CSC165
Mathematical Expression and Reasoning for Computer Science

Module 15

About Asymptotic Notation

Asymptotic Notation

- $\mathrm{O}(f(n))$ is the asymptotic upper-bound:

The set of functions that grow no faster than $f(n)$

- For example, when we say

$5n^2 + 3n + 1$ is in $\mathrm{O}(n^2)$

We mean: $5n^2 + 3n + 1$ grows no faster than n^2, asymptotically

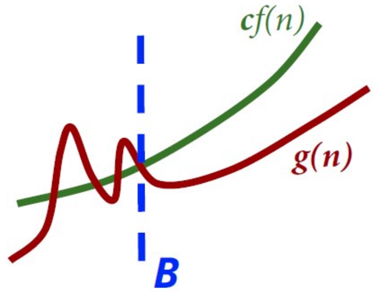
- Other bounds:
  - $\Omega(f(n))$: the asymptotic lower-bound… big Omega
  - $\Theta(f(n))$: the asymptotic tight-bound… big Theta

A High-Level Look At Asymptotic Notations

- It is a simplification of the “real” running time
- It does not tell the whole story about how fast a program runs in real life
- In real-world applications, constant factor matters! hardware matters! implementation matters!
- This simplification makes possible the development of the whole theory of computational complexity

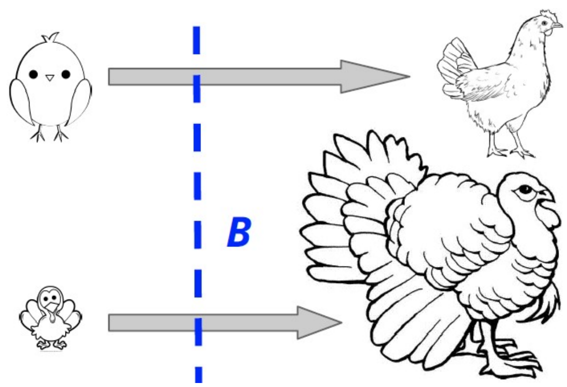
Definition of Big O

- A function $g(n)$ is in $\mathrm{O}(f(n))$ if and only if
  $$\exists c \in \mathbb{R}^{+}: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(g(n) \leq cf(n)\right)\right]\right]$$
- Beyond breakpoint B, $g(n)$ is upper-bounded by $cf(n)$, where c is some wisely chosen constant multiplier
- $cf(n)$ is an upper bound for $g(n)$



"chicken size" is in O("turkey size")

- A chicken grows slower than a turkey in the sense that, after a certain breakpoint, a chicken will always be smaller than a turkey



Definition of $\mathrm{O}(n^2)$

- A function $g(n)$ is in $\mathrm{O}(n^2)$ if and only if
  $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow (g(n) \leq cn^2)]\right]$
- Example:
  - $700n^2 \in \mathrm{O}(n^2)$?
  - Let $c = 701$, or any positive real number ≥ 700
  - Let $B = 0$, or any natural number ≥ 0
  - Then $\forall n \in \mathbb{N} : (n \geq 0) \rightarrow (700n^2 \leq 701n^2)$
  - Then $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow (700n^2 \leq cn^2)]\right]$
  - Then $700n^2 \in \mathrm{O}(n^2)$

Definition of Big Ω

- A function $g(n)$ is in $\Omega\big(f(n)\big)$ if and only if
  $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : \left[\forall n \in \mathbb{N} : (n \geq B) \rightarrow \big(g(n) \geq cf(n)\big)\right]\right]$
- Beyond breakpoint B, $g(n)$ is an upper bound for $cf(n)$, where c is some wisely chosen constant multiplier
- $cf(n)$ is a lower bound for $g(n)$

Definition of $\Omega(n^2)$

- A function $g(n)$ is in $\Omega(n^2)$ if and only if
  $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow (g(n) \geq cn^2)]\right]$
- Example:
  - $700n^2 \in \Omega(n^2)$?
  - Let $c = 699$, or any positive real number ≤ 700
  - Let $B = 0$, or any natural number ≥ 0
  - Then $\forall n \in \mathbb{N} : (n \geq 0) \rightarrow (700n^2 \geq 699n^2)$
  - Then $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow (700n^2 \geq cn^2)]\right]$
  - Then $700n^2 \in \Omega(n^2)$

