the Theorectial Minimum Note Series

REPOSITORY

undergraduate

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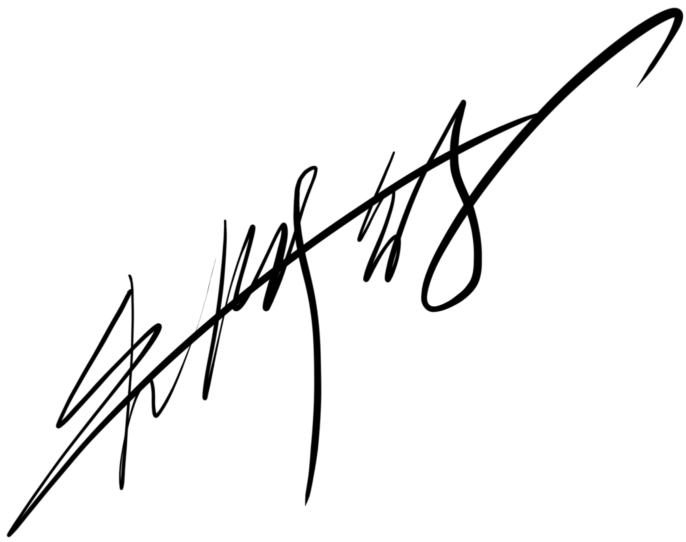
Abstract

A repository of undergraduate physics including Classical Mechanics, Classical Field Theory, Statistical Mechanics, General Relativity, Quantum Mechanics & Quantum Field Theory.

Main Reference:
Mechanics, Landau & Lifshitz
Classical Mechanics, Goldstein
the Classic Theory of Fields, Landau & Lifshitz
Introduction to Quantum Mechanics, Griffiths
Quantum Mechanics(Non-Relativistic Theory), Landau & Lifshitz
Statistical Mechanics of Particles, Kardar
Statistical Mechanics of Fields, Kardar
Spacetime and Geometry, Carroll
General Relativity and its Applications, Ferrari
General Relativity, Wald
Gravitation and Cosmology, Weinberg
Introduction to Elementary Particles, Griffiths
Field Quantization, Greiner
Quantum Field Theory, Srednicki
the Quantum Theory of Fields, Weinberg

The detailed list of reference will be shown at the beginning of each part.



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Part I
Classical Mechanics

1 Lagrangian Formalism

1.1 From D'Alembert's Principle to Lagrangian

Mathematical interlude: variation of a function

To state for simplicity: the variation of a functional means the infinitesimal change of its input/output, where inputs are functions and outputs are numbers. It is similar to the differential of a function, where inputs and outputs would always be numbers. The difference between them can be shown as follow:

$$f(t+dt) = f(t) + df(t) + o(d^2f(t)) \tag{1.1.1}$$

the differential is intrigued by the change of independent variable t. While

$$(f+\delta f)(t) = f(t) + \delta f(t) \tag{1.1.2}$$

the variation is intrigued by the change of the function.

A rule can be extremely useful:

$$\delta(df) = d(\delta f) \tag{1.1.3}$$

D'Alembert's Principle: the virtual work of the total force in equilibrium gives

$$F_i \delta r_i = 0 \tag{1.1.4}$$

This is a constraint of symmetry, implies the unique property of the field at one point: for $F = -\frac{\partial V}{\partial r}$, the place with such 0 total virtual work tend to be the place with the potential minimum neighborhood. Also we can think it as a clear result of $\sum F_i = 0$: for scalar muitplication, the inner product of the two must be 0 when one of the scalar is defined to be 0. Think of a situation where the force of constraint is always prependicular to the surface, i.e. the possible directions of any virtual displacement. Decomposing the total force to applied force and constraint force, we have

$$F_i^{(a)}\delta r_i + f_i \delta r_i = F_i^a \delta r_i = 0 \tag{1.1.5}$$

where $F_i^{(a)}$ denotes the applied force and f_i denotes the constraint force. Only the system with the net work virtual work of constraint force is 0 is studied.

Take $r_i = r_i(q_1, ...q_n, t)$ we have

$$v_i = \frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \tag{1.1.6}$$

also

$$\delta r_i = \frac{\partial r_i}{\partial q_j}\delta q_j \tag{1.1.7}$$

Consider the relation

$$m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}) - m_i \dot{r}_i \cdot \frac{d}{dt}(\frac{\partial r_i}{\partial q_j}) \tag{1.1.8}$$

noticing

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \tag{1.1.9}$$

we have

$$m_i \ddot{r}_i \cdot \frac{\partial r_i}{q_j} = \frac{d}{dt}(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j}) - m_i \dot{r}_i \cdot \frac{\partial v_i}{\partial q_j} \tag{1.1.10}$$

define the generalized force $Q_j := F_i \cdot \frac{\partial r_i}{\partial q_j}$ and using the equation of motion

$$(F_i - \dot{p_i})\delta r_i = 0 \rightarrow (Q_j - m_i \ddot{r_i} \cdot \frac{\partial r_i}{\partial q_j})\delta q_j = 0 \tag{1.1.11}$$

the term can be

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q_j}}) - \frac{\partial T}{\partial q_j} - Q_j]\delta q_j = 0 \tag{1.1.12}$$

if we denote $T = \frac{1}{2}m_i v_i^2$.

For $Q_j = -\frac{\partial V}{\partial q_j}$, if the potential is not velocity dependent, we can derive

$$[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q_j}}) - \frac{\partial (T-V)}{\partial q_j}]\delta q_j = [\frac{d}{dt}(\frac{\partial (T-V)}{\partial \dot{q_j}}) - \frac{\partial (T-V)}{\partial q_j}]\delta q_j = 0 \tag{1.1.13}$$

and thus

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_j}}) - \frac{\partial L}{\partial q_j} = 0 \tag{1.1.14}$$

where we define $L = T - V$ as the Lagrangian, the equation is called Lagrange's equation.

Here we can use the generalized force to give the equation of motion, which is

$$Q_j = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q_j}}) - \frac{\partial T}{\partial q_j} \tag{1.1.15}$$

1.2 Velocity Dependent Potentials

1.3 From Principle of Least Action to Lagrangian

Every mechanical system is characterized by a definite function $L(q, \dot{q}, t)$, at the instants t_1 & t_2, we have the system moving in a way that the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \tag{1.3.1}$$

takes the minimum value, where the function L is called the Legrangian and the integral is called the Action.

A variation of the function $q(t)$ is denoted $\delta q(t)$, we have

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{1.3.2}$$

Since S takes the minimum,which is

$$\int_{t_1}^{t_2} [L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t)]dt = 0 \tag{1.3.3}$$

we have

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q})dt = 0 \tag{1.3.4}$$

Since $\delta \dot{q} = d\delta q/dt$, using the part integrating, we have

$$\delta S = \frac{\partial L}{\partial \dot{q}}\delta q|_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q dt = 0 \tag{1.3.5}$$

which goes

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0 \tag{1.3.6}$$

Multiplying the Lagrangian of a mechanic system by any constant has no effect on the equations of motion, which corresponds to the natural arbitrariness in the choice of the unit of measurement. But the Lagrangian can also differ from a total time derivative of a function of time and coordinates. This suggeests that the Lagrangian is not a typical physical quantity which will be called observable.

To write explicitly, there could be

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}f(q, t) \tag{1.3.7}$$

then the action

$$S' = S + \int_{t_1}^{t_2} \frac{d}{dt}f(q, t)dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1) \tag{1.3.8}$$

the last two part will naturally disappear when variationed.

1.4 Lagrangian Compared to Newtonian

The law of inertia

From $\partial L/\partial \vec{r} = 0$, we have the Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = 0 \to \vec{v} = const \tag{1.4.1}$$

which states that in a inertia frame any free motion takes place with a constant velocity, both in magnitude and direction, which is the famous law of inertia.

The kinetic energy: Lagrangian of a free particle

Take a variation in velocity to be $\vec{v'} = \vec{v} + \vec{\varepsilon}$. For a free particle in a inertia frame, for homogeneity, the Lagrangian cannot explicitly depend on the coordinates or time, and the Lagrangian can only explicitly depend on the velocity. Since the space is isotropic, the Lagrangian is a function of the magnitude of the velocity. Take it v^2, we have

$$L' = L(v'^2) = L(v^2 + 2\vec{v} \cdot \vec{\varepsilon} + \varepsilon^2) \tag{1.4.2}$$

for expansion up to 1 order we get

$$L' = L(v'^2) = L(v^2 + 2\vec{v} \cdot \vec{\varepsilon}) = L(v^2) + \frac{\partial L}{\partial v^2}2\vec{v} \cdot \varepsilon \tag{1.4.3}$$

The second term on the right of this equation is a total time derivative if and only if it is a linear function of $\vec{v}$, recalling that the difference of the Lagrangian can be up to a total time derivative of coordinates and time. Thus the Lagrangian is proportional to the square of velocity, which we will denote

$$L = \frac{1}{2}mv^2 \tag{1.4.4}$$

where the constant m will later be the mass defined to make Lagrangian additive.

For arc in any given system it is useful to have

$$v^2 = (\frac{dl}{dt})^2 = (dl)^2/(dt)^2 \tag{1.4.5}$$

In Cartesian coordinates, $dl^2 = dx^2 + dy^2 + dz^2$, we have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{1.4.6}$$

In cylindrical coordinates $dl^2 = dr^2 + r^2d\phi^2 + dz^2$, and we have

$$L = \frac{m}{2}(\dot{r}^2 + r^2 d\dot{\phi}^2 + \dot{z}^2) \tag{1.4.7}$$

In spherical coordinates $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$, and we have

$$L = \frac{2}{m}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta d\dot{\phi}^2) \tag{1.4.8}$$

The Complete Lagrangian

Now we focus on the system which has interaction between but not with other systems and we introduce a function describing the interaction with only the coordinates, denoted $-U$, and we have

$$L = \frac{1}{2} m_a v_a^2 - U(r_1, r_2, \cdots) \tag{1.4.9}$$

and we denote

$$T = \frac{1}{2} m_a v_a^2 \tag{1.4.10}$$

as the kinetic energy of a system.

Again, the Lagrange equation here is

$$\frac{d}{dt}\frac{\partial L}{\partial v_a} = \frac{\partial L}{\partial r_a} \tag{1.4.11}$$

plugging in the Lagrangian here we defined with a U, actually the potential, which is a function only about coordinates, gives

$$m_a \frac{dv_a}{dt} = -\frac{\partial U}{\partial r_a} \tag{1.4.12}$$

which is actually Newton equation. The righthand side $F_a = \frac{\partial U}{\partial r_a}$ is the applied force on a-th particle.

The conservation of energy

The conservation of energy is derived under the condition of the homogeneity of time, which is to say, the Legrangian of a closed system should not depend explicitly on time, $\frac{\partial L}{\partial t} = 0$. Thus we have

$$\frac{dL}{dt} = \frac{\partial L}{\partial q} d\dot{q} + \frac{\partial L}{\partial \dot{q}} d\ddot{q} \tag{1.4.13}$$

Using the Euler-Lagrangian Equation, we substitute $\frac{\partial L}{\partial q}$ by $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ and we get

$$\frac{dL}{dt} = \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}\dot{q}) \tag{1.4.14}$$

which gives

$$\frac{d}{dt}(\dot{q}\frac{\partial L}{\partial \dot{q}} - L) \equiv \frac{d}{dt}E = 0 \tag{1.4.15}$$

where $E = const$ is defined as the Energy of a closed system.

The conservation of momentum and Newton's Third Law

The conservation of momentum follows from the homogeneity of space. By virtue of this, the mechanical properties of a closed system are invariant under any parallel displacement. It is clear that the variation of Lagrangian under such displacement $r_i \to r_i + \varepsilon$ should be

$$\delta L = \frac{\partial L}{\partial r_i} \cdot \delta r_i = \varepsilon \cdot \frac{\partial L}{\partial r_i} = 0 \to \sum \frac{\partial L}{\partial r_i} = 0 \tag{1.4.16}$$

applying

$$\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial r} \tag{1.4.17}$$

we immediately get

$$\frac{d}{dt}\sum\frac{\partial L}{\partial v_i} = 0 \to \sum\frac{\partial L}{\partial v_i} := P = const \tag{1.4.18}$$

Here we difined the invariant momentum.
For $\sum\frac{\partial L}{\partial r_i} = 0$, we can derive

$$\frac{\partial L}{\partial r_i} = -\frac{\partial U}{\partial r_i} = F_i \to \sum F_i = 0 \tag{1.4.19}$$

When system consists only two particles, $F_1 + F_2 = 0$, which gives the typical Newton's Third Law.

The conservation of angular momentum
The constraint of angular momentum is derived from the isotropy of space.
We introduce vector $\delta\vec{\varphi}$ to represent an infinitesimal rotation and gives

$$\delta\vec{r} = \delta\vec{\varepsilon}\times\vec{r} \tag{1.4.20}$$

where the direction of $\delta\vec{\varepsilon}$ is pointing out the surface that the rotation is anticlockwise looking outside. Since the radius here is fixed, we also have

$$\delta\vec{v} = \delta\vec{\varphi}\times\vec{v} \tag{1.4.21}$$

If the Lagrangian remains unchanged, then we have

$$\delta L = \frac{\partial L}{\partial r_a}\delta r_a + \frac{\partial L}{\partial v_a}\delta v_a = 0 \tag{1.4.22}$$

then take $\partial L/\partial v_a = p_a,\ \partial L/\partial r_a = \dot{p_a}$, we have

$$\dot{p_a}\cdot(\delta\varphi\times r_a) + p_a\cdot(\delta\varphi\times v_a) = 0 \tag{1.4.23}$$

and by further rearranging,

$$\delta\varphi\cdot(r_a\times\dot{p_a} + v_a\times p_a) = \delta\varphi\cdot\frac{d}{dt}(r_a\times p_a) = 0 \tag{1.4.24}$$

since $\delta\varphi$ is arbitrary, we have

$$\frac{d}{dt}(r_a\times p_a) = 0 \tag{1.4.25}$$

which is the (psuedo)vector,

$$M = r_a\times p_a = const \tag{1.4.26}$$

is conserved and called the angular momentum of the system.
Angular Momentum at z Axis
Since

$$\delta L = \delta\varphi\frac{d}{dt}M \tag{1.4.27}$$

we have

$$\frac{\partial L}{\partial\varphi} = \frac{d}{dt}M \tag{1.4.28}$$

and by Euler-Lagrange equation

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} \tag{1.4.29}$$

so that

$$M = \frac{\partial L}{\partial \dot{\varphi}} \tag{1.4.30}$$

Cautious: here $\delta\varphi$ is restricted in the $x - y$ plane, therefore the M here is actually M_z. Or, we can writing it out explicitly,

$$M_z = r_a \times p_a = m_a(x_a + y_a) \times (\dot{x_a} + \dot{y_a}) = m_a(x_a\dot{y_a} - y_a\dot{x_a}) = m_a r_a^2 \dot{\varphi}_a \tag{1.4.31}$$

here

$$\frac{d\varphi}{dt} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \tag{1.4.32}$$

or we can use the Lagrangian

$$L = \frac{1}{2}m_a(\dot{r_a}^2 + r_a^2\dot{\varphi_a}^2 + \dot{z_a}^2) - U \tag{1.4.33}$$

to derive the above euqation.

The Conservation of Laplace-Runge-Lenz Vector
For a general central force, Newton's second law can be written in a vectorial form

$$\dot{\mathbf{p}} = f(r)\frac{\mathbf{r}}{r} \tag{1.4.34}$$

so that the cross product of $\dot{\mathbf{p}}$ with the constant angular momentum $\mathbf{L}$ should be

$$\dot{\mathbf{p}} \times \mathbf{L} = \frac{mf(r)}{r}[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})] = \frac{mf(r)}{r}(\mathbf{r}r\dot{r} - r^2\dot{\mathbf{r}}) \tag{1.4.35}$$

since we have

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = r\dot{r} \tag{1.4.36}$$

since $\mathbf{L}$ is a constant, we have

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dot{\mathbf{p}} \times \mathbf{L} = -mf(r)r^2(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}) = -mf(r)r^2\frac{d}{dt}(\frac{\mathbf{r}}{r}) \tag{1.4.37}$$

if $f(r)$ is inversely proportional to r^2 by what we denote $-k$ so that k would be a positive constant, we can immediately write down

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}(\frac{mk\mathbf{r}}{r}) \tag{1.4.38}$$

we therefore define a vector $\mathbf{A}$, so that

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r} \tag{1.4.39}$$

is conserved, which showed that once the $1/r^2$ field is determined, the shape of its orbit is determined.

From the definition, we can soon derive

$$\mathbf{A} \cdot \mathbf{L} = 0 \tag{1.4.40}$$

by writing out the matrix form.

1.5 Mechanical Similarity

Since Multiplication of the Lagrangian by any constant does not affect the equation of motion, it makes possible that in a number of important cases, some useful inferences concerning the properties of the motion, without the necesity of actually integrating and solving the equations.

When the potential energy is a homogeneous function of the coordinates, which is

$$U(\alpha r_1, \alpha r_2, \cdots, \alpha r_n) = \alpha^k U(r_1, r_2, \cdots, r_n) \tag{1.5.1}$$

where k is the degree of homogeneity.

Introducing a transformation

$$r \to \alpha r \qquad t \to \beta t \tag{1.5.2}$$

so that all components of velocity are changed by a factor α/β and the kinetic energy by a factor α^2/β^2, and the potential energy α^k. if

$$\frac{\alpha^2}{\beta^2} = \alpha^k \to \beta = \alpha^{1-k/2} \tag{1.5.3}$$

the euqation of motion remains the same.

We soon derive that under such transformation

$$\frac{t'}{t} = (\frac{l'}{l})^{1-k/2} \tag{1.5.4}$$

where l is the linear dimension. This suggeests mechanical quantaties to be written in a linear dimension form and thus to derive the ratio.

An important example is the harmonic oscillator where $k = 2$. From above euqation we can see that the period of such oscillations is independent of its amplitude.

For $k = 1$, which is a uniform force field, we have

$$\frac{t'}{t} = \sqrt{\frac{l'}{l}} \tag{1.5.5}$$

this is when in fall under gravity, the time of fall is as the square root of the initial altitude.

For $k = -1$, which is the Newtonian attraction or Coulomb interaction, the potential energy is inversely proportional to the distance apart, we have

$$\frac{t'}{t} = (\frac{l'}{l})^{3/2} \tag{1.5.6}$$

which is Kepler's Third Law, states that the square of time of revolution in the orbit is as the cube of the size of the orbit.

If the potential energy is a homogeneous function of the velocities, we have by Euler's Theorem that

$$\frac{\partial T}{\partial v_a} v_a = 2T \tag{1.5.7}$$

or, using $\partial T/\partial v_a = p_a$, to give

$$2T = p_a v_a = \frac{d}{dt} p_a r_a - p \dot{+} a r_a \tag{1.5.8}$$

for a long time average value, since $p_a r_a$ is bounded, we have

$$2\overline{T} = \overline{r_a \cdot \frac{\partial U}{\partial r_a}} \tag{1.5.9}$$

which is called the Virial Theorem. Here we used

$$m_a \frac{dv_a}{dt} = -\frac{\partial U}{\partial r_a} \tag{1.5.10}$$

If the potential energy is a homogeneous function of degree k in the radius r_a, we have

$$2\overline{T} = k\overline{U} \tag{1.5.11}$$

with $\overline{T} + \overline{U} = \overline{E} = E$, we have

$$\overline{U} = \frac{2}{k+2}E \qquad \overline{T} = \frac{k}{k+2}E \tag{1.5.12}$$

For $k = 2$, the oscillations, we have $\overline{T} = \overline{U}$, which states the average value of kinetic energy and potential energy is the same.

For $k = -1$, the Newtonian gravity, we have $2\overline{T} = -\overline{U}$ and $E = -\overline{T}$, which states that the motion takes place in a finite region only if the total energy is negative.

1.6 Noether's Theorem in Particle Theory

1.7 Central Force Problem

Motion in a Central Field

In a central field, the potential is only related to the distance from the particle to a fixed point. The force is

$$F = -\frac{U(r)}{\partial \mathbf{r}} = -\frac{dU}{dr}\vec{e_r} \tag{1.7.1}$$

We have proved that in a central field the angular momentum is conserved.

Introducing the polar coordinates, we have the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) \tag{1.7.2}$$

which do not depend explicitly on φ, so that φ is called a cyclic coordinate of the euqation. For cyclic equation, we have the Euler-Lagrange euqation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} = 0 \tag{1.7.3}$$

so that the corresponding generalized momentum $p_i = \partial L/\partial q_i$ is an integral of motion.

Here the generalized momentum, associated with the generalized coordinate φ, is

$$p_\varphi = mr^2\dot{\varphi} \tag{1.7.4}$$

which happen to be the momentum at z axis.

The conservation has a vivid geometrical interpretation: since $1/2r^2 d\varphi$, denoted df is the area of the sector bounded by 2 neighboring radius vectors and an element of path. By above equation we find

$$M = 2m\dot{f} \tag{1.7.5}$$

as another form of the euqation of conserved angular momentum, that $\dot{f}$ is called the sectorial velocity.

By this we will introduce the effective potential, which is

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) = \frac{m}{2}\dot{r}^2 + \frac{M}{2mr^2} + U(r) \equiv \frac{m}{2}\dot{r}^2 + U_e(r) \tag{1.7.6}$$

here e is for effective.

The general solution of this function can be given as follow. Rewriting it we have

$$\dot{r} \equiv \frac{dr}{dt} = \sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}} \tag{1.7.7}$$

so that

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}}} + const \tag{1.7.8}$$

by introducing $d\varphi = \frac{M}{mr^2}dt$, we soon derive

$$\varphi = \int \frac{M/r^2 \ dr}{\sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}}} + const \tag{1.7.9}$$

Attractive Field
We first consider an attractive field. The potential is

$$U = -\alpha/r \tag{1.7.10}$$

where α is defined positive. The effective potential is

$$U_e = -\frac{\alpha}{r} + \frac{M}{2mr^2} \tag{1.7.11}$$

When $r = M^2/\alpha$ the effective potential takes the minimum $-m\alpha^2/2M^2$.
By substituting $U = -\alpha/r$ we have

$$\varphi = \arccos \frac{M/r - m\alpha/M}{\sqrt{2mE + m^2\alpha^2/M^2}} + const \tag{1.7.12}$$

by choosing the initial φ to make $const = 0$, and introducing

$$p = \frac{M^2}{m\alpha} \qquad e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}} \tag{1.7.13}$$

the euqation of orbit can be written

$$\frac{p}{r} = 1 + e\cos\varphi \tag{1.7.14}$$

When $E < 0$, $e < 1$, we have

$$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|} \qquad b = \frac{p}{\sqrt{1-e^2}} = \frac{M}{\sqrt{2m|E|}} \tag{1.7.15}$$

so that

$$r_{min} = \frac{p}{1+e} = a(1-e) \qquad r_{max} = \frac{p}{1-e} = a(1+e) \tag{1.7.16}$$

1.8 Rigid Body

2 Hamiltonian Formalism

2.1 Hamilton Equation

Through Legendre Transformation, we can change the set of independent variables to another. In the present case, we can write

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \tag{2.1.1}$$

From Euler-Lagrange Equation we have

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \quad p_i := \frac{\partial L}{\partial \dot{q}_i} \qquad \rightarrow \qquad \frac{\partial L}{\partial q_i} = \dot{p}_i \tag{2.1.2}$$

so that the total differential of Lagrangian above can be written in a more symmetric form

$$dL = \dot{p}_i dq_i + \dot{q}_i dp_i \tag{2.1.3}$$

then we have by definition of the action, the total time derivative of action is given

$$S = \int_{t_1}^{t_2} L dt \quad \rightarrow \quad \frac{dS}{dt} = L \tag{2.1.4}$$

of total derivative $p_i d\dot{q}_i = d(p_i \dot{q}_i) - \dot{q}_i dp_i$, to help us define

$$d(p_i \dot{q}_i - L) := dH = -\dot{p}_i dq_i + \dot{q}_i dp_i \tag{2.1.5}$$

thus the quantaty Hamiltonian is defined. We can immediately write out

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.1.6}$$

which are called Hamilton Equations, written in variable p and q. Hamilton Equations form a set of $2s$ first-order differential equations, replacing the s second-order equations in Lagrangian Formalism. They are also called Canonical Equations for their simplicity and symmetry of form.

2.2 the Routhian

2.3 Poisson Bracket

Let f to be a function of coordinates, momenta and time. The total time derivative should be

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\frac{\partial p_i}{\dot{p}_i} + \frac{\partial f}{\partial q_i}\dot{q}_i) := \frac{\partial f}{\partial t} + [H, f] \tag{2.3.1}$$

which gives the definition of a Poisson Bracket

$$[f, g] = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \tag{2.3.2}$$

where the multiplying partial derivatives with index implies summation.

Following properties are useful, and easy to check out:

$$[f, g] = -[g, f] \tag{2.3.3}$$

$$[f, const] = 0 \tag{2.3.4}$$

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g] \tag{2.3.5}$$

$$\frac{\partial}{\partial t}[f, g] = [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}] \tag{2.3.6}$$

$$[f, q_i] = \frac{\partial f}{\partial p_i}, [f, p_i] = -\frac{\partial f}{\partial q_k} \to [q_i, q_k] = 0, [p_i, p_k] = 0, [p_i, q_k] = \delta_{ik} \tag{2.3.7}$$

and the important Jacobi's Identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \tag{2.3.8}$$

An important property of the Poisson Bracket is the Poisson Theorem: two integrals of motion, their Poisson Bracket is likewise an intergral of motion, which is

$$[f, g] = const \tag{2.3.9}$$

this is the result of the Jacobi's Identity by putting H, f, g as the inputs.

2.4 the Action as a Function of the Coordinates

Previous derivation has given

$$\delta S = \frac{\partial L}{\partial \dot{q}}\delta q|_{t_1}^{t_2} + \int_{t_1}^{t_2}(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q dt = 0 \tag{2.4.1}$$

since the actual orbit satisfy Euler-Lagrange Equation, the intergration is 0. If we are to define $\delta q(t_1) = 0$, and to denote $\delta q(t_2) = \delta q$, we can focus on the variation of the action represented by the variation of q, which is to say, take the action as a function of the coordinates. For $\partial L/\partial \dot{q} = p$, we get

$$\delta S = p_i \delta q_i \quad \to \quad \frac{\partial S}{\partial q_i} = p_i \tag{2.4.2}$$

From the definition of the action, the total time derivative of action is given

$$S = \int_{t_1}^{t_2} L dt \quad \to \quad \frac{dS}{dt} = L \tag{2.4.3}$$

By previous derivation, we regard S as a function of coordinates and time. We have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\dot{q}_i = \frac{\partial S}{\partial t} + p_i \dot{q}_i \tag{2.4.4}$$

which is

$$\frac{\partial S}{\partial t} = L - p_i \dot{q}_i = -H \tag{2.4.5}$$

Together with previous $\frac{\partial S}{\partial q_i} = p_i$ we are to find

$$dS = p_i dq_i - H dt \tag{2.4.6}$$

for the total differential of the action as a function of coordinates and time by the upper limit defined and under limit set to 0. If not set, the equation should be the difference

$$dS = p_i' dq_i' - H' dt' - p_i dq_i - H dt \tag{2.4.7}$$

where the prime means the end of the path and the unprimed a beginning one.

Another intresting property of regarding the action as a function of coordinates and time is that the Hamilton Equation can be derived.

For $S = \int(p_i dq_i - Hdt)$, we have

$$\delta S = \int \delta p(dq - \frac{\partial H}{\partial p}dt) + p\delta q| - \int \delta q(dp + \frac{\partial H}{\partial q}dt) \tag{2.4.8}$$

where the second part should be 0 since $\delta q = 0$ can be set. Thus the variational principle gives $dq = \frac{\partial H}{\partial p}dt, dp = -\frac{\partial H}{\partial q}dt$.

2.5 Canonical Transformation and Liouville's Theorem

2.6 Hamilton-Jacobi Function

3 Introduction to Sympletic Geometry

Part II
Classical Field Theory

1 Fundementals of Special Relativity

1.1 The Principle of Relativity

The Special Theory of Relativity obeys two basic postulates:

• The principle of relativity: Physics is the same in all inertial frames of reference, i.e. frames moving with constant velocity relative to each other.

• The invariance of the speed of light: The velocity of light has the same value c in all inertial frames of reference.

Here, a reference frame, or a system of reference, includes 1 time axis and 3 space axis, which all actions will later be denoted as the Lie group of $SO(1,3)$. To state vividly, a reference here is a position of a particular particle in space, with a clock fixed in the system serving to indicate the time. Later we call them spacetime altogether to show the equality between classical space and time.

These two postulates leads to a invariance of a property which we name it interval. An interval is defined to be the amount of distance in the spacetime between 2 events, which is denoted by the position of a particle in spacetime $(1,3)$.

Let the first event consists of sending out a signal which propagates with light speed, from a point (t_1, x_1, y_1, z_1) to (t_2, x_2, y_2, z_2). The distance covered is clearly $c(t_2 - t_1)$ or $\vec{x}^2 := (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$.

By convention we denote $ct = x_0, x = x_1, y = x_2, z = x_3$.

The invariant interval(actually its sqaure) is thus given

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{1.1.1}$$

and a proper time, is defined

$$d\tau = ids = \sqrt{d^2x_0 - d\vec{x}^2} = \sqrt{(cdt)^2 - \frac{d\vec{x}}{dt}dt} = cdt\sqrt{1 - \frac{V^2}{c^2}} = dx_0\sqrt{1 - \frac{V^2}{c^2}} \tag{1.1.2}$$

where V is a constant, literally the speed of the moving reference K'. Here after we will use a prime sign to define the reference moving with a constant speed. The form here actually implies the Lorentz Transformation.

A proper time, here should be τ/c, is the time read from the given object(usually moving with a constant speed).

Now the interval of two events gives

$$-(ct)^2 + \vec{x}^2 = const \tag{1.1.3}$$

by this we can write out the explicit relationship between the position given by difference inertial frames. If we take advantage of equation $\cosh^2\phi - \sinh^2 = 1$, which is similar to the expression of interval above, we can immediately give

$$x = x'\cosh\phi + ct'\sinh\phi \qquad ct = x'\sinh\phi + ct'\cosh\phi \tag{1.1.4}$$

to be explicit

$$\begin{aligned} x - (ct)^2 &= (\cosh\phi + ct'\sinh\phi)^2 - (x'\sinh\phi + ct'\cosh\phi)^2 \\ &= x'(\cosh^2\phi - \sinh^2\phi) - (ct)^2(\cosh^2\phi - \sinh^2\phi) = x'^2 - (ct')^2 \end{aligned} \tag{1.1.5}$$

For simplicity the Lorentz Transformation is defined if the reference is moving along one of the space axis, while $(1,3) = (0,\vec{0})$, that is when $t = 0$ the origin of the moving reference K' happen to be where the origin of K. If x axis, then we can write

$$x = ct'\sinh\phi \quad ct = ct'\cosh\phi \to \frac{x}{ct} = \tanh\phi = \frac{V}{c} \tag{1.1.6}$$

that is

$$\sinh\phi = \frac{V/c}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \cosh\phi = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{1.1.7}$$

at last we get

$$x = \frac{x' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad t = \frac{t' + (V/c)^2 x'}{\sqrt{1-\frac{V^2}{c^2}}} \tag{1.1.8}$$

at same t' or x'

$$\Delta x|_{t'} = \frac{\Delta x'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \Delta t|_{x'} = \frac{\Delta t'}{\sqrt{1-\frac{V^2}{c^2}}} \tag{1.1.9}$$

which is the typical expression of the Lorentz Transformation. The rule of adding velocity is derived from

$$dx = \frac{dx' + V dt'}{\sqrt{1-\frac{V^2}{c^2}}} \quad dt = \frac{dt' + (V/c)^2 dx'}{\sqrt{1-\frac{V^2}{c^2}}} \quad dy = dy' \quad dz = dz' \tag{1.1.10}$$

applying $v_i = dx_i/dt$ (of course i run from 1 to 3)we get

$$v_x = \frac{v'_x + V}{1 + v'_x(V/c^2)} \quad v_y = \frac{v'_y\sqrt{1-\frac{V^2}{c^2}}}{1 + v'_x(V/c^2)} \quad v_z = \frac{v'_z\sqrt{1-\frac{V^2}{c^2}}}{1 + v'_x(V/c^2)} \tag{1.1.11}$$

if the speed is much smaller than c, we can expand it by V/c and write to the first order

$$v_x = v'_x + V(1 - \frac{{v'_x}^2}{c^2}) \quad v_y = v'_y - v'_x v'_y \frac{V}{c^2} \quad v_z = v'_z - v'_x v'_z \frac{V}{c^2} \tag{1.1.12}$$

It is also convenient to define the 4-velocity. To define the velocity, every component of the $(1,3)$ spacetime should be diveded by the propertime

$$u^i = \frac{dx^i}{d\tau} = (\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1-\frac{V^2}{c^2}}}) \tag{1.1.13}$$

where the propertime is defined $d\tau = cdt\sqrt{1-\frac{V^2}{c^2}}$

we can immediately derive

$$u^i u_i = -1 \tag{1.1.14}$$

1.2 Relativistic Mechanics

By the spirit of approaching the Classical Limit, the action and the Lagrangian can be derived as follow.

