First Name:

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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(6 Marks)
Consider the following Python code:

```
def mystery(L):
    '''
    :param L: List of size n
    :return: A mystery number
    '''
    sum1 = 0
    sum2 = 0
    bound = 1
    while bound <= len(L):
        i = 0
        while i < bound:
            j = 0
            while j < len(L):
                if L[j] > L[i]:
                    sum1 = sum1 + L[j]
                j = j + 2
            j = 1
            while j < len(L):
                sum2 = sum2 + L[j]
                j = j*2
            i = i + 1
        bound = bound * 2
    return sum1 + sum2
```

1. (3 Marks) Denote the time complexity of the given code $T(n)$ as a function of n where n is the size of the list L. Compute $T(n)$. Justify all steps.

2. (3 Marks) Prove that $T(n) \in O(n^{\frac{5}{2}})$. Hint: You can use without proof the following: $\forall \alpha \in \mathbb{R}^+ : \log_2 n \in O(n^\alpha)$.

Solution

1. **Sample Solution**

    - The value of variable `bound` on completion of iteration b is 2^b.

Proof by induction on the number of iterations

Let b denote the number of iterations.
Initial Step : Let $b = 1$.
Then `bound = 2`.
Inductive Step: Assume `bound`$= 2^b$ on iteration b.
Then `bound = bound*2` so `bound` $= 2^b 2 = 2^{b+1}$ on iteration $b + 1$.

- The `bound` loop terminates when the loop count is $\lfloor \log_2 n \rfloor + 1$.
  Proof
  Let B be the count of last loop.
  Then `bound`$=2^B$.
  Also `bound` $> n$.
  Then $B > \log_2 n$.
  Also B is the smallest integer to satisfy this condition.
  Then $B - 1 \leq \log_2 n$ and $B - 1$ is the largest integer to satisfy this condition.
  Then $B - 1 = \lfloor \log_2 n \rfloor$.
  Then $B = \lfloor \log_2 n \rfloor + 1$.

- Clearly, the number of steps of the i-loop is `bound`.
- The number of steps of the first j-loop is clearly $C = \lfloor \frac{n}{2} \rfloor$.
- The number of steps of the second j-loop is $D = \lceil \log_2 n \rceil$.

-
$$
\begin{aligned}
T(n) &= 5 + \sum_{b=1}^{B} (4 + \sum_{i=0}^{2^b} (6 + \sum_{j=0}^{C} 4 + \sum_{j=1}^{D} 3)) \\
&= 5 + \sum_{b=1}^{B} (4 + \sum_{i=0}^{2^b} (6 + 4(C+1) + 3D)) \\
&= 5 + \sum_{b=1}^{B} (4 + (6 + 4(C+1) + 3D) 2^b) \\
&= 5 + 4B + (6 + 4(C+1) + 3D) \sum_{b=1}^{B} 2^b
\end{aligned}
$$

$$= 5 + 4B + 2(6 + 4(C + 1) + 3D)(2^B - 1)$$
$$= 5 + 4(\lfloor \log_2 n \rfloor + 1 + 2(6 + 4(\lfloor \frac{n}{2} \rfloor + 1) + 3\lceil \log_2 n \rceil)(2^{\lfloor \log_2 n \rfloor + 1} - 1)$$

2. Before moving forward with the required proof, let's observe that we can always assume $n > 2$ (in general n greather than a fixed number). The reason why is that we seek B (in the definition of big-O) such that for all natural numbers greather than B, the big-O inequality is satisfied. That said, if we chose some natural number B that satisfies other conditions, we can always replace it with $B' = \max(B, 2) + 1$ and rename the variable back to B. This way we guarantee that:

$$\log_2 n \geq 1$$

so

$$\lceil \log_2 n \rceil \leq \log_2 n + 1 \leq 2 \log_2 n.$$

Also we know it is always true that

$$\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}, \quad \lfloor \log_2 n \rfloor \leq \log_2 n.$$

So, writing informally, the estimation for $T(n)$ (assuming $n > 2$) is:

$$T(n) \leq 5 \log_2 n + 4(\log_2 n + \log_2 n + 2(6 \log_2 n + 2n + \log_2 n) + 6 \log_2 n)(2 \cdot 2^{\log_2 n} - 1)$$

Dropping the -1 and doing all other operations, we have

$$T(n) \leq 5 \log_2 n + 152n \log_2 n + 32n^2$$

We know for all $n > 2$, we have $1 \leq n^2$ and $n \leq n^2$ so finally,

$$T(n) \leq 189n^2 \log_2 n.$$

The formal proof.
We know $\log_2 n \in O(n^{\frac{1}{2}})$ # From the hint
Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies \log_2 n \leq cn^{\frac{1}{2}}$.
Let $c_1 \in \mathbb{R}^+, B_1 \in \mathbb{N}$ be the numbers that satisfy the definition in the previous line.