Summary

- $O(n^2)$: set of functions that grow no faster than n^2
- $\Omega(n^2)$: set of functions that grow no slower than n^2
- $\Theta(n^2)$: set of functions that are in both $O(n^2)$ and $\Omega(n^2)$ (functions growing as fast as n^2)

Growth Rate of Typical Functions

$f(n) = n^n$	Grows Fast
$f(n) = 2^n$	
$f(n) = n^3$	
$f(n) = n^2$	
$f(n) = n \log n$	
$f(n) = n$	
$f(n) = \sqrt{n}$	
$f(n) = \log n$	
$f(n) = 1$	Grows Slow

Examples

- $7n \in \mathrm{O}(n^2)$
- $7n \notin \Omega(n^2)$

- $7n^3 \notin \mathrm{O}(n^2)$
- $7n^3 \in \Omega(n^2)$

- $7n^2 \in \mathrm{O}(n^2)$
- $7n^2 \in \Omega(n^2)$
- $7n^2 \in \Theta(n^2)$

- $\mathrm{O}(n^2)$: set of functions that grow no faster than n^2
- $\Omega(n^2)$: set of functions that grow no slower than n^2
- $\Theta(n^2)$: set of functions that are in both $\mathrm{O}(n^2)$ and $\Omega(n^2)$ (functions growing as fast as n^2)

Over-Estimation and Under-Estimation

Over-Estimation and Under-Estimation

- Simplify the function without changing the highest degree
- Use under-estimation to find a smaller function
- Under-estimation tricks:
  - Remove a positive term ($3n^2 + 2n \geq 3n^2$)
  - Multiply a negative term ($5n^2 - n \geq 5n^2 - n \times n = 4n^2$)
- Use over-estimation to find a larger function
- Over-estimation tricks:
  - Remove a negative term ($3n^2 - 2n \leq 3n^2$)
  - Multiply a positive term ($5n^2 + n \leq 5n^2 + n \times n = 6n^2$)

Over-Estimation and Under-Estimation

- Want to prove $g(n) \leq cf(n)$ where $g(n)$ and $f(n)$ are functions with many terms
- Use under-estimation and over-estimation to find simpler functions
  - Find $g'(n)$ such that $g(n) \leq g'(n)$ (note: g' is not the derivative of g, it is another related, simpler function)
  - Find $f'(n)$ such that $f'(n) \leq f(n)$ (note: f' is not the derivative of f, it is another related, simpler function)
  - Find c that works for $g'(n) \leq cf'(n)$
  - Consequently, $g(n) \leq g'(n) \leq cf'(n) \leq cf(n)$

Over-Estimation and Under-Estimation

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  - Find c that works for $g'(n) \geq cf'(n)$
  - Consequently, $g(n) \geq g'(n) \geq cf'(n) \geq cf(n)$

Examples

Example

- Prove that $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)$
- Thoughts:
  - Prove $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : \left[\forall n \in \mathbb{N} : (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq cn^2\right)\right]\right]$
  - $c \geq 3$ works
  - $B \geq 0$ works

n	$\frac{3}{2}n^2 + \frac{9}{2}n - 4$	$(1)n^2$	$(2)n^2$	$(3)n^2$	$(4)n^2$
0	-4	0	0	0	0
1	2	1	2	3	4
2	11	4	8	12	16
3	23	9	18	27	36
4	38	16	32	48	64
5	56	25	50	75	100

Proof: $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)$

Let $c_0 = 6$. Then $c_0 \in \mathbb{R}^+$.
Let $B_0 = 0$. Then $B_0 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Assume $n \geq 0$.
Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 < \frac{3}{2}n^2 + \frac{9}{2}n$
$\leq \frac{3}{2}n^2 + \frac{9}{2}n^2 = 6n^2$. # $g'(n)$
Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \leq c_0 n^2$. # $g(n) \leq g'(n) \leq cf(n)$
Then $(n \geq B_0) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq c_0 n^2\right)$.
Then $\forall n \in \mathbb{N}: (n \geq B_0) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq c_0 n^2\right)$.
Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq cn^2\right)\right]\right]$.
Therefore, $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)$.