The action should take the form

$$S = -\alpha \int_a^b d\tau \tag{1.2.1}$$

where $d\tau = \sqrt{c^2 dt^2 - d\vec{x}^2} = cdt\sqrt{1-\frac{V^2}{c^2}}$. Then we can write down

$$S = \int -\alpha c\sqrt{1-\frac{V^2}{c^2}} dt \to L = -\alpha c\sqrt{1-\frac{V^2}{c^2}} \tag{1.2.2}$$

from the Classical Limit where $c \to \infty$, we have

$$L \to const + \frac{mv^2}{2} \tag{1.2.3}$$

one way to approach it is to expand it by series of v/c

$$L = -\alpha c\sqrt{1 - \frac{V^2}{c^2}} \approx -\alpha c + \frac{\alpha v^2}{2c} \tag{1.2.4}$$

which gives $\alpha = mc$ and thus

$$S = -mc \int_a^b d\tau \tag{1.2.5}$$

the Lagrangian is therefore derived

$$L = -mc^2\sqrt{1 - \frac{V^2}{c^2}} \tag{1.2.6}$$

the classical momentum $\vec{p}$ is by definition

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{1.2.7}$$

Notice: here $\vec{p}$ is given by definition $\partial L/\partial \vec{v}$ but after we introduced the general momentum P the definition from vaccum do not stand more and $\vec{p}$ is identically the arguement $\frac{m\vec{v}}{\sqrt{1-\frac{V^2}{c^2}}}$.

and the classical energy

$$E = \vec{p} \cdot \vec{v} - L = \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{1.2.8}$$

for $\vec{v} = 0$, $E = mc^2$ gives the rest energy of a massive particle. For velocity much smaller than lightspeed c, we can expand energy in series in powers of v/c

$$E = mc^2 + \frac{mv^2}{2} \tag{1.2.9}$$

which is derived previously. Apart from the rest energy is th typical kinetic energy.

Attention: in Relativistic Mechanics the energy is always positive, diffrent from energy defined in Classical Mechanics, which is diffrent up to a constant and therefore can be set to be negative.

Also

$$p^2 + m^2c^2 = \frac{m^2c^4}{c^2 - v^2} = \frac{E^2}{c^2} \tag{1.2.10}$$

then the Hamiltonian can be defined

$$H = E = c\sqrt{p^2 + m^2c^2} \tag{1.2.11}$$

For velocity much smaller than lightspeed c, we can expand Hamiltonian in series in powers of v/c

$$H = c\sqrt{p^2 + m^2c^2} \approx mc^2 + \frac{p^2}{2m} \tag{1.2.12}$$

For a free particle, we have its equation of momentum, energy and velocity as

$$\vec{p} = \frac{E\vec{v}}{c^2} \tag{1.2.13}$$

if the particle reaches lightspeed it is simply

$$\vec{p} = \frac{E}{c} \tag{1.2.14}$$

As the least action principle suggests,

$$\delta S = mc\int_a^b \delta\tau = -imc\int_a^b \frac{dx_i \delta dx^i}{ds} \tag{1.2.15}$$

this is from $\delta y = \frac{dy}{dx}\delta x$, and $d\tau = -i\sqrt{dx^i dx_i}$ thus

$$\delta\sqrt{dx^i dx_i} = \frac{d\sqrt{dx^i dx_i}}{d(dx^i)}\delta dx^i + \frac{d\sqrt{dx^i dx_i}}{d(dx_i)}\delta dx_i = \frac{dx_i}{2\sqrt{dx^i dx_i}}\delta dx^i + \frac{dx^i}{2\sqrt{dx^i dx_i}}\delta dx_i \tag{1.2.16}$$

and it is $\frac{dx_i \delta dx^i}{\sqrt{dx^i dx_i}}$ by means of symmetry. So we have further

$$\delta S = -mc\int_a^b u_i d\delta x^i = -mcu_i\delta x^i|_a^b - mc\int_a^b \delta x^i \frac{du_i}{ds} ds \tag{1.2.17}$$

To get the actual equation of motion, we must have at the boundary $\delta x_a^i = \delta x_b^i = 0$, and an actual orbit δS=0, thus gives $\frac{du_i}{ds} = 0$, which is actually

$$\frac{du_i}{d\tau} = 0 \tag{1.2.18}$$

which gives a real equation of motion is of no 4-acceleration.

By previous derivation $dS = p_i dq_i - Hdt$, we have the 4-momentum

$$p_i = (-\frac{E}{c}, \vec{p}) \qquad p^i = (\frac{E}{c}, \vec{p}) \tag{1.2.19}$$

To see the variation as a function of coordinates, we set $\delta x_a^i = 0$, and denote $\delta x_b^i = \delta x^i$, then we find

$$p_i = mcu_i \tag{1.2.20}$$

where p_0 is by definition $-\frac{\partial S}{\partial x^0}$. This gives

$$p_i p^i = -m^2c^2 \tag{1.2.21}$$

that is exactly previous $E = c\sqrt{p^2 + m^2c^2}$. By using the metric tensor we can write it

$$g^i k \frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = -m^2c^2 \tag{1.2.22}$$

The classical Hamilton-Jacobi equation mentioned in Classical Mechanics is

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q}, q, t) = 0 \tag{1.2.23}$$

as a derivation of $\frac{\partial S}{\partial t} + H(p, q, t) = 0$.

Here the relativistic version goes

$$-\frac{1}{c^2}(\frac{\partial S}{\partial t})^2 + (\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2 + (\frac{\partial S}{\partial z})^2 = -m^2c^2 \tag{1.2.24}$$

thus we need to make a replacement

$$S' = S + mc^2 t \tag{1.2.25}$$

which will give $E = -\frac{\partial S}{\partial t}$, ahd the equation can be classical when $c \to \infty$

2 Tensor Analysis: Mathematical Interlude

2.1 Before Introducing the Upper and Lower Indices

First we will introduce the right-hand coordinates. They are basically coordinates with $\vec{e_1}$ pointing out from the papersheet, $\vec{e_2}$ to the right, and $\vec{e_3}$ pointing straight up. The $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are called coordinate basis, each a unit length along the axis and defined as a vector. For a particular vector with arbitrary value, we define it in form

$$\vec{a} = (a_1, a_2, a_3) := a^1\vec{e_1} + a^2\vec{e_2} + a^3\vec{e_3} = a^1 e_1 + a^2 e_2 + a^3 e_3 \tag{2.1.1}$$

where a^1, a^2, a^3 are simply numbers and hereafter we emit the vector arrow above the unit vectors to simplify the equations. When 1-forms are introduced, the vector arrow would be brought back for clearity.

The inner product, or the scalar product, the dot product, of two unit vectors are defined using the 2-order symmetric tensor

$$e_i \otimes e_j = e_i \cdot e_j := \delta_{ij} \tag{2.1.2}$$

where the $\otimes$ means to take the inner product and always denoted a $\cdot$ for simplicity.

The outer product, or the vector product, the cross product, of two unit vectors are defined by using the 3-order antisymmetric tensor

$$e_i \times e_j = \varepsilon_{ijk} e_k \tag{2.1.3}$$

above definitions pave the way to give inner product and outer product of arbitrary vectors, e.g.

$$\vec{a} \cdot \vec{b} = a_i b_j \delta_{ij} \tag{2.1.4}$$

$$\vec{a} \times \vec{b} = \varepsilon_{ijk} a_i b_j e_k \tag{2.1.5}$$

From here we can derive the scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$ as

$$\vec{a} \cdot (\vec{b} \times \vec{c}) := a_i(\vec{b} \cdot \vec{c})_i = \varepsilon_{ijk} a_i b_j c_k \tag{2.1.6}$$

which will remind us of a representation of 3-order determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k \tag{2.1.7}$$

A relationship between 2-order symmetric tensor and 3-order antisymmetric tensor is extremely useful:

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{2.1.8}$$

to be explicit, we will analyze the equation side by side.

To make the 2 tensors left side none-zero, we have $i \neq j \neq k \quad k \neq l \neq m$. To make the product of 2 tensor none-zero, we must have $i = l \quad j = m$ to make left side 1 or $i = m \quad j = l$ to make left side -1. Notice that it can not be 1 and -1 at the same time since $i \neq j$ has been mentioned. So we can represent it in $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

This gives the useful

$$\vec{a} \times (\vec{b} \times \vec{c}) = [\vec{a} \times (\vec{b} \times \vec{c})]_i \cdot e_i = e_i \varepsilon_{ijk} a_j (\vec{b} \times \vec{c})_k = e_i \varepsilon_{ijk} \varepsilon_{klm} a_j b_l c_m \tag{2.1.9}$$

and therefore

$$e_i \varepsilon_{ijk}\varepsilon_{klm} a_j b_l c_m = e_i a_j b_l c_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \tag{2.1.10}$$

2.2 Upper and Lower Indices

2.3 the Nabla Operator

The gradiant of a scalar field is defined

$$\text{grad } \varphi = e_i \cdot \frac{\partial}{\partial x^i}\varphi \tag{2.3.1}$$

and is therefore a vector

The divergence of a vector field is defined

$$\text{div } \vec{a} = \sum \frac{\partial a_i}{\partial x^i} \tag{2.3.2}$$

ans is therefore a scalar.

The curl od a vector field is defined

$$\text{curl } \vec{a} = e_i \cdot \epsilon_{ijk}\frac{\partial a^j}{\partial x^k} \tag{2.3.3}$$

and is therefore a psuedo-vector.

By this similarity with vector manipulation, it is useful to define a nabla operator, to represent the gradiant, the divergence and the curl:

$$\nabla := e_i \cdot \frac{\partial}{\partial x^i} \tag{2.3.4}$$

these can be

$$\text{grad } \varphi = \nabla\varphi \tag{2.3.5}$$

which is a vector

$$\text{div } \vec{a} = e_i \times \frac{\partial}{\partial x^i}\vec{a} = e_i \cdot \frac{\partial(a^j \cdot e_j)}{\partial x^i} = \sum \frac{\partial a^i}{\partial x^i} = \nabla \cdot \vec{a} \tag{2.3.6}$$

which is a scalar

$$\text{curl } \vec{a} = e_i \cdot \varepsilon_{ijk}\frac{\partial a^j}{\partial x^k} = \nabla \times \vec{a} \tag{2.3.7}$$

which is a psuedo-vector.

As a operator the nabla sign cannot be put on the right side of a formation. For example, a diretional derivative is a scalar's partial differential with respect to a unit vector $\vec{b}$, e.g.

$$\frac{\partial\varphi}{\partial\vec{b}} = (\vec{b} \cdot \nabla)\varphi \tag{2.3.8}$$

without φ rightside the ∇ means a operator and generates nothing at all.

After we introduced the nabla operator, we should be careful about the previous formation, since a nabla operator operates on all elements individually since it is actually a partial derivative. For example,

$$\nabla \times (\vec{a} \times \vec{b}) = e_i\varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x^j}(a_l b_m) = e_i(\frac{\partial a_l}{\partial x^j}b_m + \frac{\partial b_m}{\partial x^j}a_l)(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \tag{2.3.9}$$

and remembering that there must be some element rightside of the nabla operator, we have

$$\nabla \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} - \vec{b}(\nabla \cdot \vec{a}) + \vec{a}(\nabla \cdot \vec{b}) - (\vec{a} \cdot \nabla)\vec{b} \tag{2.3.10}$$

Hereafter some other useful formulas are:

2.4 Orthogonal Transformation

An orthogonal transformation is a transformation that satisfies

$$\vec{a} \cdot \vec{b} = a_i b_i = a_{i'} b_{i'} \tag{2.4.1}$$

There are several kinds of orthogonal transformation, like rotation(spatial), mirror reflection and (spatial) coordinate inversion.

Orthogonal means distance-preserving and the distance is given by the inner product.

2.5 Pseudotensor and Pseudovector

A psuedotensor or a psuedovector is a tensor or vector that, under coordinate inversion they change their signs, for example, the angular momentum, which is derived by the outer product of 2 vectors

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.5.1}$$

since both coordinate bases of $\vec{r}$ and $\vec{p}$ are inversed the components of them all get a minus sign and thus vanishes when forming the new components, thus pointing to a inverse direction after coordinate inversion, to keep its direction we defined a extra minus sign.

In other words, the outer product of 2 real vectors should be a psuedovector. We call the real vectors polar, and the psuedovector axial. Since a coordinates inversion can always be considered a time inversion, we will find other axial vectors under symmetries, for example, the magnetic field strength.

The tensor of electromagnetic field can be written

where we should see E is a real vector and H is a psuedovector.

2.6 Useful Lie Algebra

3 Particle in Fields and Maxwell Equations

3.1 4-Potential and Equations of Motion

For a particle moving in a given electromagnetic field, the action is made up of 2 parts: the action for a free particle, given previously, and a term describing the interaction of the particle with the field. The latter term should contain quantaties characterizing: (i) the particle, which is the charge of it, can be positive, negative or 0; (ii) the field, the 4-potential which contains all information of the electromagnetic field.

The interaction part of the action can be given by convention

$$q\int_a^b A_i dx^i \tag{3.1.1}$$

where A_i is devided into the time part– the scalar potential, and the space part– the vector potential, defined

$$A_i = (-\varphi/c, \vec{A}) \tag{3.1.2}$$

Then the action should be

$$S = \int(-mcd\tau + q\vec{A}\cdot d\vec{r} - q\varphi dt) \tag{3.1.3}$$

by introducing $d\vec{r}/dt = \vec{v}$ and changing it to an integration over t we have

$$S = \int_{t_1}^{t_2}(-mc^2\sqrt{1-\frac{v^2}{c^2}} + q\vec{A}\cdot\vec{v} - q\varphi)dt \tag{3.1.4}$$

which explicitly gives the Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + q\vec{A}\cdot\vec{v} - q\varphi \tag{3.1.5}$$

Thus, the generalized momentum $\vec{P}$ is given

$$\vec{P} = \frac{\partial L}{\partial\vec{v}} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} + q\vec{A} = \vec{p} + q\vec{A} \tag{3.1.6}$$

where $\vec{p}$ stands for the ordinary momentum of the particle. The generalized Hamiltonian is therefore

$$\mathcal{H} = \vec{v}\cdot\frac{\partial L}{\partial\vec{v}} - L = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + q\varphi \tag{3.1.7}$$

by previous $\frac{E^2}{c^2} = m^2c^2 + p^2$ we can similarly derive

$$\mathcal{H} = \sqrt{m^2c^4 + c^2(\vec{P}-q\vec{A})^2} + q\varphi \tag{3.1.8}$$

Here the momentum takes the ordinary form for reason that $m\vec{v}/\sqrt{1-\frac{v^2}{c^2}}$ is the linking arguement.

For relatively low velocities, the Lagrangian goes over into

$$L = \frac{mv^2}{2} + q\vec{A}\cdot\vec{v} - q\varphi \tag{3.1.9}$$

in this approximation

$$\vec{p} = m\vec{v} = \vec{P} - q\vec{A} \rightarrow \mathcal{H} = \frac{1}{2m}(\vec{P}-q\vec{A})^2 + q\varphi \tag{3.1.10}$$

Notice that

$$dS = Pd\vec{r} - Hdt \tag{3.1.11}$$

we can give

$$(\nabla S - q\vec{A})^2 - \frac{1}{c^2}(\frac{\partial S}{\partial t} + q\varphi)^2 + m^2c^2 = 0 \tag{3.1.12}$$

It is convenient to define a 4-momentum vector by following definitions:

$$p_i = \frac{\partial S}{\partial x^i} = (-\frac{\partial S}{c\partial t}, \nabla S) = (q, \vec{p}) \tag{3.1.13}$$

and a contravariant vector $(-q, \vec{p})$

3.2 Particle in Fields

The Lagrangian is given by definition

$$L = \frac{mv^2}{2} + q\vec{A} \cdot \vec{v} - q\varphi \tag{3.2.1}$$

and a generalized momentum

$$\frac{\partial L}{\partial \vec{r}} = \nabla L = q\nabla(\vec{A} \cdot \vec{v}) - q\nabla\varphi \tag{3.2.2}$$

for vector analysis we have

$$\nabla(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a} + \vec{b} \times \mathrm{rot}\vec{a} + \vec{a} \times \mathrm{rot}\vec{b} \tag{3.2.3}$$

for the situation above we recall a $\vec{r}$ derivative of velocity is a constant then

$$\frac{d}{dt}(\vec{p} + q\vec{A}) = \frac{\partial L}{\partial \vec{r}} = q(\vec{v} \cdot \nabla)\vec{A} + q\vec{v} \times \mathrm{rot}\vec{A} - q\nabla\varphi \tag{3.2.4}$$

where we have the total time derivative of $\vec{A}$ in 2 parts, which is

$$\frac{d\vec{A}}{dt} = \frac{\partial A}{\partial t} + (\vec{v} \cdot \nabla)\vec{A} \tag{3.2.5}$$

rearranging, we can derive the equation of motion in a electromagnetic field for a particle

$$\frac{d\vec{p}}{dt} = -q\frac{\partial \vec{A}}{\partial t} - q\nabla\varphi + q\vec{v} \times \nabla \times \vec{A} \tag{3.2.6}$$

the first two part righthand is velocity-independent while the third is velocity-dependent. We define

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi \tag{3.2.7}$$

as the electric field intensity

and

$$\vec{B} = \nabla \times \vec{A} \tag{3.2.8}$$

to be the magnetic induction intensity.

then we have

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B} \tag{3.2.9}$$

to be the equation of motion represented by field intensities.

Here we should find $\vec{E}$ a polar vector and $\vec{B}$ a axial vector in a simple way. To be time imversible, we take $t \to -t$, the motion of equation should be the same(think of the symmetry between a free fall motion and a vertical projectile motion), so that the equation should be

$$\frac{d\vec{p}}{dt} = q\vec{E} + q(-\vec{v}) \times (-\vec{B}) \tag{3.2.10}$$

since as time inverses the sign of velocity should be changed. Thus to keep the equation the same, we have to take $\vec{E} \to \vec{E} \quad \vec{H} \to -\vec{H}$, which states that the electric field strength is a polar vector and the magnetic field strength a axial vector.

3.3 Gauge Invariance

From previous derivation we learn that a difference of a total derivative has no effect on the equations of motion. So we can make replacement

$$\vec{A} \to \vec{A} + \frac{\partial f}{\partial x_i} \qquad \varphi \to \varphi - \frac{\partial f}{\partial t} \tag{3.3.1}$$

that this transformation do not change the equations of motion and therefore the field itself. All equations should be invariant under this transformation and it is called gauge invariance.

3.4 Constant Electromagnetic Fields

3.5 Electromagnetic Tensor

Since we have the action

$$S = \int -mcd\tau + qA_i dx^i \tag{3.5.1}$$

then the variation of the action is

$$\delta S = \int -mc\delta(d\tau) + qdx^i \delta A_i + qA_i \delta dx^i \tag{3.5.2}$$

and by $\delta y = \frac{dy}{dx}\delta x$, we have

$$\delta(d\tau) = \delta\sqrt{-dx^i dx_i} = \frac{d\sqrt{dx^i dx_i}}{ddx^i}\delta dx^i + \frac{d\sqrt{dx^i dx_i}}{ddx_i}\delta dx_i = -\frac{dx_i}{d\tau}\delta dx^i = -u_i \delta dx^i \tag{3.5.3}$$

so that the variation is

$$\delta S = \int mcu_i \delta dx^i + qdx^i \delta A_i + qA_i \delta dx^i \tag{3.5.4}$$

and further by integration by parts, we have

$$\delta S = mcu_i \delta x^i - \int mc\delta x^i du_i + qA_i \delta x^i - \int q\delta x^i dA_i + \int qdx^i \delta A_i \tag{3.5.5}$$

rearranging gives

$$\int (mcdu_i \delta x^i + q\delta x^i dA_i - qdx^i \delta A_i) - (mcu_i + qA_i)\delta x^i = 0 \tag{3.5.6}$$

since the boundary is fixed, $\delta x^i = 0$ must be satisfied, so the equation reduces to

$$mcdu_i\delta x^i + q\delta x^i dA_i - qdx^i\delta A_i = 0 \tag{3.5.7}$$

which is

$$mc\frac{du_i}{d\tau}d\tau\delta x^i + q\frac{\partial A_i}{\partial x^k}u^k d\tau\delta x^i - q\frac{\partial A_k}{\partial x^i}u^k d\tau\delta x^i \tag{3.5.8}$$

so it is

$$mc\frac{du_i}{d\tau} = q(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k})u^k \tag{3.5.9}$$

here we can define

$$F_{ik} \equiv (\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}) \tag{3.5.10}$$

so that the equation is

$$mc\frac{du_i}{d\tau} = qF_{ik}u^k \tag{3.5.11}$$

The tensor here is just the electromagnetic tensor defined before, which is

3.6 First Pair of Maxwell Equations

From equations of field strength

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi \tag{3.6.1}$$

we are to find equations represented only by field strength. That is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.6.2}$$

this is by 2 part: first, the order of taking the partial derivatives can be exchanged, then $\nabla\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t}\nabla\vec{A}$; second, the curl of any gradient should be 0,that is $\nabla \times (\nabla\varphi) \equiv 0$, altogether comes the equation above.

By taking divergence of both side and remembering the divergence of curl is 0, that is $\nabla \cdot (\nabla \times \vec{A}) = 0$, we have

$$\nabla \cdot \vec{B} = 0 \tag{3.6.3}$$

these are the first pair of Maxwell Equations.

It is more often to take the integration form. By Gauss's theorem

$$\int \nabla \cdot \vec{H} dV = \oint \vec{H} \cdot d\vec{S} \tag{3.6.4}$$

here $\vec{S}$ is the normal vector pointing outwards from the unit area, and the integration domain is the closed surface enclosing the entire volume. The relationship gives

$$\oint \vec{H} \cdot d\vec{S} = 0 \tag{3.6.5}$$

the equation states that

By the Stokes's theorem

$$\int \nabla \times \vec{E} \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{l} \tag{3.6.6}$$

and that we have

$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{H} \cdot d\vec{S} \tag{3.6.7}$$

3.7 the Action of a Electromagnetic Field

In SI units, we have the action of a free electromagnetic field as

$$S = -\frac{1}{4\mu_0 c}\int F^{ik}F_{ik}d^4x \tag{3.7.1}$$

and a Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0}F^{ik}F_{ik} \tag{3.7.2}$$

This is because only the field distribution should be included and the total action should be a scalar.

Since $F_{ik}F^{ik} = 2(B^2 - E^2/c^2)$, the $3D$ form is

$$\mathcal{L} = \frac{1}{2\mu_0}(E^2/c^2 - B^2) = \frac{1}{2}(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2) \tag{3.7.3}$$

The action of a free electromagnetic field is actually the field part of the system including the field and the particle. We can seperate the action into 3 parts, the particle, the field, and the interaction, which is denoted

$$S = S_m + S_f + S_{mf} \tag{3.7.4}$$

as discussed earlier, the interaction part is

$$S_{mf} = \int qA_i dx^i \tag{3.7.5}$$

and obviously the particle part

$$S_m = \int -mcd\tau \tag{3.7.6}$$

altogether gives

$$S = S_m + S_f + S_{mf} = \int -mcd\tau + \int qA_i dx^i - \int \frac{1}{4\mu_0}F^{ik}F_{ik} \tag{3.7.7}$$

Now we should introduce the 4-current vector, which is defined

$$j^i = \rho\frac{dx^i}{dt} = (c\rho, \vec{j}) \tag{3.7.8}$$

where $\rho = q_a\delta(r - r_a)$ and $dq = \rho dV$, so that

$$qA_i dx^i = \rho A_i dx^i dV = \rho\frac{dx^i}{dt}A_i dV dt = \frac{1}{c}A_i j^i d^4x \tag{3.7.9}$$

the action can be taken in the form

$$S = S_m + S_f + S_{mf} = \int -mcd\tau + \int \frac{1}{c}A_i j^i d^4x - \int \frac{1}{4\mu_0 c}F^{ik}F_{ik}d^4x \tag{3.7.10}$$

3.8 Second Pair of Maxwell Equations

Since we have

$$S = S_m + S_f + S_{mf} = \int -mcd\tau + \int \frac{1}{c} A_i j^i d^4x - \int \frac{1}{4\mu_0 c} F^{ik} F_{ik} d^4x \tag{3.8.1}$$

and $F^{ik}\delta F_{ik} = F_{ik}\delta F^{ik}$

$$\delta S = \frac{1}{c} \int (j^i \delta A_i + \frac{1}{2\mu_0} F^{ik} \delta F_{ik}) d^4x \tag{3.8.2}$$

and

$$\delta S = \frac{1}{c} \int [j^i \delta A_i + \frac{1}{2\mu_0} F^{ik} \delta (\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k})] d^4x = \frac{1}{c} \int (j^i \delta A_i - \frac{1}{\mu_0} F^{ik} \frac{\partial}{\partial x^k} \delta A_i) d^4x \tag{3.8.3}$$

by Gauss Theorem

$$F^{ik} \delta A_i dS_k = \frac{\partial}{\partial x^k} (F^{ik} \delta A_i) d^4x = \frac{\partial F^{ik}}{\partial x^k} \delta A_i d^4x + F^{ik} \frac{\partial}{\partial x^k} \delta A_i d^4x \tag{3.8.4}$$

so that

$$\delta S = \frac{1}{c} \int [(j^i \delta A_i) + \frac{1}{\mu_0} \frac{\partial F^{ik}}{\partial x^k} \delta A_i] d^4x - \frac{1}{c} \int \frac{1}{\mu_0} F^{ik} \delta A_i dS_k \tag{3.8.5}$$

$$j^i = -\frac{1}{\mu_0} \frac{\partial F^{ik}}{\partial k} \tag{3.8.6}$$

take $i = 1$, we have

4 Static Electromagnetic Fields

4.1 Classical Approach to Maxwell Equations

4.2 the Principle of Uniqueness

4.3 Method of Images

4.4 Method of Particular Solution

4.5 Green's Functions

4.6 Appendix: Laplace Operator in Different Coordinates

Laplace Operator in Polar Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \tag{4.6.1}$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \tag{4.6.2}$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \tag{4.6.3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \tag{4.6.4}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} \tag{4.6.5}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2y}{r^3}\frac{\partial r}{\partial x} = \frac{2xy}{r^4} \tag{4.6.6}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}, \quad \frac{\partial\theta}{\partial y} = \frac{x}{r^2}, \quad \frac{\partial^2\theta}{\partial y^2} = -\frac{2xy}{r^4} \tag{4.6.7}$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{4.6.8}$$

$$\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} = 0 \tag{4.6.9}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} \tag{4.6.10}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\theta}\right)\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial x^2} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial\theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&+ \left(\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right)\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}
\end{aligned} \tag{4.6.11}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\partial\theta}{\partial y}\right)^2 + \frac{\partial u}{\partial\theta}\frac{\partial^2\theta}{\partial y^2} \tag{4.6.12}$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=\frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]+2\frac{\partial^2 u}{\partial r\partial\theta}\left[\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}\right]\\
&+\frac{\partial u}{\partial r}\left[\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right]+\frac{\partial^2 u}{\partial\theta^2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right]+\frac{\partial u}{\partial\theta}\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right]\\
&=\frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]+\frac{\partial u}{\partial r}\left[\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right]\\
&+\frac{\partial^2 u}{\partial\theta^2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right] \qquad (1.2.28)
\end{aligned}
\tag{4.6.13}
$$

$$
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}\left(\frac{x^2}{r^2}+\frac{y^2}{r^2}\right)+\frac{\partial u}{\partial r}\left(\frac{x^2}{r^3}+\frac{y^2}{r^3}\right)+\frac{\partial^2 u}{\partial\theta^2}\left(\frac{x^2}{r^4}+\frac{y^2}{r^4}\right)
\tag{4.6.14}
$$

$$
\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}
\tag{4.6.15}
$$

$$
\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}
\tag{4.6.16}
$$

$$
\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}
\tag{4.6.17}
$$

Laplace Operator in Cylindrical Coordinates

$$
x=\rho\cos\phi,\quad y=\rho\sin\phi,\quad z=z
\tag{4.6.18}
$$

$$
\nabla^2 u=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}
\tag{4.6.19}
$$

$$
\nabla^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}
\tag{4.6.20}
$$

Laplace Operator in Spherical Coordinates

$$
x=r\cos\phi\sin\theta,\quad y=r\sin\phi\sin\theta,\quad z=r\cos\theta,\quad r^2=x^2+y^2+z^2
\tag{4.6.21}
$$

$$
\rho=r\sin\theta,\quad x=\rho\cos\phi,\quad y=\rho\sin\phi,\quad \rho^2=x^2+y^2
\tag{4.6.22}
$$

$$
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}
\tag{4.6.23}
$$

$$
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}
\tag{4.6.24}
$$

$$
\frac{\partial\theta}{\partial\rho}=\frac{1}{1+(\rho/z)^2}\frac{1}{z}=\frac{z}{z^2+\rho^2}=\frac{z}{r^2}=\frac{\cos\theta}{r}
\tag{4.6.25}
$$

$$
1=\frac{\partial r}{\partial\rho}\sin\theta+r\cos\theta\frac{\partial\theta}{\partial\rho}=\frac{\partial r}{\partial\rho}\sin\theta+\cos^2\theta
\tag{4.6.26}
$$

$$
\frac{\partial r}{\partial\rho}=\frac{1-\cos^2\theta}{\sin\theta}=\sin\theta
\tag{4.6.27}
$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial \rho} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial \rho} \tag{4.6.28}$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \tag{4.6.29}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \tag{4.6.30}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \tag{4.6.31}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{4.6.32}$$

4.7 Multiple Expansion

5 Electromagnetic Waves and Radiation

Part III
Quantum Mechanics

1 Schrödinger Equation, Observables and Operators

1.1 From Action to Schrödinger Equation and Wave Function

The least action principle gives

$$S = \int_{t_1}^{t_2} L(\dot{q}, q, t)dt \tag{1.1.1}$$

While a typical form of a electromagnetic wave can be written as $u = ae^{i\varphi}$ where a is the amplitude and φ is the phase.

Comparing the Fermat's principle to the typical changing phase of a wave, we may find out that S should be some of a constant times φ, while both of them represent a unified change of some kind of distance travelled. Therefore, a semi-classical wave function of a system can be somehow written as

$$\Phi = ae^{iS/\hbar} \tag{1.1.2}$$

where the reduced Planck constant is the constant mentioned.

As this form suggests, together with the former relationship $\frac{\partial S}{\partial t} = -H$ where H is the Hamiltonian we have the partial time derivative

$$\frac{\partial \Phi}{\partial t} = \frac{i}{\hbar}\frac{\partial S}{\partial t}\Phi = -\frac{i}{\hbar}H\Phi \tag{1.1.3}$$

which implies the wave function a part

$$\Phi = a \cdot f(S) \cdot e^{-\frac{i}{\hbar}E \cdot t} \tag{1.1.4}$$

where E is the energy of the wave.

Also take former $\frac{\partial S}{\partial q_i} = p_i$, if we denote $\nabla = \sum \frac{\partial}{\partial q_i}$ when i runs from 1 to 3, we have $\nabla S = \vec{p}$, which will give

$$\nabla \Phi = \frac{i}{\hbar}\nabla S \Phi = \frac{i}{\hbar}\vec{p}\Phi \tag{1.1.5}$$

with $H = \frac{p^2}{2\mu}$(Attention: only for semi-classical particles), it gives

$$-\frac{\hbar^2}{2\mu}\nabla^2\Phi = H\Phi \tag{1.1.6}$$

which implies the wave function a part

$$\Phi = a \cdot g(S) \cdot e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \tag{1.1.7}$$

here we use μ to denote mass.

