CSC165
Mathematical Expression and Reasoning for Computer Science

Module 14

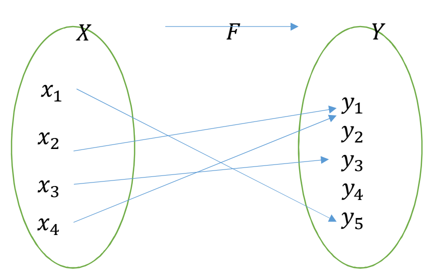
Proofs About Functions

Functions

- A function F from a set X to a set Y is a relation (correspondence) from X to Y
- Notation:
  - $F: X \to Y$ (note: $\to$ does not mean "imply" here)
  - X is called the domain
  - Y is called the image (or co-domain)
- $F: X \to Y$ needs to satisfy:
  - Every element in X is related to some element in Y
  - No element in X is related to more than one element in Y

Functions

- Range of F is the image of X under F: $\{y \in Y: y = F(x)\}$ for $x \in X$
- Inverse image of $y \in Y$: $\{x \in X: F(x) = y\}$



Function Equality

- Let $F: X \to Y$ and $G: X \to Y$ be functions
- $F = G \leftrightarrow \forall x \in X: [F(x) = G(x)]$
- Example:
  - $F: \mathbb{R} \to \mathbb{R}$ and $G: \mathbb{R} \to \mathbb{R}$
  - Define $F + G: \mathbb{R} \to \mathbb{R}$ as $\forall x \in \mathbb{R}: [(F + G)(x) = F(x) + G(x)]$
  - Define $G + F: \mathbb{R} \to \mathbb{R}$ as $\forall x \in \mathbb{R}: [(G + F)(x) = G(x) + F(x)]$
  - $(F + G)(x) = F(x) + G(x) = G(x) + F(x) = (G + F)(x)$
  - Consequently, $F + G = G + F$

Not Well-Defined "Functions"

- $F: X \to Y$ is called well defined function if:
  - Every element in X is related to some element in Y
  - No element in X is related to more than one element in Y
- If not, F is not well-defined… actually F is not a function
- Example:
  - Define $F: \mathbb{R} \to \mathbb{R}$ as $\forall x \in \mathbb{R}$: $F(x)$ is the real number y such that $x^2 + y^2 = 1$
  - For $x = 2$, there is no real number y such that $2^2 + y^2 = 1$
  - Also, for $x = 0$, both $y = 1$ and $y = -1$ satisfy $0^2 + y^2 = 1$

One-to-One Functions

One-to-One Functions

- $F: X \to Y$ is called one-to-one function if and only if:
  - For all elements x_1 and x_2 in X: if $F(x_1) = F(x_2)$, then $x_1 = x_2$, or
  - For all elements x_1 and x_2 in X: if $x_1 \neq x_2$ then $F(x_1) \neq F(x_2)$
- $F: X \to Y$ is one-to-one function $\leftrightarrow$

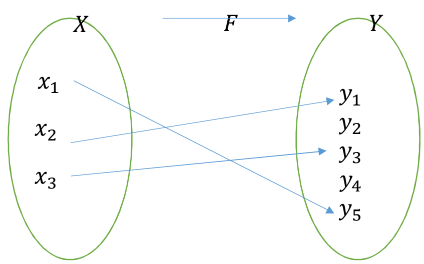
$$\forall x_1, x_2 \in X: \left[\left(F(x_1) = F(x_2)\right) \to (x_1 = x_2)\right]$$

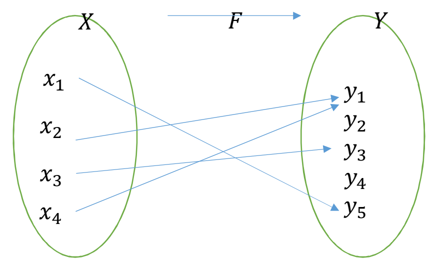
- A one-to-one function is also called an injective function
- $F: X \to Y$ is NOT one-to-one function $\leftrightarrow$

$$\exists x_1, x_2 \in X: \left[\left(F(x_1) = F(x_2)\right) \land (x_1 \neq x_2)\right]$$

One-to-One Functions

- One-to-one function: distinct elements in the domain are mapped to distinct elements in the co-domain
- Not one-to-one function: at least two elements in the domain are mapped to the same element in the co-domain





Example

- Let function $f: \mathbb{R} \to \mathbb{R}$ be defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$
- Prove that f is one-to-one
- Thoughts:
  - One-to-one function: distinct elements in the domain are mapped to distinct elements in the co-domain
  - Domain: $\mathbb{R}$
  - Co-domain: $\mathbb{R}$
  - Need to prove $\forall x_1, x_2 \in \mathbb{R}: [(f(x_1) = f(x_2)) \to (x_1 = x_2)]$
  - $f(x_1) = 2x_1 - 1$
  - $f(x_2) = 2x_2 - 1$
  - If $f(x_1) = 2x_1 - 1 = f(x_2) = 2x_2 - 1$, can we prove that $x_1 = x_2$?

