Uncertainty Quantification and Sensitivity Analysis in Dynamical Systems

Matthew Rocklin

University of Chicago

May 18th, 2011

Outline

1. Uncertainty
2. Sampling
3. Adjoint
4. Future Work
5. End

Uncertainty

Windspeed
30 km/hr

Uncertainty

Windspeed
30 km/hr

Wind Power
50 KW

Uncertainty

Windspeed
30 km/hr

Wind Power
50 KW

Electricity
40 KW

Uncertainty

Windspeed
30 km/hr

Wind Power
50 KW

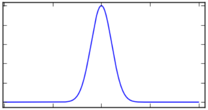
Electricity
40 KW

Lights are on!

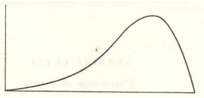
Uncertainty

Measurements are Distributions, not Scalars

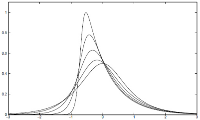
WindSpeed: Around 30 km/hr



WindPower:



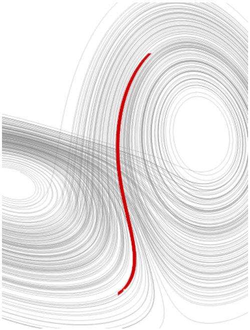
Electricity:



Lights stay on?

Dynamical Systems

Focus on time evolution of dynamical systems : $x(t) = M_t(x_0)$

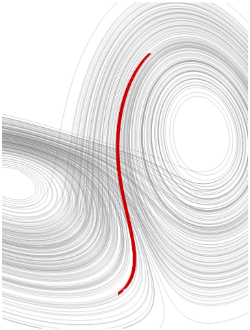


Lorenz Oscillator[Lor63]

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

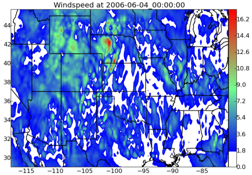
Dynamical Systems

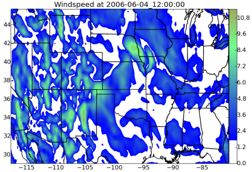
Focus on time evolution of dynamical systems : $x(t) = M_t(x_0)$



Lorenz Oscillator[Lor63]

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z
\end{aligned}
$$

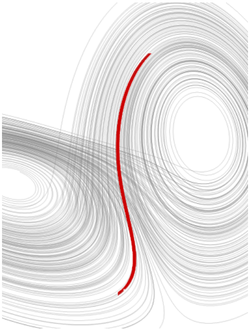




Weather Research and Forecasting Model (WRF)[SKD+08]

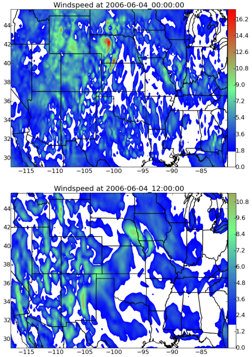
Dynamical Systems

Focus on time evolution of dynamical systems : $x(t) = M_t(x_0)$



Lorenz Oscillator[Lor63]

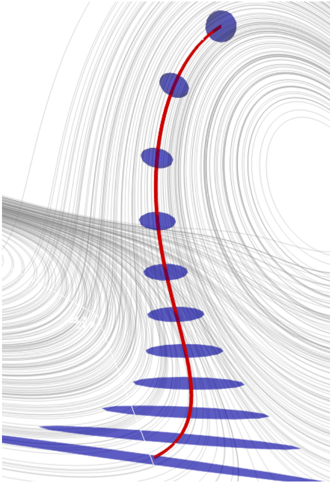
$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}$$



Weather Research and Forecasting Model (WRF)[SKD$^+$08]

– Big Mess of Fortran –

Evolving Distributions

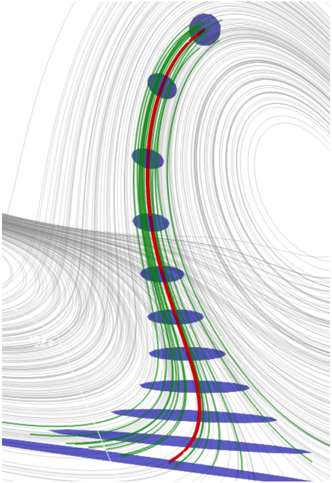


Evolving distributions is challenging.

What can we do without looking at the code?