Example

- Prove that $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)$
- Thoughts:
  - Prove $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq cn^2\right)\right]\right]$
  - $c \leq 3/2$ works
  - $B \geq 1$ works

n	$\frac{3}{2}n^2 + \frac{9}{2}n - 4$	$(\frac{3}{2})n^2$	$(1)n^2$	$(\frac{1}{2})n^2$	$(\frac{1}{4})n^2$
0	-4	0	0	0	0
1	2	1.5	1	0.5	0.25
2	11	6	4	2	1
3	23	13.5	9	4.5	2.25
4	38	24	16	8	4
5	56	37.5	25	12.5	6.25

Proof: $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)$

Let $c_1 = 1$. Then $c_1 \in \mathbb{R}^+$.
Let $B_1 = 1$. Then $B_1 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
  Assume $n \geq 1$.
    Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \geq \frac{3}{2}n^2 + \frac{9}{2}n - 4n = \frac{3}{2}n^2 + \frac{1}{2}n$
      $> \frac{3}{2}n^2 > n^2$.
    Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \geq c_1 n^2$. # $g(n) \geq g'(n) \geq cf(n)$
  Then $(n \geq B_1) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq c_1 n^2\right)$.
Then $\forall n \in \mathbb{N}: (n \geq B_1) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq c_1 n^2\right)$.
Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq cn^2\right)\right]\right]$.
Therefore, $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)$.

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Example

- Prove that $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Theta(n^2)$
- Thoughts:
  - Prove $\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)\right) \wedge \left(\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)\right)$

  - $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)$
  - Choose $c \geq 3$ and $B \geq 0$

  - $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)$
  - Choose $c \leq 3/2$ and $B \geq 1$

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Proof: $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Theta(n^2)$

Let $c_0 = 6$. Then $c_0 \in \mathbb{R}^+$.

Let $B_0 = 0$. Then $B_0 \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Assume $n \geq 0$.

Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 < \frac{3}{2}n^2 + \frac{9}{2}n \leq \frac{3}{2}n^2 + \frac{9}{2}n^2 = 6n^2$.

Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \leq c_0 n^2$.

Then $(n \geq B_0) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq c_0 n^2\right)$.

Then $\forall n \in \mathbb{N}: (n \geq B_0) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq c_0 n^2\right)$.

Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \leq cn^2\right)\right]\right]$.

Then, $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)$.

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Proof: $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Theta(n^2)$

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Let $c_1 = 1$. Then $c_1 \in \mathbb{R}^+$.

Let $B_1 = 1$. Then $B_1 \in \mathbb{N}$.

Let $n \in \mathbb{N}$.

Assume $n \geq 1$.

Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \geq \frac{3}{2}n^2 + \frac{9}{2}n - 4n = \frac{3}{2}n^2 + \frac{1}{2}n > \frac{3}{2}n^2 > n^2$.

Then $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \geq c_1 n^2$.

Then $(n \geq B_1) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq c_1 n^2\right)$.

Then $\forall n \in \mathbb{N}: (n \geq B_1) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq c_1 n^2\right)$.

Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4\right) \geq cn^2\right)\right]\right]$.

Then, $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)$.

Then, $\left(\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in O(n^2)\right) \wedge \left(\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Omega(n^2)\right)$.

Therefore, $\frac{3}{2}n^2 + \frac{9}{2}n - 4 \in \Theta(n^2)$.

More Examples

Example

- Prove $3n^2 + 2n \in O(n^2)$
- Thoughts:
  - Prove $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : \left[\forall n \in \mathbb{N} : (n \geq B) \rightarrow \left((3n^2 + 2n) \leq cn^2\right)\right]\right]$
  - c should probably be larger than 3 (the constant factor of the highest-order term)
  - See what happens when $n = 1$
  - If $n = 1$
    - $3n^2 + 2n = 3 + 2 = 5 = 5n^2$
    - So $c = 5$ and $B = 1$ is a good combination
    - Double check for $n = 2,3,4,\dots$

Proof: $3n^2 + 2n \in O(n^2)$

Let $c_0 = 5$.
Then $c_0 \in \mathbb{R}^+$.
Let $B_0 = 1$.
Then $B_0 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
  Assume $n \geq 1$.
    Then $3n^2 + 2n \leq 3n^2 + 2n \times n = 5n^2$. $\# \ g'(n)$
    Then $3n^2 + 2n \leq c_0 n^2$. $\# \ g(n) \leq g'(n) \leq cf(n)$
  Then $(n \geq B_0) \rightarrow \left((3n^2 + 2n) \leq c_0 n^2\right)$.
Then $\forall n \in \mathbb{N}: (n \geq B_0) \rightarrow \left((3n^2 + 2n) \leq c_0 n^2\right)$.
Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left((3n^2 + 2n) \leq cn^2\right)\right]\right]$.
Therefore, $3n^2 + 2n \in O(n^2)$.

