CSC165
Mathematical Expression and Reasoning for Computer Science

Module 2

About Sets

Sets

- Set is a collection of distinct objects
- These objects are called elements of the set
- The set, itself, is considered an object
- Example:
  - $S = \{1,2,3\}$
  - $S = \{0,2,4,6,\dots\}$
  - $S = \{1,\{1\}\}$
- Notation:
  - $x \in S$: x is an element in set S
  - $x \notin S$: x is not an element in set S
  - $S = \{x \in D: P(x)\}$: S is the set of all x in domain (or set) D such that $P(x)$ is true

Set Notation

- $\mathbb{R}$: set of real numbers
- $\mathbb{Z}$: set of integer numbers $\{\dots,-2,-1,0,1,2,\dots\}$
- $\mathbb{N}$: set of natural numbers $\{0,1,2,\dots\}$
- $\mathbb{Q}$: set of rational numbers

Definitions

- Let X and Y be subsets of universal set U
- X subset of Y: $X \subseteq Y \leftrightarrow \forall x \in U: (x \in X) \rightarrow (x \in Y)$
- X not subset of Y: $X \nsubseteq Y \leftrightarrow \exists x \in U: (x \in X) \wedge (x \notin Y)$
- X proper subset of Y $(X \subset Y)$:
  - $X \subseteq Y$
  - There is at least one element in Y that is not in X
- X equals Y:
  - Every element of X is in Y
  - Every element of Y is in X
  - $X = Y \leftrightarrow (X \subseteq Y) \wedge (Y \subseteq X)$

Definitions

- Union of X and Y: set of all elements that are in at least one of X or Y
  - $X \cup Y: \{x \in U: x \in X \text{ or } x \in Y\}$
- Intersection of X and Y: set of all elements that are common to both X and Y
  - $X \cap Y: \{x \in U: x \in X \text{ and } x \in Y\}$
- Difference of X and Y (X minus Y): set of all elements that are in X and not in Y
  - $X - Y: \{x \in U: x \in X \text{ and } x \notin Y\}$
- Complement of X: set of all elements in U that are not in X
  - $X^{c}: \{x \in U: x \notin X\}$

Empty Set

- Empty (null) set (Φ): a set with no elements
- Example:
  - $\{1,3\} \cap \{0,2\}$
  - $\{x \in \mathbb{R}: x^2 < -2\}$
- X and Y are disjoint sets if and only if they have no common elements
  - X and Y are disjoint $\leftrightarrow X \cap Y = \Phi$

Properties of Sets

- Let X, Y, Z be subsets of universal set U
- $X \cap Y \subseteq X$
- $X \cap Y \subseteq Y$
- $X \subseteq X \cup Y$
- $Y \subseteq X \cup Y$
- If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$

Properties of Sets

- Let X, Y be subsets of universal set U, and let $x \in U$
- $x \in X \cup Y \leftrightarrow x \in X$ or $x \in Y$
- $x \in X \cap Y \leftrightarrow x \in X$ and $x \in Y$
- $x \in X - Y \leftrightarrow x \in X$ and $x \notin Y$
- $x \in X^c \leftrightarrow x \notin X$

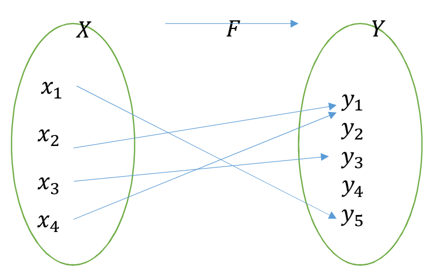
About Functions

Functions

- A function F from a set X to a set Y is a relation (correspondence) from X to Y
- Notation:
  - $F: X \rightarrow Y$ (note: $\rightarrow$ does not mean "imply" here)
  - X is called the domain
  - Y is called the image (or co-domain)
- $F: X \rightarrow Y$ needs to satisfy:
  - Every element in X is related to some element in Y
  - No element in X is related to more than one element in Y

Functions

- Range of F is the image of X under F: $\{y \in Y: y = F(x)\}$ for $x \in X$
- Inverse image of $y \in Y$: $\{x \in X: F(x) = y\}$