- Evolve a Sample
- Evolve a Gradient
  From automatic differentiation

Introduction to Sampling



- Draw samples from the initial conditions $\{x_0^1, x_0^2, ..., x_0^k\} \sim X_0$
- Evolve sample forward to time t, $\{x_t^1, x_t^2, ..., x_t^k\} = \{M_t(x_0^1), M_t(x_0^2), ..., M_t(x_0^k)\}$
- Approximate X_t by sample mean $\mu = \frac{1}{k}\sum_i x_t^i$ and covariance $\Sigma = \frac{1}{k-1}\sum_{i,j}(x_t^i - \mu) \wedge (x_t^j - \mu)^T$
- How many samples are necessary? Only 5-10 for local weather[PHK+01]

A Sampling Problem

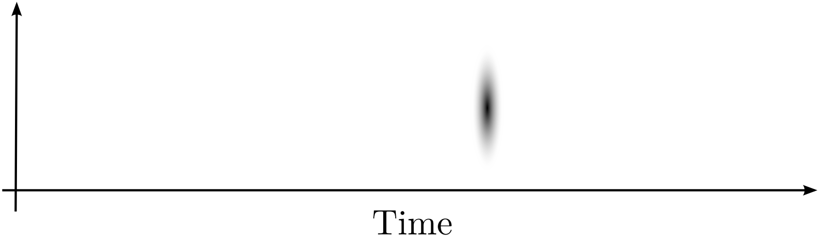
Want to quantify uncertainty of specific windspeed forecasts.

Given:

Initial Distribution

Want:

Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



A Sampling Problem

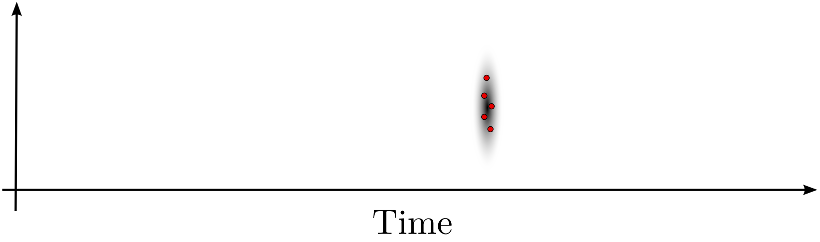
Want to quantify uncertainty of specific windspeed forecasts.

Given:

Initial Distribution

Want:

Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



A Sampling Problem

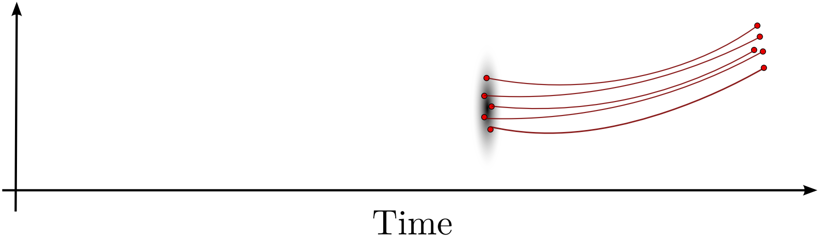
Want to quantify uncertainty of specific windspeed forecasts.

Given:

Initial Distribution

Want:

Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



A Sampling Problem

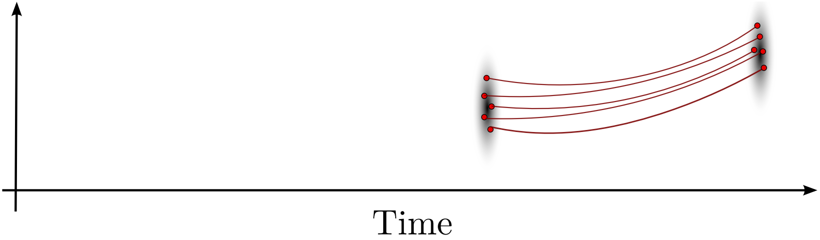
Want to quantify uncertainty of specific windspeed forecasts.

Given:

Initial Distribution

Want:

Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



A Sampling Problem

Want to quantify uncertainty of specific windspeed forecasts.

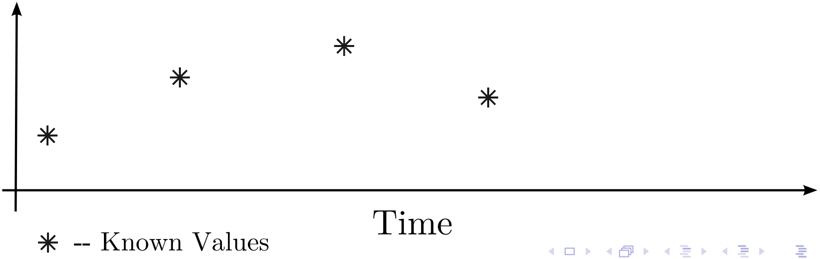
Given:

~~Initial Distribution~~

Expectation values distributed in time

Want:

Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



A Sampling Problem

Want to quantify uncertainty of specific windspeed forecasts.