Altogether we have the complete Schrödinger Equation of a free semi-classical particle

$$-\frac{\hbar^2}{2\mu}\nabla^2\Phi = i\hbar\frac{\partial}{\partial t}\Phi \tag{1.1.8}$$

and a form of semi-classical wave

$$\Phi = a \cdot e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - E\cdot t)} \tag{1.1.9}$$

For a particle in potential, the E stands for its total energy, thus the equation should be changed into

$$-\frac{\hbar^2}{2\mu}\nabla^2\Phi + V\Phi = i\hbar\frac{\partial}{\partial t}\Phi \tag{1.1.10}$$

where the V given is simply the function of potential.

By the relativistic form of Action, we come up with the Klein-Gordon equation

1.2 Normalization and Statistical Interpretation

It is obvious that a wave function times a constant fits the same Schrödinger Equation. By Born's statistical interpretation, in the Representation of space, the inner product of the wave function gives the probability of observing a particle at one point. Writing out explicitly, the probability of finding the particle between some point a and b, at time t, is precisely

$$\int_a^b \Phi^*(x,t)\Phi(x,t)dx \tag{1.2.1}$$

By convention the sum of all probabilities should be 1. This is called the normalization of wave function. Take the constant a, it should give

$$\int_{-\infty}^{\infty} a\Phi^* \cdot a\Phi dx = a^2 \int_{-\infty}^{\infty} \Phi^* \cdot \Phi dx = 1 \tag{1.2.2}$$

which states that a physically realizable pure state corresponds to the sqaure-integrable solutions to Schrödinger Equation. A typical property is that, when $x \to \infty$, $\Phi \to 0$. Scattering and periodical wave function will be later discussed.

By Born's statistical interpretation, we can also apply what is called the superposition principle: for a state $a\psi_1 + b\psi_2$, it means that there are of probability a^2 to be in state ψ_1 and of probability b^2 to be in state ψ_2, and which state the particle is in is judged by the detected eigenvalue and really is a part of the expected value.

Here we find a semi-classical particle $a = 1$. The constant is therefore called the normalization constant, which is always taken real for simplicity, since the difference can be up to a phase factor. As another important example, the classical 1D Gaussian wave package is given

$$\phi(x,t) = Aexp(-\frac{1}{2}ax^2)e^{-\frac{i}{\hbar}E\cdot t} \tag{1.2.3}$$

$$\int_{-\infty}^{\infty} \phi^*\phi dx = A^2\sqrt{\frac{\pi}{a}} \to A = (\frac{a}{\pi})^{1/4} \tag{1.2.4}$$

From the form $e^{-\frac{i}{\hbar}E\cdot t}$, we find its inner product equals to 1, which is a property will later be defined unitarity. The form is defined a time evolution operator, turns any stationary state into a time-dependent one; which is to say time as a independent variant only appears here. Then, it is easy to give

$$\frac{d}{dt}\int_{-\infty}^{\infty} \phi^*(x,t)\phi(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \phi^*(x,t)\phi(x,t) = 0 \tag{1.2.5}$$

That is to say, when a system is normalized at a specific time, it keeps normalized all the time. This is called the probability conservation law. That's also why many consider unitarity equlvalent to probability conservation.

Another method to approach probability conservation will be given as follow. Here we will define the continuity function, which is similar to the one in Electromagnetism.

First, we write down the Schrödinger equation in the form of

$$\frac{\partial \Psi}{\partial t} = \frac{\mathrm{i}\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\mathrm{i}}{\hbar}V(x)\Psi \tag{1.2.6}$$

else we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \mid \Psi(x,t) \mid^2 \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \mid \Psi(x,t) \mid^2 \mathrm{d}x \tag{1.2.7}$$

and, obviously,

$$\frac{\partial}{\partial t} \mid \Psi \mid^2 = \frac{\partial}{\partial t}(\Psi^* \Psi) = \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi \tag{1.2.8}$$

with the complex conjugate

$$\frac{\partial \Psi^*}{\partial t} = -\frac{\mathrm{i}\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{\mathrm{i}}{\hbar}V(x)\Psi^* \tag{1.2.9}$$

here we assume the potential function $V(x)$ is real. In other words, complex potential function can be used to represent particle creation and annihilation, in later chapters we will see how exactly it works. Now here we get

$$\frac{\partial}{\partial t} \mid \Psi \mid^2 = \frac{\mathrm{i}\hbar}{2m}\Big(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\Big) = \frac{\partial}{\partial x}\Big[\frac{\mathrm{i}\hbar}{2m}\Big(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\Big)\Big] \tag{1.2.10}$$

we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \mid \Psi(x,t) \mid^2 \mathrm{d}x = \frac{\mathrm{i}\hbar}{2m}\left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right]_{-\infty}^{\infty} = 0 \tag{1.2.11}$$

where we have used the boundary condition from infinity. Here this equation shows that once the wave function is normalized, it would keep normalized all the time. To make clear its physical meaning, we further define

$$\rho \equiv \mid \Psi \mid^2 \tag{1.2.12}$$

and

$$J \equiv -\frac{\mathrm{i}\hbar}{2m}\Big(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\Big) \tag{1.2.13}$$

so that previous equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{1.2.14}$$

which is exactly the continuity equation. With what we previously defined that the inner product of wave function is the probability density at certain position, we name the defined J the current density of probability. This would be very useful when we come to barrier penetration.

1.3 Operators and Eigenvalues

With Born's statistical interpretation, we may easily find out by definition, that the expected value of a physical quantity Y is

$$\langle Y \rangle = \int_{-\infty}^{\infty} Y(x)\phi^* \phi dx \tag{1.3.1}$$

We have Ehrenfest's theorem which states that the expected value of a quantum system obeys classcial rules, which means, an approach to degrade from quantum system to classical system is to take its expected values.

Here we consider the situation where the wave functions are described by a set of close set.

The values which a given physical quantity can take are called eigenvalues, and the set is called the spectrum of eigenvalues of the given quantity. Those spectra with discrete eigenvalues are called discrete spectra.

Here we consider the discrete spectrum at first. We use ψ_n to represent the n-th wave function, each normalized, with determined eigenvalue f_n, and is therefore called eigenfuction:

$$\int \psi_n^* \psi_n dq = 1 \tag{1.3.2}$$

and by the superposition principle we realize that all wave function can be expressed in a superposition of all n eigenfuctions, which is

$$\Psi = a_n \psi_n \tag{1.3.3}$$

here the same index infers summation.

As a conclusion, any wave function can be expanded in terms of the eigenfuctions of any physical quantity. A set of functions in terms of which such an expansion can be made is called a complete set or a closed set.

Again, by Born's statistical interpretation, the sum fo all probabilities should be 1, which is

$$a_n^* a_n = 1 = \int \Psi^* \Psi dq = 1 \tag{1.3.4}$$

it further gives

$$\int \Psi^* \Psi dq = a_n^* \int \Psi_n^* \Psi dq = a_n^* a_n \tag{1.3.5}$$

that is

$$a_n = \int \Psi \Psi_n^* dq \tag{1.3.6}$$

which gives the coefficients a_n of the expansion of eigenfuctions.

Further,

$$a_n = a_m \int \Psi_m \Psi_n^* dq \quad \rightarrow \quad \int \Psi_m \Psi_n^* dq = \delta_{mn} \tag{1.3.7}$$

which is called the orthogonality of eigenfuctions. Thus the eigenfuctions are called orthonormal functions.

By this we give the expected value of a physical quantity f, which is

$$\bar{f} = f_n a_n a_n^* \tag{1.3.8}$$

Here we should define the opreator, which gives

$$\bar{f} = \int \Psi^* \hat{F} \Psi dq \tag{1.3.9}$$

we can soon derive

$$\bar{f} = f_n a_n a_n^* = \int \Psi^* a_n f_n \Psi_n dq \tag{1.3.10}$$

which gives

$$\hat{F}\Psi = a_n f_n \Psi_n \tag{1.3.11}$$

and of course

$$\hat{F}\Psi_n = f_n \Psi_n \tag{1.3.12}$$

since Ψ and Ψ^* can be interchanged, we find all operators corresponding to a real physical quantity should be the complex conjugate of it self. The complex conjucate is always denoted with a dagger †, so that a operator corresponding to a physical quantity is

$$\hat{F}^{\dagger} = \hat{F} \tag{1.3.13}$$

The commutator of operators is

$$[\hat{F}, \hat{G}] = \hat{F}\hat{G} - \hat{G}\hat{F} \tag{1.3.14}$$

1.4 Continuous Spectrum

We take momentum spectrum as a example of continuous spectra.
First, we will deal with the eigenfuction of momentum operator, which is

$$-i\hbar\nabla\psi_p(r) = \hat{p}\psi_p(r) \tag{1.4.1}$$

which, seperating its variables gives

$$-i\hbar\frac{\partial}{\partial x_i}\psi_{p_i}(x_i) = p_i\psi_{p_i}(x_i) \tag{1.4.2}$$

and their solution

$$\psi_{p_i}(x_i) = A_i \exp(\frac{i}{\hbar}p_i x_i) \tag{1.4.3}$$

where A_i are the constants to be determined, altogether

$$\psi_p(\vec{r}) = A\exp(\frac{i}{\hbar}\vec{p}\cdot\vec{r}) \tag{1.4.4}$$

here $A = \prod A_i$.
The equation thus imply a probability density

$$|\psi_p(r)|^2 = \left|A\exp\left(\frac{\mathrm{i}}{\hbar}p\cdot r\right)\right|^2 = |A|^2 = const \tag{1.4.5}$$

which is the same in all position. We soon derive

$$\begin{aligned}
\int_{-\infty}^{\infty}\psi^*_{p'}(x)\psi_p(x)\mathrm{d}x &= \int_{-\infty}^{\infty}A^*\exp\left(-\frac{\mathrm{i}}{\hbar}p'x\right)A\exp\left(\frac{\mathrm{i}}{\hbar}px\right)\mathrm{d}x \\
&= |A|^2\int_{-\infty}^{\infty}\exp\left[\frac{\mathrm{i}}{\hbar}(p-p^{'})x\right]\mathrm{d}x \\
&= \quad |A|^2[2\pi\hbar\delta(p-p')] \Bigm| \left[\text{with } A = \tfrac{1}{\sqrt{2\pi\hbar}}\right] \\
&= \delta(p-p')
\end{aligned} \tag{1.4.6}$$

Here we use the common relationship

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}px}dp \tag{1.4.7}$$

since

$$\int_{-\infty}^{\infty}\delta(x)e^{-\mathrm{i}px}dx = 1 \tag{1.4.8}$$

so that the normalized eigenfuctions are

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{\mathrm{i}}{\hbar}px\right) \tag{1.4.9}$$

This is the orthogonormal equation of momentum eigenfuctions, which represented by Dirac delta function rather than the Kronecker delta function we used for discrete spectrum. Since momentum and position is symmetric in the equation above, we can write down

$$\begin{aligned}\int_{-\infty}^{\infty}\psi_p^*(x^{'})\psi_p(x)\mathrm{d}p &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\Big(-\frac{\mathrm{i}}{\hbar}px'\Big)\exp\Big(\frac{\mathrm{i}}{\hbar}px\Big)\mathrm{d}p \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\Big[\frac{\mathrm{i}}{\hbar}(x-x')p\Big]\mathrm{d}p \\ &= \delta(x-x')\end{aligned} \tag{1.4.10}$$

1.5 Dirac Notation

The Dirac notation is used to simplify inner products as well to show important properties of Quantum Mechanics.

Basic Forms

• A wave function Φ is denoted $|\Phi\rangle$. The complex conjugate is therefore $\langle\Phi|$.

• A eigenstate, or a eigenfuction, is denoted $|\phi_n\rangle$, and called a eigenvector. Its orthonormality is represented

$$\int \phi_m^*\phi_n dq = \langle\phi_m|\phi_n\rangle = \delta_{mn} \tag{1.5.1}$$

or, even more concisely,

$$\langle m|n\rangle = \delta_{mn} \tag{1.5.2}$$

for continuous spectrum, it is

$$\int \phi_\lambda^*\phi_{\lambda'} dq = \langle\phi_\lambda|\phi_{\lambda'}\rangle = \delta(\lambda-\lambda') \tag{1.5.3}$$

• The inner product of 2 arbitrary eigenstates is

$$\int \Psi^*\Phi dq = \langle\Psi|\Phi\rangle \tag{1.5.4}$$

• Taking a inner product is taking a projection from one eigenvector to another.

• The expected value of a operator is

$$\int \Phi^*\hat{F}\Phi dq = \langle\Phi|\hat{F}|\Phi\rangle \equiv \langle\hat{F}\rangle \tag{1.5.5}$$

• The projection operator is defined

$$|n\rangle\langle n| \tag{1.5.6}$$

and by applying it we have

$$|n\rangle\langle n|\Phi\rangle = \langle n|\Phi\rangle|n\rangle \tag{1.5.7}$$

1.6 Uncertainty Principle and CSCO

Uncertainty Principle
We define a commutation relationship

$$[A, B] = \mathrm{i}\hbar C \tag{1.6.1}$$

and still we use $\langle A\rangle$ to denote the expected value of a operator.
To simplify the equations, we define the deviation operator

$$\delta A \equiv A - \langle A\rangle, \quad \delta B \equiv B - \langle B\rangle \tag{1.6.2}$$

and easily we find

$$\begin{aligned}
[\delta A, \delta B] &= \delta A\delta B - \delta B\delta A \\
&= (A - \langle A\rangle)(B - \langle B\rangle) - (B - \langle B\rangle)(A - \langle A\rangle) \\
&= (AB - \langle A\rangle B - A\langle B\rangle + \langle A\rangle\langle B\rangle) - (BA - \langle B\rangle A - B\langle A\rangle + \langle B\rangle\langle A\rangle) \\
&= AB - BA = \big[A, B\big] = \mathrm{i}\hbar C
\end{aligned} \tag{1.6.3}$$

Now we construct

$$I(\beta) = \int \mid (\beta\delta A - \mathrm{i}\delta B)\Psi \mid^2 \mathrm{d}\tau \geqslant 0 \tag{1.6.4}$$

expanding it gives

$$\begin{aligned}
I(\beta) &= \int \mid (\beta\delta A - \mathrm{i}\delta B)\Psi \mid^2 \mathrm{d}\tau \\
&= \int \Psi^* \left[\beta\,(\delta A)^\dagger + \mathrm{i}\,(\delta B)^\dagger\right](\beta\delta A - \mathrm{i}\delta B)\,\Psi \mathrm{d}\tau
\end{aligned} \tag{1.6.5}$$

and is further

$$\begin{aligned}
&\int \Psi^*\,(\beta\delta A + \mathrm{i}\delta B)\,(\beta\delta A - \mathrm{i}\delta B)\,\Psi\mathrm{d}\tau \\
&= \beta^2 \int \Psi^*(\delta A)^2\Psi\mathrm{d}\tau - \mathrm{i}\beta \int \Psi^*(\delta A\delta B - \delta B\delta A)\Psi\mathrm{d}\tau + \int \Psi^*(\delta B)^2\Psi\mathrm{d}\tau \\
&(\delta A\delta B - \delta B\delta A = \mathrm{i}\hbar C) \\
&= \beta^2\langle\Psi \mid (\delta A)^2 \mid \Psi\rangle + \beta\langle\Psi \mid \hbar C \mid \Psi\rangle + \langle\Psi \mid (\delta B)^2 \mid \Psi\rangle \\
&= \beta^2\langle(\delta A)^2\rangle + \hbar\beta\langle C\rangle + \langle(\delta B)^2\rangle
\end{aligned} \tag{1.6.6}$$

Now we have

$$\beta^2\langle(\delta A)^2\rangle + \hbar\beta\langle C\rangle + \langle(\delta B)^2\rangle \geqslant 0 \tag{1.6.7}$$

to make this correct, we must have

$$\hbar^2\langle C\rangle^2 - 4\langle(\delta A)^2\rangle\langle(\delta B)^2\rangle \leqslant 0 \tag{1.6.8}$$

which is

$$\langle(\delta A)^2\rangle\langle(\delta B)^2\rangle \geqslant \frac{\hbar^2\langle C\rangle^2}{4} \tag{1.6.9}$$

if we consider

$$\langle(\delta A)^2\rangle = \langle(A - \langle A\rangle)^2\rangle = \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle = \langle A^2\rangle - \langle A\rangle^2 \tag{1.6.10}$$

we are to define

$$\Delta A \equiv \sqrt{\langle A^2\rangle - \langle A\rangle^2}, \quad \Delta B \equiv \sqrt{\langle B^2\rangle - \langle B\rangle^2} \tag{1.6.11}$$

so that

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2, \quad (\Delta B)^2 = \langle B^2\rangle - \langle B\rangle^2 \tag{1.6.12}$$

we finally get

$$\Delta A \Delta B \geqslant \frac{\hbar\langle C\rangle}{2} \tag{1.6.13}$$

which is exactly the uncertainty principle of Hermitian operators.
Completeness of Eigenvectors

1.7 The Evolution of Expected Value

We take the expected value as

$$\langle \hat{F}\rangle = \langle \Psi|\hat{F}|\Psi\rangle \tag{1.7.1}$$

and now focusing on how the system evolves. First it is

$$\frac{\mathrm{d}\langle \hat{F}\rangle}{\mathrm{d}t} = \langle \mathbf{\Psi} \mid \hat{T}^{+}\hat{F} \mid \mathbf{\Psi}\rangle + \left\langle \mathbf{\Psi} \middle| \frac{\partial \hat{F}}{\partial t} \middle| \mathbf{\Psi} \right\rangle + \langle \mathbf{\Psi} \mid \hat{F}\hat{T} \mid \mathbf{\Psi}\rangle \tag{1.7.2}$$

where

$$\hat{T} = \frac{\partial}{\partial t}, \quad \hat{T}^{+} = \left(\frac{\partial}{\partial t}\right)^{+} \tag{1.7.3}$$

so that the Schrödinger equation should be

$$\hat{T} \mid \Psi\rangle = -\frac{\mathrm{i}}{\hbar}\hat{H} \mid \Psi\rangle, \quad \langle \Psi \mid \hat{T}^{+} = \frac{\mathrm{i}}{\hbar}\langle \Psi \mid \hat{H} \tag{1.7.4}$$

$$\begin{aligned}
\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}t} &= \frac{\mathrm{i}}{\hbar}\langle \mathbf{\Psi}|\hat{H}\hat{F}|\mathbf{\Psi}\rangle + \left\langle \mathbf{\Psi} \middle| \frac{\partial \hat{F}}{\partial t} \middle| \mathbf{\Psi} \right\rangle - \frac{\mathrm{i}}{\hbar}\langle \mathbf{\Psi}|\hat{F}\hat{H}|\mathbf{\Psi}\rangle \\
&= \frac{\mathrm{i}}{\hbar}\langle \Psi \mid \hat{H}\hat{F} - \hat{F}\hat{H} \mid \Psi\rangle + \left\langle \Psi \middle| \frac{\partial \hat{F}}{\partial t} \middle| \Psi \right\rangle \\
&= \frac{\mathrm{i}}{\hbar}\left\langle \left[\hat{H}, \hat{F}\right] \right\rangle + \left\langle \frac{\partial \hat{F}}{\partial t} \right\rangle
\end{aligned} \tag{1.7.5}$$

if the operator is not time dependent, then $\frac{\partial \hat{F}}{\partial t} = 0$, so that we get

$$\frac{\mathrm{d}\langle \hat{F}\rangle}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\langle [\hat{H}, \hat{F}]\rangle \tag{1.7.6}$$

which is called the Heisenberg equation of motion.
further, if the operator commutes with H, we have

$$\frac{\mathrm{d}\langle \hat{F}\rangle}{\mathrm{d}t} = 0 \tag{1.7.7}$$

and we call the corresponding mechanical quantity conservative under motion or simply a conserved quantity.

A very useful result is the Virial Theorem. If we take $\hat{F} = xp$, then by plugging in the commutation relationship we have

$$\frac{\mathrm{d}\langle xp\rangle}{\mathrm{d}t} = 2\langle T\rangle - \langle x\frac{\mathrm{d}V}{\mathrm{d}x}\rangle \tag{1.7.8}$$

which in its stationary state that

$$2\langle T\rangle = \langle x\frac{\mathrm{d}V}{\mathrm{d}x}\rangle \tag{1.7.9}$$

1.8 Different Pictures

1.9 Appendix: Fourier Transform and Delta Function

2 1D Application

2.1 Prepositions

There are 9 prepositions which will help us simplify the problem in 1D applications. Before actually stating these prepositions, we need to introduce the Wronskian.

Take ψ_1, ψ_2 as 2 functions, and their Wronskian is defined to be

$$W(\psi_1, \psi_2) = \psi_1\psi_2' - \psi_1'\psi_2 \tag{2.1.1}$$

which has following properties:
- asymmetry: $W(\psi_1, \psi_2) = -W(\psi_2, \psi_1)$
- bi-linearity: $W(\psi_1, C_2\psi_2 + C_3\psi_3) = C_2W(\psi_1, \psi_2) + C_3W(\psi_1, \psi_3)$
- Jacobi identity: $W(\psi_1, W(\psi_2, \psi_3)) + W(\psi_2, W(\psi_3, \psi_1)) + W(\psi_3, W(\psi_1, \psi_2)) = 0$

the Jacobi identity can be easily checked by expanding the Wronskian.

The Wronskian is used to check if 2 functions are linerarly independent. If 2 functions are linerarly dependent, then their Wronskian must be 0.

Preposition 1

If both ψ_1, ψ_2 are solutions for a stationary Schrödinger equation, then

$$W(\psi_1, \psi_2) \equiv \phi_1\phi_2' - \phi_2\phi_1' = const \tag{2.1.2}$$

Proof

$$\psi_1'' + C\psi_1 = 0 \to \psi_1''\psi_2 + \psi_1\psi_2 = 0 \tag{2.1.3}$$

$$\psi_2'' + C\psi_2 = 0 \to \psi_2''\psi_1 + \psi_1\psi_2 = 0 \tag{2.1.4}$$

$$\psi_2''\psi_1 - \psi_1''\psi_2 = (\psi_1\psi_2' - \psi_2\psi_1')' = W'(\psi_1, \psi_2) = 0 \tag{2.1.5}$$

Preposition 2

We call the number of independent solutions of the stationary Schrödinger equation a degeneracy, $D(E)$. Then we have

$$D(E) \leqslant 2 \tag{2.1.6}$$

Proof by contradiction. Let there be 3 independent solutions, from preposition 1,

$$W(\psi_1, \psi_2) = C_2, \quad W(\psi_1, \psi_3) = C_3 \tag{2.1.7}$$

then

$$0 = C_3W(\psi_1, \psi_2) - C_2W(\psi_1, \psi_3) = W(\psi_1, C_3\psi_2 - C_2\psi_3) \tag{2.1.8}$$

which states that the 3 solutions are not independent.

Preposition 3

For a given energy E, the real and the imaginary part of its solution are solutions respectively.

Proof for $\psi = u + iv$, u, v both real functions, the Schrödinger equation is given

$$(n'' + iv'') + C(u + iv) = 0 \tag{2.1.9}$$

seperating the real and the imaginary part gives the Schrödinger equation.

Preposition 4
If a particle only move in finite space, the state of the particle is called a bound state. For a 1D bound state,

$$D(E) = 1 \tag{2.1.10}$$

Proof at infinity

$$\psi_1 = 0, \psi_2 = 0 \rightarrow W(\psi_1, \psi_2) = const = 0 \tag{2.1.11}$$

since the Wronskian is a constant by preposition 1, if it is 0 at infinity, it is 0 everywhere.

Preposition 5
1D bound states can be represented by a real wave function.
Proof for $\psi = u + iv$, u, v both real functions, from preposition 3, they are the solutions of a same state, and by preposition 4, they must be linearly dependent, which is $v = Cu$, then C can be chosen 0 so that the function can be real.

Preposition 6
The energy of a particle in a bound state is not less than the potential minimum.
Proof

$$E = \int dx\psi \frac{d\psi}{dx}|_{-\infty}^{\infty} + \frac{\hbar^2}{2m}\int dx(\psi')^2 + \int dx V\psi^2 \geqslant V_{min}\int dx^2\psi \geqslant V_{min} \tag{2.1.12}$$

Preposition 7
1D bound states that correspond to different energy are orthogonal to each other.
Proof
Let ψ_1 and ψ_2 be states with energy E_1 and E_2, which is

$$\psi_1'' + \frac{2m}{\hbar^2}(E_1 - V)\psi_1 = 0 \tag{2.1.13}$$

$$\psi_2'' + \frac{2m}{\hbar^2}(E_1 - V)\psi_2 = 0 \tag{2.1.14}$$

and is suggested by preposition 5 to be real functions. Using the similar method used in preposition 1, we get

$$W(\psi_1, \psi_2)' = \frac{2m}{\hbar^2}(E_2 - E_1)\psi_1\psi_2 \tag{2.1.15}$$

then

$$\frac{2m}{\hbar^2}(E_2 - E_1)\int dx\psi_1\psi_2 = \int dx W' = W|_{\infty}^{\infty} = 0 \tag{2.1.16}$$

with the boundary condition. When the energy is different, we have

$$(E_2 - E_1) \neq 0 \quad \rightarrow \quad \int dx\psi_1\psi_2 = 0 \tag{2.1.17}$$

Preposition 8
If the system have a symmetric potential, which is $V(-x) = V(x)$, then bound states have a determined parity, either a symmetric wave function or a asymmetric one.
Proof

Preposition 9
Bound state node theorem:
- states with no node corresponds to energy minimum.
-
-

2.2 Bound States: Square Well

As a clear conclusion, the condition to determine the wave function is as follow:

• The amplitude should be, at least asymtotically, 0 at infinity, since the wave function is square integrable, and the integration of all squared amplitude should be normalized to 1.

• The wave function should be continuous.

• The first derivative of the wave function should be continuous, if there is no delta function locally since the second derivative represents energy and must exist.

the Infinite Square Well

The potential field can be given

$$V(x) = \begin{cases} 0, & (0 < x < a) \\ \infty, & (x \leqslant 0 \ or \ x \geqslant a) \end{cases} \tag{2.2.1}$$

the static Schrödinger equation goes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = E\psi \tag{2.2.2}$$

denote $k = \sqrt{\frac{2mE}{\hbar^2}}$, we have the differential equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0 \tag{2.2.3}$$

the general solution should be

$$\psi(x) = A\sin(kx) + B\cos(kx) \tag{2.2.4}$$

by applying $\phi(0) = 0$, which is the boundary condition given by the potential field, we have $B = 0$, and the solution is left to $\psi(x) = A\sin(kx)$.

Another boundary condition is $\phi(a) = 0$, which gives $k = \frac{n\pi}{a}$, and we have the eigenvalue of energy

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{2.2.5}$$

To determine a, we can integrate

$$\int_0^a A^2\sin^2\left(\frac{n\pi}{a}x\right)\mathrm{d}x = A^2\frac{a}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{a}} \tag{2.2.6}$$

and here

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{2m}{\hbar^2}\frac{p^2}{2m}} = \frac{p}{\hbar} = \frac{1}{\hbar}\frac{h}{\lambda} = \frac{2\pi}{\lambda} \tag{2.2.7}$$

we find k is the de Brogile wave number.

Semi-infinite Square Well

The potential of a semi-infinite square well is

$$V(x) = \begin{cases} \infty, & (x \leqslant 0) \\ 0, & (0 < x < a) \\ V_0, & (x \geqslant a) \end{cases} \tag{2.2.8}$$

Here we will focus on the bound state, so we have $0 < E < V_0$. Inside the well where $0 < x < a$, we have

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\phi = 0 \tag{2.2.9}$$

where $k^2 = 2mE/\hbar^2$.

We again get

$$\psi(x) = A\sin(kx) \tag{2.2.10}$$

but here A is not determined and is correlated with the new boundary condition.

Now for $x \geqslant a$, we have

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + V_0\phi = E\phi \tag{2.2.11}$$

which can be written

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - K^2\phi = 0 \tag{2.2.12}$$

and here $K^2 = 2m(V_0 - E)/\hbar^2$. The minus sign is introduced to make sure K is positive. So the general solution here should be

$$\phi(x) = \mathrm{C}\exp(-Kx) + \mathrm{D}\exp(Kx) \tag{2.2.13}$$

and the boundary condition suggests

$$A\sin(ka) = \mathrm{C}\exp(-Ka), \quad Ak\cos(ka) = -CK\exp(-Ka) \tag{2.2.14}$$

which is a set of linear homogeneous equations with determinant

$$\begin{vmatrix} \sin(ka) & -\exp(-Ka) \\ k\cos(ka) & K\exp(-Ka) \end{vmatrix} = 0 \tag{2.2.15}$$

expanding it gets what we call it a secular equation

$$[K\sin(ka) + k\cos(ka)]\exp(-Ka) = 0 \tag{2.2.16}$$

that is a transcendental equation

$$\cot(ka) = -\frac{K}{k} \tag{2.2.17}$$

since the exponential part cannot be 0.

