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I have read the declaration on the cover sheet and confirm my agreement with it.

1. A and B are independent, $\mathbb{P}(B) = \mathbb{P}(B|A) = 0.5$

   (a) FALSE. $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B) = 0.8 \times 0.5 = 0.4 \neq 0$
   Thus, these two events are not mutually exclusive.

   (b) FALSE. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.8 + 0.5 - 0.4 = 0.9$
   $\mathbb{P}(A \cap (A \cup B)) = \mathbb{P}(A) = 0.8 \neq \mathbb{P}(A \cap B) \times \mathbb{P}(A)$

   (c) FALSE. A, B are independt, $\mathbb{P}(A|B) = \mathbb{P}(A) \neq \mathbb{P}(B)$

   (d) TRUE.
   $\mathbb{P}(A \mid C) = \dfrac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)}$ $\mathbb{P}(B \mid C) = \dfrac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$. Because they have the same denominator, we are actually comparing $\mathbb{P}(A \cap C)$ and $\mathbb{P}(B \cap C)$.
   We could calculate the range of each. When $A \subset C, \mathbb{P}_{max}(A \cap C) = \mathbb{P}(A) = 0.8$ while the minimum probability of $\mathbb{P}(A \cap C) = \mathbb{P}(A) + \mathbb{P}(C) - 1 = 0.7$. Therefore, $\mathbb{P}(A \cap C) \in [0.7, 0.8]$
   Same calculate for $\mathbb{P}_{max}(B \cap C) = \mathbb{P}(B) = 0.5$ when $B \subset C$, and $\mathbb{P}_{min}(B \cap C) = \mathbb{P}(B) + \mathbb{P}(C) - 1 = 0.4$. Therefore, $\mathbb{P}(B \cap C) \in [0.4, 0.5]$.
   We can tell $\mathbb{P}(A \cap C) > \mathbb{P}(B \cap C)$, therefore $\mathbb{P}(A \mid C) > \mathbb{P}(B \mid C)$

   (e) TRUE. We calculate that $\mathbb{P}(B) = 0.5 < \mathbb{P}(A)$ which satisfy the claim.

2. (a) (i) Let 'Trail' = hockey match
   Let 'Success' = winning, so $\mathbb{P}(\text{Success}) = \dfrac{1}{2}$
   $X \sim \text{Geometric}(p = \dfrac{1}{2})$, because X is the number of failures before the first success .

   (ii) $\mathbb{E}(X) = \dfrac{1-p}{p} = \dfrac{1-\frac{1}{2}}{\frac{1}{2}} = 1$

   (iii) $\mathbb{P}(X = 3) = (1-p)^3 p = \frac{1}{2^4} = \frac{1}{16}$

   (b) (i) Each match is independent, so, the first win won't have influence on the next loss.
   Let 'Success' be the loss, then $\mathbb{P}(\text{Success}) = \frac{1}{3}$
   $X \sim \text{Geometric}(p = \frac{1}{3})$, because X is the number of failures before the first success.

   (ii) $\mathbb{E}(X) = \dfrac{1-p}{p} = \frac{1-\frac{1}{3}}{\frac{1}{3}} = 2$

   (iii) $\mathbb{P}(X = 0) = (1-p)^0 p = (1 - \frac{1}{3})^0 \times \frac{1}{3} = \frac{1}{3}$

   (c) (i) $X \sim \text{Binomial}(n = 5, p = \frac{1}{3})$, because X is the number of losses within 5 matches.

   (ii) $\mathbb{E}(X) = np = 5 \times \frac{1}{3} = \frac{5}{3}$

   (iii) $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 0.868$
   $\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 0.539$
   $\mathbb{P}(X \geq 2|X \geq 1) = \dfrac{\mathbb{P}(X \geq 2)}{\mathbb{P}(X \geq 1)} = \frac{0.539}{0.868} = 0.621$

   (d) (i) Let 'Trail' = hockey match
   Let 'Success' = winning, so $\mathbb{P}(\text{Success}) = \frac{1}{2}$
   $X \sim \text{NegBin}(k = 3, p = \frac{1}{2})$, because X is the number of failures before their kth successes.

(ii) $\mathbb{E}(X) = \frac{k(1-p)}{p} = \frac{3\times(1-\frac{1}{2})}{\frac{1}{2}} = 3$

(iii) $\mathbb{P}(X = 1) = \binom{3+1-1}{1}(\frac{1}{2})^3(1-\frac{1}{2})^1 = \frac{3}{16}$

(e) (i) X is the number of draws from first 5 matches, $X \sim$ Binomial$(n = 5, p = \frac{1}{6})$
Y is the number of losses from the last 5 matches, $Y \sim$ Binomial$(n = 5, p = \frac{1}{3})$
Z is a joint probability distribution of sum of two Binomial distribution with different probability, therefore it's other distribution that it's not covered in class.

(ii) $\mathbb{E}(Z) = \mathbb{E}(X) + \mathbb{E}(Y) = 5 \times \frac{1}{6} + 5 \times \frac{1}{3} = \frac{5}{2}$

(iii) $\mathbb{P}(Z = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0) = \binom{5}{0}(\frac{1}{6})^0(\frac{5}{6})^5 * \binom{5}{2}(\frac{1}{3})^2(\frac{2}{3})^3 + \binom{5}{1}(\frac{1}{6})^1(\frac{5}{6})^4 * \binom{5}{1}(\frac{1}{3})^1(\frac{2}{3})^4 + \binom{5}{2}(\frac{1}{6})^2(\frac{5}{6})^3 * \binom{5}{0}(\frac{1}{3})^0(\frac{2}{3})^5 = 0.286$

(f) We assume that each penalty stroke is a Bernoulli trial with $P_i$with success rate$p_i = 0.8^{i-1}$.

$$\mathbb{E}\left[\sum_i P_i\right] = \sum_i \mathbb{E}(P_i) = \sum_i p_i = 3.3616$$

$$\text{Var}\left[\sum_i P_i\right] = \sum_i \text{Var}(P_i) = \sum_i p_i(1-p_i) = 0.882$$

3. (a) Because $X \sim$ Poisson$(\lambda_1), Y \sim$ Poisson(λ_2), X and Y are independent. Let $Z = X+Y$, then we have $Z \sim$ Possion$(\lambda_1 + \lambda_2)$. We are given Var(X) + Var$(Y) = 8$, and in Poisson distribution Var$(X) = \lambda_1$, Var$(Y) = \lambda_2$. So, $Z \sim$ Poisson(8)
$\mathbb{P}(X + Y \geq 3) = 1 - \mathbb{P}(Z = 0) - \mathbb{P}(Z = 1) - \mathbb{P}(Z = 2) = 1 - (\frac{8^0}{0!} + \frac{8^1}{1!} + \frac{8^2}{2!})e^{-8} = 0.986$

(b) $\hat{\lambda}_2 = 5$ and Var(X) + Var$(Y) = 8$, $\hat{\lambda}_1 = 8 - 5 = 3$
We have $X \sim$ Poisson(3) and $Y \sim$ Poisson(5), independently, and $Z = X+Y$. Then the conditional distribution of $X \mid Z$ is Binomial(z, p), where $p = \dfrac{3}{8}$
We are calculate $X \mid Z = 12$, therefore it's Binomial$(12, \dfrac{3}{8})$.
$\mathbb{P}(X = 6 \mid Z = 12) = \binom{12}{6}(\frac{3}{8})^6(\frac{5}{8})^6 = 0.153$