Proof: f is One-to-One

Let $x_1, x_2 \in \mathbb{R}$.
  Assume $f(x_1) = f(x_2)$.
    Then $2x_1 - 1 = 2x_2 - 1$.
    Then $2x_1 = 2x_2$.
    Then $x_1 = x_2$.
  Then $\big(f(x_1) = f(x_2)\big) \to (x_1 = x_2)$.
Then, $\forall x_1, x_2 \in \mathbb{R}: \big[\big(f(x_1) = f(x_2)\big) \to (x_1 = x_2)\big]$.
Therefore, $f: \mathbb{R} \to \mathbb{R}$ defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$ is one-to-one.

Example

- Let function $g: \mathbb{Z} \to \mathbb{Z}$ be defined as $\forall n \in \mathbb{Z}: g(n) = n^2$
- Prove that g is not one-to-one
- Thoughts:
  - Not one-to-one function: at least two elements in the domain are mapped to the same element in the co-domain
  - Domain: $\mathbb{Z}$
  - Co-domain: $\mathbb{Z}$
  - Need to prove $\exists n_1, n_2 \in \mathbb{Z}: \big[\big(g(n_1) = g(n_2)\big) \wedge (n_1 \neq n_2)\big]$
  - $g(n_1) = n_1^2$
  - $g(n_2) = n_2^2$
  - Can we find $n_1, n_2 \in \mathbb{Z}$ such that $g(n_1) = n_1^2 = g(n_2) = n_2^2$ and $n_1 \neq n_2$?

Proof: g is NOT One-to-One

Let $n_{1'} = 3$.
Then $n_{1'} \in \mathbb{Z}$.
Let $n_{2'} = -3$.
Then $n_{2'} \in \mathbb{Z}$.
Then $n_{1'} \neq n_{2'}$.
Then $g(n_{1'}) = 3^2 = 9$.
Then $g(n_{2'}) = (-3)^2 = 9$.
Then $g(n_{1'}) = g(n_{2'})$.
Then $\big(g(n_{1'}) = g(n_{2'})\big) \wedge (n_{1'} \neq n_{2'})$.
Then, $\exists n_1, n_2 \in \mathbb{Z}: \big[\big(g(n_1) = g(n_2)\big) \wedge (n_1 \neq n_2)\big]$.
Therefore, $g: \mathbb{Z} \to \mathbb{Z}$ defined as $\forall n \in \mathbb{Z} g(n) = n^2$ is not one-to-one.

Onto Functions

Onto Functions

- $F: X \to Y$ is called onto function if and only if:
  - For every element y in Y, it is possible to find an element x in X with the property $y = F(x)$
- Every element in the co-domain is an image of some element in the domain
- $F: X \to Y$ is onto function $\leftrightarrow$

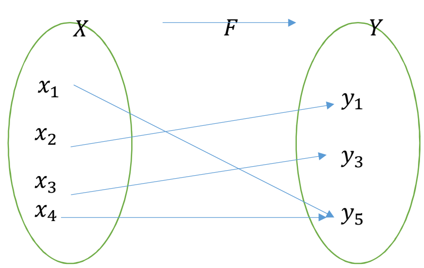
$$\forall y \in Y: [\exists x \in X: F(x) = y]$$

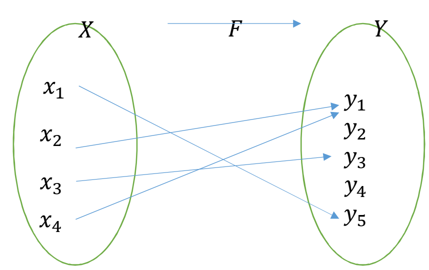
- An onto function is also called an surjective function
- $F: X \to Y$ is NOT onto function $\leftrightarrow$

$$\exists y \in Y: [\forall x \in X: F(x) \neq y]$$

Onto Functions

- Onto function: each element in the co-domain is mapped to from (an) element(s) in the domain
- Not onto function: at least one element in the co-domain is not mapped to from elements in the domain