To solve the transcendental equation, we must apply the graphical method.

the Symmetric Finite Square Well

The potential of a symmetric finite square well is

$$V(x) = \begin{cases} V_0, & (x \leqslant 0) \\ 0, & (0 < x < a) \\ V_0, & (x \geqslant a) \end{cases} \tag{2.2.18}$$

which we will have

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} - K^2\psi_1 = 0 (x \leqslant 0) \tag{2.2.19}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} + k^2\psi_2 = 0 (0 < x < a) \tag{2.2.20}$$

$$\frac{\mathrm{d}^2\psi_3}{\mathrm{d}x^2} - K^2\psi_3 = 0(x \geqslant a) \tag{2.2.21}$$

with solution in each area should be

$$\psi_1(x) = C\exp(Kx) \tag{2.2.22}$$

$$\psi_2(x) = A\sin(kx) + B\cos(kx) \tag{2.2.23}$$

$$\psi_3(x) = \mathrm{D}\exp(-Kx) \tag{2.2.24}$$

the boundary condition gives

$$C = B, \quad KC = kA \tag{2.2.25}$$

and

$$A\sin(ka) + B\cos(ka) = \mathrm{D}\exp(-Ka), kA\cos(ka) - kB\sin(ka) = -KD\exp(-Ka) \tag{2.2.26}$$

we will get

$$(K^2 - k^2)\sin(ka) + 2kK\cos(ka) = 0 \tag{2.2.27}$$

the Asymmetric Finite Square Well
The potential asymmetric square well is

$$V(x) = \begin{cases} V_1 & (x\leqslant 0) \\ 0 & (0<x<a) \\ V_0 & (x\geqslant a) \end{cases} \tag{2.2.28}$$

2.3 Scattering

Step Potential
Consider a step potential

$$V(x) = \begin{cases} 0, & (x < 0) \\ V_0, & (x \geqslant 0) \end{cases} \tag{2.3.1}$$

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} + k^2\psi_1 = 0 \quad (x < 0) \tag{2.3.2}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} + \rho^2\psi_2 = 0 \quad (x \geqslant 0) \tag{2.3.3}$$

where

$$k^2 = \frac{2mE}{\hbar^2}, \quad \rho^2 = \frac{2m\,(E - V_0)}{\hbar^2} \tag{2.3.4}$$

$$\psi_1(x)=A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-ikx} \tag{2.3.5}$$

$$\psi_2(x) = \mathrm{C}\mathrm{e}^{\mathrm{i}\rho x} + D\mathrm{e}^{-\mathrm{i}\rho x} \tag{2.3.6}$$

where at $x \geqslant 0$ there should be no reflected wave so that D=0.
Using the boundary condition, we soon get

$$\frac{B}{A} = \frac{k - \rho}{k + \rho}, \quad \frac{C}{A} = \frac{2k}{k + \rho} \tag{2.3.7}$$

Now we come to calculate the current density of probability at $x < 0$ to resolve the reflection coefficient. By definition

$$\begin{aligned} J_1(x) &= -\frac{\mathrm{i}\hbar}{2m}\left(\varphi_1^* \frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1^*}{\partial x}\psi_1\right) \\ &= -\frac{\mathrm{i}\hbar}{2m}\big\{\big[A^*\mathrm{e}^{-ikx}(\mathrm{i}kA\mathrm{e}^{\mathrm{i}k}) - (-\mathrm{i}kA^*\mathrm{e}^{-ikx})A\mathrm{e}^{\mathrm{i}kx}\big] \\ &- \big[B^*\mathrm{e}^{\mathrm{i}kx}(-\mathrm{i}kB\mathrm{e}^{-\mathrm{i}kx}) - (\mathrm{i}kB^*\mathrm{e}^{ikx})B\mathrm{e}^{-ikx}\big]\big\} \\ &= \frac{\hbar k}{m}(|A|^2 - |B|^2) \end{aligned} \tag{2.3.8}$$

here the two items are the current density of probability of incident wave and reflected wave respectively. Thus, the reflection coefficient is

$$R = \frac{\frac{\hbar k}{m} \mid B \mid^2}{\frac{\hbar k}{m} \mid A \mid^2} = \frac{\mid B \mid^2}{\mid A \mid^2} = \frac{(k - \rho)^2}{(k + \rho)^2} = 1 - \frac{4k\rho}{(k + \rho)^2} \tag{2.3.9}$$

thus obviously the current density of probability for outgoing wave is therefore $\frac{4k\rho}{(k+\rho)^2}$. Still, we can calculate it explicitly by writing out

$$\begin{aligned} J_2\,(x) &= -\frac{\mathrm{i}\hbar}{2m}\left(\psi_2^* \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_2^*}{\partial x}\psi_2\right) \\ &= -\frac{\mathrm{i}\hbar}{2m}\left[\mathrm{C}^*\mathrm{e}^{-\mu\pi}(\mathrm{i}\rho\mathrm{C}\mathrm{e}^{\mu\pi}) - (-\mathrm{i}\rho\mathrm{C}^*\mathrm{e}^{-\mu\pi})\mathrm{C}\mathrm{e}^{\mu\pi}\right] \\ &= \frac{\hbar\rho}{m} \mid C \mid^2 \end{aligned} \tag{2.3.10}$$

at $x > 0$, so that the outgoing coefficient should be

$$T = \frac{\frac{\hbar\rho}{m} \mid C \mid^2}{\frac{\hbar k}{m} \mid A \mid^2} = \frac{\rho \mid C \mid^2}{k \mid A \mid^2} = \frac{\rho}{k}\frac{(2k)^2}{(k + K)^2} = \frac{4k\rho}{(k + \rho)^2} \tag{2.3.11}$$

so that

$$R + T = 1 \tag{2.3.12}$$

stands for the probability current conservation.
To find out the direct relationship between E, V_0 and R, we can have the previous equation in the form

$$R = 1 - \frac{4\sqrt{1 - \frac{V_0}{E}}}{\left(1 + \sqrt{1 - \frac{V_0}{E}}\right)^2} \tag{2.3.13}$$

it is clear that when $E > V_0$, or to say, $V_0/E < 1$, the greater V_0/E become, the particle is more likely to be reflected. When it equals 1 the particle is completely reflected.
for $E < V_0$, the wave function becomes

$$\psi_1(x) = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}, \qquad \psi_2(x) = D\mathrm{e}^{-Kx} \tag{2.3.14}$$

here

$$k^2 = \frac{2mE}{\hbar^2}, \quad K^2 = \frac{2m(V_0 - E)}{\hbar^2} \tag{2.3.15}$$

which shows that the probability of particle at $x > 0$ is nonzero. We get

$$\frac{B}{A} = \frac{k - \mathrm{i}K}{k + \mathrm{i}K}, \quad \frac{D}{A} = \frac{2k}{k + \mathrm{i}K} \tag{2.3.16}$$

now

$$\frac{B}{A} = \frac{k^2 - K^2}{k^2 + K^2} - \mathrm{i}\frac{2kK}{k^2 + K^2} = \mathrm{e}^{\mathrm{i}\delta} \tag{2.3.17}$$

and

$$\tan\delta = -\frac{2kK}{k^2 - K^2} = \frac{2\sqrt{\frac{E}{V_0} - \left(\frac{E}{V_0}\right)^2}}{1 - 2\left(\frac{E}{V_0}\right)} \tag{2.3.18}$$

Square Potential

Consider an 1D square barrier, which is

$$V(x) = \begin{cases} 0 & (x < 0) \\ V_0 & (0 \leqslant x \leqslant a) \\ 0 & (x > a) \end{cases} \tag{2.3.19}$$

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} + k^2\psi_1 = 0 \quad (x < 0) \tag{2.3.20}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} - K^2\psi_2 = 0 \quad (0 \leqslant x \leqslant a) \tag{2.3.21}$$

$$\frac{\mathrm{d}^2\psi_3}{\mathrm{d}x^2} + k^2\psi_3 = 0 \quad (x > a) \tag{2.3.22}$$

with $0 \leqslant x \leqslant a$, we can write the wave function as

$$\psi_2(x) = C_1\cosh(Kx) + C_2\sinh(Kx) \tag{2.3.23}$$

the boundary condition gives

$$A + B = C_1, \quad i\eta(A - B) = C_2, \quad \eta \equiv k/K \tag{2.3.24}$$

soon we have

$$C_1\cosh(Ka) + C_2\sinh(Ka) = D\mathrm{e}^{\mathrm{i}ka} \tag{2.3.25}$$

$$C_1\sinh(Ka) + C_2\cosh(Ka) = \mathrm{i}\eta D\mathrm{e}^{\mathrm{i}ka} \tag{2.3.26}$$

plugging in A and B, we have

$$\left(1 + \frac{B}{A}\right)\cosh(Ka) + \mathrm{i}\eta\left(1 - \frac{B}{A}\right)\sinh(Ka) = \frac{D}{A}\mathrm{e}^{\mathrm{i}ka} \tag{2.3.27}$$

and

$$\left(1 + \frac{B}{A}\right)\sinh(Ka) + \mathrm{i}\eta\left(1 - \frac{B}{A}\right)\cosh(Ka) = \mathrm{i}\eta\frac{D}{A}\mathrm{e}^{\mathrm{i}ka} \tag{2.3.28}$$

the solution should be

$$\frac{B}{A} = \frac{-(1+\lambda^2)\sinh(Ka)}{(1-\lambda^2)\sinh(Ka) - 2\mathrm{i}\eta\cos(Ka)} \tag{2.3.29}$$

$$\frac{D}{A}\mathrm{e}^{ika} = \frac{-2\mathrm{i}\lambda}{(1-\lambda^2)\sinh(Ka) - 2\mathrm{i}\eta\cos(Ka)} \tag{2.3.30}$$

finally we have

$$R = \frac{\mid B \mid^2}{\mid A \mid^2} = \frac{(k^2+K^2)^2\sinh^2(Ka)}{(k^2+K^2)^2\sinh^2(Ka) + 4k^2K^2} \tag{2.3.31}$$

and

$$T = \frac{\mid D \mid^2}{\mid A \mid^2} = \frac{4k^2K^2}{(k^2+K^2)^2\sinh^2(Ka) + 4k^2K^2} \tag{2.3.32}$$

where again $R + T = 1$ and the probability current density takes the form $\frac{\hbar k}{m} \mid A \mid^2$ and $\frac{\hbar k}{m} \mid D \mid^2$.

Barrier Penetration

2.4 Delta Bound States and Scattering

The Definition of Delta Function
The Delta function is given

$$\int_{\infty}^{-\infty} \delta(x-x_0)dx = 1 \tag{2.4.1}$$

Delta Potential Well
The delta potential well is given

$$V(x) = -\frac{\hbar^2\Omega}{\mu}\delta(x) \tag{2.4.2}$$

where Ω is a positive constant named delta potential strength and μ is the mass. It is actually taking its form from setting

$$\psi'(0^+) - \psi'(0^-) = -2\Omega\psi(0) \tag{2.4.3}$$

This is from the Schrödinger euqation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \frac{\hbar^2\Omega}{m}\delta(x)\psi = E\psi \tag{2.4.4}$$

and by integrating both sides,

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\mathrm{d}\Big[\psi'(x)\Big] - \frac{\hbar^2\Omega}{m}\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\mathrm{d}x = E\int_{-\epsilon}^{\epsilon}\psi(x)\mathrm{d}x \tag{2.4.5}$$

where ϵ is a sufficiently small positive number, and by approximation

$$\int_{-\epsilon}^{\epsilon}\psi(x)\mathrm{d}x \approx 2\epsilon\psi(0) \xrightarrow{\epsilon\to 0} 0 \tag{2.4.6}$$

since the wave function is not changing rapidly in this segment. Notice: if we have different but finite potential on each side, the integration can still be taken 0 since ϵ is sufficiently small. Then we can have

$$-\frac{\hbar^2}{2m}[\psi^{'}(\varepsilon)-\psi^{'}(-\varepsilon)]-\frac{\hbar^2\Omega}{m}\psi(0)=0 \tag{2.4.7}$$

which is the previous setting.
We may take another form like

$$V(x)=a\delta(x), \qquad \psi'(0^+)-\psi'(0^-)=\frac{2\mu a}{\hbar^2}\psi(0) \tag{2.4.8}$$

The Delta potential well can also be solved using Fourier transformation, which is very useful for fields with multiple delta potential well. Now consider

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}-k^2\psi=-2\Omega\delta(x)\psi \tag{2.4.9}$$

as the Schrödinger equation simplified, where $k=\sqrt{-\frac{2mE}{\hbar^2}}$. Then Fourier transform both sides of the equation

$$-\omega^2\Psi(\omega)-k^2\Psi(\omega)=-2\Omega\int_{-\infty}^{\infty}\delta(x)\psi(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x=-2\Omega\psi(0) \tag{2.4.10}$$

here

$$\Psi(\omega)=\int_{-\infty}^{\infty}\psi(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x \tag{2.4.11}$$

We soon get

$$\Psi(\omega)=\frac{2\Omega\psi(0)}{k^2+\omega^2} \tag{2.4.12}$$

then we take the inverse transformation

$$\begin{aligned}\psi(x)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega\\&=\frac{\Omega\psi(0)}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\omega x)+\mathrm{i}\sin(\omega x)}{k^2+\omega^2}\mathrm{d}\omega\\&=\frac{2\Omega\psi(0)}{\pi}\int_{0}^{\infty}\frac{\cos(\omega x)}{k^2+\omega^2}\mathrm{d}\omega\\&=\frac{\Omega\psi(0)}{k}\mathrm{e}^{-k|x|}=\psi(0)\mathrm{e}^{-\Omega|x|} \quad (k=\Omega)\end{aligned} \tag{2.4.13}$$

Delta Barrier Scattering

$$V(x)=\frac{\hbar^2\Omega}{m}\delta(x) \tag{2.4.14}$$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\frac{\hbar^2\Omega}{m}\delta(x)\psi=E\psi \tag{2.4.15}$$

$$\psi'(0^+)-\psi'(0^-)=2\Omega\psi(0) \tag{2.4.16}$$

2.5 1D Harmonic Oscillator: Analytic

The Hamiltonian of a harmonic oscillator is given

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \tag{2.5.1}$$

and the Schrödinger equation should therefore be

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \tag{2.5.2}$$

To solve this equation, we will introduce the dimensionless displacement and the dimensionless energy,

$$\xi = \alpha x, \quad \alpha = \sqrt{\frac{m\omega}{\hbar}} \tag{2.5.3}$$

$$\lambda = \frac{2E}{\hbar\omega} \tag{2.5.4}$$

then the equation can be written

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + (\lambda - \xi^2)\psi = 0 \tag{2.5.5}$$

for $\lambda \ll \xi^2$, which is at infinite ξ, we get a asymptotic solution

$$\psi(\xi) = A\mathrm{e}^{-\xi^2/2} \tag{2.5.6}$$

here the boundary condition is applied.
Now the solution should be given by a Hermite equation

$$\psi(\xi) = y(\xi)\mathrm{e}^{-\xi^2/2} \tag{2.5.7}$$

where $y(\xi)$ is still to be determined.
By solving the equation we get the analytic solution of $y(\xi)$

$$\mathrm{H}_n(\xi) = \sum_{m=0}^{M}\frac{(-1)^n n!}{m!\,(n-2m)!}(2\xi)^{n-2m} \tag{2.5.8}$$

and we will have

$$\lambda = 2n + 1 \tag{2.5.9}$$

to make sure it is finite. By this we soon obtain the eigenvalue of energy,

$$E_n = \hbar\omega(n + \frac{1}{2}) \quad (n = 0, 1, 2, \cdots) \tag{2.5.10}$$

the ground state where $n = 0$ is also called a vacuum state.
The eigenfuction is therefore

$$\psi_n(\xi) = (\frac{\alpha}{2^n n!\sqrt{\pi}})^{1/2}\mathrm{H}_n(\xi)\exp(-\frac{1}{2}\xi^2) \quad (n = 0, 1, 2, \cdots) \tag{2.5.11}$$

2.6 1D Harmonic Oscillator: Algebraic

For 1D harmonic oscillator, we have an algebraic method which will be shown as follow. First, we define two useful operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}) \tag{2.6.1}$$

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}) \tag{2.6.2}$$

so that
since

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hbar\omega}{4}\Big(\frac{2m\omega}{\hbar}\hat{x}^2 + \frac{2}{m\hbar\omega}\hat{p}^2\Big) \tag{2.6.3}$$

we have

$$\hat{H} = \frac{\hbar\omega}{2}(a^{\dagger}a + aa^{\dagger}) \tag{2.6.4}$$

by plugging in

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar \tag{2.6.5}$$

we find

$$[a, a^{\dagger}] = 1 \tag{2.6.6}$$

$$\begin{aligned}
\hat{H}(a^{\dagger}\psi) =& \hbar\omega(a^{\dagger}a + \frac{1}{2})(a^{\dagger}\psi) = \hbar\omega(a^{\dagger}aa^{\dagger} + \frac{1}{2}a^{\dagger})\phi \\
=& \hbar\omega a^{\dagger}(aa^{\dagger} + \frac{1}{2})\psi = a^{\dagger}[\hbar\omega(a^{\dagger}a + 1 + \frac{1}{2})\psi] \\
=& a^{\dagger}(\hat{H} + \hbar\omega)\psi = a^{\dagger}(E + \hbar\omega)\psi = (E + \hbar\omega)(a^{\dagger}\psi)
\end{aligned} \tag{2.6.7}$$

and similarly

$$\hat{H}\,(a\psi) = (E - \hbar\omega)\,(a\psi) \tag{2.6.8}$$

If we apply the annihilation operator on the ground state, by its physical meaning, we should have

$$a\psi_0 = 0 \tag{2.6.9}$$

by plugging in its original definition, which is

$$\frac{\alpha}{\sqrt{2}}\Big(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\Big)\psi_0 = 0 \tag{2.6.10}$$

here as before $\alpha = \sqrt{\frac{m\omega}{\hbar}}$.
The solution is

$$\psi_0\,(x) = A_0 \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \tag{2.6.11}$$

and we can do the normalization

$$A_0^2 \int_{-\infty}^{\infty} \mathrm{e}^{-a^2x^2}\mathrm{d}x = A_0^2\frac{\sqrt{\pi}}{\alpha} = 1 \Rightarrow A_0 = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \tag{2.6.12}$$

so that finally

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \tag{2.6.13}$$

which is consistent with previous derivations, and the energy of ground state can be given

$$\hbar\omega\Big(a^+ a + \frac{1}{2}\Big)\psi_0 = E_0\psi_0 \tag{2.6.14}$$

and

$$E_0 = \frac{1}{2}\hbar\omega \tag{2.6.15}$$

which is also the same as what we derived using the analytical method.
By comparison, we should find

$$a^+ a\psi_n = n\psi_n \tag{2.6.16}$$

and by applying the commutation relationship

$$aa^+\psi_n = (n+1)\psi_n \tag{2.6.17}$$

therefore

$$a\psi_n = \sqrt{n}\psi_{n-1} \tag{2.6.18}$$

$$a^+\psi_n = \sqrt{n+1}\psi_{n+1} \tag{2.6.19}$$

and further

$$\psi_n = \frac{1}{\sqrt{n!}}(a^+)^n\psi_0 \tag{2.6.20}$$

here we assume all eigenfuctions are normalized.

2.7 Periodic Field

Planar Rotor

Let us consider a particle with mass μ and constained to move in a circle radius a. The classcial Hamiltonian can be written

$$H = \frac{p^2}{2\mu} = \frac{(pa)^2}{2\mu a^2} = \frac{L^2}{2\mu a^2} = \frac{L^2}{2I} \tag{2.7.1}$$

where the corresponding operator should be

$$\hat{H} = \frac{\hat{p}^2}{2\mu} = \frac{1}{2\mu}(\mathrm{i}\hbar\frac{\partial}{\partial(a\varphi)})^2 = -\frac{\hbar^2}{2\mu a^2}\frac{d^2}{d\varphi^2} = \frac{\hat{L}^2}{2I} \tag{2.7.2}$$

so that $\hat{L} = -i\hbar\frac{d}{d\varphi}$. Here the minus sign is to be consistent with what we will define in 3D applications. Since in 1D planar rotor the momentum operator always come together, it is casual to take both sign here actually.

Now the stationary Schrödinger euqation should be

$$-\frac{\hbar}{2I}\frac{d^2\Psi(\varphi)}{d\varphi^2} = E\Psi(\varphi) \tag{2.7.3}$$

the solution should be

$$\frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}m\varphi) \tag{2.7.4}$$

since we find $\int_0^{2\pi} d\varphi\varphi^*\varphi = 1$
the energy is positive, so we can get

$$E = E_m = \frac{\hbar^2 m^2}{2I} = \frac{\hbar^2}{2\mu a^2}m^2, \quad m = 0, \pm 1, \pm 2, \cdots \tag{2.7.5}$$

Floquet Theorem
Bloch Theorem

2.8 Appendix: Hermite equation

2.9 Appendix: Useful Integration

First we start from calculating

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \tag{2.9.1}$$

where we have

$$I^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y \tag{2.9.2}$$

so that

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi} \tag{2.9.3}$$

since the integrated function is an even function, we also get

$$\int_0^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{2.9.4}$$

Then we can compute the Gamma Function, which is defined

$$\Gamma(n) = \int_0^{\infty} \mathrm{e}^{-x}x^{n-1}\mathrm{d}x \tag{2.9.5}$$

integrating by parts gives

$$\Gamma(n) = -(\mathrm{e}^{-x}x^{n-1})|_0^{\infty} + (n-1)\int_0^{\infty} \mathrm{e}^{-x}x^{n-2}\mathrm{d}x = (n-1)\Gamma(n-1) \tag{2.9.6}$$

which is the recursion relationship of Gamma function. Notice that

$$\Gamma(1) = \int_0^{\infty} \mathrm{e}^{-x}\mathrm{d}x = 1 \tag{2.9.7}$$

and

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} \mathrm{e}^{-x}x^{-\frac{1}{2}}\mathrm{d}x = 2\int_0^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y = \sqrt{\pi} \tag{2.9.8}$$

here $y^2 = x$. We find if n denotes a positive integer, we have

$$\Gamma(n) = (n-1)(n-2)\cdots 1 \cdot \Gamma(1) = (n-1)! \tag{2.9.9}$$

and

$$\begin{aligned}\Gamma(n+\frac{1}{2}) &= (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2}) \\ &= (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}\cdot\sqrt{\pi}\end{aligned} \tag{2.9.10}$$

and we move on to further calculate

$$I(n) = \int_0^\infty \mathrm{e}^{-ax^2} x^n \mathrm{d}x \tag{2.9.11}$$

where n is set to be 0 or a positive integer.
First, denote $y = \alpha^{1/2}x$, then

$$I(0) = \alpha^{-\frac{1}{2}} \int_0^\infty \mathrm{e}^{-y^2}\mathrm{d}y = \frac{\sqrt{\pi}}{2\alpha^{1/2}} \tag{2.9.12}$$

and

$$I(1) = \alpha^{-1} \int_0^\infty \mathrm{e}^{-y^2} y\mathrm{d}y = \frac{1}{2\alpha} \tag{2.9.13}$$

and we have our recursion relationship

$$I(n) = -\frac{\partial}{\partial\alpha}\int_0^\infty \mathrm{e}^{-\alpha x^2} x^{n-2}\mathrm{d}x = -\frac{\partial}{\partial\alpha}I(n-2) \tag{2.9.14}$$

so that we can calculate all the integration for any given n. To examplify,

$$I(2) = \int_0^\infty \mathrm{e}^{-\alpha x^2} x^2\mathrm{d}x = \frac{\sqrt{\pi}}{4\alpha^{3/2}} \tag{2.9.15}$$

$$I(3) = \int_0^\infty \mathrm{e}^{-\alpha x^2} x^3\mathrm{d}x = \frac{1}{2\alpha^2} \tag{2.9.16}$$

$$I(4) = \int_0^\infty \mathrm{e}^{-ax^2} x^4\mathrm{d}x = \frac{3\sqrt{\pi}}{8\alpha^{5/2}} \tag{2.9.17}$$

$$I(5) = \int_0^\infty \mathrm{e}^{-\alpha x^2} x^5\mathrm{d}x = \frac{1}{\alpha^3} \tag{2.9.18}$$

Then we come to calculate

$$I(n) = \int_0^\infty \frac{x^{n-1}}{\mathrm{e}^s - 1}\mathrm{d}x \left(n = 2, 3, 4, \frac{3}{2}, \frac{5}{2}\right) \tag{2.9.19}$$

Since

$$\frac{x^{n-1}}{\mathrm{e}^x - 1} = \frac{x^{n-1}\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} = x^{n-1}\mathrm{e}^{-x}(1 + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \cdots) = \sum_{k=1}^{\infty} x^{n-1}\mathrm{e}^{-kx} \tag{2.9.20}$$

so we have

$$I(n) = \int_0^\infty \frac{x^{n-1}\mathrm{d}x}{\mathrm{e}^x - 1} = \sum_{k=1}^{\infty}\int_0^\infty x^{n-1}\mathrm{e}^{-kx}\mathrm{d}x = \sum_{k=1}^{\infty}\frac{1}{k^n}\int_0^\infty y^{n-1}\mathrm{e}^{-y}\mathrm{d}y \tag{2.9.21}$$

At last, we calculate

$$I = \int_0^\infty \frac{x\mathrm{d}x}{\mathrm{e}^x + 1} \tag{2.9.22}$$

Since

$$\frac{x}{\mathrm{e}^x + 1} = \frac{x\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}} = x\mathrm{e}^{-x}\left(1 - \mathrm{e}^{-x} + \mathrm{e}^{-2x} - \cdots \right) = \sum_{k=1}^{\infty} \left(-1 \right)^{k-1} x\mathrm{e}^{-kx} \tag{2.9.23}$$

so that

$$\begin{aligned} \mathrm{I} &= \int_0^\infty \frac{x\mathrm{d}x}{\mathrm{e}^x + 1} = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^\infty x\mathrm{e}^{-kx}\mathrm{d}x \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} \int_0^\infty y\mathrm{e}^{-y}\mathrm{d}y \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} = \frac{\pi^2}{12} \end{aligned} \tag{2.9.24}$$

3 Angular Momentum

3.1 Angular Momentum Operator

The angular momentum opreator is

$$\hat{L} = \hat{r} \times \hat{p} = -i\hbar\hat{r} \times \nabla \tag{3.1.1}$$

and it is given by

$$\begin{vmatrix} i & j & k \\ x & y & z \\ \hat{p_x} & \hat{p_y} & \hat{p_z} \end{vmatrix} = \varepsilon_{ijk} e_i x_j \hat{p_k} \tag{3.1.2}$$

and thus gives

$$\hat{L}_{x_i} = -i\hbar(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}) \tag{3.1.3}$$

which helps to define a squared angular momentum operator

$$\hat{L^2} = \hat{L_x^2} + \hat{L_y^2} + \hat{L_z^2} = -\hbar^2[(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})^2 + (z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})^2 + (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})^2] \tag{3.1.4}$$

To represent all these under spherical coordinates, we are to make such substitutions:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta \tag{3.1.5}$$

$$r^2 = x^2 + y^2 + z^2 \quad \cos\theta = \frac{z}{r} \quad \tan\phi = \frac{y}{x} \quad \rho^2 = x^2 + y^2 \tag{3.1.6}$$

Here θ is the latitude angle and ϕ is the longitude angle. ρ is the projected length of r in the $x - y$ plain.

By $r^2 = x^2 + y^2 + z^2$ we have

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{x}{\rho}\frac{\rho}{r} = \cos\phi\sin\theta \tag{3.1.7}$$

similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\phi\sin\theta \quad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta \tag{3.1.8}$$

by $\cos\theta = z/r$ and $r^2 = x^2 + y^2 + z^2$ we have

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sin\theta}\frac{z}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r}\cos\theta\cos\phi \tag{3.1.9}$$

and similarly

$$\frac{\partial\theta}{y} = \frac{1}{r}\cos\theta\sin\phi \quad \frac{\partial\theta}{\partial z} = -\frac{1}{r}\sin\theta \tag{3.1.10}$$

and by $\tan\phi = y/x$ we have

$$\frac{\partial\phi}{\partial x} = -\frac{y}{x^2}\cos^2\phi = -\frac{1}{r}\frac{\sin\phi}{\sin\theta} \tag{3.1.11}$$

similarly

$$\frac{\partial\phi}{\partial y} = \frac{1}{r}\frac{\cos\phi}{\sin\theta}, \quad \frac{\partial\phi}{\partial z} = 0 \tag{3.1.12}$$

and finally we are to give

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{3.1.13}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{3.1.14}$$

$$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \tag{3.1.15}$$

the angular momentum operators are

$$\hat{L_x}=i\hbar(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}) \tag{3.1.16}$$

$$\hat{L_y}=i\hbar(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}) \tag{3.1.17}$$

$$\hat{L_z}=-i\hbar\frac{\partial}{\partial\phi} \tag{3.1.18}$$

by these above we get the squared angular momentum operator

$$\hat{L^2}=-\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}] \tag{3.1.19}$$

3.2 3D Bound States

Here after we will denote the mass μ since a useful constant m will be introduced.

By using a spherical coordinate, we write the Schrödinger equation with a central force field as

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi \tag{3.2.1}$$

where by denoting $V(r)$ we mean the potential here is only a function of distance.

If we take a seperation variable, which is $\psi(\boldsymbol{r})=R(r)Y(\theta,\phi)$, then we can have

$$-\frac{1}{Y}[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}]=l(l+1) \tag{3.2.2}$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\mathrm{d}R}{\mathrm{d}r})+\frac{2mr^2}{\hbar^2}\big[E-V(r)\big]=l(l+1) \tag{3.2.3}$$

are called the angular equation and the radial equation.

Here l is the seperation constant. Later we will come to realize, only with this form of seperation constant, the solution of the wave function can be finite at the boundary, and l as a eigenvalue is the angular momentum of a particle. In fact, by comparing with the squared momentum operator, we have

$$L^2=-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sinh\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{3.2.4}$$

suggesting a dimensionless squared momentum operator. Thus

$$Y(\theta, \phi) = \Theta(\theta)\Phi(\phi) \tag{3.2.5}$$

and that is why we call l the angular momentum number of a particle.

By again seperating the variable, which is $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$, we give

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + m^2\Phi = 0 \tag{3.2.6}$$

$$\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + [l(l+1)\sin^2\theta - m^2]\Theta = 0 \tag{3.2.7}$$

where m is a constant which will be later defined a magnetic quantum number.

The above 2 equations are called longitude angular equation and latitude angular equation.

The latitude angular equation follows the periodical boundary condition, which is

$$\Phi(\phi + 2\pi) = \Phi(\phi) \tag{3.2.8}$$

that gives the constant before Φ non-negative, part of the reason why we take it m^2. Then the general solution will take the form

$$\Phi(\phi) = c\mathrm{e}^{im\phi} + d\mathrm{e}^{-im\phi} \tag{3.2.9}$$

And the solution of the latitude equation is given by

$$\Theta(\theta) = P_l^m(\cos\theta), m = 0, 1, 2, ..., l \tag{3.2.10}$$

where the m order asscotiated Legendre function is defined from the m order derivative of Legendre function,

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}P_l^{(m)}(x) \tag{3.2.11}$$

and the form of asscotiated Legendre equation is given by the Rodrigues formula

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l \tag{3.2.12}$$

Now the angular equation can be written

$$Y_{lm}(\theta, \phi) = A_{lm}P_l^m(\cos\theta)e^{im\phi} \tag{3.2.13}$$

also, since the dimensionless angular momentum opreator suggests $L_z = -i\frac{\partial}{\partial\phi}$, we come to realize that

$$L_z Y_{lm}(\theta, \phi) = m[A_{lm}P_l^m(\cos\theta)e^{im\phi}] = mY_{lm} \tag{3.2.14}$$

with previous

$$L^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sinh\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{3.2.15}$$

altogether suggests the function Y_{lm} to be a eigen function of $\hat{L^2}$ and L_z, with eigenvalue $l(l+1)\hbar^2$ and $m\hbar$,

3.3 Angular Momentum Operator Algebraically

From previous definition

$$\hat{L}_{x_i} = -i\hbar(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}) \tag{3.3.1}$$

and the important commutation relationship

$$[x_i, \hat{p}_{x_i}] = i\hbar \tag{3.3.2}$$

and all other commutators between position and momentum is 0.

$$\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right] &= \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x \\
&= (y\hat{p}_z - z\hat{p}_y)(z\hat{p}_z - x\hat{p}_z) - (z\hat{p}_z - x\hat{p}_z)(y\hat{p}_z - z\hat{p}_y) \\
&= \hat{p}_z zy\hat{p}_x + z\hat{p}_z x\hat{p}_y - z\hat{p}_z y\hat{p}_x - \hat{p}_z zx\hat{p}_y \\
&= (z\hat{p}_z - \hat{p}_z z)(x\hat{p}_y - y\hat{p}_x) \\
&= i\hbar\hat{L}_z
\end{aligned} \tag{3.3.3}$$

and thus generally

$$[\hat{L}_{x_i}, \hat{L}_{x_j}] = \varepsilon_{ijk}i\hbar\hat{L}_{x_k} \tag{3.3.4}$$

we further have

$$\begin{aligned}
\hat{L}\times\hat{L} &= \begin{vmatrix} i & j & k \\ \hat{L_x} & \hat{L_y} & \hat{L_z} \\ \hat{L_x} & \hat{L_y} & \hat{L_z} \end{vmatrix} \\
&= (\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)\boldsymbol{i} + (\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z)\boldsymbol{j} + (\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)\boldsymbol{k} \\
&= [\hat{L_y}, \hat{L_z}]\boldsymbol{i} + [\hat{L_z}, \hat{L_x}]\boldsymbol{j} + [\hat{L_x}, \hat{L_y}]\boldsymbol{k} \\
&= \mathrm{i}\hbar(\hat{L}_x\boldsymbol{i} + \hat{L}_y\boldsymbol{j} + \hat{L}_z\boldsymbol{k}) = \mathrm{i}\hbar\hat{\boldsymbol{L}}
\end{aligned} \tag{3.3.5}$$

and

$$[\hat{L}_{x_i}, x_j] = \varepsilon_{ijk}i\hbar x_k \tag{3.3.6}$$

further we define the squared momentum operator

$$\hat{L^2} = \hat{L_x^2} + \hat{L_y^2} + \hat{L_z^2} \tag{3.3.7}$$

hereafter we remove the hat above the operator and we use upper case for operator while lower case for eigenvalues. Energy is exceptional, with eigenvalues E and operator H.

and we have

$$L^2\mathrm{Y}_{lm}(\theta,\phi) = l(l+1)\hbar^2\mathrm{Y}_{lm}(\theta,\phi) \tag{3.3.8}$$

$$L_z\mathrm{Y}_{lm}(\theta,\phi) = m\hbar\mathrm{Y}_{lm}(\theta,\phi) \tag{3.3.9}$$

here

$$\mathrm{Y}_{lm}\left(\theta,\phi\right) = \mathrm{A}_{lm}\mathrm{P}_l^m\left(\cos\theta\right)\mathrm{e}^{\mathrm{i}\omega\phi}_{m=0,\pm1,\pm2,\ldots,\pm l} \tag{3.3.10}$$

Now we can define the creation and annihilation operator of angular momentum, which is similar to the harmonic oscillator creation and annihilation operator,

$$L^{\pm} \equiv L_x \pm \mathrm{i}L_y \tag{3.3.11}$$

then

$$\left[L_z, L^{\pm}\right] = \pm\hbar L^{\pm} \tag{3.3.12}$$

and

$$[L^2, L^{\pm}] = 0 \tag{3.3.13}$$

We have known that the function Y_{lm} is the eigenfuction of both L^2 and L_z, now we denote it briefly by f and seek more relationship between them.