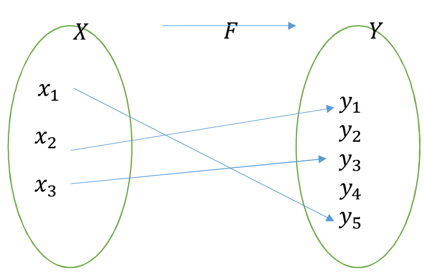


One-to-One Functions

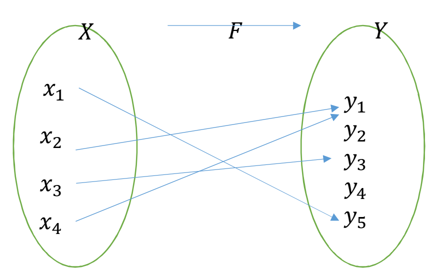
- $F: X \to Y$ is called one-to-one function if and only if:
  - For all elements x_1 and x_2 in X: if $F(x_1) = F(x_2)$, then $x_1 = x_2$, or
  - For all elements x_1 and x_2 in X: if $x_1 \neq x_2$ then $F(x_1) \neq F(x_2)$
- $F: X \to Y$ is one-to-one function $\leftrightarrow$
  $$\forall x_1, x_2 \in X: \left[\left(F(x_1) = F(x_2)\right) \to (x_1 = x_2)\right]$$
- A one-to-one function is also called an injective function
- $F: X \to Y$ is NOT one-to-one function $\leftrightarrow$
  $$\exists x_1, x_2 \in X: \left[\left(F(x_1) = F(x_2)\right) \wedge (x_1 \neq x_2)\right]$$

One-to-One Functions

- One-to-one function: distinct elements in the domain are mapped to distinct elements in the co-domain



- Not one-to-one function: at least two elements in the domain are mapped to the same element in the co-domain

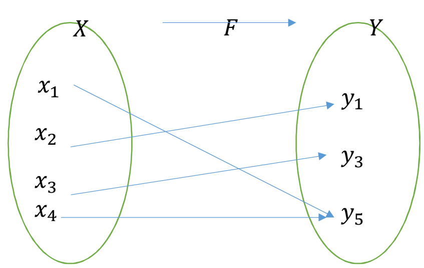


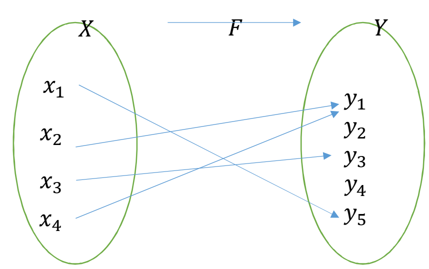
Onto Functions

- $F: X \to Y$ is called onto function if and only if:
  - For every element y in Y, it is possible to find an element x in X with the property $y = F(x)$
- Every element in the co-domain is an image of some element in the domain
- $F: X \to Y$ is onto function $\leftrightarrow$
$$\forall y \in Y: [\exists x \in X: F(x) = y]$$
- An onto function is also called an surjective function
- $F: X \to Y$ is NOT onto function $\leftrightarrow$
$$\exists y \in Y: [\forall x \in X: F(x) \neq y]$$

Onto Functions

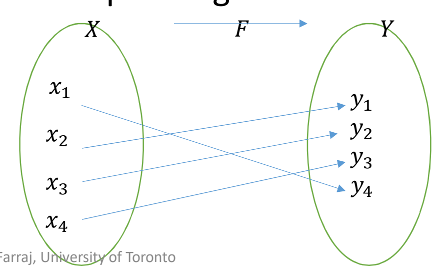
- Onto function: each element in the co-domain is mapped to from (an) element(s) in the domain
- Not onto function: at least one element in the co-domain is not mapped to from elements in the domain





One-to-One Correspondences

- $F: X \to Y$ is called one-to-one correspondence if and only if:
  - F is one-to-one
  - F is onto
- Any element y in Y has a corresponding element x in X such that $y = F(x)$
- Any element x in X has a unique corresponding element in Y such that $y = F(x)$
- F is called bijection



Inverse Functions

- Let $F: X \to Y$ be a one-to-one correspondence
- Let $F^{-1}: Y \to X$ be defined as:
  - Given any element y in Y: $F^{-1}(y)$ is the unique element x in X such that $F(x) = y$
  - $F^{-1}(y) = x \leftrightarrow y = F(x)$
- F^{-1} is called the inverse function of F

- If $F: X \to Y$ is one-to-one correspondence, then F^{-1} is also one-to-one correspondence
- Can you prove that?