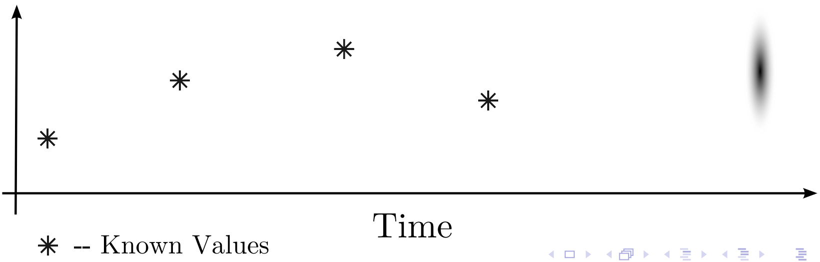
Given:

~~Initial Distribution~~

Expectation values distributed in time

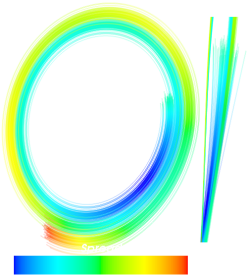
Want:

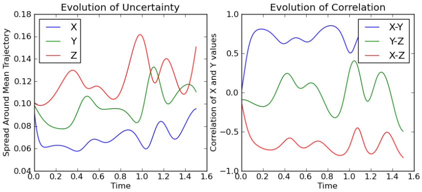
Distribution for 12-hour forecasts of windspeeds, solar irradiance, etc... at targeted locations



Naive Distributions

Issue with not having an initial distribution
Lorenz collapse images

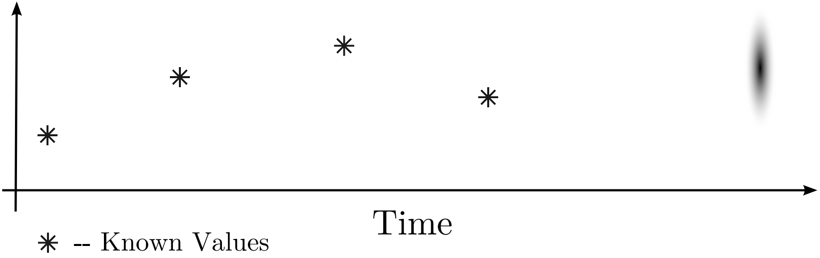




- Start with isotropic distribution $x \sim \mathcal{N}(\mu, \alpha I)$
- After brief evolution we see low-dimensionality and strong correlations.
- Model has intrinsic distribution at this state "Errors of the Day" [Kal02, CKP$^+$03] .

Breeding an Ensemble

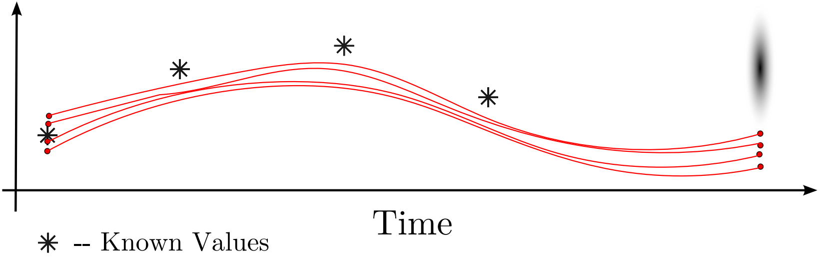
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

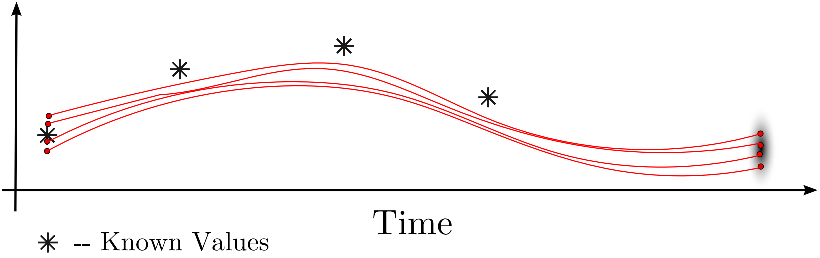
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

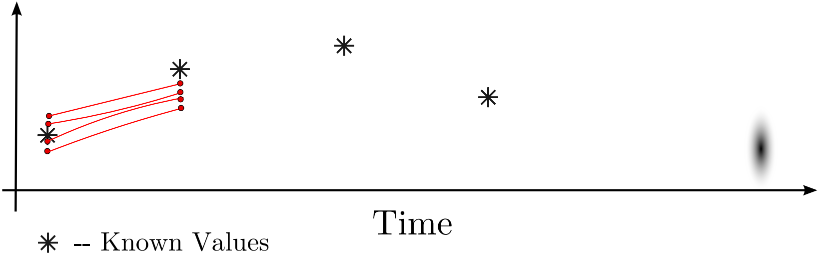
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

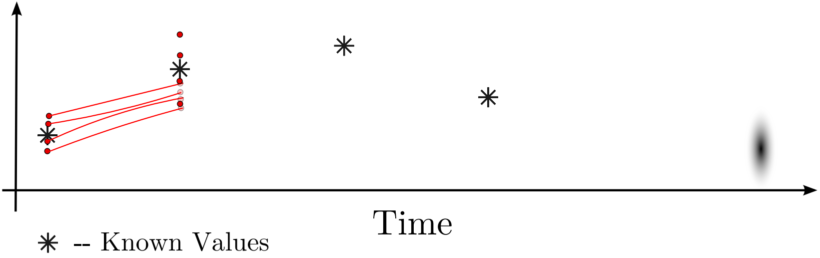
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

