1. Sample Solution

(a) If $x > 100$ then $\frac{100}{3-2x} > -1$.
$\forall x \in \mathbb{R}$: if $x > 100$ then $\frac{100}{3-2x} > -1$.
Let $x \in \mathbb{R}$.
Assume $x > 100$.
Then $2x > 200$.
Then $-2x < -200$.
Then $3 - 2x < -197$.
Then $\frac{1}{3-2x} > \frac{-1}{197}$.
Then $\frac{100}{3-2x} > \frac{-100}{197} = -0.5076$.
Then $\frac{100}{3-2x} > -1$.
Then if $x > 100$ then $\frac{100}{3-2x} > -1$.
Therefore, $\forall x \in \mathbb{R}$: if $x > 100$ then $\frac{100}{3-2x} > -1$.

Converse Problem:
$\forall x \in \mathbb{R}$: if $\frac{100}{3-2x} > -1$ then $x > 100$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (\frac{100}{3-2x} > -1) \wedge \neg(x > 100)$.
Let $x_0 = -100$.
Then $x_0 \in \mathbb{R}$.
Then $\frac{100}{3-2x_0} = 0.4926 > -1$.
However, $x_0 \not> 100$ (i.e., $\neg(x_0 > 100)$ is True).
Then $(\frac{100}{3-2x_0} > -1) \wedge \neg(x_0 > 100)$.
Therefore, $\exists x \in \mathbb{R} : (\frac{100}{3-2x} > -1) \wedge \neg(x > 100)$.

(b) If $\frac{3}{x^2-1} < \frac{1}{100}$ then $x \geqslant 20$.
$\forall x \in \mathbb{R}$: if $\frac{3}{x^2-1} < \frac{1}{100}$ then $x \geqslant 20$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (\frac{3}{x^2-1} < \frac{1}{100}) \wedge \neg(x \geqslant 20)$.
Let $x_0 = -20$.
Then $x_0 \in \mathbb{R}$.
Then $\frac{3}{x_0^2-1} = 0.0075 < \frac{1}{100}$.
However, $x \ngeqslant 20$ (i.e., $\neg(x_0 \geqslant 20)$ is True).
Then $(\frac{3}{x_0^2-1} < \frac{1}{100}) \wedge \neg(x_0 \geqslant 20)$.
Therefore, $\exists x \in \mathbb{R} : (\frac{3}{x^2-1} < \frac{1}{100}) \wedge \neg(x \geqslant 20)$.

Converse Problem:
$\forall x \in \mathbb{R}$: if $x \geqslant 20$ then $\frac{3}{x^2-1} < \frac{1}{100}$.
Let $x \in \mathbb{R}$.
Assume $x \geqslant 20$.
Then $x^2 \geqslant 400$.
Then $x^2 - 1 \geqslant 399$.
Then $\frac{1}{x^2-1} \leqslant \frac{1}{399}$.
Then $\frac{3}{x^2-1} \leqslant \frac{3}{399}$.
Then $\frac{3}{x^2-1} < \frac{3}{300} = \frac{1}{100}$.

Then $\frac{3}{x^2-1} < \frac{1}{100}$.
Then if $x \geqslant 20$ then $\frac{3}{x^2-1} < \frac{1}{100}$.
Therefore, $\forall x \in \mathbb{R}$: if $x \geqslant 20$ then $\frac{3}{x^2-1} < \frac{1}{100}$.

(c) $x > 10$ when $\frac{x^5-2}{3x^2+7} < 100$.
$\forall x \in \mathbb{R}$: if $\frac{x^5-2}{3x^2+7} < 100$ then $x > 10$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (\frac{x^5-2}{3x^2+7} < 100) \wedge \neg(x > 10)$
Let $x_0 = 0$.
Then $x_0 \in \mathbb{R}$.
Then $\frac{x_0^5-2}{3x_0^2+7} = \frac{-2}{7} < 100$.
However, $x \not> 10$ (i.e., $\neg(x_0 > 10)$ is True).
Then $(\frac{x_0^5-2}{3x_0^2+7} < 100) \wedge \neg(x_0 > 10)$.
Therefore, $\exists x \in \mathbb{R} : (\frac{x^5-2}{3x^2+7} < 100) \wedge \neg(x > 10)$.