We first find

$$L_z(L^{\pm}f) = \pm\hbar L^{\pm}f + L^{\pm}L_z f = (\mu \pm \hbar)(L^{\pm}f) \tag{3.3.14}$$

and

$$\begin{aligned} L_z[(L^+)^2 f] &= (\hbar L^+ + L^+ L_z)(L^+ f) = \hbar L^+ L^+ f + L^+ L_z L^+ f \\ &= \hbar[(L^+)^2 f] + L^+(\mu + \hbar)(L^+ f) \\ &= (\mu + 2\hbar)[(L^+)^2 f] \end{aligned} \tag{3.3.15}$$

$$\mu + N\hbar \equiv l\hbar \tag{3.3.16}$$

$$L^+ f_l = 0 \tag{3.3.17}$$

$$L^2 f_l = \lambda f_l \tag{3.3.18}$$

$$L_z f_l = l\hbar f_l \tag{3.3.19}$$

$$\begin{aligned} L^{\pm}L^{\mp} &= (L_x \pm \mathrm{i}L_y)(L_x \mp \mathrm{i}L_y) = L_x^2 + L_y^2 \mp \mathrm{i}(L_x L_y - L_y L_x) \\ &= L^2 - L_z^2 \mp \mathrm{i}(\mathrm{i}\hbar L_z) \\ &= L^2 - L_z^2 \pm \hbar L_z \end{aligned} \tag{3.3.20}$$

$$L^2 = L^+L^- + L_z^2 - \hbar L_z \tag{3.3.21}$$

$$L^2 = L^-L^+ + L_z^2 + \hbar L_z \tag{3.3.22}$$

$$L^2 f_l = (L^-L^+ + L_z^2 + \hbar L_z) f_l = (0 + l^2\hbar^2 + l\hbar^2) f_l = l(l+1)\hbar^2 f_l \tag{3.3.23}$$

$$L^2 f_{lm} = l(l+1)\hbar^2 f_{lm} \tag{3.3.24}$$

$$L_z f_{lm} = m\hbar f_{lm} \tag{3.3.25}$$

$$L^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle \tag{3.3.26}$$

$$L_z \mid l, m\rangle = m\hbar \mid l, m\rangle \tag{3.3.27}$$

$$L^+ \mid l, m\rangle = A_{lm}^+ \mid l, m+1\rangle \tag{3.3.28}$$

$$L^- \mid l, m\rangle = A_{lm}^- \mid l, m-1\rangle \tag{3.3.29}$$

$$A_{lm}^{\pm} = \hbar\sqrt{l(l+1) - m(m \pm 1)} = \hbar\sqrt{(l \mp m)(l \pm m + 1)} \tag{3.3.30}$$

$$L_z L^+ = L^+(L_z + \hbar) \tag{3.3.31}$$

$$L_z L^- = L^-(L_z - \hbar) \tag{3.3.32}$$

$$L_z L^+ \mid l, m\rangle = L^+(L_z + \hbar) \mid l, m\rangle = (m+1)\hbar L^+ \mid l, m\rangle \tag{3.3.33}$$

$$L_z \mid l, m+1\rangle = (m+1)\hbar \mid l, m+1\rangle \tag{3.3.34}$$

$$L^+ \mid l, m\rangle = A_{lm}^+ \mid l, m+1\rangle \tag{3.3.35}$$

$$L^- \mid l, m\rangle = A_{lm}^- \mid l, m-1\rangle \tag{3.3.36}$$

$$(L^{\pm})^+ = (L_x \pm \mathrm{i}L_y)^+ = L_x^+ \mp \mathrm{i}L_y^+ = L_x \mp \mathrm{i}L_y = L^{\mp} \tag{3.3.37}$$

$$\langle l, m \mid L^- L^+ \mid l, m\rangle = \langle l, m \mid (L^+)^+ L^+ \mid l, m\rangle = |A_{lm}^+|^2 \tag{3.3.38}$$

$$\langle l, m \mid L^2 - L_z^2 - \hbar L_z \mid l, m\rangle = l(l+1)\hbar^2 - m(m+1)\hbar^2 \tag{3.3.39}$$

$$A_{lm}^+ = \hbar\sqrt{l(l+1) - m(m+1)} = \hbar\sqrt{(l-m)(l+m+1)} \tag{3.3.40}$$

$$A_{lm}^- = \hbar\sqrt{l(l+1) - m(m-1)} = \hbar\sqrt{(l+m)(l-m+1)} \tag{3.3.41}$$

3.4 Appendix: Legendre Function

3.5 Appendix: Generalized Laguerre Polynomial

3.6 Appendix: Confluent Hypergeometric Function

4 3D Application

4.1 Central Force Problem

First, we review what we derived from last chapter and is called the Radial Schrödinger equation

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\mathrm{d}R}{\mathrm{d}r}) + \frac{2mr^2}{\hbar^2}\big[E - V(r)\big] = l(l+1) \tag{4.1.1}$$

Where R stands for the Radial wave function. Hereafter, we simplify this function by changing the variable

$$u(r) \equiv rR(r) \tag{4.1.2}$$

so that we have $R = u/r, dR/dr = \left[r(du/dr) - u\right]/r^2, (d/dr)\left[r^2(dR/dr)\right] = rd^2u/dr^2$, hence

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu \tag{4.1.3}$$

Now we consider its property when $r \to 0$, and a potential when $r \to 0$ have the property $r^2V(r) \to 0$. Almost every central field we consider will suffice, for example, the potential of harmonic oscillator, the linear central force, the Coulomb potential, the Yukawa potential.

4.2 Spherical Square Potential Well

4.3 3D Harmonic Oscillator

4.4 The Hydrogen Atom

For Hydrogen atom, the potential energy of the electron in SI units is given

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \tag{4.4.1}$$

so the radial equation should be

$$-\frac{\hbar^2}{2m_e}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m_e}\frac{\ell(\ell+1)}{r^2}\right]u = Eu \tag{4.4.2}$$

we denote

$$\kappa \equiv \frac{\sqrt{-2m_eE}}{\hbar}. \tag{4.4.3}$$

since we are considering a bound state and κ should be real. Then we have

$$\frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{m_ee^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{(\kappa r)} + \frac{\ell(\ell+1)}{(\kappa r)^2}\right]u. \tag{4.4.4}$$

which suggests

$$\rho \equiv \kappa r, \quad \rho_0 \equiv \frac{m_ee^2}{2\pi\epsilon_0\hbar^2\kappa} \tag{4.4.5}$$

so that ρ would be the dimensionless radius and ρ_0 some reference radius.

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u \tag{4.4.6}$$

and we will examine this first by checking the asymptotic solution to this equation.b As $\rho \to \infty$, we have

$$\frac{d^2 u}{d\rho^2} = u \tag{4.4.7}$$

and the general solution should be

$$u(\rho) = Ae^{-\rho} \tag{4.4.8}$$

for reason it does not blow up at infinity.

On the other hand, as $\rho \to 0$, the angular momentum part is taking advantage since it has lower order of ρ, which will be

$$\frac{d^2 u}{d\rho^2} = \frac{\ell(\ell+1)}{\rho^2} u. \tag{4.4.9}$$

Then the general solution should be

$$u(\rho) = C\rho^{\ell+1} + D\rho^{-\ell} \tag{4.4.10}$$

since when $\rho \to 0$ the wave function should be finite, we take the $C\rho^{\ell+1}$ part. Now the solution can be written in the form

$$u(\rho) = \rho^{\ell+1} e^{-\rho} v(\rho), \tag{4.4.11}$$

then we have

$$\frac{du}{d\rho} = \rho^{\ell} e^{-\rho} \left[(\ell + 1 - \rho) v + \rho \frac{dv}{d\rho} \right] \tag{4.4.12}$$

$$\frac{d^2 u}{d\rho^2} = \rho^{\ell} e^{-\rho} \left\{ \left[-2\ell - 2 + \rho + \frac{\ell(\ell+1)}{\rho} \right] v + 2(\ell + 1 - \rho) \frac{dv}{d\rho} + \rho \frac{d^2 v}{d\rho^2} \right\} \tag{4.4.13}$$

$$\rho \frac{d^2 v}{d\rho^2} + 2(\ell + 1 - \rho) \frac{dv}{d\rho} + [\rho_0 - 2(\ell+1)] v = 0 \tag{4.4.14}$$

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \tag{4.4.15}$$

$$\frac{dv}{d\rho} = \sum_{j=0}^{\infty} j c_j \rho^{j-1} = \sum_{j=0}^{\infty} (j+1) c_{j+1} \rho^j \tag{4.4.16}$$

$$\sum_{j=0}^{\infty} j(j+1) c_{j+1} \rho^j + 2(\ell+1) \sum_{j=0}^{\infty} j(j+1) c_{j+1} \rho^j - 2 \sum_{j=0}^{\infty} j c_j \rho^j + [\rho_0 - 2(\ell+1)] \sum_{j=0}^{\infty} c_j \rho^j = 0 \tag{4.4.17}$$

$$j(j+1) c_{j+1} + 2(\ell+1)(j+1) c_{j+1} - 2j c_j + [\rho_0 - 2(\ell+1)] c_j = 0 \tag{4.4.18}$$

$$c_{j+1} = \left\{ \frac{2(j + \ell + 1) - \rho_0}{(j+1)(j+2\ell+2)} \right\} c_j \tag{4.4.19}$$

$$c_{j+1} \approx \frac{2j}{j(j+1)} c_j = \frac{2}{j+1} c_j \tag{4.4.20}$$

$$c_j \approx \frac{2^j}{j!} c_0 \tag{4.4.21}$$

$$v(\rho) = c_0 \sum_{j=0}^{\infty} \frac{2^j}{j!} \rho^j = c_0 e^{2\rho} \tag{4.4.22}$$

$$u(\rho) = c_0 \rho^{l+1} e^{\rho} \tag{4.4.23}$$

$$c_{N-1} \neq 0 \quad \text{but} \quad c_N = 0 \tag{4.4.24}$$

$$2\left(N + \ell\right) - \rho_0 = 0 \tag{4.4.25}$$

$$n \equiv N + \ell \tag{4.4.26}$$

$$\rho_0 = 2n \tag{4.4.27}$$

$$E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{m_e e^4}{8\pi^2 \epsilon_0^2 \hbar^2 \rho_0^2} \tag{4.4.28}$$

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1, 2, 3, \ldots \tag{4.4.29}$$

4.5 Hellmann-Feynmann Theorem

4.6 2D Central Force Problem

4.7 1D Hydrogen Atom

4.8 Electromagnetic Fields

From previous classical theory of fields we derived that the relativistic Hamiltonian for a particle in an Electromagnetic field should be

$$\mathcal{H} = \sqrt{m^2 c^4 + c^2 (\vec{P} - q\vec{A})^2} + q\varphi \tag{4.8.1}$$

by writing it in a easier and non-relativistic form, we expand the first item by series of $1/mc$, and then omit the self-energy part,finally denote the mass in μ as we used to in Quantum Mechanics to get

$$H = \frac{1}{2\mu} (\vec{P} - q\vec{A})^2 + q\varphi \tag{4.8.2}$$

5 Representation

5.1 Representation of Continuous Spectrum

We start from the Schrödinger euqation in the position space and then to find out its form in the momentum space.

First we write

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi_x + V(x)\psi_x = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi_x \tag{5.1.1}$$

by plugging in

$$\psi(r,t) = \frac{1}{(2\pi\hbar)^{3/2}}\int_{-\infty}^{+\infty}\varphi(p,t)\mathrm{e}^{\mathrm{i}p\cdot r/\hbar}\mathrm{d}^3p \tag{5.1.2}$$

we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi(p,t) = \frac{p^2}{2\mu}\varphi(p,t) + \int_{-\infty}^{+\infty}V_{pp'}\varphi(p',t)\mathrm{d}^3p' \tag{5.1.3}$$

where

$$V_{pp'} = \frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}(p-p')\cdot r/\hbar}V(r,t)\mathrm{d}\tau \tag{5.1.4}$$

5.2 Density Opreator

$$\mathrm{Tr}\hat{F} = \sum_k\langle k \mid \hat{F} \mid k\rangle \tag{5.2.1}$$

$$\sum_k\langle k \mid \hat{F} \mid k\rangle = \sum_i\langle j \mid \hat{F} \mid j\rangle \tag{5.2.2}$$

$$\sum_k \mid k >< k \mid = 1, \quad \sum_j \mid j >< j \mid = 1 \tag{5.2.3}$$

plugging in, we have

$$\sum_j\langle j \mid \hat{F} \mid j\rangle = \sum_j\langle j \mid \sum_k \mid k\rangle\langle k \mid \hat{F} \mid j\rangle = \sum_j\sum_k\langle j \mid k\rangle\langle k \mid F \mid j\rangle \tag{5.2.4}$$

$$\sum_k\langle k \mid \hat{F} \mid k\rangle = \sum_k\langle k \mid \hat{F}\sum_j \mid j\rangle\langle j \mid k\rangle = \sum_k\sum_j\langle k \mid \hat{F} \mid j\rangle\langle j \mid k\rangle \tag{5.2.5}$$

$$\mathrm{Tr}\hat{F} = \sum_n\langle n \mid \hat{F} \mid n\rangle = \sum_n\lambda_n \tag{5.2.6}$$

$$\langle\hat{F}\rangle \equiv \langle\Psi \mid \hat{F} \mid \Psi\rangle = \sum_n \mid c_n \mid^2 \lambda_n \tag{5.2.7}$$

$$\langle\hat{F}\rangle = \mathrm{Tr}(\hat{F}\hat{\rho}) = \mathrm{Tr}(\hat{\rho}\hat{F}) \tag{5.2.8}$$

5.3 Representation of Discrete Q Representation

In this section we focus on time-dependent vector $|\Psi(t)\rangle$ in terms of wave function under any given discrete spectrum. First we denote the operator $\hat{Q}$ for certain mechanical quantity Q, then we can denote the eigenfuction as

$$\hat{Q} \mid \phi_n\rangle = q_n \mid \phi_n\rangle (n = 1, 2, 3, \cdots) \tag{5.3.1}$$

with orthogonality and completeness

$$\langle\phi_m \mid \phi_m\rangle = \delta_{mn}, \quad \sum_n \mid \phi_n\rangle\langle\phi_n \mid = 1 \tag{5.3.2}$$

so we can expand it

$$\mid \psi(t)\rangle = \sum_n \mid \phi_n\rangle\langle\phi_n \mid \Psi(t)\rangle = \sum_n \langle\phi_n \mid \Psi(t)\rangle \mid \phi_n\rangle = \sum_n a_n(t) \mid \phi_n\rangle \tag{5.3.3}$$

where $a_n(t)$ is the time-dependent probability amplitude

$$a_n(t) = \langle\phi_n \mid \Psi(t)\rangle = \int \phi_n^*(x)\Psi(x, t)\mathrm{d}x (n = 1, 2, 3, \cdots) \tag{5.3.4}$$

assume that $\mid \Psi(t)\rangle$ is normalized, so that

$$\begin{aligned}
\langle\Psi(t) \mid \Psi(t)\rangle &= \langle\Psi(t) \mid \sum_n \mid \phi_n\rangle\langle\phi_n \mid \sum_m \mid \phi_m\rangle\langle\phi_m \mid \Psi(t)\rangle \\
&= \sum_n \sum_m \langle\Psi(t) \mid \phi_n\rangle\langle\phi_n \mid \phi_m\rangle\langle\phi_m \mid \Psi(t)\rangle \\
&= \sum_n \sum_m a_n^*(t) a_m(t) \delta_{nm} \\
&= \sum_n a_n^*(t) a_n(t) \\
&= \sum \mid a_n(t) \mid^2 = 1
\end{aligned} \tag{5.3.5}$$

$$\langle\phi_1 \mid \Psi(t)\rangle, \langle\phi_2 \mid \Psi(t)\rangle, \cdots \langle\phi_n \mid \Psi(t)\rangle, \cdots = a_1(t), a_2(t), \cdots a_n(t), \cdots \tag{5.3.6}$$

$$\boldsymbol{\psi} = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{bmatrix} \tag{5.3.7}$$

$$\boldsymbol{\Psi}^+ = (a_1^*(t), a_2^*(t), \cdots a_n^*(t), \cdots) \tag{5.3.8}$$

$$\boldsymbol{\Psi}^+\boldsymbol{\Psi} = (a_1^*(t), a_2^*(t), \cdots a_n^*(t), \cdots) \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{bmatrix} = \mid a_1(t) \mid^2 + \mid a_2(t) \mid^2 + \cdots \mid a_n(t) \mid^2 + \cdots = 1 \tag{5.3.9}$$

$$\mid \Phi(t)\rangle = \hat{F} \mid \Psi(t)\rangle \tag{5.3.10}$$

$$| \Psi(t)\rangle = \sum_n | \phi_n\rangle\langle\phi_n | \Psi(t)\rangle = \sum_n \langle\phi_n | \Psi(t)\rangle | \phi_n\rangle = \sum_n a_n(t) | \phi_n\rangle \tag{5.3.11}$$

$$| \Phi(t)\rangle = \sum_k | \phi_k\rangle\langle\phi_k | \Phi(t)\rangle = \sum_k \langle\phi_k | \Phi(t)\rangle | \phi_k\rangle = \sum_k b_k(t) | \phi_k\rangle \tag{5.3.12}$$

$$\sum_k b_k(t) | \phi_k\rangle = \hat{F} \sum_n a_n(t) | \phi_n\rangle \tag{5.3.13}$$

$$\langle\phi_m | \sum_k b_k(t) | \phi_k\rangle = \langle\phi_m | \hat{F} \sum_n a_n(t) | \phi_n\rangle \tag{5.3.14}$$

$$b_m(t) = \sum_n \langle\phi_m | \hat{F} | \phi_n\rangle a_n(t) = \sum_n F_m a_n(t) \quad (m, n = 1, 2, 3, \cdots) \tag{5.3.15}$$

$$F_{mn} = \langle\phi_m | \hat{F} | \phi_n\rangle \quad (m, n = 1, 2, 3, \cdots) \tag{5.3.16}$$

$$\begin{aligned}
b_1(t) &= \sum_n F_{1n} a_n(t) = F_{11} a_1(t) + F_{12} a_2(t) + \cdots + F_{1n} a_n(t) + \cdots \\
b_2(t) &= \sum_n F_{2n} a_n(t) = F_{21} a_1(t) + F_{22} a_2(t) + \cdots + F_{2n} a_n(t) + \cdots \\
b_m(t) &= \sum_n F_{mn} a_n(t) = F_{m1} a_1(t) + F_{m2} a_2(t) + \cdots + F_{mn} a_n(t) + \cdots \\
&\vdots
\end{aligned} \tag{5.3.17}$$

$$\begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_m(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1n} & \cdots \\ F_{21} & F_{22} & \cdots & F_{2n} & \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ F_{m1} & F_{m2} & \cdots & F_{mn} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{bmatrix} \tag{5.3.18}$$

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1n} & \cdots \\ F_{21} & F_{22} & \cdots & F_{2n} & \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ F_{m1} & F_{m2} & \cdots & F_{mn} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{5.3.19}$$

$$\boldsymbol{\Phi} = \boldsymbol{F\Psi} \tag{5.3.20}$$

$$F_{mn}^* = \langle\phi_m | \hat{F} | \phi_n\rangle^* = \langle\phi_n | \hat{F}^+ | \phi_m\rangle = \langle\phi_n | \hat{F} | \phi_m\rangle = F_{nm} \tag{5.3.21}$$

$$F_{mn} = \langle\phi_m | \hat{Q} | \phi_n\rangle = q_n \delta_{mn} = \begin{cases} q_n & (m = n) \\ 0 & (m \neq n) \end{cases} \tag{5.3.22}$$

5.4 Matrix Mechanics

5.5 Transformation of Representations

$$|\,\Psi(t)\rangle = \sum |\,a\rangle\langle a\,|\,\Psi(t)\rangle = \sum C_a(t)\,|\,a\rangle \quad (C_a(t) = \langle a\,|\,\Psi(t)\rangle) \tag{5.5.1}$$

$$|\,\Psi(t)\rangle = \sum_b |\,b\rangle\langle b\,|\,\Psi(t)\rangle = \sum_b D_b(t)\,|\,b\rangle \quad (D_b(t) = \langle b\,|\,\Psi(t)\rangle) \tag{5.5.2}$$

where

$$\boldsymbol{C}_a = \begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_a(t) \\ \vdots \end{bmatrix}, \quad \boldsymbol{D}_b = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_b(t) \\ \vdots \end{bmatrix} \tag{5.5.3}$$

$$\sum_a C_a(t)\,|\,a\rangle = \sum_b D_b(t)\,|\,b\rangle \tag{5.5.4}$$

$$\langle\alpha\,|\sum_a C_a(t)\,|\,a\rangle = \langle\alpha\,|\sum_b D_b(t)\,|\,b\rangle \tag{5.5.5}$$

$$C_a(t) = \sum_b \langle\alpha\,|\,D_b(t)\,|\,b\rangle = \sum_b D_b(t)\langle\alpha\,|\,b\rangle \tag{5.5.6}$$

$$C_a(t) = \sum_b D_b(t)\langle a\,|\,b\rangle \equiv \sum_b S_{ab} D_b(t) \tag{5.5.7}$$

$$\begin{aligned} C_1(t) &= \sum_b \mathrm{S}_{1b} D_b(t) = \mathrm{S}_{11} D_1(t) + \mathrm{S}_{12} D_2(t) + \cdots + \mathrm{S}_{1b} D_b(t) + \cdots \\ C_2(t) &= \sum_b \mathrm{S}_{2b} D_b(t) = \mathrm{S}_{21} D_1(t) + \mathrm{S}_{22} D_2(t) + \cdots + \mathrm{S}_{2b} D_b(t) + \cdots \\ C_a(t) &= \sum_b \mathrm{S}_{ab} D_b(t) = \mathrm{S}_{a1} D_1(t) + \mathrm{S}_{a2} D_2(t) + \cdots + \mathrm{S}_{ab} D_b(t) + \cdots \\ &: \end{aligned} \tag{5.5.8}$$

$$\begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_a(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1b} & \cdots \\ S_{21} & S_{22} & \cdots & S_{2b} & \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ S_{a1} & S_{a2} & \cdots & S_{ab} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_b(t) \\ \vdots \end{bmatrix} \tag{5.5.9}$$

$$\delta_{\alpha a} = \langle\alpha\,|\,a\rangle = \langle\alpha\,|\sum_b |\,b\rangle\langle b\,|\,a\rangle = \sum_b \langle a\,|\,b\rangle\langle b\,|\,a\rangle = \sum_b S^*_{ba} S_{ba} = (S^+S)_{\alpha a} = \begin{cases} 1 & (\alpha = a) \\ 0 & (\alpha \neq a) \end{cases} \tag{5.5.10}$$

$$\delta_{\beta b} = \langle\beta\,|\,b\rangle = \langle\beta\,|\sum_a |\,a\rangle\langle a\,|\,b\rangle = \sum_a \langle\beta\,|\,a\rangle\langle a\,|\,b\rangle = \sum_a S_{\beta a} S^*_{ba} = (SS^+)_{\beta b} = \begin{cases} 1 & (\beta = b) \\ 0 & (\beta \neq b) \end{cases} \tag{5.5.11}$$

$$\mathbf{S}^+\mathbf{S} = \mathbf{S}\mathbf{S}^+ = \mathbf{I} \tag{5.5.12}$$

$$S^{+} = S^{-1} \tag{5.5.13}$$

$$\begin{aligned}
F_{\alpha\alpha} &= \langle \alpha \mid \hat{F} \mid a \rangle = \langle \alpha \mid \sum_{b} \mid b \rangle \langle b \mid \hat{F} \sum_{\beta} \mid \beta \rangle \langle \beta \mid a \rangle \\
&= \sum_{b} \sum_{\beta} \langle \alpha \mid b \rangle \langle b \mid \hat{F} \mid \beta \rangle \langle \beta \mid a \rangle \\
&= \sum_{b} \sum_{\beta} S_{ab} F_{b\beta} S_{\beta\alpha} \\
&= \sum \sum S_{ab} F_{b\beta} S^{*}_{a\beta} = (S F_b S^{+})_{aa}
\end{aligned} \tag{5.5.14}$$

$$F_a = S F_b S^{+} \tag{5.5.15}$$

$$F_a = S F_b S^{-1} \tag{5.5.16}$$

5.6 the Coherent States

5.7 Pauli Matrices

6 Approximation

6.1 Nondegenerate Pertubation Theory

The determined time-independent Schrödinger equation unperturbed is denoted

$$H^0\psi_n^0 = E_n^0\psi_n^0 \tag{6.1.1}$$

with the orthonormal condition $<\psi_m^0|\psi_m^0>=\delta_{mn}$. We now introduce the pertubation by

$$H = H^0 + \lambda H', \lambda \ll 1 \tag{6.1.2}$$

also the wave function and the energy altered, which can be written in power series of λ,

$$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \cdots \tag{6.1.3}$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \cdots \tag{6.1.4}$$

where E_n^1 is the first-order correction to the nth eigenvalue, and ϕ_n^1 the first-order correction to the nth eigenfuction; with superscript 2 is the second-order correction.

Plugging in, we have

$$(H^0+\lambda H')[\psi_n^0+\lambda\psi_n^1+\lambda^2\psi_n^2+\cdots] = (E_n^0+\lambda E_n^1+\lambda^2 E_n^2+\cdots)[\psi_n^0+\lambda\psi_n^1+\lambda^2\psi_n^2+\cdots] \tag{6.1.5}$$

collecting in powers of λ, we should have

$$H^0\psi_n^0 + \lambda(H^0\psi_n^1 + H'\psi_n^0) + \lambda^2(H^0\psi_n^2 + H'\psi_n^1) + \cdots \tag{6.1.6}$$

equals to

$$E_n^0\psi_n^0 + \lambda(E_n^0\psi_n^1 + E_n^1\psi_n^0) + \lambda^2(E_n^0\psi_n^2 + E_n^1\psi_n^1 + E_n^2\psi_n^0) + \cdots \tag{6.1.7}$$

which gives theory of different orders,

$$H^0\phi_n^0 = E_n^0\phi_n^0 \tag{6.1.8}$$

$$H^0\psi_n^1 + H'\psi_n^0 = E_n^0\psi_n^1 + E_n^1\psi_n^0 \tag{6.1.9}$$

$$H^0\psi_n^2 + H'\psi_n^1 = E_n^0\psi_n^2 + E_n^1\psi_n^1 + E_n^2\psi_n^0 \tag{6.1.10}$$

The first-order theory should be given as follow. By taking the inner product with ϕ_n^0, we have

$$\langle\psi_n^0|H^0\psi_n^1\rangle + \langle\psi_n^0|H'\psi_n^0\rangle = E_n^0\langle\psi_n^0|\psi_n^1\rangle + E_n^1\langle\psi_n^0|\psi_n^0\rangle \tag{6.1.11}$$

and remembering H^0 is hermitian further suggests

$$\langle\psi_n^0|H^0\psi_n^1\rangle = \langle H^0\psi_n^0|\psi_n^1\rangle = \langle E_n^0\psi_n^0|\psi_n^1\rangle = E_n^0\langle\psi_n^0|\psi_n^1\rangle \tag{6.1.12}$$

so that

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle \tag{6.1.13}$$

which states that the first-order correction to the energy is the expectation value of the pertubation in the unpertubated state.

To find the first-order correction to the wave function, we have

$$(H^0 - E_n^0)\psi_n^1 = -(H' - E_n^1)\psi_n^0 \tag{6.1.14}$$

from previous second-order equation. Since the unpertubed wave function constitute a complete set, we can write

$$\psi_n^1 = \sum_{m \neq n} c_m^{(n)} \psi_m^0 \tag{6.1.15}$$

here the $m = n$ in the sum is excluded for valid calculation nextstep; also by substituding ϕ_n^1 by $\phi_n^1 + \alpha\phi_n^0$ in the above equation it still holds(which will require a different λ in the original equation but here do not appear), so that excluding $m = n$ in the sum is identical as including it and thus carries the same information.

By plugging in we have

$$\sum_{m \neq n} (E_m^0 - E_n^0) c_m^{(n)} \psi_m^0 = -(H' - E_n^1)\psi_n^0 \tag{6.1.16}$$

To take a inner product with ϕ_l^0, we have

$$\sum_{m \neq n} (E_m^0 - E_n^0) c_m^{(n)} \langle \phi_l^0 | \phi_m^0 \rangle = -\langle \phi_l^0 | H' | \phi_n^0 \rangle + E_n^1 \langle \phi_l^0 | \phi_n^0 \rangle \tag{6.1.17}$$

for $l \neq n$, we are to find

$$(E_m^0 - E_n^0) c_l^{(n)} = -\langle \phi_l^0 | H' | \phi_n^0 \rangle \tag{6.1.18}$$

or

$$c_m^{(n)} = \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \tag{6.1.19}$$

so that

$$\psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{(E_n^0 - E_m^0)} \psi_m^0 \tag{6.1.20}$$

For the second-order theory, we only focus on the energy. We have

$$\langle \psi_n^0 | H^0 \psi_n^2 \rangle + \langle \psi_n^0 | H' \psi_n^1 \rangle = E_n^0 \langle \psi_n^0 | \psi_n^2 \rangle + E_n^1 \langle \psi_n^0 | \psi_n^1 \rangle + E_n^2 \langle \psi_n^0 | \psi_n^0 \rangle \tag{6.1.21}$$

The first term of both side cancels since the hermicity suggests

$$\langle \psi_n^0 | H^0 \psi_n^2 \rangle = \langle H^0 \psi_n^0 | \psi_n^2 \rangle = E_n^0 \langle \psi_n^0 | \psi_n^2 \rangle \tag{6.1.22}$$

here it is valid for reason that ϕ_n^0 is a set of eigenfuction under H^0.

It further gives

$$E_n^2 = \langle \psi_n^0 | H' | \psi_n^1 \rangle - E_n^1 \langle \psi_n^0 | \psi_n^1 \rangle \tag{6.1.23}$$

and we have

$$\langle \psi_n^0 | \psi_n^1 \rangle = \sum_{m \neq n} c_m^{(n)} \langle \psi_n^0 | \psi_m^0 \rangle = 0 \tag{6.1.24}$$

so that

$$E_n^2 = \langle \psi_n^0 | H' | \psi_n^1 \rangle = \sum_{m \neq n} c_m^{(n)} \langle \psi_n^0 | H' | \psi_m^0 \rangle \tag{6.1.25}$$

which is finaly

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0} \tag{6.1.26}$$

6.2 Degenerate Pertubation Theory

For denenerate pertubation, we focus on a 2-fold pertubation for simplicity.

Suppose that

$$H^0\psi_a^0 = E^0\psi_a^0, \quad H^0\psi_b^0 = E^0\psi_b^0, \quad \langle\psi_a^0|\psi_b^0\rangle = 0 \tag{6.2.1}$$

which states a and b 2 are distinct unpertubed states but with same eigenvalue E^0, both normalized.

With $H = H^0 + \lambda H', \quad E = E^0 + \lambda E^1 + \lambda^2 E^2 + \cdots, \quad \phi = \phi^0 + \lambda\phi^1 + \lambda^2\phi^2 + \cdots$, we rewrite it in powers of λ, which is

$$H^0\psi^0 + \lambda(H'\psi^0 + H^0\psi^1) + \cdots = E^0\psi^0 + \lambda(E^1\psi^0 + E^0\psi^1) + \cdots \tag{6.2.2}$$

but for degenerate states, we have $H^0\phi^0 = E^0\phi^0$, the first term at each side canceled. Then to the first order, we have

$$H^0\psi^1 + H'\psi^0 = E^0\psi^1 + E^1\psi^0 \tag{6.2.3}$$

taking a inner product with ψ_a^0 gives

$$\langle\psi_a^0|H^0\psi^1\rangle + \langle\psi_a^0|H'\psi^0\rangle = E^0\langle\psi_a^0|\psi^1\rangle + E^1\langle\psi_a^0|\psi^0\rangle \tag{6.2.4}$$

since H^0 is hermitian, the first term of each side coincide, which by further expanding $\psi^0 = \alpha\psi_a^0 + \beta\psi_b^0$, we have

$$\alpha\langle\psi_a^0|H'\Big|\psi_a^0\rangle + \beta\langle\psi_a^0|H'\Big|\psi_b^0\rangle = \alpha E^1 \tag{6.2.5}$$

writing it more compactly

$$\alpha W_{aa} + \beta W_{ab} = \alpha E^1 \tag{6.2.6}$$

a inner product with ψ_b^0 gives

$$\alpha W_{ba} + \beta W_{bb} = \beta E^1 \tag{6.2.7}$$

here we use W_{ab} to represent $\langle\psi_a^0|H'\Big|\psi_b^0\rangle$ and with other inner product we span a matrix W.