Example

- Prove $3n^2 + 2n + 5 \in O(n^2)$
- Thoughts:
  - Prove $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left((3n^2 + 2n + 5) \leq cn^2\right)\right]\right]$
  - c should probably be larger than 3 (the constant factor of the highest-order term)
  - See what happens when $n = 1$
  - If $n = 1$
    - $3n^2 + 2n + 5 = 3 + 2 + 5 = 10 = 10n^2$
    - So $c = 10$ and $B = 1$ is a good combination
    - Double check for $n = 2,3,4, \ldots$

Proof: $3n^2 + 2n + 5 \in O(n^2)$

Let $c_0 = 10$.
Then $c_0 \in \mathbb{R}^+$.
Let $B_0 = 1$.
Then $B_0 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
  Assume $n \geq 1$.
    Then $3n^2 + 2n + 5 \leq 3n^2 + 2n \times n + 5 \times n^2 = 10n^2$. # $g'(n)$
    Then $3n^2 + 2n + 5 \leq c_0 n^2$. # $g(n) \leq g'(n) \leq cf(n)$
  Then $(n \geq B_0) \rightarrow \big((3n^2 + 2n + 5) \leq c_0 n^2\big)$.
Then $\forall n \in \mathbb{N}: (n \geq B_0) \rightarrow \big((3n^2 + 2n + 5) \leq c_0 n^2\big)$.
Then, $\exists c \in \mathbb{R}^+: \Big[\exists B \in \mathbb{N}: \big[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \big((3n^2 + 2n + 5) \leq cn^2\big)\big]\Big]$.
Therefore, $3n^2 + 2n + 5 \in O(n^2)$.

Example

- Prove $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$
- Thoughts:
  - Prove
    $\exists c \in \mathbb{R}^+: \Big[\exists B \in \mathbb{N}: \big[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \big((7n^6 - 5n^4 + 2n^3) \leq c(6n^8 - 4n^5 + n^2)\big)\big]\Big]$
  - Upper-bound the left side $(7n^6 - 5n^4 + 2n^3)$ by over-estimating
  - Lower-bound the right side $(6n^8 - 4n^5 + n^2)$ by under-estimating
  - Choose a c that connects the two bounds

Prove: $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$

- Assume $n \geq 1$
  - Then $7n^6 - 5n^4 + 2n^3 < 7n^6 + 2n^3$
    $\leq 7n^6 + 2n^6 = 9n^6$
  - Then $7n^6 - 5n^4 + 2n^3 < 9n^6$ $\quad$ # $g(n) \leq g'(n)$
  - Then $6n^8 - 4n^5 + n^2 > 6n^8 - 4n^5$
    $\geq 6n^8 - 4n^8 = 2n^8$
  - Then $2n^8 < 6n^8 - 4n^5 + n^2$ $\quad$ # $f'(n) \leq f(n)$
  - Find c such that $g(n) \leq g'(n) \leq cf'(n) \leq cf(n)$
    - $9n^6 \leq c(2n^8)$
    - $n^6 \leq \frac{2}{9}c\, n^8$
    - If $c \geq \frac{9}{2}$ then $n^6 \leq n^8$
  - $7n^6 - 5n^4 + 2n^3 < 9n^6 = \frac{9}{2}2n^6 \leq \frac{9}{2}2n^8 < \frac{9}{2}(6n^8 - 4n^5 + n^2)$
- Let $c = \frac{9}{2}$ and $B = 1$