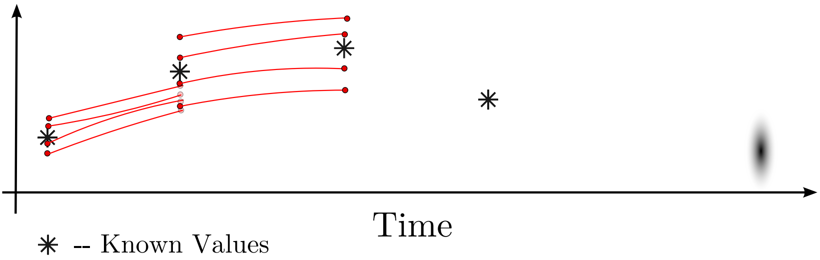
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

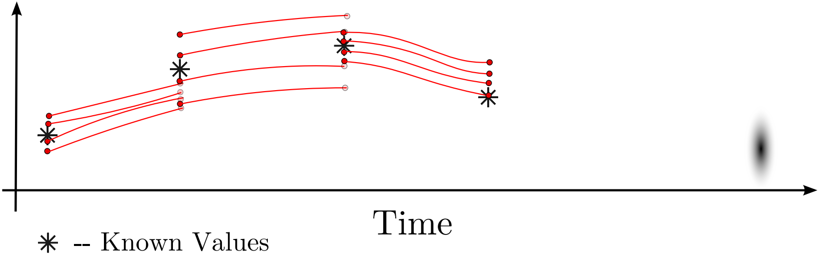
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

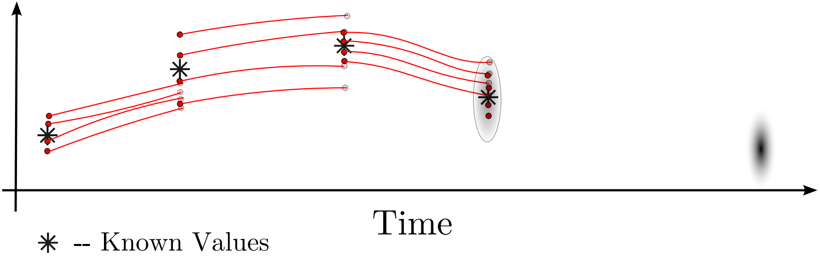
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

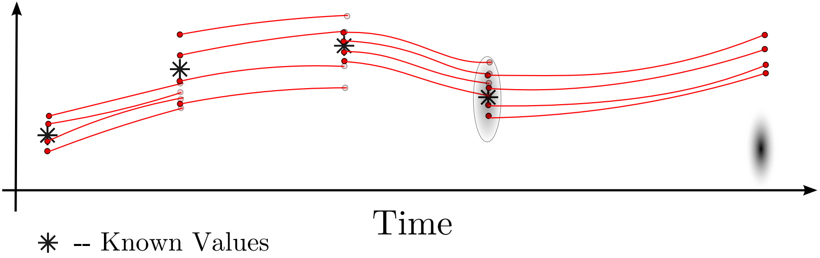
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Breeding an Ensemble

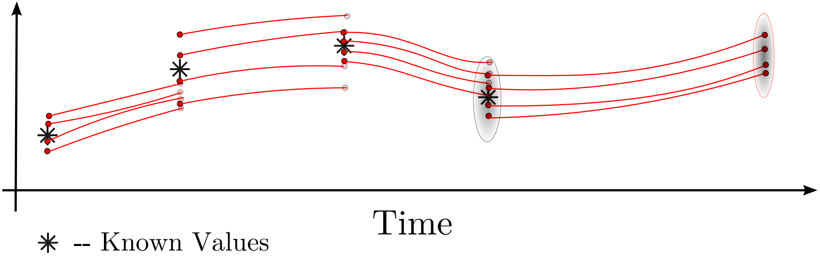
Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

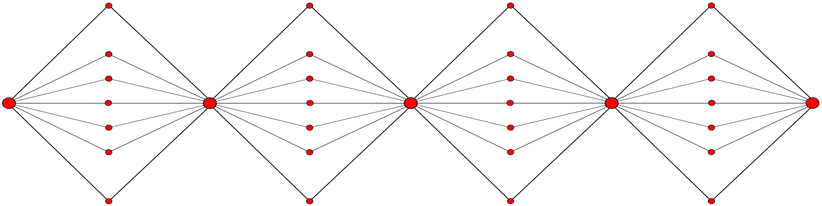
Breeding an Ensemble

Bred Vectors[Kal02]- Breed an ensemble of states with periodic rescaling of error around a known mean



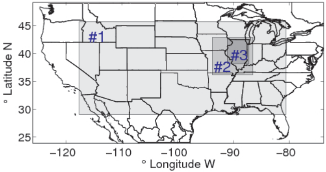
- How to check that distribution is reasonable?
- Does the recentering/spreading disturb this?
- Does this process converge? Is it sensitive to seed vectors? Sensitive to choice of norm?