Let $c_0 = 189c_1, B_0 = \max(B_1, 2) + 1$. # From the informal estimation.
Let $n \in \mathbb{N}$.
Then $n > 2$.
Then $T(n) \leq 189n^2 \log_2 n$. # See the informal estimation.
Also $n > B_1$.
Then $\log_2 n \leq c_1 n^{\frac{1}{2}}$.
Then $T(n) \leq 189c_1 n^{\frac{5}{2}}$.
Then $T(n) \leq c_0 n^{\frac{5}{2}}$.
Then $n > B_0 \implies T(n) \leq c_0 n^{\frac{5}{2}}$.
Then $\forall n \in \mathbb{N} : n > B_0 \implies T(n) \leq c_0 n^{\frac{5}{2}}$.
Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies T(n) \leq cn^{\frac{5}{2}}$.
Then $T(n) \in O(n^{5/2})$.

Problem 2.

(6 Marks) Using the appropriate definitions, prove the following:

1. (3 Marks)
$$7n^2 + 77n + 1 \in \Theta(n^2 + n + 165).$$

2. (3 Marks)
$$n \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}}).$$

Solution

1. We will solve the problem by showing separately $7n^2+77n+1 \in \Omega(n^2+n+165)$ and $7n^2 + 77n + 1 \in O(n^2 + n + 165)$.

   Prove $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$.
   Let $c_0 = 1, b_0 = 3$.
   Then $77n + 1 > 165$.
   Also $n^2 > n$.
   Then $7n^2 + 77n + 1 > 1 \cdot (n^2 + n + 165)$.
   Then $n > B_0 \implies 7n^2 + 77n + 1 \geq c_0 \cdot (n^2 + n + 165)$.
   Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies 7n^2+77n+1 \geq c\cdot(n^2+n+165)$.
   Then $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$.

   Prove $7n^2 + 77n + 1 \in O(n^2 + n + 165)$.
   Let $B_0 = 0, c_0 = 77$.
   Then $7n^2 + 77n + 1 < 77n^2 + 77n + 77 \cdot 165 = 77(n^2 + n + 165)$.
   Then $n > B_0 \implies 7n^2 + 77n + 1 \leq c_0(n^2 + n + 165)$.
   Then $\forall n \in \mathbb{N} : n > B_0 \implies 7n^2 + 77n + 1 \leq c_0(n^2 + n + 165)$.
   Then$\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies 7n^2+77n+1 \leq c(n^2+n+165)$.
   Then $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$.

2. We know $\log n \in O(n^{\frac{5}{2}})$.# From the hint,prob.1
   Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies \log n \leq cn^{\frac{5}{2}}$.
   Let $c_1 \in \mathbb{R}^+, B_1 \in \mathbb{N}$ be the elements that satisfy the above.
   Let $c_0 = 7c_1 + 1, B_0 = B_1$.
   Then $c_0 \in \mathbb{R}^+, B_0 \in \mathbb{N}$.
   Let $n > B_0$.

Then $n \log n^7 + n^{\frac{7}{2}} = 7n \log n + n^{\frac{7}{2}}$
$\leq 7n(c_1 n^{\frac{5}{2}}) + n^{\frac{7}{2}} = (7c_1 + 1)n^{\frac{7}{2}}$
Then $n > B_0 \implies n \log n^7 + n^{\frac{7}{2}} \leq c_0 n^{\frac{7}{2}}$.
Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies n \log n^7 + n^{\frac{7}{2}} \leq cn^{\frac{7}{2}}$.
Then $n \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$.

Problem 3.

(6 Marks) Let $\mathcal{F} = \{f | f : \mathbb{N} \to \mathbb{R}^+\}$. Using the appropriate definitions, prove or disprove the following:

1. (3 Marks)
$$\forall f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \implies f(n) \in O(3^{g(n)}).$$