Proof: $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$

Let $c_0 = 9/2$. Then $c_0 \in \mathbb{R}^+$.
Let $B_0 = 1$. Then $B_0 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Assume $n \geq 1$.
Then $7n^6 - 5n^4 + 2n^3 < 7n^6 + 2n^3 \leq 7n^6 + 2n^6 = 9n^6$
$= \frac{9}{2}2n^6 = c_0 2n^6$
$\leq c_0 2n^8 = c_0(6n^8 - 4n^8) \leq c_0(6n^8 - 4n^5) \leq c_0(6n^8 - 4n^5 + n^2)$.
Then $7n^6 - 5n^4 + 2n^3 \leq c_0(6n^8 - 4n^5 + n^2)$.
Then $(n \geq B_0) \rightarrow \left((7n^6 - 5n^4 + 2n^3) \leq c_0(6n^8 - 4n^5 + n^2)\right)$.
Then $\forall n \in \mathbb{N}: (n \geq B_0) \rightarrow \left((7n^6 - 5n^4 + 2n^3) \leq c_0(6n^8 - 4n^5 + n^2)\right)$.
Then, $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left((7n^6 - 5n^4 + 2n^3) \leq c(6n^8 - 4n^5 + n^2)\right)\right]\right]$.
Therefore, $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$.

Example

- Prove that $n^2 + n \in \Omega(15n^2 + 3)$
- Thoughts:
  - Prove $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : \left[\forall n \in \mathbb{N} : (n \geq B) \rightarrow \left((n^2 + n) \geq c(15n^2 + 3)\right)\right]\right]$
  - Try under-estimation and over-estimation
  - Let's pick $B = 1$
  - Try to pick c small enough to make the right side a lower bound
- Assume $n \geq 1$
  - $n^2 + n > n^2$ # $g(n) \geq g'(n)$
  - $15n^2 + 3 \leq 15n^2 + 3n^2 = 18n^2$ # $f'(n) \geq f(n)$
  - Find c such that $g(n) \geq g'(n) \geq cf'(n) \geq cf(n)$
  - $n^2 + n > n^2 = \frac{1}{18} 18\, n^2 = \frac{1}{18}(15n^2 + 3n^2) \geq \frac{1}{18}(15n^2 + 3)$
- Let $c = \frac{1}{18}$ and $B = 1$

Proof: $n^2 + n \in \Omega(15n^2 + 3)$

Let $c_1 = \frac{1}{18}$. Then $c_1 \in \mathbb{R}^+$.
Let $B_1 = 1$. Then $B_1 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
  Assume $n \geq 1$.
    Then $n^2 + n > n^2 = \frac{1}{18} 18n^2$
$$= \frac{1}{18}(15n^2 + 3n^2) = c_1(15n^2 + 3n^2)$$
$$\geq c_1(15n^2 + 3).$$
    Then $n^2 + n \geq c_1(15n^2 + 3)$.
  Then $(n \geq B_1) \rightarrow \left((n^2 + n) \geq c_1(15n^2 + 3)\right)$.
Then $\forall n \in \mathbb{N} : (n \geq B_1) \rightarrow \left((n^2 + n) \geq c_1(15n^2 + 3)\right)$.
Then, $\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : \left[\forall n \in \mathbb{N} : (n \geq B) \rightarrow \left((n^2 + n) \geq c_1(15n^2 + 3)\right)\right]\right]$.
Therefore, $n^2 + n \in \Omega(15n^2 + 3)$.

Disproof of Big O

Example

- Prove that $n^3 \notin O(3n^2)$
- Thoughts:
  - Prove $\neg\left(\exists c \in \mathbb{R}^+ : \left[\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow (n^3 \leq c(3n^2))]\right]\right)$
  - Prove $\forall c \in \mathbb{R}^+ : \left[\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge \neg(n^3 \leq c(3n^2))]\right]$
  - Prove $\forall c \in \mathbb{R}^+ : \left[\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (n^3 > c(3n^2))]\right]$

  - Remember, we choose n after c and B
  - We want to choose n such that $n^3 > c(3n^2)$
  - That is, we want $n > 3c$
  - Similarly, we want to choose n such that $n \geq B$

