1. **Sample Solution**
   #Proof by contradiction.
   Let $n, k \in \mathbb{N}$.
   Assume $k^2 < n < (k+1)^2 \wedge n > 0$ and $\exists m \in \mathbb{N} : n = m^2$.
   Let $m_0 \in \mathbb{N} : n = m_0^2$.
   Then $k^2 < m_0^2 < (k+1)^2$.
   Then $k < m_0 < k+1$. #Contradiction: m_0 is not natural number.
   Then $\forall m \in \mathbb{N} : n \neq m^2$.
   Therefore $\forall n \in \mathbb{N} : \forall k \in \mathbb{N} : (n > 0 \wedge k^2 < n < (k+1)^2) \Rightarrow (\forall m \in \mathbb{N} : n \neq m^2)$.

2. **Sample Solution**
   Let $n \in \mathbb{N}$.
   Assume $n > 0$.
   Let $k = \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$.
   Then $k \in \mathbb{N}$.
   Then $k \leqslant \sqrt{n} + \frac{1}{2} < k+1$.
   Then $k - \frac{1}{2} \leqslant \sqrt{n} < k + \frac{1}{2}$..
   Then $k^2 - k + \frac{1}{4} \leqslant n < k^2 + k + \frac{1}{4}$.
   Then $k^2 - k < n < k^2 + k + 1$. #n is natural number.
   Then $k^2 < n + k < (k+1)^2$.
   Then $n + k = n + \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$ cannot be a square (see above).
   Therefore, $\forall n \in \mathbb{N} : n + \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$ cannot be a square.

3. **Sample Solution**

   Let $i_0 = 3$.
   Then $i_0 \in \mathbb{N}$.
   Let $j \in \mathbb{N}$.
   Assume $a_j \neq a_{i_0}$.
   Then $a_j \neq a_3 = 3$.
   Then $a_j = 0$ or $a_j = 1$ or $a_j = 2$. # By inspection.
   Let $k_0 = j + 1$.
   Then $k_0 \in \mathbb{N}$. # Since $j \in \mathbb{N}$.
   Then $a_{k_0} = a_{j+1} = a_j + 1$. # By inspection, since $a_j \neq 3$.
   Then $\exists k \in \mathbb{N} : a_k = 1 + a_j$.
   Then $a_j \neq a_{i_0} \Rightarrow [\exists k \in \mathbb{N} : a_k = 1 + a_j]$.
   Then $\forall j \in \mathbb{N} : [a_j \neq a_{i_0} \Rightarrow [\exists k \in \mathbb{N} : a_k = 1 + a_j]]$.
   Therefore, $\exists i \in \mathbb{N} : [\forall j \in \mathbb{N} : [a_j \neq a_i \Rightarrow [\exists k \in \mathbb{N} : a_k = 1 + a_j]]]$

4. **Sample Solution**

   Let us start by defining the predicate

$$P(n) : \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$$

   We need to prove that $\forall n \in \mathbb{N}, P(n)$.

   Base case:
   let $n = 0$. #We want to prove $P(0)$.

Then we can calculate:

$$\begin{aligned}
\sum_{j=0}^{n} T_j &= \sum_{j=0}^{0} T_j \\
&= t_0 \\
&= \frac{0(0+1)}{2} \\
&= 0
\end{aligned}$$

And also $\frac{0(0+1)(0+2)}{6} = 0$.
Then $P(0)$.

Induction step:
Let $k \in \mathbb{N}$
Assume $P(k)$.
Then $\sum_{j=0}^{k} T_j = k(k+1)(k+2)/6$.
We want to prove $P(k+1)$, i.e., that $\sum_{j=0}^{k} T_j = (k+1)(k+2)(k+3)/6$.

We'll calculate starting from the left side and show that it equals the right side.

$$\begin{aligned}
\sum_{j=0}^{k+1} T_j &= \left(\sum_{k=0}^{k} T_j\right) + T_{k+1} \\
&= \frac{k(k+1)(k+2)}{6} + T_{k+1} && \text{(by our assumption of } P(k)\text{)} \\
&= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} && \text{(by the definition of } T_{k+1}\text{)} \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6} \\
&= \frac{(k+1)(k+2)(k+3)}{6}
\end{aligned}$$

Then $P(k+1)$.
Then $P(k) \Rightarrow P(k+1)$.
Then $\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1)$.

Therefore, $\forall n \in \mathbb{N}, P(n)$.