tutorial after lecture tonight... from 8 to 9... there is a brief quiz at the end.

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complete induction

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use Introduction to the Theory of Computation, Section 1.3

Outline

Principle of complete induction

Examples of complete induction

Complete Induction

another flavour needed

Every natural number greater than 1 has a prime factorization

write them as products of 1 or more primes.

Try some examples

2: 2
3: 3
4: 2x2
5: 5
6: 2x3
7: 7
8: 2x2x2

9: 3x3

?

How does the factorization of 8 help with the factorization of 9?

notational convenience...

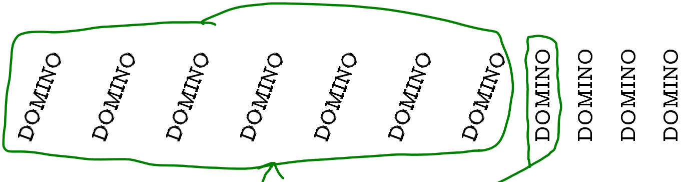
I will use (though you don't have to) the following:

$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$$

More dominoes



$$\left(\forall n \in \mathbb{N}, \left[\bigwedge_{k=0}^{k=n-1} P(k)\right] \Rightarrow P(n)\right) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true

complete induction outline

inductive step: state inductive hypothesis $H(n)$

derive conclusion $C(n)$: show that $C(n)$ follows from $H(n)$, indicating where you use $H(n)$ and why that is valid

verify base case(s): verify that the claim is true for any cases not covered in the inductive step

sometimes I embed the base(s) in the inductive step. Also, some people go from assuming from start to n, then show this implies P(n+1)

Wait! isn't that the same outline as simple induction?

Yes, we just modify the inductive hypothesis, $H(n)$ so that it assumes the main claim for every natural number from the starting point up to $n-1$, and the conclusion, $C(n)$, is now the main claim for n.

watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ \left[f(\lfloor\sqrt{n}\rfloor)\right]^2 + 2f(\lfloor\sqrt{n}\rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture

f(0) = 1
f(1) = 1
f(2) = 3
f(3) = 3
f(4) = 15
f(5) = 15
f(6) = 15
f(7) = 15
f(8) =15
f(9) = 15

f(10, 11, 12, 13, 14, 15) = 15
f(16) = 255

For all natural numbers $n > 1$, $f(n)$ is a multiple of 3~~?~~

use the complete induction outline

P(n): f(n) is a multiple of 3.

Let n \in \N, assume n >= 4. Assume P(2) and P(3) ... P(n-1). I wish to show that P(n) follows, that is that f(n) is also multiple of 3.

We know that f(n) = f(floor(sqrt(n)))^2 + 2f(floor(sqrt(n))) # since n > 1
= f(floor(sqrt(n))) x [f(floor(sqrt(n))) + 2]
= 3k x [f(floor(sqrt(n))) + 2], for some k \in \N # by hypothesis since sqrt(n) >= 2
since n>= 4
also n > 1 implies n^2 > n
which implies n > sqrt(n)
so f(n) is also a multiple of 3.

Base case(s). f(2) = 3 = f(3), as shown previously, which verifies our base cases 2, and 3.

zero pair free binary strings, zpfbs...

Denote by $zpfbs(n)$ the number of binary strings of length n that contain no pairs of adjacent zeros. What is $zpfbs(n)$ for the first few natural numbers n?

f(0) = 1
f(1) = 2
f(2) = 3
f(3) = 5
f(4) = 8
f(5) = 13
f(6) = 21

b1 b2 1 0

break up the counting into 2 parts...

f(4) counts binary strings of length 4 that have no adjacent zeros.

binary strings of length 4 that end in 1 and have no adjacent zeros are simply a bs of length 3 with no adjacent zeros with a 1 appended.

binary strings of length 4 that end in 0 and have no adjacent zeros are simply the binary strings of length 2 with 10 appended.

f(n) = 1 if n is 0
= 2 if n is 1
= f(n-2) + f(n-1) if n > 1

what is $zpfbs(n)$?

use the complete induction outline

P(n): zpfbs(n) = 1 if n is 0, 2 if n is 1, and
is zpfbs(n-2) + zpfbs(n-1) if n > 1

Let n \in \N. Assume P(0) and P(1) and ... P(n-1). I want to show that P(n) follows, that is zpfbs(n) is what the formula on the previous page gives.

case n >= 2. Then note that zpfbs(n) counts those binary strings of length n that end in a 1 and those that end in 0. Ending in 1 doesn't introduce, or take away any pairs of zeros, so there are zpfbs(n-1) of those, by the hypthesis # since n-1 < n and n-1 >= 0. Ending in 0 is a problem, so we will only count the binary strings of length n that end in 10, and there are zpfbs(n-2) of those, by the hypothesis # since n-2 < n and n-2 >= 0.
Thus, altogether, we have zpfbs(n-2) + zpfbs(n-1) strings with no adjacent 0s of length n, as the formula claims (although it's called f rather than zpfbs).

case n < 2: If n were 0, then the only possible binary string is the empty string, and it has no pairs of zeros, so zpfbs(0) = 1, as claimed. If n were 1, then the only possible binary strings are 0 and 1, neither of which have adjacent 0s, so zpfbs(1) = 2, as claimed. These verify the base case.

Every natural number greater than 1 has a prime factorization

use the complete induction outline

P(n): n has a representation as product of 1 or more primes.

prime factorization: representation as product of 1 or more primes

Let n be an arbitrary, fixed, natural number that is no smaller than 2. Assume P(2) ... P(n-1).
We will show P(n), that n has a representation as a product of 1 or more primes.

case n is prime: we're done, n's prime factorization is itself.

case n is composite. Then by definition being composite n is the product of some f1 and f2 \in n such that 1 < f1, f2 < n. By P(f1) and P(f2), f1 and f2 have representations as products of primes, so n's representation is simply the product of these. *

* Some students worry that you must know that there exists at least one prime (alternatively that 2 is prime) independently of this proof before it is valid. Not so. The worry is that by applying the induction step enough times you will eventually encounter a number that is not composite. Yes, so a *consequence* of this proof, not a *prerequisite* of it, is that for every integer n >= 2 there is at least one prime less than or equal to n.

After a certain natural number n, every postage can be made up by combining $3-$ and $5-$ cent stamps

what is the "certain natural number"?

After a certain natural number n, every postage can be made up by combining $3-$ and $5-$ cent stamps

use the complete induction outline

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notes...