We say that the eigenvalues of the matrix W give the first-order corrections to the energy and the corresponding eigenvectors tell us the coefficients $\alpha,\ \beta$ that determine the mixed state, by

$$\begin{pmatrix} W_{aa} - E^1 & W_{ab} \\ W_{ba} & W_{bb} - E^1 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0 \tag{6.2.8}$$

there should be non-trivial solutions if and only if the determinant of W is 0, which is

$$\begin{vmatrix} W_{aa} - E^1 & W_{ab} \\ W_{ba} & W_{bb} - E^1 \end{vmatrix} = (W_{aa} - E_1)(W_{bb} - E_1) - |W_{ab}|^2 = 0 \tag{6.2.9}$$

so that the perturbated energies are

$$E_\pm^1 = \frac{1}{2}[W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4\left|W_{ab}\right|^2}] \tag{6.2.10}$$

6.3 Variational Principle

7 Spin

8 Identical Particles

9 Quantum Transition

10 Many-body Problem

11 Propagator

A propagator, or a kernel, something we defined to represent the amplitude describing a particle moving from $X(t_a) = x_a$ to $x(t_b) = x_b$, a contribution of all possible orbits from a to b, is denoted $K(a, b)$.

By definition we can give

$$K(b, a) = \sum_{all\, paths\, from\, a\, to\, b} \phi[x(t)] \tag{11.0.1}$$

where a contribution of a path defined a phase according to the action

$$\phi[x(t)] = const\, e^{(i/\hbar)S[x(t)]} \tag{11.0.2}$$

the Classical Limit

the classical limit of wave function gives $S/\hbar$ a enourmous angle which a slight change of path causes massive vibrations from positive to negative where they happened to vanish adding them up. Only for those pathes near the classical path where S takes the extreme values the slight change of path will not change S in the first order, all the close pathes can add up to contribute to the classical path of a wave, or say, a particle.

Part IV
Statistical Mechanics

1 Thermodynamics

1.1 General Introduction

Here a few terminologies are to be concisely explained.
Equations of State
The equation of state can be written

$$f(p,V,T)=0 \tag{1.1.1}$$

$$\alpha=\frac{1}{V}(\frac{\partial V}{\partial T})_p \tag{1.1.2}$$

$$\beta=\frac{1}{p}(\frac{\partial p}{\partial T})_V \tag{1.1.3}$$

$$\kappa_T=-\frac{1}{V}(\frac{\partial V}{\partial p})_T \tag{1.1.4}$$

$$(\frac{\partial T}{\partial V})_p(\frac{\partial p}{\partial T})_V(\frac{\partial V}{\partial p})_T=-1 \tag{1.1.5}$$

$$\alpha=\kappa_T\beta p \tag{1.1.6}$$

Ideal Gas Equation

$$pV=nRT, \qquad R=8.3145J\cdot mol^{-1}\cdot K^{-1} \tag{1.1.7}$$

$$(p+\frac{an^2}{V^2})(V-nb)=nRT \tag{1.1.8}$$

$$p=(\frac{nRT}{V})[1+\frac{n}{V}B(T)+(\frac{n}{V})^2C(T)+\dots] \tag{1.1.9}$$

1.2 the Zeroth Law

The zeorth law of Thermodynamics describes the transitive nature of thermal equilibrium. It states:

If two systems are seperately in equilibrium with the third system, then the former 2 systems are also in equilibrium with each other.

Or, we state, between those systems in equilibrium there is a state function of same value, which is the classcial temperature.

1.3 the First Law

Massive observation gives, the amount of work required to change the state of an otherwise adiabatically isolated system depends only on the initial and final states, and not on the means by which the work is performed, or on the intermediate states through which the system passes.

Defining a state function energy,

$$U_b-U_a=W_S \tag{1.3.1}$$

If not adiabatic, the absorbed heat is defined

$$Q=U_b-U_a-W \tag{1.3.2}$$

The equation for the first law is therefore

$$U_b - U_a = W + Q \tag{1.3.3}$$

Work
Heat Capacity and Enthalpy
The heat capacity is defined

$$C = \lim_{\Delta T \to 0} \frac{\Delta Q}{\Delta T} \tag{1.3.4}$$

$$C_v = \lim_{\Delta T \to 0} (\frac{\Delta Q}{\Delta T})_v = \lim_{\Delta U \to 0} (\frac{\Delta U}{\Delta T})_v = (\frac{\partial U}{\partial T})_v \tag{1.3.5}$$

$$C_p = \lim_{\Delta T \to 0} (\frac{\Delta Q}{\Delta T})_p = \lim_{\Delta U \to 0} (\frac{\Delta U + p\Delta V}{\Delta T})_p = (\frac{\partial U}{\partial T})_p + p(\frac{\partial V}{\partial T})_p \tag{1.3.6}$$

$$H = U + pV \tag{1.3.7}$$

$$\Delta H = \Delta U + p\Delta V \tag{1.3.8}$$

$$C_p = (\frac{\partial H}{\partial T})_p \tag{1.3.9}$$

1.4 the Second Law

The second law have 2 statements that is equivelent to each other.

Kelvin's statement: No process is possible whose sole result is the complete conversion of heat into work.

Clausius's statement: No process is possible whose sole result is the transfor of heat from a colder to a hotter body.

1.5 Thermodynamic Functions

Enthalpy
Enthalpy is defined previously and is

$$H = U + pV \tag{1.5.1}$$

In a isobaric procedure, we have

$$\Delta H = \Delta U + p\Delta V \tag{1.5.2}$$

so that the isenthalpic change is equivelent to the heat absorbed in tthe isobaric procedure. It descibes the procedure where there is no heat exchange and the system comes to mechanical equilibrium with a constant external force.

Entropy

$$dS = \frac{\delta Q}{T} \tag{1.5.3}$$

$$\int_A^B \frac{\delta Q}{T} \tag{1.5.4}$$

$$dS = \frac{dU + pdV}{T} = C_v + \frac{pV}{T}\frac{dV}{V} = C_m + nR\frac{dV}{V} \tag{1.5.5}$$

$$S = C_v \ln T + nR \ln V + S_0 \tag{1.5.6}$$

$$\ln p + \ln V = \ln R + \ln T \tag{1.5.7}$$

$$\frac{dp}{p} + \frac{dV}{V} = \frac{dT}{T} \tag{1.5.8}$$

$$C_{p,m} - C_{V,m} = R \tag{1.5.9}$$

$$dS_m = C_p T - R\frac{dp}{p} \tag{1.5.10}$$

$$S = C_p \ln T - R \ln p + S_0 \tag{1.5.11}$$

Helmholtz Free Energy

Helmholtz free energy is defined to be a function for isothermal transformations in the absence of mechanical work.

$$F = U - TS \tag{1.5.12}$$

$$\Delta Q \leqslant T\Delta S \tag{1.5.13}$$

$$\Delta U = \Delta W + \Delta Q \tag{1.5.14}$$

$$-\Delta W = \Delta Q - \Delta U \leqslant T\Delta S - \Delta U \tag{1.5.15}$$

$$-W \leqslant -F \tag{1.5.16}$$

Gibbs Free Energy

Gibbs free energy applies to isothermal transformations involving mechanical works at constant external force.

$$G = F + pV = U - TS + pV \tag{1.5.17}$$

$$p\Delta V \leqslant -F \tag{1.5.18}$$

$$G = F + pV = U - TS + pV \tag{1.5.19}$$

$$dF - pdV = dG \leqslant 0 \tag{1.5.20}$$

Generalized Forces and Displacements

1.6 Maxwell Relationships: Legendre Transformation

$$dU = TdS - pdV \tag{1.6.1}$$

$$dU = (\frac{\partial U}{\partial S})_V dS + (\frac{\partial U}{\partial V})_S dV \tag{1.6.2}$$

$$(\frac{\partial U}{\partial S})_V = T, \quad (\frac{\partial U}{\partial V})_s = -p \tag{1.6.3}$$

$$\frac{\partial^2 U}{\partial S \partial V} = \frac{\partial^2 U}{\partial V \partial S} \tag{1.6.4}$$

$$(\frac{\partial T}{\partial V})_S = -(\frac{\partial p}{\partial S})_V \tag{1.6.5}$$

$$H = U + pV, \quad dH = TdS + Vdp \tag{1.6.6}$$

$$dH = (\frac{\partial H}{\partial S})_p dS + (\frac{\partial H}{\partial p})_s dp \tag{1.6.7}$$

$$(\frac{\partial H}{\partial S})_p = T, \quad (\frac{\partial H}{\partial p})_s = V \tag{1.6.8}$$

$$\frac{\partial^2 H}{\partial S \partial p} = -\frac{\partial^2 H}{\partial p \partial S} \tag{1.6.9}$$

$$(\frac{\partial T}{\partial p})_S = (\frac{\partial V}{\partial S})_p \tag{1.6.10}$$

$$F = U - TS \tag{1.6.11}$$

$$dF = dU - TdS - SdT = TdS - pdV - TdS - SdT = -SdT - pdV \tag{1.6.12}$$

$$(\frac{\partial F}{\partial T})_V = -S, \quad (\frac{\partial F}{\partial V})_T = -p \tag{1.6.13}$$

$$(\frac{\partial S}{\partial V})_T = (\frac{\partial p}{\partial T})_V \tag{1.6.14}$$

$$G = F + pV, \quad dG = -SdT + Vdp \tag{1.6.15}$$

$$(\frac{\partial G}{\partial T})_p = -S, \quad (\frac{\partial G}{\partial p})_T = V \tag{1.6.16}$$

$$(\frac{\partial S}{\partial p})_T = -(\frac{\partial V}{\partial T})_p \tag{1.6.17}$$

$$(\frac{\partial T}{\partial V})_S = -(\frac{\partial p}{\partial S})_V \tag{1.6.18}$$

$$(\frac{\partial T}{\partial p})_S = (\frac{\partial V}{\partial S})_p \tag{1.6.19}$$

$$(\frac{\partial S}{\partial V})_T = (\frac{\partial p}{\partial T})_V \tag{1.6.20}$$

$$(\frac{\partial S}{\partial p})_T = -(\frac{\partial V}{\partial T})_p \tag{1.6.21}$$

Applications

$$dU = (\frac{\partial U}{\partial T})_V dT + (\frac{\partial U}{\partial V})_T dV \tag{1.6.22}$$

$$dU = TdS - pdV, \quad dS = (\frac{\partial S}{\partial T})_V dT + (\frac{\partial S}{\partial V})_T dV \tag{1.6.23}$$

$$dU = T(\frac{\partial S}{\partial T})_V dT + [T(\frac{\partial S}{\partial V})_T - p]dV \tag{1.6.24}$$

$$C_V = (\frac{\partial U}{\partial T})_V = T(\frac{\partial S}{\partial T})_V, \quad (\frac{\partial U}{\partial V})_T = T(\frac{\partial S}{\partial V})_T - p \tag{1.6.25}$$

$$(\frac{\partial U}{\partial V})_T = T(\frac{\partial p}{\partial T})_V - p \tag{1.6.26}$$

$$dH = TdS - Vdp = (\frac{\partial H}{\partial T})_p dT + (\frac{\partial H}{\partial p})_T dp \tag{1.6.27}$$

$$dS = (\frac{\partial S}{\partial T})_V dT + (\frac{\partial S}{\partial V})_T dV \tag{1.6.28}$$

$$dH = T(\frac{\partial S}{\partial T})_p + [T(\frac{\partial S}{\partial p})_T + V]dp \tag{1.6.29}$$

$$C_p = (\frac{\partial H}{\partial T})_p = T(\frac{\partial S}{\partial T})_p, \quad (\frac{\partial H}{\partial p})_T = T(\frac{\partial S}{\partial p})_T + V \tag{1.6.30}$$

$$C_p - C_V = T(\frac{\partial S}{\partial T})_V - T(\frac{\partial S}{\partial T})_V \tag{1.6.31}$$

1.7 Polynary System

1.8 the Third Law

2 Distribution

2.1 Preperation

2.2 Boltzmann, Bose & Fermi Distribution

$$\Omega_{m.b.} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} \tag{2.2.1}$$

$$\Omega_{b.e.} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} \tag{2.2.2}$$

$$\Omega_{f.d.} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!} \tag{2.2.3}$$

2.3 Boltzmann Distribution

$$U = \sum_l a_l \varepsilon_l = \sum_l \varepsilon_l \omega_l e^{-\alpha-\beta\varepsilon_l} \tag{2.3.1}$$

$$Z_1 = \sum_l \omega_l e^{-\beta\varepsilon_l} \tag{2.3.2}$$

$$N = \sum_l \omega_l e^{-\alpha-\beta\varepsilon_l} \tag{2.3.3}$$

$$N = e^{-\alpha} \sum_l \omega_l e^{-\beta\varepsilon_l} = e^{-\alpha} Z_1 \tag{2.3.4}$$

$$U = e^{-\alpha} \sum_l \varepsilon_l \omega_l e^{-\beta\varepsilon_l} = \frac{N}{Z_1}(-\frac{\partial}{\partial\beta})Z_1 = -N\frac{\partial}{\partial\beta}\ln Z_1 \tag{2.3.5}$$

$$dE = Ydy \rightarrow Y = \sum_l \frac{\partial\varepsilon_l}{\partial y} a_l \tag{2.3.6}$$

$$\frac{\partial}{\partial y} e^{-\beta\varepsilon_l} = -\beta\frac{\partial\varepsilon_l}{\partial y} e^{-\beta\varepsilon_l} \tag{2.3.7}$$

$$Y = \sum_l \frac{\partial\varepsilon_l}{\partial y}\omega_l e^{-\alpha-\beta\varepsilon_l} = e^{-\alpha}(-\frac{1}{\beta}\frac{\partial}{\partial y})\sum_l \omega_l e^{-\beta\varepsilon_l} = \frac{N}{Z_1}(-\frac{1}{\beta}\frac{\partial}{\partial y})Z_1 = -\frac{N}{\beta}\frac{\partial}{\partial y}\ln Z_1 \tag{2.3.8}$$

$$\frac{1}{T}dQ = \frac{1}{T}(dU - Ydy) = dS \tag{2.3.9}$$

$$dQ = dU - Ydy = -Nd(\frac{\partial \ln Z_1}{\partial\beta}) + \frac{N}{\beta}\frac{\partial \ln Z_1}{\partial y}dy \tag{2.3.10}$$

$$d\ln Z_1 = \frac{\partial \ln Z_1}{\partial\beta}d\beta + \frac{\partial \ln Z_1}{\partial y}dy \tag{2.3.11}$$

$$\beta(dU - Ydy) = Nd(\ln Z_1 - \beta\frac{\partial}{\partial\beta}\ln Z_1) \tag{2.3.12}$$

$$\beta = \frac{1}{kT} \tag{2.3.13}$$

$$dS = Nkd(\ln Z_1 - \beta \frac{\partial}{\partial \beta} \ln Z_1) \tag{2.3.14}$$

$$S = Nk(\ln Z_1 - \beta \frac{\partial}{\partial \beta} \ln Z_1) \tag{2.3.15}$$

$$N = e^{-\alpha} Z_1 \rightarrow \ln Z_1 = \ln N + \alpha \tag{2.3.16}$$

$$S = k(N \ln N + \alpha N + \beta U) \tag{2.3.17}$$

$$a_l = \omega_l e^{-\alpha - \beta \varepsilon_l} \rightarrow \alpha + \beta \varepsilon_l = \ln \frac{\omega_l}{a_l} \tag{2.3.18}$$

$$S = k(N \ln N + \sum a_l \ln \omega_l - \sum a_l \ln a_l) \tag{2.3.19}$$

$$\ln \Omega = N \ln N + \sum a_l \ln \omega_l - \sum a_l \ln a_l \tag{2.3.20}$$

$$S = k \ln \Omega \tag{2.3.21}$$

$$S = Nk(\ln Z_1 - \beta \frac{\partial}{\partial \beta} \ln Z_1) - k \ln N! \tag{2.3.22}$$

$$S = k \ln \frac{\Omega_{m.b.}}{N!} \tag{2.3.23}$$

2.4 Fermi & Bose Distribution

$$\overline{N} = \sum a_l = \sum \frac{\omega_l}{exp(\alpha + \beta \varepsilon_l) - 1} \tag{2.4.1}$$

$$\Xi = \prod \Xi_l = \prod (1 - e^{-\alpha - \beta \varepsilon_l})^{-\omega_l} \tag{2.4.2}$$

$$\ln \Xi = -\sum \omega_l \ln(1 - e^{-\alpha - \beta \varepsilon_l}) \tag{2.4.3}$$

$$\frac{\partial}{\partial \alpha} \ln \Xi = -\sum \omega_l \frac{e^{-\alpha - \beta \varepsilon_l}}{1 - e^{-\alpha - \beta \varepsilon_l}} = -\overline{N} \tag{2.4.4}$$

$$\overline{N} = -\frac{\partial}{\partial \alpha} \ln \Xi \tag{2.4.5}$$

$$\frac{\partial}{\partial y} \ln \Xi = \frac{\partial \varepsilon_l}{\partial y} \frac{\partial}{\partial \varepsilon_l} [-\sum \omega_l \ln(1 - e^{-\alpha - \beta \varepsilon_l})] = -\beta \sum \frac{\partial \varepsilon_l}{\partial y} a_l \tag{2.4.6}$$

$$Y = \sum \frac{\partial \varepsilon_l}{\partial y} a_l = -\frac{1}{\beta} \frac{\partial}{\partial y} \ln \Xi \tag{2.4.7}$$

$$\beta(dU - Y dy + \frac{\alpha}{\beta} d\overline{N}) = -\beta d(\frac{\partial \ln \Xi}{\partial \beta}) + \frac{\partial \ln \Xi}{\partial y} dy - \alpha d(\frac{\partial \ln \Xi}{\partial \alpha}) \tag{2.4.8}$$

$$d \ln \Xi = \frac{\partial \ln \Xi}{\partial \alpha} d\alpha + \frac{\partial \ln \Xi}{\partial \beta} d\beta + \frac{\partial \ln \Xi}{\partial y} dy \tag{2.4.9}$$

$$\beta(dU - Ydy + \frac{\alpha}{\beta}d\overline{N}) = d(\ln \mathit{\Xi} - \alpha\frac{\partial \ln \mathit{\Xi}}{\partial \alpha} - \beta\frac{\partial \ln \mathit{\Xi}}{\partial \beta}) \tag{2.4.10}$$

$$\frac{1}{T}(dU - Ydy + \frac{\alpha}{\beta}d\overline{N}) = dS \tag{2.4.11}$$

$$\beta = \frac{1}{kT} \qquad \alpha = -\frac{\mu}{kT} \tag{2.4.12}$$

$$dS = kd(\ln \mathit{\Xi} - \alpha\frac{\partial \ln \mathit{\Xi}}{\partial \alpha} - \beta\frac{\partial \ln \mathit{\Xi}}{\partial \beta}) \tag{2.4.13}$$

$$S = k(\ln \mathit{\Xi} - \alpha\frac{\partial \ln \mathit{\Xi}}{\partial \alpha} - \beta\frac{\partial \ln \mathit{\Xi}}{\partial \beta}) = k(\ln \mathit{\Xi} + \alpha\overline{N} + \beta U) \tag{2.4.14}$$

$$\ln \Omega = \sum[(\omega_l + a_l)\ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l] \tag{2.4.15}$$

$$\mathit{\Xi} = \prod \mathit{\Xi}_l = \prod (1 + e^{-\alpha-\beta\varepsilon_l})^{\omega_l} \tag{2.4.16}$$

$$J = U - TS - \overline{N}\mu \tag{2.4.17}$$

$$J = -kT \ln \mathit{\Xi} \tag{2.4.18}$$

3 Ensembles

3.1 the Microcanonical Ensemble

The microcanocial systems are those systems that both mechanically and adiabatically isolated.

$$\Omega^{(0)}(E_1, E^{(0)} - E_1) = \Omega_1(E_1)\Omega_2(E^{(0)} - E_1) \tag{3.1.1}$$

$$\frac{\partial \Omega^{(0)}}{\partial E_1} = 0 \tag{3.1.2}$$

$$\frac{\partial \Omega_1}{\partial E_1}\Omega_2 + \Omega_1 \frac{\partial \Omega_2}{\partial E_2}\frac{\partial E_2}{\partial E_1} = 0 \tag{3.1.3}$$

$$\frac{\partial E_2}{\partial E_1} = -1 \tag{3.1.4}$$

$$[\frac{\partial \ln \Omega_1}{\partial E_1}]_{N_1,V_1} = [\frac{\partial \ln \Omega_2}{\partial E_2}]_{N_2,V_2} = \beta \tag{3.1.5}$$

3.2 the Canonical Ensemble

$$E + E_r = E^{(0)} \qquad E \ll E^{(0)} \tag{3.2.1}$$

$$\rho_s \propto E_r \tag{3.2.2}$$

$$\ln \Omega_r(E^{(0)} - E_s) = \ln \Omega_r(E^{(0)}) + (\frac{\partial \ln \Omega_r}{\partial E_r})_{E_r = E^{(0)}}(-E_s) = \ln \Omega_r(E^{(0)}) - \beta E_s \tag{3.2.3}$$

$$\rho_s = \frac{1}{Z}e^{-\beta E_s}, Z = \sum_s e^{-\beta E_s} \tag{3.2.4}$$

$$\rho_l = \frac{1}{Z}\Omega_l e^{-\beta E_l} \qquad Z = \sum_l \Omega_l e^{-\beta E_l} \tag{3.2.5}$$

$$\rho(q,p)d\Gamma = \frac{1}{N!h^{Nr}}\frac{e^{-\beta E(q,p)}}{Z}d\Gamma \tag{3.2.6}$$

$$Z = \frac{1}{N!h^{Nr}}\int e^{-\beta E(q,p)}d\Gamma \tag{3.2.7}$$

$$U = \overline{E} = \frac{1}{Z}\sum E_s e^{-\beta E_s} = \frac{1}{Z}(-\frac{\partial}{\partial \beta})\sum e^{-\beta E_s} = -\frac{1}{\beta}\ln Z \tag{3.2.8}$$

$$Y = -\frac{1}{\beta}\frac{\partial}{\partial y}\ln Z \tag{3.2.9}$$

3.3 the Gibbs Canonical Ensemble

3.4 the Grand Canonical Ensemble

Part V
General Relativity

1 Differential Geometry

1.1 Mapping

A map f from a space $\mathbf{M}$ to a space $\mathbf{N}$ is a rule which associates an element x out of $\mathbf{M}$ to a unique element $y = f(x)$ out of $\mathbf{N}$.

The statement "f maps $\mathbf{M}$ into $\mathbf{N}$" is indicated as $f : \mathbf{M} \to \mathbf{N}$. and "$f$ maps a particular element x in $\mathbf{M}$ to y in $\mathbf{N}$" is indicated as $f : x \mapsto y$.

Following definitions are useful:

• If every point of $\mathbf{N}$ has at least one inverse image, it is a map from $\mathbf{M}$ onto $\mathbf{N}$, and is called a surjective map.

• If for any point of $\mathbf{N}$ which has an inverse image in $\mathbf{M}$, the inverse image is unique, the map is said to be injective.

• A map of $\mathbf{M}$ and $\mathbf{N}$, both surjective and injective, is called bijective, or a one-to-one map. Bijective maps are invertible. The inverse map of f is denoted f^{-1}.

• Composition of maps is under 2 maps $f : \mathbf{M} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{P}$ we maps $\mathbf{M}$ to $\mathbf{P}$ $g \circ f : \mathbf{M} \to \mathbf{P}$.

1.2 Vectors, One-Forms, and Tensors

A contravariant vector

$$\vec{V} \to_o \left\{V^i\right\}_{i=1,2,\ldots,n} \tag{1.2.1}$$

indicates that $\vec{V}$ has components with respect to a given frame O, can be difined as a collection of n ordered numbers which transform under the coordinate transformation

$$V^i = \frac{\partial x^{i'}}{\partial x_j} V^j = \Lambda^{i'}_j V^j \tag{1.2.2}$$

In addition the covariant vectors are defined

$$V_i = \frac{\partial x^j}{\partial x^{i'}} A_j = \Lambda^j_{i'} A_j \tag{1.2.3}$$

where $\Lambda^j_{i'}$ is the inverse matrix of $\Lambda^{i'}_j$.

A path $\mathcal{C}$ is a connected sequence of points in a manifold and a curve γ is a mapping from an interval $I = [a, b] \subset R$ to a path in a manifold, associates a real number to each point of the path, and we say it a parametrization of the path

$$\gamma : s \in [a, b] \mapsto \gamma(s) \in \mathcal{C} \subset \mathbf{M} \tag{1.2.4}$$

the path is then the image of the real interval I in the manifold.

Path can be parametrized in different ways, called reparamitrization, there will be new curves though the path is still the same.

The tangent vector of a curve with parameter λ can be given

$$\vec{V} \to_O \left\{\frac{dx^i}{d\lambda}\right\} \tag{1.2.5}$$

which means the set of number $\frac{dx^i}{d\lambda}$ are the components of the tangent vector to the curve in the point p. It has clear geometrical meanings that it measures how the components along each coordinate basis change after a slight change of the parameter and is vividly components of the tangent vector from that point. There are 2 opposite directions of tangent vector geometrically; from the definition we restric it to the direction where the slight change of parameter happens.

Caution: while the curve belongs to the manifold M, the tangent vector belongs to the tangent space of manifold M at some point p, so we denote the tangent space T_p.

And here comes the directional derivative along a curve:

a real, differentiable function Φ defined in U maps the point in U into a real number

$$\Phi : \mathbf{U} \to R \tag{1.2.6}$$

the directional derivative of Φ in point p along the curve is defined as a real number

$$\frac{d\Phi}{d\lambda} = \frac{\partial \Phi}{\partial x^i}\frac{dx^i}{d\lambda} \tag{1.2.7}$$

and here where Φ is arbitrary

$$\frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} \tag{1.2.8}$$

that is the directional derivative opreator in p acting on the space of C^1 functions (namely the Φ) to real numbers. The generic directional derivative is a linear combination of the directional derivatives along the coordinate lines, and the quantities $\left\{\frac{dx^i}{d\lambda}\right\}$ are the components of the vector in the basis x^i. In other words, the directional derivative operator is a vector. It is easy to find the directional derivative operators obey vector rules like associativity and commutativity of vector sum, associativity and distributivity of multiplication by real numbers, existance of identity and so on.

Vectors as geometrical objects

A geometrical object is something will not change under coordinate transformations. Represented in different coordinates, the components of a geometrical object may change, but the object itself will not.

For directional derivatives, they are vectors so we may write down them in a coordinate basis

$$\vec{V} = \frac{d}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} = V^i\frac{\partial}{\partial x^i} \tag{1.2.9}$$

If we apply $\vec{V}$ to a generic function Φ, we find

$$\frac{d\Phi}{d\lambda} = \vec{V}(\Phi) = V^i\frac{\partial \Phi}{\partial x^i} \tag{1.2.10}$$

to be the directional derivative of Φ along $\vec{V}$. This is what we mentioned before: vectors map C^1 functions to real numbers.

We denote the vectors of coordinate basis

$$\vec{e}_{(i)} \equiv \frac{\partial}{\partial x^i} \tag{1.2.11}$$

Then a vector can be expressed as a linear combination of basis vectors

$$\vec{A} = A^i\vec{e}_{(i)} \tag{1.2.12}$$

The transformation law of vectors can be easily derived since

$$V^i\vec{e}_{(i)} = V^{i'}\vec{e}_{(i')} = V^{i'}\Lambda^i_{i'}\vec{e}_{(i)} \tag{1.2.13}$$

thus

$$V^i = \Lambda^i_{i'}V^{i'} \to V^{i'} = \Lambda^{i'}_i V^i \tag{1.2.14}$$

where the definition goes

$$\Lambda^{i'}_{i} = \frac{\partial x^{i'}}{\partial x^{i}} \tag{1.2.15}$$

One forms as geometrical objects

A one-form is a linear, real-valued function of vectors

$$\tilde{q} : T_p \to R \qquad \vec{V} \mapsto \tilde{q}(\vec{V}) \tag{1.2.16}$$

which means that the one-form $\tilde{q}$ at the point p associates to any vector $\vec{V}$ a real number $\tilde{q}(\vec{V})$.

The one form $\tilde{q}$ at point p belongs to the cotangent space of the manifold, which is the dual space of the tangent space, thus denoted T^*_p. The cotangent space can be considered as the normal plane of the vector at the point p while the tangent space is the tangent plane of the manifold at the point p. In a geometrical sense, through associating a vector to a real number, 1-form is able to capture the linearity of the neigborhood of a pint p in the manifold, can be considered the gradient of a field, which is closely related to the geometrical meaning of a vector.

The basis of one-forms are defined as follow: applied to a vector gives as a result of the i-th component of the vector

$$\tilde{\omega}^{(i)}(\vec{V}) = V^i \tag{1.2.17}$$

by the definition of coordinate basis we have

$$\tilde{\omega}^{(i)} \vec{e_{(j)}} = \delta^i_j \tag{1.2.18}$$

and by this we can write down any one-form by its components

$$\tilde{q} = q_j \tilde{\omega}^{(j)} \tag{1.2.19}$$

and then we have

$$\tilde{q}(\vec{V}) = q_j \tilde{\omega}^{(j)}(V^i \vec{e}_{(i)}) = q_j V^i \delta^j_i = q_j V^j \tag{1.2.20}$$

Due to the symmetry we can have the vectors to map one-forms into a real numbers by

$$\vec{V}(\tilde{q}) = \tilde{q}(\vec{V}) = q_j V^j \tag{1.2.21}$$

clearly the transformation matrix is the inverse of that used to transform the basis vectors since their product is a Kronecker Delta.

The transformation rules for one-forms can be similarly given by

$$q_{j'} = \tilde{q}(\vec{e}_{(j')}) = \tilde{q}(\Lambda^k_{j'} \vec{e}_{(k)}) = \Lambda^k_{j'} q_k \tag{1.2.22}$$

Differential as one-forms

The differential $d\Phi$ of a real function Φ is the variation of the function in an unspecified direction at first order of displacement. if we specify the direction as given by an arbitrary vector $\vec{V}$, we get a real number

$$d\Phi(\vec{V}) := V^j \frac{\partial \Phi}{\partial x^i} \tag{1.2.23}$$

according to the definition, those who map vectors into real numbers are one-forms, thus the components of $d\Phi$ are

$$d\Phi_i = d\Phi(\vec{e}_{(i)}) = e^j_{(i)} \frac{\partial \Phi}{\partial x^j} = \delta^j_i \frac{\partial \Phi}{\partial x^j} = \frac{\partial \Phi}{\partial x^i} \tag{1.2.24}$$

the components of the one-form $d\Phi$ are the components of the gradient of the function.

And the definition suggests that the differential of any coordinate x^i is the one form dx^i that

$$dx^i(\vec{V}) = V^j \frac{x^i}{x^j} = V^j \delta^i_j = V^i \tag{1.2.25}$$

the one-form dx^i associates to any vector $\vec{V}$ the component V^i, therefore the differentials dx^i are the coordinate basis one-forms.

The coordinate seperation on the path parameterized by λ can be given

$$x^i(\lambda + \Delta\lambda) - x^i(\lambda) = \frac{dx^i}{d\lambda}\Delta\lambda + \mathcal{O}(\Delta\lambda^2) = \delta x^i + \mathcal{O}(\Delta\lambda^2) \tag{1.2.26}$$

which says if the term $\mathcal{O}$ can be neglected, then the vector $\vec{\delta x}$ should be the infinitesimal displacement along the parameterized path, where we can apply basis one-forms to $\vec{\delta x}$, then we have

$$dx^i(\vec{\delta x}) = \delta x^i \tag{1.2.27}$$

thus the basis one-forms can be considered as the components of the infinitesimal displacement along a generic direction.

Then we can conclude the property of a typical 1-form. Given a vector, a inner product can show the properties of the vector. To be precise, measuring the component of a vector in a given direction.