Computation

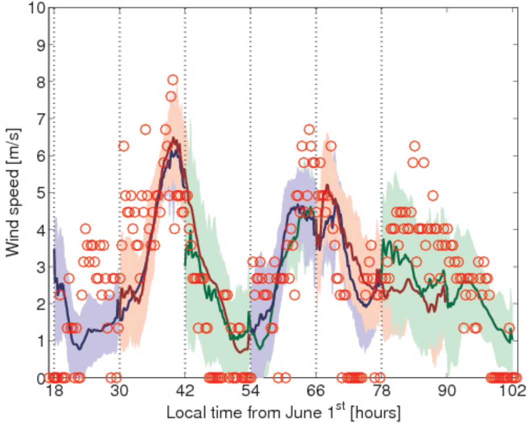


- Sampling is Embarrassingly Parallel - Breeding process requires periodic communication at a central node

- Condor - Mostly harvested free cycles on departmental workstations
- Multi-CPU jobs for larger simulations



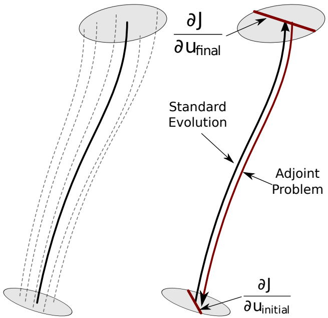
Results



- Solid Line - Mean
- Colored Bands - 2σ spread
- Circles - Observed values
- Taken from [CZR$^+$11]

Wasteful Computation

Computed uncertainty everywhere to determine uncertainty in one location



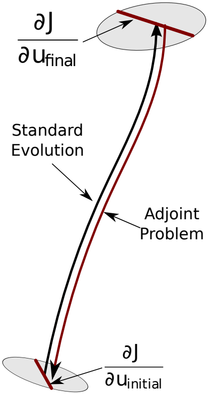
Efficiency

Can we efficiently focus our computation around a single question?

Adaptive Observation

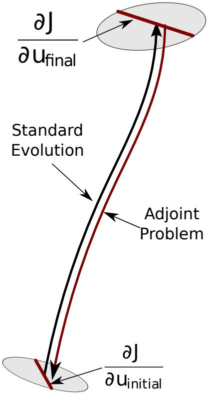
What should we observe now to reduce this specific uncertainty?
To what variables is this question sensitive?

Derivatives



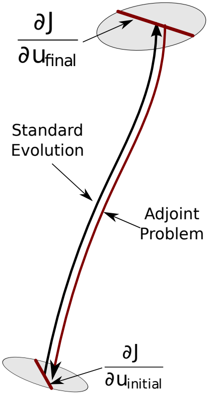
- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State

Derivatives



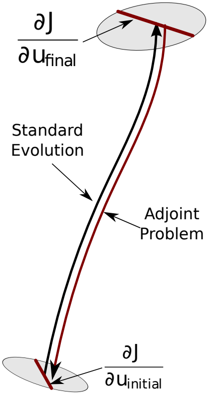
- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state. *"Windspeed in Chicago"*

Derivatives



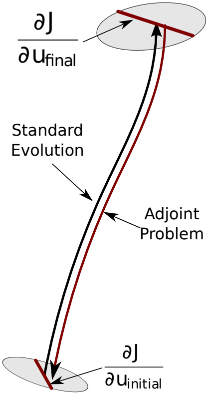
- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state

Derivatives



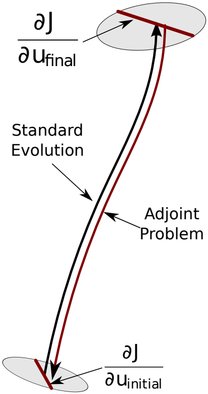
- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state
- $\delta \mathbf{x}_t$: Change in $\mathbf{x}_t$

Derivatives



- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state
- $\delta \mathbf{x}_t$: Change in $\mathbf{x}_t$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Change in windspeed

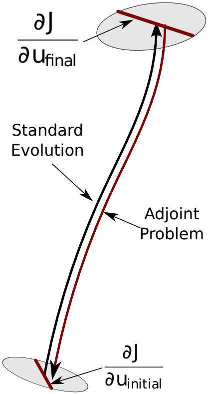
Derivatives



- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state
- $\delta \mathbf{x}_t$: Change in $\mathbf{x}_t$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Change in windspeed

$$X_t = \mathbf{x}_t + \delta \mathbf{x}_t \ : \ \delta \mathbf{x}_t \sim \text{Random Var.}$$

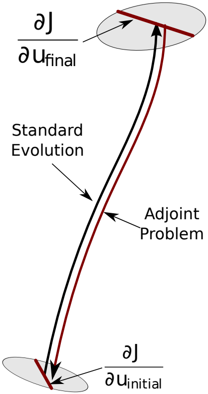
Derivatives



- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state
- $\delta \mathbf{x}_t$: Random Variable - Error in $\mathbf{x}_t$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in windspeed

$$X_t = \mathbf{x}_t + \delta \mathbf{x}_t \ : \ \delta \mathbf{x}_t \sim \text{Random Var.}$$