Prove: $n^3 \notin O(3n^2)$

- Thoughts:
  - Prove $\forall c \in \mathbb{R}^+ : \left[\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (n^3 > c(3n^2))]\right]$
  - So, we want both $n > 3c$ and $n \geq B$
  - Need $n > \max(3c, B)$
  - Note: $n \in \mathbb{N}$, $B \in \mathbb{N}$, but $3c \in \mathbb{R}^+$
  - However, $\lceil 3c \rceil \in \mathbb{N}$
  - Choose $n = \max(\lceil 3c \rceil, B) + 1 \in \mathbb{N}$
  - Why the "+1"?
  - In this case, $(\max(\lceil 3c \rceil, B) + 1 > 3c) \wedge (\max(\lceil 3c \rceil, B) + 1 > B)$
  - Note: $n_0 = \lceil 3c \rceil + B + 1$ also works

Proof: $n^3 \notin O(3n^2)$

Let $c \in \mathbb{R}^+$.
  Let $B \in \mathbb{N}$.
    Let $n_0 = \max(\lceil 3c \rceil, B) + 1$. # note: $n_0 = \lceil 3c \rceil + B + 1$ works
    Then $n_0 \in \mathbb{N}$.
    Then $n_0 \geq B$.
    Then $n_0 > 3c$.
    Then $n_0 n_0^2 > 3cn_0^2$.
    Then $n_0^3 > c3n_0^2$.
    Then $(n_0 \geq B) \wedge \left(n_0^3 > c(3n_0^2)\right)$.
    Then $\exists n \in \mathbb{N} : (n \geq B) \wedge (n^3 > c(3n^2))$.
  Then $\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (n^3 > c(3n^2))]$.
Then $\forall c \in \mathbb{R}^+ : \left[\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (n^3 > c(3n^2))]\right]$.
Therefore, $n^3 \notin O(3n^2)$.

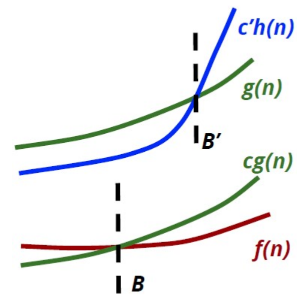
Big O and Generic Functions

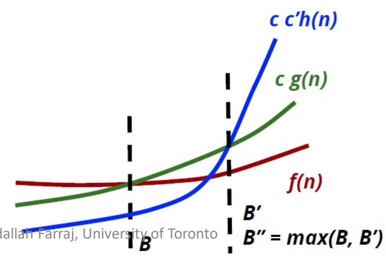
Introduction

- All statements we have proven so far
  - $3n^2 + 2n \in O(n^2)$
  - $3n^2 + 2n + 5 \in O(n^2)$
  - $7n^6 - 5n^4 + 2n^3 \in O(6n^8 - 4n^5 + n^2)$
  - $n^2 + n \in \Omega(15n^2 + 3)$
  - $n^3 \notin O(3n^2)$
- These are statements about specific functions. Let's prove some general statements about Big O
- Let $\mathcal{F}: \{f: \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}\}$ be the set of all functions that take a natural number as input and return a non-negative real number

Example

- Prove that $\forall f, g, h \in \mathcal{F}: [(f \in O(g) \wedge g \in O(h)) \rightarrow f \in O(h)]$
- Thoughts:
  - If f grows no faster than g and g grows no faster than h, then f must grow no faster than h
  - $f(n) \leq cg(n)$ for $n \geq B$
  - $g(n) \leq c'h(n)$ for $n \geq B'$
  - Find B'' and c'', so that $f(n) \leq c''h(n)$ for $n \geq B''$
  - Beyond B'': Beyond both B and B'
  - Let $B'' = \max(B, B')$
  - Note: $B'' = B + B'$ also works
  - Want $f(n) \leq c''h(n)$
  - $f(n) \leq cg(n) \leq c(c'h(n))$
  - Let $c'' = cc'$





Proof: $\forall f, g, h \in \mathcal{F}: [(f \in O(g) \wedge g \in O(h)) \rightarrow f \in O(h)]$

Let $f, g, h \in \mathcal{F}$.

Assume $f \in O(g) \wedge g \in O(h)$.

Then $f \in O(g)$.

Then $\exists c \in \mathbb{R}^+: \big[\exists B \in \mathbb{N}: [\forall n \in \mathbb{N}: (n \geq B) \rightarrow (f(n) \leq cg(n))]\big]$.

Let $c_1 \in \mathbb{R}^+ \wedge B_1 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}: (n \geq B_1) \rightarrow (f(n) \leq c_1 g(n))$.