Example

- Let function $f: \mathbb{R} \to \mathbb{R}$ be defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$
- Prove that f is onto
- Thoughts:
  - Onto function: each element in the co-domain is mapped to from (an) element(s) in the domain
  - Domain: $\mathbb{R}$
  - Co-domain: $\mathbb{R}$
  - Need to prove $\forall y \in \mathbb{R}: [\exists x \in \mathbb{R}: f(x) = y]$
  - $y = 2x - 1$
  - $y + 1 = 2x$
  - $x = \frac{y+1}{2}$

Proof: f is Onto

Let $y \in \mathbb{R}$.

Let $x_0 = \frac{y+1}{2}$.

Then $x_0 \in \mathbb{R}$.

Then $f(x_0) = f\left(\frac{y+1}{2}\right)$

$= 2\frac{y+1}{2} - 1$

$= (y + 1) - 1$

$= y.$

Then $\exists x \in \mathbb{R}: f(x) = y$.

Then, $\forall y \in \mathbb{R}: [\exists x \in \mathbb{R}: f(x) = y]$.

Therefore, $f: \mathbb{R} \to \mathbb{R}$ defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$ is onto.

Example

- Let function $h: \mathbb{Z} \to \mathbb{Z}$ be defined as $\forall n \in \mathbb{Z}: h(n) = 2n + 3$
- Prove that h is not onto
- Thoughts:
  - Not onto function: at least one element in the co-domain is not mapped to from elements in the domain
  - Domain: $\mathbb{Z}$
  - Co-domain: $\mathbb{Z}$
  - Need to prove $\exists m \in \mathbb{Z}: [\forall n \in \mathbb{Z}: h(n) \neq m]$
  - Try $m = 0$
  - We cannot find any $n \in \mathbb{Z}$ where $h(n) = m = 0$
  - This works for all even integers m

Proof: h is NOT Onto

Let $m_0 = 0$.

Then $m_0 \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$.

If $0 = 2n + 3$, then $n = -\frac{3}{2} \notin \mathbb{Z}$.

Then $h(n) \neq 0$.

Then $\forall n \in \mathbb{Z}: h(n) \neq m_0$.

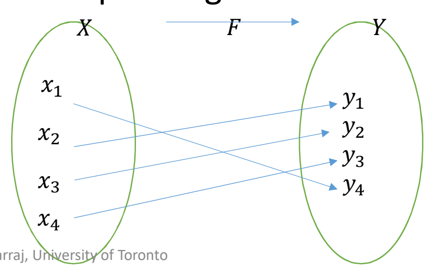
Then, $\exists m \in \mathbb{Z}: [\forall n \in \mathbb{Z}: h(n) \neq m]$.

Therefore, $h: \mathbb{Z} \to \mathbb{Z}$ defined as $\forall n \in \mathbb{Z}: h(n) = 2n + 3$ is not onto.

One-to-One Correspondences

One-to-One Correspondences

- $F: X \to Y$ is called one-to-one correspondence if and only if:
  - F is one-to-one
  - F is onto
- Any element y in Y has a corresponding element x in X such that $y = F(x)$
- Any element x in X has a unique corresponding element in Y such that $y = F(x)$
- F is called bijection



Inverse Functions

- Let $F: X \to Y$ be a one-to-one correspondence
- Let $F^{-1}: Y \to X$ be defined as:
  - Given any element y in Y: $F^{-1}(y)$ is the unique element x in X such that $F(x) = y$
  - $F^{-1}(y) = x \leftrightarrow y = F(x)$
- F^{-1} is called the inverse function of F

- If $F: X \to Y$ is one-to-one correspondence, then F^{-1} is also one-to-one correspondence
- Can you prove that?

Example

- Let function $f: \mathbb{R} \to \mathbb{R}$ be defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$
- Find f^{-1}
  - Given any element y in $\mathbb{R}$: $f^{-1}(y)$ is the unique element x in $\mathbb{R}$ such that $f(x) = y$
  - $y = f(x) \leftrightarrow f^{-1}(y) = x$
  - $y = 2x - 1$
  - $y + 1 = 2x$
  - $\frac{y+1}{2} = x$
  - $f^{-1}(y) = \frac{y+1}{2}$
- $f^{-1}: \mathbb{R} \to \mathbb{R}$ is defined as $\forall y \in \mathbb{R}: f^{-1}(y) = \frac{y+1}{2}$