Derivatives



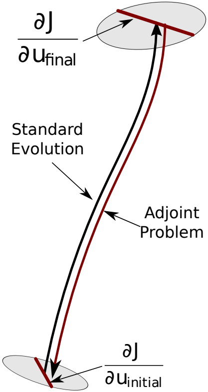
- $\mathbf{x}_t = M_t(\mathbf{x}_0)$: Final State
- $J(\mathbf{x}_t)$: Functional on final state.
  "Windspeed in Chicago"
- $\frac{\partial J}{\partial \mathbf{x}_t}$: Sensitivity of J to changes in final state
- $\delta \mathbf{x}_t$: Random Variable - Error in $\mathbf{x}_t$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in windspeed

$$X_t = \mathbf{x}_t + \delta \mathbf{x}_t \ : \ \delta \mathbf{x}_t \sim \text{Random Var.}$$

Still need to find $\delta \mathbf{x}_t$

Derivatives

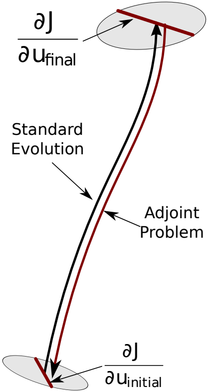
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed

Derivatives

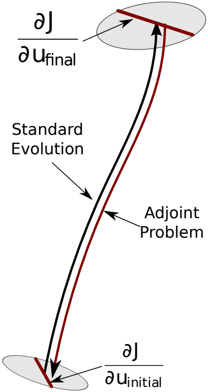
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$

Derivatives

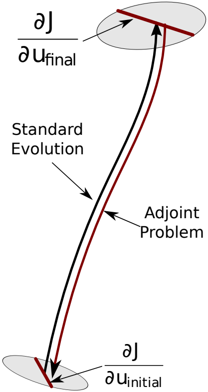
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$

Derivatives

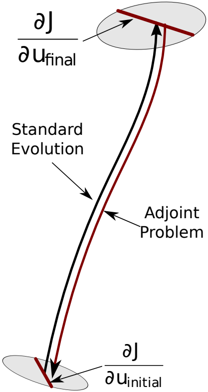
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$

Derivatives

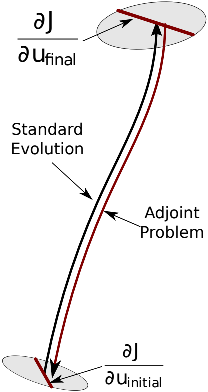
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$ – Difficult

Derivatives

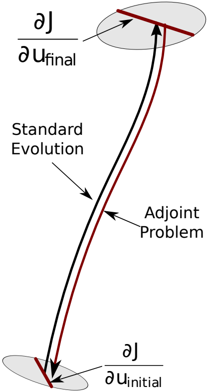
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$ – Difficult
- $\delta J = \langle {\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}}^* \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_0 \rangle$

Derivatives

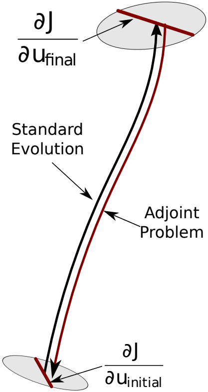
Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$ – Difficult
- $\delta J = \langle \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}^* \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_0 \rangle$ – Easy

Derivatives

Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$

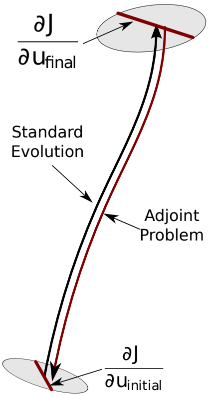


- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$ – Difficult
- $\delta J = \langle \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}^* \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_0 \rangle$ – Easy

$$\frac{\partial J}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}^T \frac{\partial J}{\partial \mathbf{x}_t}$$

Derivatives

Automatic Differentiation lets us apply $\frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}$



- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_t \rangle$: Uncertainty in Windspeed
- Know $\delta \mathbf{x}_0$
- $\delta \mathbf{x}_t = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 + O(\delta \mathbf{x}_0^2)$
- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_t}, \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0} \delta \mathbf{x}_0 \rangle$ – Difficult
- $\delta J = \langle \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}^* \frac{\partial J}{\partial \mathbf{x}_t}, \delta \mathbf{x}_0 \rangle$ – Easy

$$\frac{\partial J}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_0}^T \frac{\partial J}{\partial \mathbf{x}_t}$$

- $\delta J = \langle \frac{\partial J}{\partial \mathbf{x}_0}, \delta \mathbf{x}_0 \rangle$