Conclusion:

• The vector basis is

$$\left\{\vec{e}_{(i)}\right\} \equiv \left\{\frac{\partial}{\partial x^i}\right\}, \quad e^j_i = (\frac{\partial}{\partial} x^i)^j = \delta^j_i \tag{1.2.28}$$

• The 1-form basis

$$\left\{\tilde{\omega}^{(i)}\right\} \equiv \left\{dx^i\right\}, \quad \omega^{(i)}_j = (dx^i)_j = \delta^i_j \tag{1.2.29}$$

Altogether gives

$$\tilde{\omega}^{(i)}(\vec{e}_{(j)}) = \vec{e}_{(j)}(\tilde{\omega}^{(i)}) = \delta^i_j \tag{1.2.30}$$

Tensors

A tensor of type (M, N) is a linear, real-valued function, which associates to M one-forms and N vectors a real number. Here are some most frequently used types of tensor:

• A $(0, 1)$ tensor is a function that takes a vector as argument, and returns a number, that is to say, a $(0, 1)$ tensor is a one-form:

$$\tilde{q}(\vec{V}) = q_i V^i \tag{1.2.31}$$

• A $(1, 0)$ tensor is a function that takes a one-form as argument, and returns a number, that is to say, a $(1, 0)$ tensor is a vector:

$$\vec{V}(\tilde{q}) = V^i q_i \tag{1.2.32}$$

• A $(0, 2)$ tensor is a function that takes 2 vectors as arguments and returns a number,

$$\vec{V}, \vec{W} \rightarrow F(\vec{V}, \vec{W}) \subset R \tag{1.2.33}$$

where we define

$$F_{ij} = F(\vec{e}_{(i)}, \vec{e}_{(j)}) \tag{1.2.34}$$

since there are n basis vectors, the quantaties F_{ij} are components of an $n \times n$ real matrix. Using the linear property, we may find

$$F(\vec{A},\vec{B}) = F(A^i\vec{e}_{(i)}, B^j\vec{e}_{(j)}) = A^iB^jF_{ij} \tag{1.2.35}$$

thus $A^iB^jF_{ij}$ is the real number which results from the application of the tensor to any pair of vectors.

Basis of tensor space

The coordinate basis $\{\omega^{(i)(j)}\}$ for $(0,2)$ tensors can be constructed as follows.

First we define

$$F = F_{ij}\omega^{(i)(j)} \tag{1.2.36}$$

where F_{ij} are the components of the tensor defined earlier. Then we have

$$F(\vec{A},\vec{B}) = F_{ij}\omega^{(i)(j)}(\vec{A},\vec{B}) \tag{1.2.37}$$

On the other hand, we have $F(\vec{A},\vec{B}) = F_{ij}A^iB^j$, we get

$$F(\vec{A},\vec{B}) = F_{ij}\tilde{\omega}^{(i)}(\vec{A})\tilde{\omega}^{(j)}(\vec{B}) \tag{1.2.38}$$

we find

$$\omega^{(i)(j)}(\vec{A},\vec{B}) = \tilde{\omega}^{(i)}(\vec{A})\tilde{\omega}^{(j)}(\vec{B}) \tag{1.2.39}$$

now we define $\omega^{(i)(j)}$ as the outer product, indicated by $\otimes$, of the 2 basis one-forms

$$\omega^{(i)(j)} = \tilde{\omega}^{(i)} \otimes \tilde{\omega}^{(j)} \tag{1.2.40}$$

again the equation means explicitly

$$\omega^{(i)(j)}(\vec{A},\vec{B}) = \tilde{\omega}^{(i)}(\vec{A})\tilde{\omega}^{(j)}(\vec{B}) \tag{1.2.41}$$

Similarly we have the basis for $(2,0)$ tensor

$$e_{(i)(j)} = \vec{e}_{(i)} \otimes \vec{e}_{(j)} \tag{1.2.42}$$

which shows

$$e_{(i)(j)}(\tilde{\alpha}^{(i)}, \tilde{\sigma}^{(j)}) = \vec{e}_{(i)}(\tilde{\alpha})\vec{e}_{(j)}(\tilde{\sigma}) \tag{1.2.43}$$

and consequently

$$T = T^{ij}\vec{e}_{(i)} \otimes \vec{e}_{(j)} \tag{1.2.44}$$

and the transformation rules for tensors are merely the same: we transform each basis vectors or basis one-forms to transform a tensor.

the Metric Tensor

The metric tensor helps to definea scalar product between 2 vectors.

A scalar product is a mapping from 2 vector space to a real number:

$$T_p \times T_p \to R \qquad (\vec{U},\vec{V}) \mapsto \vec{U}\cdot\vec{V} \tag{1.2.45}$$

the metric tensor is a symmetric $(0,2)$ tensor which defines the scalar product

$$g(\vec{A},\vec{B}) = \vec{A}\cdot\vec{B} \tag{1.2.46}$$

and its components go

$$g_{ij} = g(\vec{e}_{(i)}, \vec{e}_{(j)}) = \vec{e}_{(i)}\cdot\vec{e}_{(j)} \tag{1.2.47}$$

thus

$$g(\vec{A},\vec{B}) = g_{ij}A^iB^j \tag{1.2.48}$$

A particular case of psuedo-Reimannian manifold is the Lorentzian spacetime, in which

$$g_{\alpha\beta} = \eta_{\alpha\beta} = diag(-1, 1, 1, 1) \tag{1.2.49}$$

and that gives

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.2.50}$$

Also it is by another point of view that the metric tensor maps vectors into one-forms.

1.3 the Covariant Derivative of Vectors

Considering a vector field $\vec{V} = V^\alpha \vec{e}_{(\alpha)}$ on a manifold M and x^α to be a chosen coordinate system. By applying Leibniz's rule we shall find the derivative of $\vec{V}$ with respect to the coordinate to be

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_{(\alpha)} + V^\alpha \frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} \tag{1.3.1}$$

by definition of Minkowski's spacetime, where we can define a global Minkowskian coordinate system

$$\frac{\partial \vec{e}_{M(\alpha)}}{\partial x^\beta} = \vec{0} \tag{1.3.2}$$

where $e_{\vec{M(\alpha)}}$ is a coordinate basis belongs to the tangent space T_p and is same for any p. This rule is called the affine connection of Minkowski's spacetime.

For basic vectors $\vec{e}_{(\alpha)}$ of the generic frame x^α, and the basic vectors $e_{\vec{M(\mu')}}$ of the LIF $\xi^{\mu'}$ we translate them between by $\Lambda^{\mu'}_{\alpha} = \frac{\partial \xi^{\mu'}}{\partial x^\beta}$ we find that

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = (\frac{\partial}{\partial x^\beta}\Lambda^{\mu'}_{\alpha})\vec{e}_{M(\mu')} = (\frac{\partial}{\partial x^\beta}\Lambda^{\mu'}_{\alpha})\Lambda^{\gamma}_{\mu'}\vec{e}_{(\gamma)} \tag{1.3.3}$$

defining

$$\Gamma^\gamma_{\alpha\beta} = (\frac{\partial}{\partial x^\beta}\Lambda^{\mu'}_{\alpha})\Lambda^{\gamma}_{\mu'} = \frac{\partial x^\gamma}{\partial \xi^{\mu'}}\frac{\partial^2 \xi^{\mu'}}{\partial x^\alpha \partial x^\beta} \tag{1.3.4}$$

we find to the first order in the displacement from p

$$\frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} e_{\vec{(\gamma)}} \tag{1.3.5}$$

the indices:
α the basic vector differentiating.
β the coordinate with respect to which the differentiation performed.
γ the dummy index of summation.
now we can write

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_{(\alpha)} + V^\alpha \frac{\partial \vec{e}_{(\alpha)}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_{(\alpha)} + V^\alpha \Gamma^\gamma_{\alpha\beta} e_{\vec{(\gamma)}} \tag{1.3.6}$$

by relabelling the dummy index we have

$$\frac{\partial \vec{V}}{\partial x^\beta} = (\frac{\partial V^\alpha}{\partial x^\beta} + V^\sigma \Gamma^\alpha_{\beta\sigma})\vec{e}_{(\alpha)} \tag{1.3.7}$$

then, introducing the following notation, we have

$$V^\alpha_{,\beta} \equiv \frac{\partial V^\alpha}{\partial x^\beta} \qquad V^\alpha_{;\beta} \equiv V^\alpha_{,\beta} + V^\sigma \Gamma^\alpha_{\beta\sigma} \tag{1.3.8}$$

we finally get

$$\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha_{;\beta} \vec{e}_{(\alpha)} \tag{1.3.9}$$

Now we can define the covariant derivative of a vector

$$\nabla \vec{V} \equiv V^\alpha_{;\beta} \vec{e}_{(\alpha)} \otimes \tilde{\omega}^{(\beta)} = \frac{\partial \vec{V}}{\partial x^\beta} \otimes \tilde{\omega}^{(\beta)} \tag{1.3.10}$$

which means $\nabla \vec{V}$ is a $(1,1)$ tensor which map vectors to vectors, with components $V^\alpha_{;\beta}$ while $\tilde{\omega}^{(\beta)}(\vec{V}) = V^\beta$

we get

$$F(\quad , \vec{t}) = F^\alpha_\beta \vec{e}_{(\alpha)} \tilde{\omega}^{(\beta)}(\vec{t}) = F^\alpha_\beta t^\beta \vec{e}_{(\alpha)} = V^\alpha_{;\beta} t^\beta \vec{e}_{(\alpha)} = \nabla \vec{V}(\vec{t}) \tag{1.3.11}$$

where F is a $(1,1)$ tensor and here represents $\nabla \vec{V}$ and we have applied it to a tangent vector $\vec{t}$ of a curve in a manifold. The components of the vector therefore fits the equation $t^\mu = \frac{dx^\mu}{d\lambda}$ since $dx^i(\vec{t}) = \frac{dx^i}{d\lambda} = t^i$ and then we have

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial \vec{V}}{\partial x^\beta} \frac{\partial x^\beta}{\partial \lambda} = V^\alpha_{;\beta} t^\beta \vec{e}_{(\alpha)} = \nabla \vec{V}(\vec{t}) \tag{1.3.12}$$

In a LIF covariant derivative and ordinary derivative coincides since its components are the same.

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} \tag{1.3.13}$$

1.4 the Covariant Derivative of Scalars and One-Forms

Since a scalar function does not depend on the basis vectors, the covariant derivative of a scalar field on a manifold coincides with the ordinary derivative, which is

$$\nabla_\mu \Phi \equiv \frac{\partial \Phi}{\partial x^\mu} \tag{1.4.1}$$

while

$$d\Phi_i = d\Phi(\vec{e}_{(i)}) = e^j_{(i)} \frac{\partial \Phi}{\partial x^j} = \frac{\partial \Phi}{\partial x^i} \tag{1.4.2}$$

we come to realize that $\nabla_\mu \Phi$ are the components of a $(0,1)$ tensor.

Assuming that the first derivatives vanish

$$\frac{\partial \tilde{\omega}^{(\mu')}_M}{\partial \xi^\beta} = \tilde{0} \tag{1.4.3}$$

then we can derive the covariant derivative. But for a way easier approach, we can define the covariant derivative of a one-form field by suggesting the covariant derivative opreators must fit the Leibniz's rule. Now we define

$$\Phi = \tilde{q}(\vec{V}) = q_\alpha V^\alpha \tag{1.4.4}$$

then we have

$$\nabla_\mu \Phi = \frac{\partial \Phi}{\partial x^\mu} = \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + \frac{\partial V^\alpha}{\partial x^\mu} q_\alpha \tag{1.4.5}$$

$$\nabla_\mu \Phi = \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + V^\alpha_{,\mu} q_\alpha = \frac{\partial q_\alpha}{\partial x^\mu} V^\alpha + q_\alpha (V^\alpha_{;\mu} - V^\beta \Gamma^\alpha_{\mu\beta}) \tag{1.4.6}$$

relabeling the indices

$$\nabla_\mu \Phi = (\frac{\partial q_\alpha}{\partial x^\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha}) V^\alpha + q_\sigma V^\sigma_{;\mu} = V^\alpha q_{\alpha;\mu} + q_\sigma V^\sigma_{;\mu} \tag{1.4.7}$$

thus

$$q_{\alpha;\mu} = \frac{\partial q_\alpha}{\partial x^\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha} \equiv \nabla_\mu q_\alpha \tag{1.4.8}$$

are the components of

$$\frac{\partial \tilde{q}}{\partial x^\mu} = (\frac{\partial q_\alpha}{\partial x^\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha}) \tilde{\omega}^{(\mu)} \tag{1.4.9}$$

and

$$\nabla \tilde{q} = q_{\alpha;\mu} \vec{e}_{(\alpha)} \otimes \tilde{\omega}^{(\mu)} = \frac{\partial \tilde{q}}{\partial x^\mu} \otimes \tilde{\omega}^{(\mu)} \tag{1.4.10}$$

should be a $(0, 2)$ tensor which maps one-forms to one-forms, is defined to be the covariant derivative of a one-form.

The components of $\nabla \tilde{q}$ should be

$$(\nabla \tilde{q})_{\alpha\mu} = \nabla_\mu q_\alpha = q_{\alpha;\mu} \tag{1.4.11}$$

Notice that by the definition of Christoffel's Symbols they are symmetric in the lower indices.

1.5 the Covariant Derivative of Tensors

given a tensor $\mathbf{T} = T_{\alpha\beta} \tilde{\omega}^{(\alpha)} \otimes \tilde{\omega}^{(\beta)}$, let $\tilde{q}$ be the one-form obtained by contracting $\mathbf{T}$ with $\vec{V}$, in components

$$q_\alpha = T_{\alpha\beta} V^\beta \tag{1.5.1}$$

the covariant derivative should be with components

$$q_{\alpha;\mu} = \frac{\partial q_\alpha}{\partial x^\mu} - q_\sigma \Gamma^\sigma_{\mu\alpha} \equiv \nabla_\mu q_\alpha \tag{1.5.2}$$

by substituding we have

$$\nabla_\mu (T_{\alpha\beta} V^\beta) = (T_{\alpha\beta} V^\beta)_{,\mu} - T_{\sigma\beta} V^\beta \Gamma^\sigma_{\mu\alpha} \tag{1.5.3}$$

by expanding and replacing with components of ordinary derivative, relabeling, we have

$$\nabla_\mu (T_{\alpha\beta} V^\beta) = T_{\alpha\beta,\mu} V^\beta + T_{\alpha\beta} V^\beta_{;\mu} - \Gamma^\beta_{\delta\mu} V^\delta - \Gamma^\sigma_{\mu\alpha} T_{\sigma\beta} V^\beta \tag{1.5.4}$$

$$\nabla_\mu (T_{\alpha\beta} V^\beta) = [T_{\alpha\beta,\mu} - \Gamma^\sigma_{\beta\mu} T_{\alpha\sigma} - \Gamma^\sigma_{\mu\alpha} T_{\sigma\beta}] V^\beta + T_{\alpha\beta} V^\beta_{;\mu} \tag{1.5.5}$$

thus we have the component of a covariant derivative of a tensor

$$(\nabla T_{\alpha\beta})_\mu \equiv \nabla_\mu T_{\alpha\beta} \equiv T_{\alpha\beta;\mu} = T_{\alpha\beta,\mu} - \Gamma^\sigma_{\beta\mu} T_{\alpha\sigma} - \Gamma^\sigma_{\mu\alpha} T_{\sigma\beta} \tag{1.5.6}$$

Here we should give the definition of tensor equations.

A equation is a tensor equation, if following properties are satisfied:

• All the tensor have the same rank, individually upper indices and lower indices;

• The equation should be invariant under coordinate transformation.

Acutually the latter property suggests a covariant derivative rather than ordinary derivative since only covariant derivative considers the effect of changing coordinate system and thus invariant under coordinate transformation. We conclude that covariant derivative is metric-sensitive.

Caution: here the vector basis is only a local vector, attached to a particular point p (and its neigborhood) in the manifold with a LIF and is thus not a tensor. Consequently the ordinary derivative is not invariant under coordinate transformation, as mentioned before.

1.6 Christoffel's Symbols in Terms of a Metric Tensor

the spacetime metric in a LIF
Using Taylor expansion we have

$$g_{\alpha\beta;\mu} = \eta_{\alpha\beta;\mu} = 0 \tag{1.6.1}$$

since the first derivatives of a metric vanish
then we have

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu} g_{\mu\beta} - \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} = 0 \tag{1.6.2}$$

which gives

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu} g_{\mu\beta} + \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} \tag{1.6.3}$$

and by relabeling the indices

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2\Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} \tag{1.6.4}$$

multiply it by $g^{\alpha\gamma}$ and remembering $g^{\alpha\gamma} g_{\alpha\nu} = \delta^{\gamma}_{\mu}$
we finally get

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\gamma\alpha} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{1.6.5}$$

this can be remembered in a simple way: since the Christoffel's symbol is symmetric from its 2 lower indices, and the upper indices remains upper indices, the lower remains the lower, and a summation index is needed.

1.7 Parallel Transport

noticing by parallel transport the vector do not change along the curve

$$\frac{d\vec{V}}{d\lambda} = 0 \tag{1.7.1}$$

while $\vec{t}$ is the tangent vector on a curve we get

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial \vec{V}}{\partial \xi^{\beta}} \frac{d\xi^{\beta}}{d\lambda} = V^{\alpha}_{;\beta} t^{\beta} \vec{e}_{(\alpha)} = 0 \tag{1.7.2}$$

that is

$$V^{\alpha}_{;\beta} = 0 \rightarrow \nabla_{\vec{t}} \vec{V} = 0 \tag{1.7.3}$$

which reads the covariant derivative of the vector along the tangent direction of the curve is 0

then in a curved space where exists Γ

$$\frac{dV^{\alpha}}{d\lambda} = -\Gamma^{\alpha}_{\beta\nu} V^{\nu} t^{\beta} \tag{1.7.4}$$

1.8 Geodesic Equation

in a LIF the distance between two neigboring points is

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \tag{1.8.1}$$

where $\{\xi^\alpha\}$ stands for the coordinates of a LIF
the eqation of motion is given

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{1.8.2}$$

where τ is the proper time
in a new frame $\{x^\alpha\}$that need not to be a LIF, the distance is given

$$ds^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu \tag{1.8.3}$$

while

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial\xi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\tau} \tag{1.8.4}$$

the equation can be rewritten

$$\frac{d}{d\tau}(\frac{\partial\xi^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\tau}) = \frac{d^2x^\gamma}{d\tau^2}\frac{\partial\xi^\alpha}{\partial x^\gamma} + \frac{\partial^2\xi^\alpha}{\partial x^\beta\partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{1.8.5}$$

multiply it by $\frac{\partial x^\sigma}{\partial\xi^\alpha}$, we shall find

$$\frac{d^2x^\sigma}{d\tau^2} + [\frac{\partial x^\sigma}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\beta\partial x^\gamma}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{1.8.6}$$

if we consider a free particle which moves along its own worldline with four-velocity $\vec{u}$ which is the tangent vector to the worldline with components $u^\mu = \frac{dx^\mu}{d\lambda}$

by the equivalence principle, we can define a LIF in any point of the worldline making the four-acceleration zero

$$\frac{du^{\mu'}}{d\lambda} = \frac{\partial u^{\mu'}}{\partial\xi^{\alpha'}}\frac{d\xi^{\alpha'}}{d\lambda} = u^{\alpha'}u^{\mu'}_{,\alpha'} = 0 \tag{1.8.7}$$

in a LIF the ordinary and covariant derivatives coincide, which gives

$$u^{\alpha'}u^{\mu'}_{;\alpha'} = 0 \tag{1.8.8}$$

that holds in any frame
expanding gives

$$u^{\alpha'}u^{\mu'}_{;\alpha'} = u^{\alpha'}u^{\mu'}_{,\alpha'} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1.8.9}$$

which is the geodesic equation

that is to say, geodesics are those curves which parallel-transport their own tangent vectors, the four-velocities.

there can be different affine parameters to parameterize one curve

1.9 the Curvature Tensor

Given a generic vector $\vec{V}$ parallely transported along a infinitesimal and closed loop $ABCD$ whose boundaries are portions of coordinate lines

the total variation of the vector components is denoted

$$\delta V^{\alpha} = \delta V^{\alpha}_{AB} + V^{\alpha}_{BC} + V^{\alpha}_{CD} + V^{\alpha}_{DA} \tag{1.9.1}$$

the equation of parallel transport gives

$$AB\&CD: \quad \nabla_{\vec{e}_{(1)}}\vec{V} = e^{\mu}_{(1)}V^{\alpha}_{;\mu} = 0 \rightarrow \frac{\partial V^{\alpha}}{\partial x^1} + \Gamma^{\alpha}_{\beta 1}V^{\beta} = 0 \tag{1.9.2}$$

$$BC\&DA: \quad \nabla_{\vec{e}_{(2)}}\vec{V} = e^{\mu}_{(2)}V^{\alpha}_{;\mu} = 0 \rightarrow \frac{\partial V^{\alpha}}{\partial x^2} + \Gamma^{\alpha}_{\beta 2}V^{\beta} = 0 \tag{1.9.3}$$

the integrals should be

$$\delta V^{\alpha}_{AB} = -\int_A^B \Gamma^{\alpha}_{\beta 1}V^{\beta}dx^1 \tag{1.9.4}$$

$$\delta V^{\alpha}_{BC} = -\int_B^C \Gamma^{\alpha}_{\beta 2}V^{\beta}dx^2 \tag{1.9.5}$$

$$\delta V^{\alpha}_{CD} = -\int_C^D \Gamma^{\alpha}_{\beta 1}V^{\beta}dx^1 \tag{1.9.6}$$

$$\delta V^{\alpha}_{DA} = -\int_D^A \Gamma^{\alpha}_{\beta 2}V^{\beta}dx^2 \tag{1.9.7}$$

if we Taylor-expand δV^{α}_{BC} and δV^{α}_{CD} for

the final expression of δV^{α} can be written as

$$\delta V^{\alpha} = \delta a\delta b[-\frac{\partial}{\partial x^1}(\Gamma^{\alpha}_{\beta 2}V^{\beta}) + \frac{\partial}{\partial x^2}(\Gamma^{\alpha}_{\beta 1}V^{\beta})] \tag{1.9.8}$$

which we can further expand

$$\delta V^{\alpha} = \delta a\delta b[\frac{\partial \Gamma^{\alpha}_{\beta 1}}{\partial x^2} - \frac{\partial \Gamma^{\alpha}_{\beta 2}}{\partial x^1} - \Gamma^{\alpha}_{\sigma 1}\Gamma^{\sigma}_{\beta 2} + \Gamma^{\alpha}_{\sigma 2}\Gamma^{\sigma}_{\beta 1}]V^{\beta} \tag{1.9.9}$$

the terms in the square bracket should be the components of a $(1,3)$ tensor, which is under the definition a Reimann tensor

$$R^{\alpha}_{\beta\mu\nu} \equiv \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} \tag{1.9.10}$$

by contracting the Riemann tensor with the metric tensor we have the $(0,2)$ Ricci tensor

$$R_{\mu\nu} \equiv g^{\alpha\beta}R_{\alpha\beta\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{1.9.11}$$

the scalar Ricci curvature

$$R \equiv g^{\alpha\beta}R_{\alpha\beta} \tag{1.9.12}$$

Later we should see, the Reimann tensor gives description of spacetime curvature caused by mass-energy distribution; the Ricci tensor combines the source of gravity through stress-energy tensor and and the geometrical properties of the gravitational field with the scalar curvature, which further gives a general curvature of spacetime.

in a LIF the Reimann tensor is

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} \tag{1.9.13}$$

replacing the the expression of Christoffel symbol by metrics, we find

$$R^{\alpha}_{\beta\mu\nu} = \frac{1}{2}g^{\alpha\sigma}[g_{\alpha\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}] \tag{1.9.14}$$

or, lowering the indices,

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda}R^{\lambda}_{\beta\mu\nu} = \frac{1}{2}[g_{\alpha\nu,\beta\mu} - g\alpha\mu, \beta\nu + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}] \tag{1.9.15}$$

and consequently gives the Ricci tensor its components

$$R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu} = \frac{1}{2}g^{\alpha\sigma}[g_{\sigma\nu,\beta\alpha} - g_{\sigma\alpha,\beta\nu} + g_{\beta\alpha,\sigma\nu} - g_{\beta\nu,\sigma\alpha}] \tag{1.9.16}$$

commutator of covariant derivatives

$$[\nabla_\alpha, \nabla_\beta]V^\mu = R^{\mu}_{\nu\alpha\beta}V^\nu \tag{1.9.17}$$

From the definition of a Reimann tensor we can easily give antisymmetric relations

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \qquad R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{1.9.18}$$

and it is symmetric under exchange of the 2 pairs

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{1.9.19}$$

then it satisfies the Ricci Identities,

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \tag{1.9.20}$$

Also we can give the Bianchi Identities, by differentiating the Reimann tensor given by metric tensors,

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}[g_{\alpha\nu,\beta\mu} - g\alpha\mu, \beta\nu\lambda + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}] \tag{1.9.21}$$

using the symmetry of metrics, it is easy to find that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \tag{1.9.22}$$

and in a LIF frame the ordinary derivative and covariant derivative coincides, so that in other frames

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{1.9.23}$$

is valid since it is a tensor equation.

2 Einstein Equations

2.1 the Stress-Energy Tensor in a Flat Spacetime

the motion fo a particle of mass m and three-velocity $\mathbf{v}$ is described by the energy-momentum four-vector

$$p^\alpha = mcu^\alpha \tag{2.1.1}$$

where $u^\alpha = d\xi^\alpha/d\tau$ are the components of the 4-velocity $\vec{u}$ and τ related to the proper time τ/c.

remembering $u_\alpha u^\alpha = -1$ with the definition of four-velocity to be $u^\mu = dx^\mu/d\tau$
and by defining $\xi^0 = ct$ and $\gamma = d\xi^0/d\tau$ we obtain

$$u^0 = \gamma \quad u^i = \frac{d\xi^i}{d\tau} = \frac{d\xi^i}{dt}\frac{dt}{d\tau} = v^i\gamma/c \tag{2.1.2}$$

$$\eta_{\alpha\beta}u^\alpha u^\beta = -\gamma^2(1 - v^2/c^2) = -1 \rightarrow \gamma = (1 - \frac{v^2}{c^2})^{-1/2} \tag{2.1.3}$$

and then

$$p^\alpha = m\gamma(c, \vec{v}) \tag{2.1.4}$$

where

$$p^0 = \frac{E}{c} \qquad E = mc^2\gamma \tag{2.1.5}$$

$$\vec{p} = m\gamma\vec{v} \tag{2.1.6}$$

consider an example of non-interacting particles where they follow the worldline
we define the energy density of the system as

$$T^{00} = \sum_n cp_n^0(t)\delta^3(\xi - \xi_n(t)) = \sum_n E_n\delta^3(\xi - \xi_n(t)) \tag{2.1.7}$$

where the index n represents the n-th particle. Note that since $\delta^3(\xi - \xi_n(t))$ has the dimensions of an inverse qubit length, T^{00} therefore has the dimensions of an energy divided by a volume, which is energy density.

And we give the density of momentum T^{0i}/c

$$T^{0i} = \sum_n cp_n^i(t)\delta^3(\xi - \xi_n(t)) \tag{2.1.8}$$

the momentum current is defined

$$T^{ki} \equiv \sum_n p_n^k(t)\frac{d\xi_n^i(t)}{dt}\delta^3(\xi - \xi_n(t)) \tag{2.1.9}$$

altogether we have

$$T^{\alpha\beta} \equiv \sum_n p_n^\alpha(t)\frac{d\xi_n^\beta(t)}{dt}\delta^3(\xi - \xi_n(t)) \tag{2.1.10}$$

Further more since

$$p_n^\alpha = \frac{E_n}{c^2}\frac{d\xi_n^\alpha(t)}{dt} \tag{2.1.11}$$

the equation can also be written

$$T^{\alpha\beta} = c^2 \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\xi - \xi_n(t)) \tag{2.1.12}$$

which clearly shows that $T^{\alpha\beta}$ is symmetric in its two indices.
The equation can also be written

$$T^{\alpha\beta} = mc^2 \sum_n \int u_n^\alpha u_n^\beta \delta^4(\xi - \xi_n(\tau_n)) d\tau_n \tag{2.1.13}$$

this should be clearer when we derive backward

$$\begin{aligned}
T^{\alpha\beta} &= mc^2 \sum_n \int u_n^\alpha u_n^\beta \delta^4(\xi - \xi_n(\tau_n)) d\tau_n \\
&= c \sum_n \int [p_n^\alpha \frac{d\xi_n^\beta}{d\tau_n} \delta^3(\xi - \xi_n(\tau_n))] \delta(\xi^0 - \xi_n^0(\tau_n)) \frac{d\tau_n}{d\xi_n^0} d\xi_n^0 \\
&= c \sum_n [p_n^\alpha \frac{d\xi_n^\beta}{d\xi_n^0} \delta^3(\xi - \xi_n(\tau_n))]_{\xi_n^0(\tau_n)=\xi^0} \\
&= c \sum_n p_n^\alpha \frac{d\xi_0^\beta}{d\xi^0} \delta^3(\xi - \xi_n(\tau_n)) \\
&= \sum_n p_n^\alpha(t) \frac{d\xi_n^\beta(t)}{dt} \delta^3(\xi - \xi_n(t))
\end{aligned} \tag{2.1.14}$$

Later we will see:
• $T^{\alpha\beta}$ are the components of a tensor which satisfies a divergence-free equation.
• This equation can be generalized to the case of curved spacetimes, which is in the presence of a gravitational field.

2.2 Geodesic Equations in the Weak-Field, Stationary Limit

”Spacetime tells matter how to move, matter tells spacetime how to curve.”

–John Archibal Wheeler

Since Newtonian gravity works remarkbly well for non-relativistic Systems or in general when the gravitational field is weak, the new theory requires that in the weak-field limit the equations should reduce to Poisson’s equation for Newtonian potential Φ, namely

$$\nabla^2 \Phi = 4\pi G \rho \tag{2.2.1}$$

where ρ is the matter density, and ∇^2 is Laplace’s operator in flat space and in Cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.2.2}$$

To state clearly, first we write down the typical form

$$F = \frac{GMm}{r^2} \tag{2.2.3}$$

which is the law of gravity. Rewriting in vector potential gives

$$\vec{A} = \frac{GM}{r^3} \vec{r} \tag{2.2.4}$$

where the direction is assigned. Now in scalar potential it suggests

$$\vec{A} = \nabla\Phi \qquad \Phi = -G\frac{M}{r} \tag{2.2.5}$$

and by Gauss' Theorem the flux of $\vec{A}$ over the closed surface, here chosen to be a spheric, is the divergence of the vector over the enclosed volume, here chosen to be the 3-D ball,

$$\mathbf{\Psi} = \iint\limits_{closed} \vec{A}d\vec{S} = \frac{GM}{r^2} \cdot 4\pi r^2 = 4\pi GM = \iiint\limits_{closed} 4\pi G\rho dV \tag{2.2.6}$$

where

$$\mathbf{\Psi} = \iint\limits_{closed} \vec{A}d\vec{S} = \iiint\limits_{closed} (\nabla \cdot \vec{A})dV = \iiint\limits_{closed} \nabla^2\Phi dV \tag{2.2.7}$$

and thus gives the Poisson equation of gravity.