Then $g \in O(h)$.

Then $\exists c \in \mathbb{R}^+: \big[\exists B \in \mathbb{N}: [\forall n \in \mathbb{N}: (n \geq B) \rightarrow (g(n) \leq ch(n))]\big]$.

Let $c_2 \in \mathbb{R}^+ \wedge B_2 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}: (n \geq B_2) \rightarrow (g(n) \leq c_2 h(n))$.

Let $c'' = c_1 . c_2$.

Then $c'' \in \mathbb{R}^+$.

Let $B'' = \max(B_1, B_2)$. #Note: $B'' = B + B'$ also works

Then $B'' \in \mathbb{N}$.

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Proof: $\forall f, g, h \in \mathcal{F}: [(f \in O(g) \wedge g \in O(h)) \rightarrow f \in O(h)]$

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Let $n \in \mathbb{N}$.

Assume $n \geq B''$.

Then $f(n) \leq c_1 g(n)$.

Then $g(n) \leq c_2 h(n)$.

Then $f(n) \leq c_1 g(n) \leq c_1 c_2 h(n) = c'' h(n)$.

Then $\forall n \in \mathbb{N}: (n \geq B'') \rightarrow (f(n) \leq c'' h(n))$.

Then $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: [\forall n \in \mathbb{N}: (n \geq B) \rightarrow (f(n) \leq ch(n))]\right]$.

Then $f \in O(h)$.

Then $(f \in O(g) \wedge g \in O(h)) \rightarrow f \in O(h)$.

Therefore, $\forall f, g, h \in \mathcal{F}: [(f \in O(g) \wedge g \in O(h)) \rightarrow f \in O(h)]$.

Example

- Prove that $\forall f, g \in \mathcal{F}: [f \in O(g) \rightarrow g \in \Omega(f)]$
- Thoughts:
  - If f grows no faster than g, then g grows no slower than f
  - Assume $f \in O(g)$:
  - $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(f(n) \leq cg(n)\right)\right]\right]$
  - $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(\frac{1}{c} f(n) \leq g(n)\right)\right]\right]$
  - $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B) \rightarrow \left(g(n) \geq \frac{1}{c} f(n)\right)\right]\right]$
  - Prove $f \in \Omega(g)$
  - $\exists c' \in \mathbb{R}^+: \left[\exists B' \in \mathbb{N}: \left[\forall n \in \mathbb{N}: (n \geq B') \rightarrow \left(g(n) \geq c' f(n)\right)\right]\right]$
  - Choose $B' = B$ and $c' = \frac{1}{c}$

Proof: $\forall f, g \in \mathcal{F}: [f \in \mathrm{O}(g) \rightarrow g \in \Omega(f)]$

Let $f, g \in \mathcal{F}$.
Assume $f \in \mathrm{O}(g)$.
Then $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: [\forall n \in \mathbb{N}: (n \geq B) \rightarrow (f(n) \leq cg(n))]\right]$.
Let $c_1 \in \mathbb{R}^+ \wedge B_1 \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}: (n \geq B_1) \rightarrow (f(n) \leq c_1 g(n))$.
Let $c_2 = \frac{1}{c_1}$. Then $c_2 \in \mathbb{R}^+$.
Let $B_2 = B_1$. Then $B_2 \in \mathbb{N}$.
Let $n \in \mathbb{N}$.
Assume $n \geq B_2$.
Then $n \geq B_1$.
Then $f(n) \leq c_1 g(n)$.
Then $\frac{1}{c_1} f(n) \leq g(n)$.
Then $g(n) \geq \frac{1}{c_1} f(n) = c_2 f(n)$.
Then $\forall n \in \mathbb{N}: (n \geq B_2) \rightarrow (g(n) \geq c_2 f(n))$.
Then $\exists c \in \mathbb{R}^+: \left[\exists B \in \mathbb{N}: [\forall n \in \mathbb{N}: (n \geq B) \rightarrow (g(n) \geq cf(n))]\right]$.
Then $g \in \Omega(f)$.
Then $f \in \mathrm{O}(g) \rightarrow g \in \Omega(f)$.
Therefore, $\forall f, g \in \mathcal{F}: [f \in \mathrm{O}(g) \rightarrow g \in \Omega(f)]$.