CSC165
Mathematical Expression and Reasoning for Computer Science

Module 6

Notes

Domain

- Changing the domain or universe of the implication statement could change the True/False outcome of the statement
- Keep an eye on the domain

$\mathbb{R}$: Set of real numbers

$\mathbb{R}^+$: Set of positive reals

$\mathbb{R}^-$: Set of negative reals

Predicates

- If $\exists x \in D: P(x)$ is true:
  - Some element in the domain has the property P
- If $\forall x \in D: P(x)$ is true:
  - All elements of the domain D have the property P
- If $\exists x \in D: \neg P(x)$ is true:
  - Some element in the domain does not have the property P
- If $\forall x \in D: \neg P(x)$ is true:
  - All elements of the domain D do not have the property P

Implication

- $P(x) \rightarrow Q(x)$:
- If x has the property P, then x has the property Q
- If x does not have the property P:
  - $P(x) \rightarrow Q(x)$ does not indicate whether x has the property Q or not
- $P(x) \rightarrow Q(x)$ is false when $P(x)$ is true and $Q(x)$ is false
- Otherwise, $P(x) \rightarrow Q(x)$ is true

Implication Example

- $E(x)$: x is enrolled in CSC165
- $T(x)$: x is a student
- $E(x) \rightarrow T(x)$
  - If x has the property of being enrolled in CSC165, then x has the property of being a student
  - If x is enrolled in CSC165, then x is a student
  - This is an open sentence, we cannot verify it unless we know x
- $\forall x \in D: E(x) \rightarrow T(x)$
  - For all x in domain D: if x has the property of being enrolled in CSC165, then x has the property of being a student
  - This is a statement, it can be verified

Converse, Contrapositive…

S: $\forall x \in \mathbb{R}$: if $P(x)$ then $Q(x)$

- Write the contrapositive of S

$\forall x \in \mathbb{R}$: if $\neg Q(x)$ then $\neg P(x)$

- Write the converse of S

$\forall x \in \mathbb{R}$: if $Q(x)$ then $P(x)$

- Write the contrapositive of the converse of S

$\forall x \in \mathbb{R}$: if $\neg P(x)$ then $\neg Q(x)$

Converse, Contrapositive…

S: All Java programs passed test 1

- Write the contrapositive of S

(All) Programs that did not pass test 1 are not written in Java… or

(All) Programs that failed test 1 are not written in Java

- Write the converse of S

(All) Programs that passed test 1 are written in Java

- Write the contrapositive of the converse of S

(All) Programs that are not written in Java did not pass test 1

Converse, Contrapositive…

S: $\forall x \in \mathbb{R}$: if $x > 10$ then $x > 5$

- Contrapositive of S

$\forall x \in \mathbb{R}$: if $x \ngtr 5$ then $x \ngtr 10$… or

$\forall x \in \mathbb{R}$: if $x \leq 5$ then $x \leq 10$

- Converse of S

$\forall x \in \mathbb{R}$: if $x > 5$ then $x > 10$

- Contrapositive of the converse of S

$\forall x \in \mathbb{R}$: if $x \ngtr 10$ then $x \ngtr 5$

$\forall x \in \mathbb{R}$: if $x \leq 10$ then $x \leq 5$

Bounding

Bounding

- When we express time or memory use properties of code (in CSC108/148/..., and later in this course), it will often be simpler and more useful to express bounds that ignore "small inputs“
- Why?
- Because larger inputs will usually be where the time or memory use can become significant enough to matter

- Explore the following claims using our standard approaches:

Bounding

1. $\forall x \in \mathbb{R}$: if $x > 10$ then $x > 5$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $x > 5 + 5$.

Then $x > 5$.

Then if $x > 10$ then $x > 5$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $x > 5$.

Bounding

2. $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 10x$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $x.x > 10(x)$. # by multiplying both sides by positive value x

Then $x^2 > 10x$.

Then if $x > 10$ then $x^2 > 10x$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 10x$.

Bounding

3. $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 100$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $x^2 > 10x$.

Then $x^2 > 10(10) = 100$.

Then if $x > 10$ then $x^2 > 100$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 100$.

Bounding

4. $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 60$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $x^2 > 100 = 60 + 40$.

Then $x^2 > 60$.

Then if $x > 10$ then $x^2 > 60$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > 60$.

Bounding

5. $\forall x \in \mathbb{R}$: if $x > 10$ then $\frac{1}{x} < \frac{1}{10}$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $\frac{x}{x} > \frac{10}{x}$. # By dividing both sides by positive value x

Then $1 > \frac{10}{x}$.

Then $\frac{1}{10} > \frac{1}{x}$.

Then $\frac{1}{x} < \frac{1}{10}$.

Then if $x > 10$ then $\frac{1}{x} < \frac{1}{10}$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $\frac{1}{x} < \frac{1}{10}$.