Derivatives

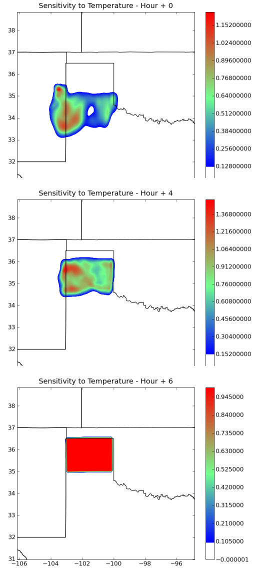
Advantages

- Efficiency : Before we evolved 30-100 samples forward. Now we evolve one state and back-evolve one sensitivity. No sampling mess.
- Statistics-Free : This process is independent of the initial distribution. Can swap alternative distributions in and out.
- Adaptive Observation : We can more easily ask questions about the effects of the initial distribution

Drawbacks

- Linearity assumption : Only works for distributions tightly clustered around the mean
- Automatic Differentiation : Easy in theory, often challenging in practice.

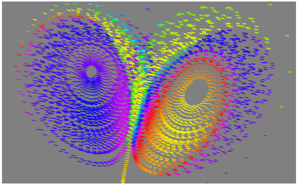
Adaptive Observation



Sensitivities $\frac{\partial J}{\partial \mathbf{x}_t}$ restricted to ground temperature.

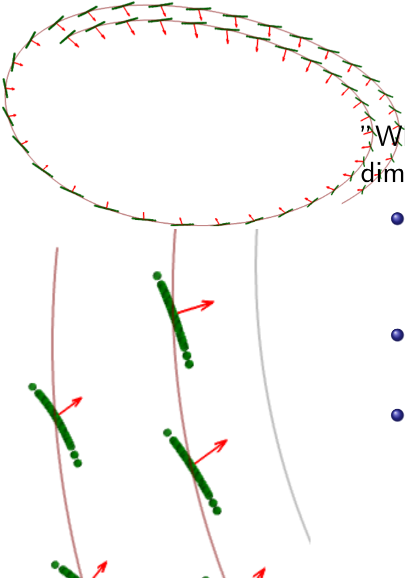
Bright areas highlight desirable observation.

Specific to this day's weather.



Adaptive Observation, Berliner [BLS99].

Comparison

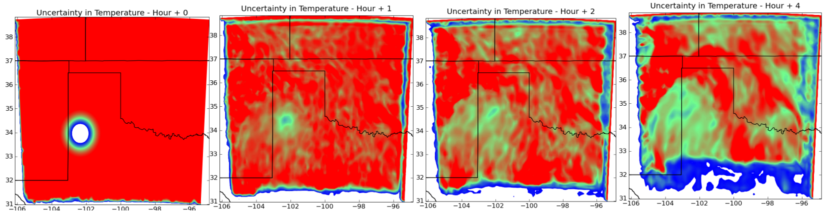


"What is the uncertainty along red dimension at forecast time"

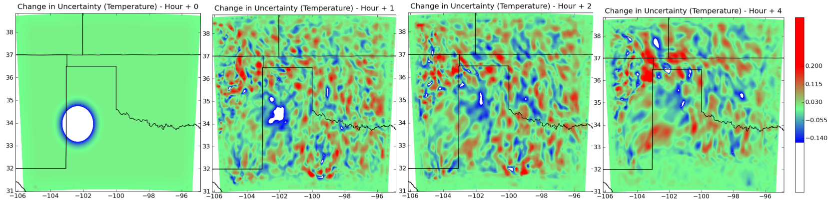
- First order: Evaluates to zero regardless of when the computation occurs.
- Sampling: Observed nonlinear effects
- Adjoint: More efficient but strictly linear

Adaptive Observation Demonstration

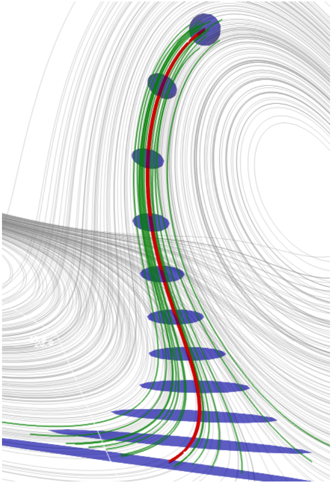
Standard deviation of temperature with an observation



Standard deviation of temperature controlled with non-observed experiment



Future Work

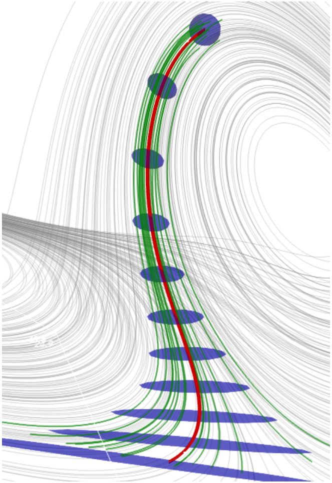


Derivatives provide lots of information cheaply. How can we use this?