2. (3 Marks)
$$\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)}).$$

Solution

1. We disprove it.
   Let $f(n) = 9^n, g(n) = n$.
   Then $f, g \in \mathcal{F}$.
   Then $\log f(n) = n \log 9$.
   Let $c_0 = \log 9, B_0 = 1$.
   Then $c_0 \in \mathbb{R}^+, B_0 \in \mathbb{N}$.
   Then $n > 1 \implies (\log 9)n \leq (\log 6)n$.
   Then $\forall n \in \mathbb{N} : n > 1 \implies (\log 9)n \leq (\log 9)n$.
   Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies \log f(n) \leq cg(n)$.
   Then $\log f(n) \in O(g(n))$.
   #Show that $f(n) \notin O(3^{g(n)})$.
   Let $c \in \mathbb{R}, B \in \mathbb{N}$.
   Let $n_0 = \max(B, \log_3 c) + 1$.
   Then $n_0 > B$.
   Then $9^{n_0} = (3^2)^{n_0} = 3^{2n_0} = 3^{n_0}3^{n_0} > c3^{n_0} = c3^{g(n_0)}$.
   Then $\exists n \in \mathbb{N} : n > B \wedge f(n) > c3^{g(n)}$.
   Then $\forall c \in \mathbb{R}^+ : \forall B \in \mathbb{N} : \exists n \in \mathbb{N} : n > B \wedge f(n) > 3^{g(n)}$.
   Then $f(n) \notin O(3^{g(n)})$.
   Then $\exists f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \wedge f(n) \notin O(3^{g(n)})$.

2. Let $f \in \mathcal{F}$.
   Let $c_0 = 1, B_0 = 0$.
   Then $c_0 \in \mathbb{R}, B_0 \in \mathbb{N}$.
   Let $k = \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor$.
   Then $k \in \mathbb{N}$.

Then $k \leq \sqrt{\lfloor f(n) \rfloor} < k+1$. # By definition of the floor
Then $k^2 \leq \lfloor f(n) \rfloor < (k+1)^2$. # All members are nonnegative
Then $k^2 \leq f(n) < (k+1)^2$. # $(k+1)^2 - k^2 \geq 1$
Then $k \leq \sqrt{f(n)} < (k+1)$ # Extract square root, all members nonnegative
Then $\lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq \sqrt{f(n)}$.
Then $n > B_0 \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq c_0 \sqrt{f(n)}$.
Then $\exists c \in \mathbb{R}^+ : \exists B \in \mathbb{N} : \forall n \in \mathbb{N} : n > B \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq c\sqrt{f(n)}$.
Then $\lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)})$.
Then $\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)})$.

Problem 4.

(6 MARKS) Recall that $n! = 1 \cdot 2 \dots n$. Also, by convention, $0! = 1$. Using the method of mathematical induction, prove the following:

1. (3 MARKS)
$$\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1.$$
2. (3 MARKS)
$$\forall n \in \mathbb{N} : n \geq 1 \to 2^n \leq 2^{n+1} - 2^{n-1} - 1.$$

Solution

1. Let $P(n) : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.
   Basis step: Prove $P(0)$.
   $\sum_{i=0}^{0} i \cdot i! = 0 \cdot 1 = 0$.
   Also $(0+1)! - 1 = 1 - 1 = 0$.
   Then $P(0)$.
   Inductive Step:
   Let $n \in \mathbb{N}$.
   Assume $P(n)$.
   Then $\sum_{i=0}^{n+1} i \cdot i!$
   $= \sum_{i=0}^{n} i \cdot i! + (n+1)(n+1)!$
   $= (n+1)! - 1 + (n+1)(n+1)!$ #using inductive hypothesis
   $= (n+2)(n+1)! - 1$ # Factoring $(n+1)!$
   $= (n+2)! - 1$. # Using the definition of factorial
   Then $P(n+1)$.
   Then $P(n) \implies P(n+1)$.
   Then $\forall n \in \mathbb{N} : n \geq 0 \implies (P(n) \implies P(n+1))$.
   Then $\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

2. Let $P(n) : 2^n \leq 2^{n+1} - 2^{n-1} - 1$.
   Basis Step:
   $2^{1+1} - 2^{1-1} - 1 = 4 - 1 - 1 = 2 \geq 2^1$.
   Then $P(1)$.
   Inductive step
   Let $n \in \mathbb{N}$.
   Assume $n \geq 1$.

Assume $2^n \leq 2^{n+1} - 2^{n-1} - 1$.
Then $2^{n+1+1} - 2^{n+1-1} - 1 = 2^{n+2} - 2^n - 1 = 2(2^{n+1} - 2^{n-1} - 1) + 1$
$\geq 2 \cdot 2^n + 1$ #Using inductive hypothesis
$\geq 2^{n+1}$ # Elementary algebra
Then $P(n+1)$.
Then $P(n) \implies P(n+1)$.
Then $n \geq 1 \implies (P(n) \implies P(n+1))$.
Then $\forall n \in \mathbb{N} : n \geq 1 \implies 2^n \leq 2^{n+1} - 2^{n-1} - 1$.