Bounding

6. $\forall x \in \mathbb{R}$: if $x > 10$ then $\frac{1}{x} < 1$

Let $x \in \mathbb{R}$.

Assume $x > 10$.

Then $\frac{1}{x} < \frac{1}{10}$.

Then $\frac{1}{x} < \frac{1}{10} + \frac{9}{10} = \frac{10}{10} = 1$.

Then if $x > 10$ then $\frac{1}{x} < 1$.

Therefore, $\forall x \in \mathbb{R}$: if $x > 10$ then $\frac{1}{x} < 1$.

Bounding

7. $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 > x$

- If $x > 10$ then $x^2 > 10\,x = x + 9x$
- Then $x^2 > x$

8. $\forall x \in \mathbb{R}$: if $x > 10$ then $-x < -5$

- If $x > 10$ then $-x < -10$
- Then $-x < -5$

9. $\forall x \in \mathbb{R}$: if $x > 10$ then $x^2 - 5x < 3x^2 - 2x$

- If $x > 10$ then $5x > 2x$
- Then $-5x < -2x$
- Then $x^2 - 5x < x^2 - 2x$
- Then $x^2 - 5x < x^2 - 2x + 2x^2$
- Then $x^2 - 5x < 3x^2 - 2x$

Bounding Example

Bounding Example

- Consider:

(S1) $\forall x \in \mathbb{R}: \text{if } x \geq 10 \text{ then } \frac{5}{x^2-7x+1} < \frac{1}{6}$

- What is S1 about?
  - S1 makes a direct claim for all numbers that are larger than or equal 10
  - Example: since 165 ≥ 10: S1 claims that $\frac{5}{165^2-7(165)+1} < \frac{1}{6}$
- Is it true?

Bounding Example

- (S1): $\forall x \in R\text{: if } x \geq 10 \text{ then } \frac{5}{x^2-7x+1} < \frac{1}{6}$
- Exploration:
  - The numerator is positive
  - The denominator looks positive
  - So the larger the denominator is, the smaller the ratio is
  - So try underestimating the denominator with something simpler:

$$\begin{aligned} 165^2 - 7(165) + 1 &> 100^2 - 7(165) + 1 \\ &> 100^2 - 10(200) + 1 \\ &> 100^2 - 10(200) = 10000 - 2000 = 8000 \end{aligned}$$

  - So: $165^2 - 7(165) + 1 > 8000$

Bounding Example

- Then $\frac{5}{165^2-7(165)+1} < \frac{5}{8000}$

$$\begin{aligned} &< \frac{5}{5000} = \frac{1}{1000} \\ &< \frac{1}{6} \end{aligned}$$

- Try to make that less dependent on the particulars of 165:

$$\begin{aligned} 165^2 - 7(165) + 1 &= 165(165-7) + 1 \\ &> 165(165-7) \\ &> 100(100) = 10000 \end{aligned}$$

- Try using just the fact that $165 \geq 10$:

$$\begin{aligned} 165^2 - 7(165) + 1 &> 165(10) - 7(165) + 1 = 10(165) - 7(165) + 1 \\ &\qquad = 3(165) + 1 \\ &> 3(10) + 1 = 31 \end{aligned}$$

Bounding Example

- (S1): $\forall x \in \mathbb{R}$: if $x \geq 10$ then $\frac{5}{x^2-7x+1} < \frac{1}{6}$

Let $x \in \mathbb{R}$.

Assume $x \geq 10$.

Then $x^2 - 7x + 1 \geq 10x - 7x + 1$
$= 3x + 1$
$\geq 30 + 1 = 31$.

Then $\frac{5}{x^2-7x+1} \leq \frac{5}{31}$
$< \frac{5}{30} = \frac{1}{6}$.

Therefore, $\forall x \in \mathbb{R}$: if $x \geq 10$ then $\frac{5}{x^2-7x+1} < \frac{1}{6}$.

Another Way to Prove $\forall x \in \mathbb{R}$: if $x \geq 10$ then $\frac{5}{x^2-7x+1} < \frac{1}{6}$

Let $x \in \mathbb{R}$.

Assume $x \geq 10$.

Then $x^2 - 7x + 1 > x^2 - 7x$
$= x(x-7)$
$\geq x(10-7)$
$= 3x$
$\geq 3(10)$
$= 5(6)$.

Then $\frac{1}{x^2-7x+1} < \frac{1}{5(6)}$.

Then $\frac{5}{x^2-7x+1} < \frac{5}{5(6)}$
$= \frac{1}{6}$.

Then if $x \geq 10$ then $\frac{5}{x^2-7x+1} < \frac{1}{6}$.

Therefore, $\forall x \in \mathbb{R}$: if $x \geq 10$ then $\frac{5}{x^2-7x+1} < \frac{1}{6}$.