$X = \mu + \delta x$, Random Variable

- Mean $\mu_i = E(X_i)$ is the simplest representation of X.
  Evolve using the model
- Covariance matrix $\Sigma_{ij} = E(\delta \mathbf{x}_i \cdot \delta \mathbf{x}_j)$ is the next higher order term
  Evolve using the derivative

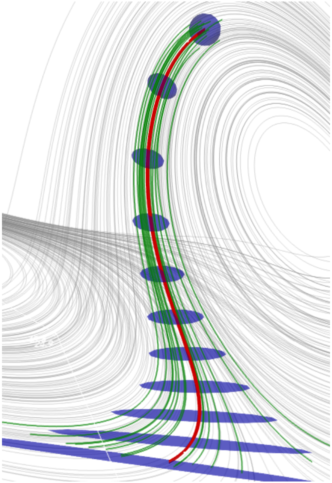
Future Work



$$
\begin{aligned}
L &= \text{Linear operator time } 0 \rightarrow t \\
\Sigma_{ij,t} &= E(\delta \mathbf{x}_{i,t} \cdot \delta \mathbf{x}_{j,t}) \\
&= E((L\delta \mathbf{x}_{i,0}) \cdot (L\delta \mathbf{x}_{j,0})) \\
&= L\Sigma_{ij,0} L^T
\end{aligned}
$$

- Low Dimensions : Distributions are cheaper than sampling.

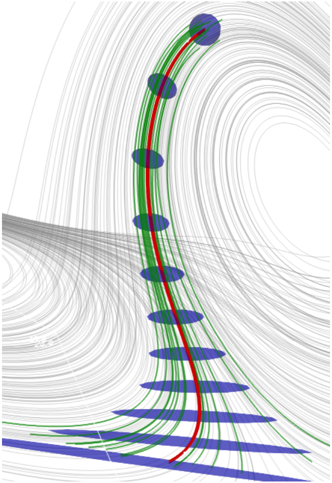
Future Work



$$
\begin{aligned}
L &= \text{Linear operator time } 0 \to t \\
\Sigma_{ij,t} &= E(\delta \mathbf{x}_{i,t} \cdot \delta \mathbf{x}_{j,t}) \\
&= E((L\delta \mathbf{x}_{i,0}) \cdot (L\delta \mathbf{x}_{j,0})) \\
&= L\Sigma_{ij,0}L^T
\end{aligned}
$$

- Low Dimensions : Distributions are cheaper than sampling.
- High Dimensionality?
  –Distributions are low-rank
  –Can apply gradient in batch on low-rank representation

Future Work



$$
\begin{aligned}
L &= \text{Linear operator time } 0 \to t \\
\Sigma_{ij,t} &= E(\delta\mathbf{x}_{i,t} \cdot \delta\mathbf{x}_{j,t}) \\
&= E((L\delta\mathbf{x}_{i,0}) \cdot (L\delta\mathbf{x}_{j,0})) \\
&= L\Sigma_{ij,0}L^T
\end{aligned}
$$

- Low Dimensions : Distributions are cheaper than sampling.
- High Dimensionality?
  –Distributions are low-rank
  –Can apply gradient in batch on low-rank representation
- What about nonlinearity?

References

L. Mark Berliner, Zhan-Qian Lu, and Chris Snyder.
Statistical Design for Adaptive Weather Observations.
Journal of the Atmospheric Sciences, 56(15):2536–2552, August 1999.

M Corazza, E Kalnay, DJ Patil, SC Yang, R Morss, Ming Cai, I Szunyogh, Brian Hunt, and James Yorke.
Use of the breeding technique to estimate the structure of the analysis "errors of the day".
Nonlinear Processes in Geophysics, pages 233–243, 2003.

Emil M. Constantinescu, Victor M. Zavala, Matthew Rocklin, Sangmin Lee, and Mihai Anitescu.
A Computational Framework for Uncertainty Quantification and Stochastic Optimization in Unit Commitment With Wind Power Generation.
IEEE Transactions on Power Systems, 26(1):431–441, February 2011.

DJ Estep.
A short course on duality, adjoint operators, Green's functions, and a posteriori error analysis, 2004.

E Kalnay.
Atmospheric Modeling, Data Assimilation and Predictability.
Cambridge University Press, 2002.

EN Lorenz.
Deterministic Nonperiodic Flow.
Atmos. Sci, 20:130–141, 1963.

D. Patil, Brian Hunt, Eugenia Kalnay, James Yorke, and Edward Ott.
Local Low Dimensionality of Atmospheric Dynamics.
Physical Review Letters, 86(26):5878–5881, June 2001.

William C. Skamarock, Joseph B. Klemp, Jimy Dudhia, David O. Gill, Dale M. Barker, Michael G. Duda, Xiang-Yu Huang, Wei Wang, and Jordan G. Powers.
A description of the Advanced Research WRF Version 3.
NCAR Technical Note, NCAR/TN47(June), 2008.

Questions?

Paper, Slides, and Code available at:

http://people.cs.uchicago.edu/~mrocklin/masters.html