Converse Problem:
$\forall x \in \mathbb{R}$: if $x > 10$ then $\frac{x^5-2}{3x^2+7} < 100$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (x > 10) \wedge \neg(\frac{x^5-2}{3x^2+7} < 100)$.
Let $x_0 = 11$.
Then $x_0 \in \mathbb{R}$.
Then $x_0 > 10$.
However, $\frac{x_0^5-2}{3x_0^2+7} = 435.2676 \not< 100$ (i.e., $\neg(\frac{x_0^5-2}{3x_0^2+7} < 100)$ is True).
Then $(x_0 > 10) \wedge \neg(\frac{x_0^5-2}{3x_0^2+7} < 100)$.
Therefore, $\exists x \in \mathbb{R} : (x > 10) \wedge \neg(\frac{x^5-2}{3x^2+7} < 100)$.

(d) $\frac{x^4+x^3+x+1}{x^2} > 200000$ implies that $x > 100$.
$\forall x \in \mathbb{R}$: if $\frac{x^4+x^3+x+1}{x^2} > 200000$ then $x > 100$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (\frac{x^4+x^3+x+1}{x^2} > 200000) \wedge \neg(x > 100)$.
Let $x_0 = -450$.
Then $x_0 \in \mathbb{R}$.
Then $\frac{x_0^4+x_0^3+x_0+1}{x_0^2} = 202050 > 200000$.
However, $x_0 \not> 100$ (i.e., $\neg(x_0 > 100)$ is True).
Then $(\frac{x_0^4+x_0^3+x_0+1}{x_0^2} = 202050 > 200000) \wedge \neg(x_0 > 100)$.
Therefore, $\exists x \in \mathbb{R} : (\frac{x^4+x^3+x+1}{x^2} > 200000) \wedge \neg(x > 100)$.

Converse Problem:
$\forall x \in \mathbb{R}$: if $x > 100$ then $\frac{x^4+x^3+x+1}{x^2} > 200000$.
False statement.
Prove the negated form: $\exists x \in \mathbb{R} : (x > 100) \wedge \neg(\frac{x^4+x^3+x+1}{x^2} > 200000)$.
Let $x_0 = 150$.
Then $x_0 \in \mathbb{R}$.
Then $x_0 > 100$.
However, $\frac{x_0^4+x_0^3+x_0+1}{x_0^2} = 22650 \not> 200000$. (i.e., $\neg(\frac{x_0^4+x_0^3+x_0+1}{x_0^2} > 200000)$ is True).

Then $(x_0 > 100) \wedge \neg(\frac{x_0^4+x_0^3+x_0+1}{x_0^2} > 200000)$.
Therefore, $\exists x \in \mathbb{R} : (x > 100) \wedge \neg(\frac{x^4+x^3+x+1}{x^2} > 200000)$.

2. Sample Solution

(a) Let x be the length of the shorter leg. The other leg has length $x + 4$. Then by Pythagoras, we have
$$x^2 + (x+4)^2 = 20^2.$$

(b)

Let x be a positive real number.
Assume $x^2 + (x+4)^2 = 20^2$.
Then $x^2 + 4x - 192 = 0$.
Then $(x+16)(x-12) = 0$.
Then $x = -16$ or $x = 12$.
Then, if $x^2 + (x+4)^2 = 20^2$ then $x = -16$ or $x = 12$.
Therefore, for any positive number x, if $x^2 + (x+4)^2 = 20^2$ then $x = -16$ or $x = 12$.

Since x has to be positive real, then $x = 12$ cm.
Thus, $\forall x \in \mathbb{R}$, if $x^2 + (x+4)^2 = 20^2$, then $x = 12$.

Prove the converse ($\forall x \in \mathbb{R}$, if $x = 12$ then $x^2 + (x+4)^2 = 20^2$).
Let $x \in \mathbb{R}$.
Assume $x = 12$.
Then $x^2 + (x+4)^2 = 12^2 + 16^2 = 400 = 20^2$.
Then if $x = 12$ then $x^2 + (x+4)^2 = 20^2$.
Therefore, $\forall x \in \mathbb{R}$, if $x = 12$ then $x^2 + (x+4)^2 = 20^2$.