Example

- Let function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $\forall x \in \mathbb{R}: f(x) = 2x - 1$
- Find f^{-1}
  - Given any element y in $\mathbb{R}$: $f^{-1}(y)$ is the unique element x in $\mathbb{R}$ such that $f(x) = y$
  - $y = f(x) \leftrightarrow f^{-1}(y) = x$
  - $y = 2x - 1$
  - $y + 1 = 2x$
  - $\frac{y+1}{2} = x$
  - $f^{-1}(y) = \frac{y+1}{2}$
- $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $\forall y \in \mathbb{R}: f^{-1}(y) = \frac{y+1}{2}$

Countable Sets

Set Cardinality

- Finite set:
  - A set that has no elements at all (i.e., an empty set), or
  - A set that can be put into one-to-one correspondence with a set $\{1,2,\ldots,n\}$ for some positive integer n
- Infinite set: A nonempty set that cannot be put into one-to-one correspondence with a set $\{1,2,\ldots,n\}$ for some positive integer n
- Cardinality of a set is a measure of the "number of elements of the set"
- X and Y are two sets:
  - X has the same cardinality as Y if and only if there is a one-to-one correspondence between X and Y
  - X has the same cardinality as Y if and only if there is a function f from X to Y that is one-to-one and onto

Examples

- $X = \{1, 2, 3\}, Y = \{$“one”, “two”, “three”$\}$
- How do their sizes compare?
- Cardinality: $|X| = |Y| = 3$

- $X = \{$natural numbers$\}, Y = \{$even natural numbers$\}$
- $|X| = |Y|$… why?
  - Let $f : X \to Y$ be $\forall n \in \mathbb{N}: f(n) = 2n$
  - Then f is well defined # $2n$ is well defined
  - Then f is one-to-one # if $f(m) = f(n) = y$, then $m = n = y/2$
  - Then f is onto # all y in Y has $x = y/2$ in X mapping to it
  - Then $\exists$ a well defined $f : X \to Y$ that is one-to-one and onto # f is one-to-one correspondence
  - Then $|X| = |Y|$

Countable Sets

- A set is called countably infinite if and only if it has the same cardinality as the set of $\mathbb{Z}^+$
- A set is called countable if and only if it is finite or countably infinite
- A set that is not countable is called uncountable

- Example:
  - $\mathbb{Z}$ (set of integer numbers): countable
  - $\mathbb{Q}$ (set of rational numbers): countable
  - $\mathbb{R}$ (set of real numbers): uncountable

About Summation

Summation Notation

- For a sequence a_k, and for integers m and n where $m \leq n$, the summation of sequence a_k elements from $k = m$ to $k = n$ is

$$\sum_{k=m}^{k=n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

- Notation:
  - k is the index of the summation
  - m is the lower limit of the summation
  - n is the upper limit of the summation
- k is a dummy variable!

Example

- Let sequence a_k be: $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1, a_5 = 2$
- $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + (-1) + 0 + 1 + 2 = 0$
- $\sum_{k=2}^{2} a_k = a_2 = -1$
- $\sum_{k=1}^{2} a_{2k} = a_{2(1)} + a_{2(2)} = a_2 + a_4 = -1 + 1 = 0$

Examples

- Find $\sum_{k=1}^{5} k^2$
  - $a_k = k^2$
  - $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$
- Find $\sum_{i=1}^{2} i$
  - $a_i = i$
  - $\sum_{i=1}^{2} i = 1 + 2 = 3$
- Find $\sum_{i=1}^{2} 3$
  - $a_i = \{3,3\}$
  - $\sum_{i=1}^{2} 3 = 3 + 3 = 6$
- Find $\sum_{i=1}^{3} k$
  - $a_i = \{k, k, k\}$
  - $\sum_{i=1}^{3} k = k + k + k = 3k$
- Find $\sum_{i=0}^{3} k$
  - $a_i = \{k, k, k, k\}$
  - $\sum_{i=0}^{3} k = k + k + k + k = 4k$

Examples

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \sum_{i=n+1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

$$2^{n+1} + \sum_{k=0}^{n} 2^k = \sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n} 2^k + \sum_{k=n+1}^{n+1} 2^k = \sum_{k=0}^{n+1} 2^k$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Examples

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Find $\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k$

$$\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k$$

$$= \sum_{k=m}^{n} (k+1) + 2\sum_{k=m}^{n} (k-1)$$

$$= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2(k-1)$$

$$= \sum_{k=m}^{n} (k+1+2(k-1))$$

$$= \sum_{k=m}^{n} (3k-1)$$

Examples

- Remember index is a dummy variable
- Find $\sum_{k=0}^{6} \frac{1}{k+1}$
- Let $j = k + 1$
- $\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}$
- Again, remember index is a dummy variable
- $\sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$
- Then $\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$