST natural number
$\exists n \in \mathbb{N}: [\forall m \in \mathbb{N}: m \geq n]$	There exists a smallest natural number
$\forall m \in \mathbb{N}: [\exists n \in \mathbb{N}: m > n]$	For every natural number there is a smaller number
$\forall n \in \mathbb{N}: [\exists m \in \mathbb{N}: m > n]$	For every natural number there is a bigger number
$\exists m \in \mathbb{N}: [\exists n \in \mathbb{N}: m > n]$	There is a natural number that is bigger than another
$\exists n \in \mathbb{N}: [\exists m \in \mathbb{N}: m > n]$	There is a natural number that is less than another
$\forall m \in \mathbb{N}: [\forall n \in \mathbb{N}: m > n]$	All natural numbers are bigger than all natural numbers
$\forall n \in \mathbb{N}: [\forall m \in \mathbb{N}: m > n]$	All natural numbers are bigger than all natural numbers

Negation Example

- Consider the following statement:
  $S3: \forall x \in X: [\exists y \in Y: P(x, y)]$
- Negate S3
- Remember: push the negation sign inside layer by layer
  $\neg S3: \neg(\forall x \in X: [\exists y \in Y: P(x, y)])$

  $\neg S3: \exists x \in X: \neg[\exists y \in Y: P(x, y)]$

  $\neg S3: \exists x \in X: \left[\forall y \in Y: \neg\left(P(x, y)\right)\right]$

  $\neg S3: \exists x \in X: [\forall y \in Y: \neg P(x, y)]$