Consider a non-relativistic particle moving in a weak and stationary gravitational field. Since $v \ll c$, it follows that

$$\frac{dx^i}{dt} \ll c \to \frac{dx^i}{d\tau} \ll c\frac{dt}{d\tau} = \frac{dx^0}{d\tau} \tag{2.2.8}$$

and thus gives the geodesic equation in this limit

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{d^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \to \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}(c\frac{dt}{d\tau})^2 = 0 \tag{2.2.9}$$

we easily find

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\sigma}(2g_{0\sigma,0} - g_{00,\sigma}) \tag{2.2.10}$$

in addition, since the field is stationary, $g_{0\sigma,0} = 0$, then is simplifies to

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\sigma}\frac{\partial g_{00}}{\partial x^\sigma} \tag{2.2.11}$$

Since we have assumed that the field is weak, we may choose a coordinate system such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{2.2.12}$$

and the inverse metric

$$g^{\mu\nu} - \eta^{\mu\nu} - h^{\mu\nu} \tag{2.2.13}$$

the equation can be

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\sigma}\frac{\partial h_{00}}{\partial x^\sigma} \tag{2.2.14}$$

and the geodesic equation can be

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\alpha}\frac{\partial h_{00}}{\partial x^\alpha}(c\frac{dt}{d\tau})^2 \tag{2.2.15}$$

the time part suggests

$$\frac{d^2ct}{d\tau^2} = -\frac{1}{2}\frac{\partial h_{00}}{\partial ct}(c\frac{dt}{d\tau})^2 \tag{2.2.16}$$

the space part suggests

$$\frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\nabla h_{00}(c\frac{dt}{d\tau})^2 \tag{2.2.17}$$

since the field is assumed to be stationary, $\frac{\partial h_{00}}{\partial t} = 0$, which makes $dt/d\tau = const$, by rescaling $cdt/d\tau = 1$, we have

$$\frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\nabla h_{00} \tag{2.2.18}$$

by recalling the Newtonian equation

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi \tag{2.2.19}$$

we get

$$h_{00} = -2\frac{\Phi}{c^2} + const \qquad g_0 0 = -(1 + 2\frac{\Phi}{c^2}) \tag{2.2.20}$$

then we have

$$\nabla^2 g_{00} = -\frac{2}{c^2}\nabla^2\Phi = -\frac{8\pi G}{c^2}\rho \tag{2.2.21}$$

in non-relativistic limit, the matter distribution is given by $T^{00} = T_{00} = \rho c^2$, which gives

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4}T^{00} \tag{2.2.22}$$

Which suggests us to define a tensor

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.2.23}$$

that satisfies

$$G_{00} = -\nabla^2 g_{00} \tag{2.2.24}$$

the tensor should have following features:
• $G_{\mu\nu}$ must be a symmetric tensor as $T_{\mu\nu}$, and also, in the weak field limit it must reduce to $G_{00}\ -\nabla^2 g_{00}$.
• It must be linear in second derivatives of the metric tensor and can contain quadratic terms of its first derivatives, and no linear terms in metric tensor should be included.
• as $T_{\mu\nu}$, $G_{\mu\nu}$ must satisfies the divergence-free equation $G^{\mu\nu}_{;\nu} = 0$

2.3 Einstein's Field Equation

As discussed previously, we have the Ricci tensor

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\beta\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{2.3.1}$$

which is clearly a symmetric tensor, and a scalar curvature

$$R = g^{\alpha\beta}R_{\alpha\beta} = R^{\alpha}_{\alpha} \tag{2.3.2}$$

This suggests a linear conbination to form $G_{\mu\nu}$, that is

$$G_{\mu\nu} = AR_{\mu\nu} + Bg_{\mu\nu}R \tag{2.3.3}$$

where A and B are constants to be determined.
Also, the divergence-free equation requires that

$$G^{\mu\nu}_{;\mu} = A(R^{\mu\nu} + \frac{B}{A}g^{\mu\nu}R) = 0 \tag{2.3.4}$$

where we should use the Bianchi Identities introduced previously

$$R_{\lambda\mu\nu\beta;\eta} + R_{\lambda\mu\eta\nu;\beta} + R_{\lambda\mu\beta\eta;\nu} = 0 \tag{2.3.5}$$

By contracting the equations with metric $g^{\lambda\nu}$, and recalling that the covariant derivative of metric vanishes, we find

$$g^{\lambda\nu}(R_{\lambda\mu\nu\beta;\eta} + R_{\lambda\mu\eta\nu;\beta} + R_{\lambda\mu\beta\eta;\nu}) = R_{\mu\beta;\eta} - R_{\mu\eta;\beta} + g^{\lambda\nu}R_{\lambda\mu\beta\eta;\nu} \tag{2.3.6}$$

contracting once more gives

$$g^{\mu\beta}(R_{\mu\beta;\eta} - R_{\mu\eta;\beta}) + g^{\lambda\nu}g^{\mu\beta}R_{\lambda\mu\beta\eta;\nu} = R_{;\eta} - R^{\beta}_{\eta;\beta} - R^{\nu}_{\eta;\nu} = 0 \tag{2.3.7}$$

and by lastly contracting $g^{\eta\alpha}$, we get

$$R^{\beta\alpha}_{;\beta} - \frac{1}{2}g^{\eta\alpha}R_{;\eta} = 0 \tag{2.3.8}$$

which relabelling the indices gives

$$(R_{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R_{;\eta})_{;\beta} = 0 \tag{2.3.9}$$

which implies $B/A = -\frac{1}{2}$ to satisfy the divergence-free equation.
Till now we have constructed

$$G_{\mu\nu} = A(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \tag{2.3.10}$$

to find A, we should remember the limit of weak, stationary field

$$G_{00} = A(R_{00} - \frac{1}{2}g_{00}R) \sim -\nabla^2 g_{00} \tag{2.3.11}$$

since the field is weak, we might as well introduce $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. The Christoffel symbol then become

$$\Gamma^{\alpha}_{\mu\nu} = \eta^{\alpha\sigma}(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma}) \tag{2.3.12}$$

the expression of the Ricci tensor is

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}_{\mu\nu} \tag{2.3.13}$$

thus the Ricci tensr can be written as

$$R_{\mu\nu} = \frac{1}{2}\eta^{\alpha\rho}(h_{\mu\alpha,\rho\nu} + h_{\rho\nu,\mu\alpha} - h_{\mu\mu,\rho\alpha} - h_{\rho\alpha,\mu\nu}) \tag{2.3.14}$$

the 00 component of it should be

$$R_{00} = \frac{1}{2}\eta^{\alpha\rho}(2h_{0\alpha,0\rho} - h_{00,\rho\alpha} - h_{\rho\alpha,00}) \tag{2.3.15}$$

here we should notice that both $\eta^{\alpha\rho}$ and $h_{\rho\nu}$ are symmetric in its 2 indices.
If the field is stationary, the time derivatives of metric tensor should all vanish, which gives

$$R_{00} = -\frac{1}{2}\eta^{ij}h_{00,ij} = -\frac{1}{2}\nabla^2 g_{00} \tag{2.3.16}$$

In the weak field limit

$$|T_{ij}| \ll |T_{00}| \tag{2.3.17}$$

which suggests a trace

$$T = g^{\mu\nu}T_{\mu\nu} \simeq \eta^{\mu\nu}T_{\mu\nu} \simeq -T_{00} \tag{2.3.18}$$

for $g^{\mu\nu}g_{\mu\nu} = 4$, which can be easily checked in LIF that it coincides with $\eta^{\mu\nu}$, and a tensor equation is valid in all frames, we have the trace

$$g^{\mu\nu}A(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = \frac{8\pi G}{c^4}T \rightarrow -AR = \frac{8\pi G}{c^4}T \tag{2.3.19}$$

by weak field limit it further suggests

$$-AR = -A(R_{00} - \frac{1}{2}\eta_{00}R) = -\frac{8\pi G}{c^4}T_{00} \rightarrow R = 2R_{00} \tag{2.3.20}$$

which gives

$$G_{00} = 2AR_{00} \rightarrow G_{00} = -A\nabla^2 g_{00} \tag{2.3.21}$$

it determines $A = 1$, then the Einstein equation is derived

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.3.22}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.3.23}$$

is called the Einstein tensor.
The alternative form is given

$$R = -\frac{8\pi G}{c^4}T \tag{2.3.24}$$

then

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{2.3.25}$$

Lovelock's theorm gives
As proved by Lovelock, one may add to the Einstein tensor given a term proportional to $g_{\mu\nu}$, such that the Einstein tensor would become

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.3.26}$$

2.4 Euler-Lagrange's Equations

Consider a tensor field $\Phi(x)$, the action for this field is a functional of it and its first derivative, written in Lagrangian density is

$$S = \int d^4x \mathcal{L}(\Phi, \partial_\mu \Phi) \tag{2.4.1}$$

henceforth we denote $\partial_\mu = \frac{\partial}{\partial x^\mu}$.
A generic variation $\delta\Phi$ which we assume to vanish on the boundary of integration volume or asymptotically if the volume is infinite. It suggests

$$\delta S = \int d^4x(\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\delta\partial_\mu\Phi) = \int d^4x(\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\mu(\delta\Phi)) \tag{2.4.2}$$

here

$$\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi \equiv \frac{\partial \mathcal{L}}{\partial \Phi_{\mu\nu}}\delta\Phi_{\mu\nu} \tag{2.4.3}$$

actually indicated a summation over the 2 indices.
the last part can be integrated by parts

$$\int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\mu(\delta\Phi) = \int d^4x \partial_\mu[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\delta\Phi] - \int d^4x \partial_\mu[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}]\delta\Phi \tag{2.4.4}$$

And by Gauss' theorem, the volume integral of the 4-divergence of $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\delta\Phi$ is equal to the integral of this quanity over the boundary of the volume

The Einstein-Hilbert action is given

$$S^{EH} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R \tag{2.4.5}$$

Einstein equations can be derived from the Euler-Lagrange equation or, simpler, by varying the Einstein-Hilbert action. We shall see

$$\delta S^{EH} = \frac{c^3}{16\pi G}\int d^4x \delta(\sqrt{-g}R) = \frac{c^3}{16\pi G}\int d^4x[\delta(\sqrt{-g}R + \sqrt{-g}\delta R)] \tag{2.4.6}$$

where the variation gives $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. To be explicit, the equation can be derived as follow.

In addition

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{2.4.7}$$

then the equation becomes

$$\delta S^{EH} = \frac{c^3}{16\pi G}[\int (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d^4x + \int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x] \tag{2.4.8}$$

in order to compute the right part, we need to compute the variation of the Ricci tensor. Using Leibniz's rule and the definition of Ricci tensor, we have

$$\delta R_{\mu\nu} = \delta\Gamma^\lambda_{\mu\nu,\lambda} - \delta\Gamma^\lambda_{\mu\lambda,\nu} + \delta\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \delta\Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\lambda} + \Gamma^\alpha_{\mu\nu}\delta\Gamma^\lambda_{\alpha\lambda} - \Gamma^\lambda_{\alpha\nu}\delta\Gamma^\alpha_{\mu\lambda} \tag{2.4.9}$$

by the fact $\delta g^{\lambda\sigma} = -g^{\rho\lambda}g^{\sigma\delta}\delta g_{\rho\sigma}$, which will be discussed later, we give

3 Symmetries

A spacetime isometry is a diffeomorphism under which the metric tenor is invariant. In many cases, an isometry defines a submanifold which is mapped into itself. For example, in a stationary spacetime the invariant submanifold is the time axis and in a spherically symmetric spacetime a 2-sphere.

3.1 Killing Vector Fields

Consider a vector field $\vec{\xi}(x^\mu)$ defined at every point in the spacetime, where the $\vec{\xi}$ identifies a symmetry where an infinitesimal translation along it leaves the line element unchanged,

$$\delta(ds^2) = \delta(g_{\alpha\beta}dx^\alpha dx^\beta) = 0 \tag{3.1.1}$$

which to be precise, is

$$\delta g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}[\delta(dx^\alpha)dx^\beta + dx^\alpha\delta(dx^\beta)] = 0 \tag{3.1.2}$$

since we have

$$\delta g_{\alpha\beta} = g_{\alpha\beta,\mu}\xi^\mu\delta\lambda \tag{3.1.3}$$

by introducing a curve $x^\alpha(\lambda)$ to which $\vec{\xi}$ is tangent, where an infinitesimal translation can be written $\delta x^i = \frac{dx^i}{d\lambda}\delta\lambda = \xi^i\delta\lambda$, and Taylor expand it gives the above relationship.

Since the variation operator and the differential opreator commutes, we have further

$$\delta(dx^\alpha) = d(\delta x^\alpha) = d(\xi^\alpha\delta\lambda) = d\xi^\alpha\delta\lambda = \xi^\alpha, dx^\mu\delta\lambda \tag{3.1.4}$$

which leaves

$$g_{\alpha\beta,\mu}\xi^\mu\delta\lambda dx^\alpha dx^\beta + g_{\alpha\beta}[\xi^\alpha, dx^\mu\delta\lambda dx^\beta + \xi^\beta_{,\gamma}dx^\gamma\delta\lambda dx^\alpha] = 0 \tag{3.1.5}$$

relabeling the indices gives

$$[g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta}]dx^\alpha dx^\beta\delta\lambda = 0 \tag{3.1.6}$$

which has an alternative form, covariant and thus more useful

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{3.1.7}$$

and is called the Killing equation. The more consise notation is

$$\xi_(\alpha;\beta) = 0 \tag{3.1.8}$$

We will prove that this is equivalent to the form above. It can be derived as follow:

3.2 the Lie Derivative

The variation of a (N, N') tensor under an infinitesimal translation along the direction of a vector field $\vec{\xi}$ is called a Lie derivative along the vector field, where the vector need not to be a Killing vector.

For instance, if T is a $(0, 2)$ tensor, we have

$$\mathcal{L}_{\vec{\xi}}T_{\alpha\beta} = T_{\alpha\beta,\mu}\xi^\mu + T_{\delta\beta}\xi^\delta_{,\alpha} + T_{\alpha\delta}\xi^\delta_{,\beta} \tag{3.2.1}$$

Which gives another alternative form of Killing equation

$$\mathcal{L}_{\vec{xi}}g_{\alpha\beta} = 0 \tag{3.2.2}$$

Moreover, since $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$, we may find

$$\mathcal{L}_{\vec{\xi}}(g_{\alpha\beta}g^{\beta\gamma}) = g_{\alpha\beta}\mathcal{L}_{\vec{\xi}}g^{\beta\gamma} + g^{\beta\gamma}\mathcal{L}_{\vec{\xi}}g_{\alpha\beta} = 0 \tag{3.2.3}$$

thus

$$\mathcal{L}_{\vec{\xi}}g^{\alpha\beta} = -\xi^{\alpha;\beta} - \xi^{\beta\alpha} \tag{3.2.4}$$

which also vanishes when $\vec{\xi}$ is a Killing vector.

It can also be proved that the Lie derivative of a vector field $\vec{V}$ along the direction of another vector field $\vec{\xi}$ is

$$\mathcal{L}_{\vec{\xi}}V^\alpha = [\vec{\xi}, \vec{V}]^\alpha = \xi^\mu V^\alpha_{,\mu} - V^\mu \xi^\alpha_{,\mu} \tag{3.2.5}$$

3.3 Poincaré group

Killing vector fields of flat spacetime

Since all Christoffel's symbols vanish, the Killing equation becomes

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0 \tag{3.3.1}$$

also

$$\xi_{\alpha,\beta\gamma} + \xi_{\beta,\alpha\gamma} = 0 \quad \xi_{\beta,\gamma\alpha} + \xi_{\gamma,\beta\alpha} = 0 \quad \xi_{\gamma,\alpha\beta} + \xi_{\alpha,\gamma\beta} = 0 \tag{3.3.2}$$

so that we find

$$\xi_{\alpha,\beta\gamma} = 0 \tag{3.3.3}$$

with a general solution

$$\xi_\alpha = c_\alpha + \epsilon_{\alpha\gamma}x^\gamma \tag{3.3.4}$$

which means it must contains a linear part and a constant part, and by substituding, it gives

$$\epsilon_{\alpha\gamma}x^\gamma_{,\beta} + \epsilon_{\beta\gamma}x^\gamma_{,\alpha} = \epsilon_{\alpha\gamma}\delta^\gamma_\beta + \epsilon_{\beta\gamma}\delta^\gamma_\alpha = \epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0 \tag{3.3.5}$$

and it suggests

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \tag{3.3.6}$$

3.4 Spherical Surface

Consider a sphere of unit radius, which means, with metric

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2 \tag{3.4.1}$$

recalling the original form of Killing equation

$$g_{\alpha\beta,\mu}\xi^\mu + g_{\delta\beta}\xi^\delta_{,\alpha} + g_{\alpha\delta}\xi^\delta_{,\beta} = 0 \tag{3.4.2}$$

we soon get

$$\begin{aligned}
(\alpha,\beta) &= (1,1) \quad 2g_{\delta 1}\xi^\delta_{,1} = 0 \to \xi^1_{,1} = 0 \\
(\alpha,\beta) &= (1,2) \quad g_{\delta 2}\xi^\delta_{,1} + g_{1\delta}\xi^\delta_{,2} = 0 \to \xi^1_{,2} + \sin^2\theta\xi^2_{,1} = 0 \\
(\alpha,\beta) &= (2,2) \quad g_{22,\mu}\xi^\mu + 2g_{\delta 2}\xi^\delta_{,2} = 0 \to \cos\theta\xi^1 + \sin\theta\xi^2_{,2} = 0
\end{aligned} \tag{3.4.3}$$

which the general solution is

$$\xi^1 = c\sin(\varphi + a), \quad \xi^2 = c\cos(\varphi + a)\cot\theta + b \tag{3.4.4}$$

here a, b, c are real constants.

A set of 3 independent Killing vector fields in the space part for the sphere

$$\begin{aligned} \xi^{(1)\alpha} &= (0,1) \\ \xi^{(2)\alpha} &= (\sin\varphi, \cos\varphi\cot\theta) \\ \xi^{(3)\alpha} &= (\cos\varphi, -\sin\varphi\cot\theta) \end{aligned} \tag{3.4.5}$$

here corresponds to the choices $a, b, c = (0,1,0), (0,0,1), (\frac{\pi}{2}, 0, 0)$ respectively.

3.5 Conservation Laws

$$\frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu = 0 \tag{3.5.1}$$

$$\xi_\alpha[\frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu] = \frac{d(\xi_\alpha u^\alpha)}{d\lambda} - u^\alpha \frac{d\xi_\alpha}{d\lambda} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu \xi_\alpha \tag{3.5.2}$$

$$u^\beta \frac{d\xi_\beta}{d\lambda} = u^\beta \frac{\partial\xi_\beta}{\partial x^\nu}\frac{dx^\nu}{d\lambda} = u^\beta u^\nu \frac{\partial\xi_\beta}{\partial x^\nu} \tag{3.5.3}$$

$$\frac{d(\xi_\alpha u^\alpha)}{d\lambda} - u^\beta u^\nu [\frac{\partial\xi_\beta}{\partial x^\nu} - \Gamma^\alpha_{\beta\nu}\xi_\alpha] = 0 \tag{3.5.4}$$

$$(\xi_\mu T^{\mu\nu})_{;\nu} = \xi_{\mu;\nu} T^{\mu\nu} + \xi_\mu T^{\mu\nu}_{;\nu} = 0 \tag{3.5.5}$$

Singularities

$$R^t_{rtr} = -2\frac{m}{r^3}(1 - \frac{2m}{r})^{-1} \tag{3.5.6}$$

$$R^t_{\theta t\theta} = \frac{1}{\sin^2\theta} R^t_{\varphi t\varphi} = \frac{m}{r^5} \tag{3.5.7}$$

$$R^\theta_{\varphi\theta\varphi} = 2\frac{m}{r^5}\sin^2\theta \tag{3.5.8}$$

$$R^r_{\theta r\theta} = \frac{1}{\sin^2\theta} R^r_{\varphi r\varphi} = -\frac{m}{r^5} \tag{3.5.9}$$

3.6 Hypersurface Orthogonal Vector Fields

$$\vec{t}\cdot\vec{V} = 0 \quad \to \quad t^\alpha V^\beta g_{\alpha\beta} = 0 \tag{3.6.1}$$

$$d\Sigma \to_O \left(\frac{\partial\Sigma}{\partial x^0}, \frac{\partial\Sigma}{\partial x^1}, \ldots \frac{\partial\Sigma}{\partial x^n}\right) = \{\Sigma_{,\alpha}\} \tag{3.6.2}$$

$$V_\alpha = f\Sigma_{,\alpha} \tag{3.6.3}$$

$$\frac{d\Sigma}{ds} = \frac{\partial\Sigma}{\partial x^\alpha}\frac{dx^\alpha}{ds} = \Sigma_{,\alpha} t^\alpha = 0 \tag{3.6.4}$$

$$V_\alpha t^\alpha = f\Sigma_{,\alpha} t^\alpha = 0 \tag{3.6.5}$$

$$t^{\alpha}_{(i)} V_{\alpha} = 0 \tag{3.6.6}$$

$$\begin{aligned} V_{\alpha;\beta} - V_{\beta;\alpha} = &\quad (f\Sigma_{,\alpha})_{;\beta} - (f\Sigma_{,\beta})_{;\alpha} \\ = &\quad f\left(\Sigma_{,\alpha;\beta} - \Sigma_{,\beta;\alpha}\right) + \Sigma_{,\alpha} f_{;\beta} - \Sigma_{,\beta} f_{;\alpha} = \\ = &\quad f\left(\Sigma_{,\alpha,\beta} - \Sigma_{,\beta,\alpha} - \Gamma^{\mu}_{\beta\alpha}\Sigma_{,\mu} + \Gamma^{\mu}_{\alpha\beta}\Sigma_{,\mu}\right) + \Sigma_{,\alpha} f_{,\beta} - \Sigma_{,\beta} f_{,\alpha} \\ = &\quad V_{\alpha}\frac{f,\beta}{f} - V_{\beta}\frac{f,\alpha}{f} \end{aligned} \tag{3.6.7}$$

$$V_{\alpha;\beta} - V_{\beta;\alpha} = V_{\alpha}\frac{f_{,\beta}}{f} - V_{\beta}\frac{f_{,\alpha}}{f} \tag{3.6.8}$$

$$\omega^{\delta} = 0 \tag{3.6.9}$$

$$g_{00} = g(\vec{e}_{(0)}, \vec{e}_{(0)}) = \vec{e}_{(0)} \cdot \vec{e}_{(0)} \neq 0 \qquad g_{0i} = g(\vec{e}_{(0)}, \vec{e}_{(i)}) = 0, \quad i = 1, 2 \tag{3.6.10}$$

$$ds^2 = g_{00}(dx^0)^2 + g_{ik}(dx^i)(dx^k), \quad i, k = 1, 2 \tag{3.6.11}$$

3.7 Diffeomorphism Invariance

4 Schwarzchilld Solution and Spacetime

4.1 Static and Spherically Symmetric Spacetimes

A spacetime is said to be stationary if it admits a timelike Killing vector field $\vec{k}$, abd by a subtle choice of the coordinate system the metric can be made independent of time, which is

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \tag{4.1.1}$$

And a spacetime is said to be static, if it admits a hypersurface-orthogonal, timelike Killing vector field.

In this case, we can choose the vectors of the coordinate basis in such a way that at each spacetime point $e\vec{(0})$ coincides with $\vec{k}$, and the remaining basis vectors are tangent to the surfaces at which $\vec{k}$ is orthogonal. So the line elements take the form

$$ds^2 = g_{00}(x^i)(dx^0)^2 + g_{kj}(x^i)dx^k dx^j \tag{4.1.2}$$

The 2-sphere coordinate transformation is

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = g_{\mu\nu}dx^\mu dx^\nu, \quad g_{\mu\nu} = \left(\begin{array}{cc} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{array}\right) \tag{4.1.3}$$

the sphere of unit radius gives $a = 1$. And it further gives the inverse metric

$$g^{\mu\nu} = \left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{array}\right) \tag{4.1.4}$$

so that the only non-vanishing derivative of the metric tensor is

$$g_{\varphi\varphi,\theta} = 2\sin\theta\cos\theta \tag{4.1.5}$$

The spherical coordinate transformation is

$$\begin{array}{rcl} x^1 & = & r\sin\theta\cos\varphi \\ x^2 & = & r\sin\theta\sin\varphi \\ x^3 & = & r\cos\theta \end{array} \tag{4.1.6}$$

so that differentiating them gives

$$\begin{array}{rcl} dx^1 & = & \sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi \\ dx^2 & = & \sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi \\ dx^3 & = & \cos\theta dr - r\sin\theta d\theta \end{array} \tag{4.1.7}$$

we find

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 = g_{ij}dx^i dx^j \tag{4.1.8}$$

which gives the metric matrix

$$g_{ij} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{array}\right) \tag{4.1.9}$$

and the inverse metric

$$g^{ij} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2\sin^2\theta) \end{array}\right) \tag{4.1.10}$$

Spacial symmetry
Define $x^2 = \theta$ and $x^3 = \varphi$, the line element of a 2-sphere of radius a is therefore

$$ds^2_{(2)} = g_{22}(dx^2)^2 + g_{33}(dx^3)^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.11}$$

and the surface of the sphere

$$A = \int \sqrt{g_{(2)}} d\theta d\varphi = \int_0^\pi a^2 \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi a^2 \tag{4.1.12}$$

where $g_{(2)} = a^4 \sin^2\theta$ is the determinant of the above metric.
The length of its max circumference is therefore

$$C = \int_0^{2\pi} \sqrt{g_{\varphi\varphi}(\theta = \pi/2)} d\varphi = \int_0^{2\pi} a \sin(\pi/2) d\varphi = 2\pi a \tag{4.1.13}$$

The assumption of spherical symmetry is indeed a assumption of a independent on $x^2 = \theta$ and $x^3 = \varphi$, which is

$$ds^2_{(2)} = a^2(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.14}$$

If the metric is static, a is also independent on $x^0 = ct$, that is, independent on time. so we may choose $r = a(x^1)$ as a coordinate transformation.
The line element of 3-space is

$$ds^2_{(3)} = g_{rr} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.15}$$

and, that of the spacetime,

$$ds^2 = g_{00}(dx^0)^2 + g_{rr} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.16}$$

further, if static, we have

$$ds^2 = g_{00}(r)(dx^0)^2 + g_{rr}(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.17}$$

It is more convenient, in order to write the following solution as a solution of Differential equation, to have $g_{00}(r) = -e^{2\nu(r)}$ and $g_{rr}(r) = e^{2\lambda(r)}$, so that

$$ds^2 = -e^{2\nu(r)}(dx^0)^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1.18}$$

4.2 Schwarzschild Solution, Hypersurfaces and Singularities

By the equation of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.2.1}$$

and the equation to expand Ricci tensor in terms of Christoffel symbols

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} \tag{4.2.2}$$

and the equation of the scalar curvature

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^\alpha{}_\alpha \tag{4.2.3}$$

along with Christoffel symbol in terms of metric tensor

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\gamma\alpha}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{4.2.4}$$

we can soon determine the non-vanishing Christoffel symbols,

$$\Gamma^r_{00} = e^{2(\nu-\lambda)}\nu_{,r} \quad \Gamma^0_{0r} = \Gamma^0_{r0} = \nu_{,r} \quad \Gamma^r_{rr} = \lambda_{,r} \tag{4.2.5}$$

$$\Gamma^r_{\theta\theta} = -re^{-2\lambda} \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \tag{4.2.6}$$

$$\Gamma^r_{\varphi\varphi} = -r\sin^2\theta e^{-2\lambda} \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta \tag{4.2.7}$$

we can finally determine

$$G_{00} = \frac{1}{r^2}e^{2\nu}\frac{d}{dr}[r(1-e^{-2\lambda})] \tag{4.2.8}$$

$$G_{rr} = -\frac{1}{r^2}e^{2\lambda}(1-e^{-2\lambda}) + \frac{2}{r}\nu_{,r} \tag{4.2.9}$$

$$G_{\theta\theta} = r^2e^{-2\lambda}[\nu_{,rr} + \nu_{,r}^2 + \frac{\nu_{,r}}{r} - \nu_{,r}\lambda_{,r} - \frac{\lambda_{,r}}{r}] \tag{4.2.10}$$

$$G_{\varphi\varphi} = \sin^2\theta\ G_{\theta\theta} \tag{4.2.11}$$

and the remaining components just vanish.

The vaccum solution exists when $G_{\mu\nu} = 0$. Now we should focus on the components of G. The first equation gives

$$e^{2\lambda} = \frac{1}{1-\frac{K}{r}} \tag{4.2.12}$$

where K is an integration constant.

The second equation gives

$$\nu_{,r} = \frac{1}{2}\frac{K}{r(r-K)} \tag{4.2.13}$$

the solution of which is

$$\nu = \frac{1}{2}\log(1-\frac{K}{r}) + \nu_0, \quad \to \quad e^{2\nu} = \left(1-\frac{K}{r}\right)e^{2\nu_0} \tag{4.2.14}$$

where again ν_0 is an integration constant. By rescaling the time coordinate

$$x^0 \to e^{-\nu_0}x^0, \quad dx^0 \to e^{-\nu_0}dx^0 \tag{4.2.15}$$

we have

$$g_{00}(dx^0)^2 = -e^{2\nu}(dx^0)^2 \quad \to \quad -e^{2\nu}e^{-2\nu_0}(dx^0)^2 = -\left(1-\frac{K}{r}\right)(dx^0)^2 \tag{4.2.16}$$

The final form becomes

$$ds^2 = -(1-\frac{K}{r})c^2dt^2 + \frac{dr^2}{1-\frac{K}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.2.17}$$

To determine K, we check the weak-field, stationary limit geodesic equation which reduced to Newtonian equations of motion provided

$$g_{00} = -\left(1+\frac{2\Phi}{c^2}\right) = -\left(1-\frac{2GM}{c^2r}\right) \tag{4.2.18}$$

so that

$$K = \frac{2GM}{c^2} \tag{4.2.19}$$

so the static, spherically symmetric solution of Einstein's equation is then

$$ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.2.20}$$

where the metric in the above equation is called the Schwarzschild solution, and the coordinate (t, r, θ, φ) as the Schwarzschild coordinate.

Here the constant K is equal to the physical mass of the solution multiplied by the factor $2G/c^2$ and is of dimension of length. The quantity

$$R_S = \frac{2GM}{c^2} \tag{4.2.21}$$

is the Schwarzschild radius.

By natural units the Schwarzschild solution reads

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.2.22}$$

where

$$m = \frac{GM}{c^2} \tag{4.2.23}$$

is called the geometrical mass.

4.3 the Birkhoff Theorem

4.4 Equations of Motion

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = \frac{1}{2} g_{\mu\nu}(x^\alpha) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \equiv \frac{1}{2} g_{\mu\nu}(x^\alpha) \dot{x}^\mu \dot{x}^\nu \tag{4.4.1}$$

$$\mathcal{L} = \frac{1}{2}[-(1 - \frac{2m}{r})\dot{t}^2 + \frac{\dot{r}^2}{(1 - \frac{2m}{r})} + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2] \tag{4.4.2}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad \rightarrow \quad \frac{d}{d\lambda}(r^2\dot{\theta}) = r^2 \sin\theta \cos\theta \dot{\varphi}^2 \tag{4.4.3}$$

4.5 Geodesic Motion

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = \frac{1}{2} g_{\mu\nu}(x^\alpha) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \equiv \frac{1}{2} g_{\mu\nu}(x^\alpha) \dot{x}^\mu \dot{x}^\nu \tag{4.5.1}$$

$$S = \int \mathcal{L}\left(x^\alpha, \dot{x}^\alpha\right) d\lambda = \frac{1}{2} \int g_{\mu\nu}(x^\alpha) \dot{x}^\mu \dot{x}^\nu d\lambda \tag{4.5.2}$$

$$x^\mu(\lambda) \longrightarrow x^\mu(\lambda) + \delta x^\mu(\lambda), \quad \lambda \in [\lambda_0, \lambda_1] \tag{4.5.3}$$

$$\delta x^\mu(\lambda_0) = \delta x^\mu(\lambda_1) = 0 \tag{4.5.4}$$

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x^\alpha} \delta x^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \delta(\dot{x}^\alpha) \right) d\lambda = 0 \tag{4.5.5}$$

$$\delta(\dot{x}^\alpha) = \delta\left(\frac{dx^\alpha}{d\lambda}\right) = \frac{d\delta x^\alpha}{d\lambda} \tag{4.5.6}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\delta(\dot{x}^\alpha) = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\frac{d\delta x^\alpha}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\delta x^\alpha\right) - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)\delta x^\alpha \tag{4.5.7}$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{x}^\alpha)} = 0 \tag{4.5.8}$$

$$\ddot{x}^\gamma + \Gamma^\gamma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \tag{4.5.9}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{x}^\alpha)} &= \\
&= \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - \frac{d}{d\lambda}\left[\frac{1}{2}g_{\mu\nu}(\delta^\mu_\alpha\dot{x}^\nu + \dot{x}^\mu\delta^\nu_\alpha)\right] \\
&= \frac{1}{2}\left[g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - \frac{d}{d\lambda}\left(2g_{\alpha\nu}\dot{x}^\nu\right)\right] \\
&= \frac{1}{2}\left(g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - 2g_{\alpha\nu,\beta}\dot{x}^\beta\dot{x}^\nu - 2g_{\alpha\nu}\ddot{x}^\nu\right) \\
&= \frac{1}{2}(-2g_{\alpha\nu}\ddot{x}^\nu + g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - g_{\alpha\mu,\nu}\dot{x}^\nu\dot{x}^\mu - g_{\alpha\nu,\mu}\dot{x}^\mu\dot{x}^\nu) = 0.
\end{aligned} \tag{4.5.10}$$

$$\ddot{x}^\gamma + \frac{1}{2}g^{\alpha\gamma}\left[g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right]\dot{x}^\mu\dot{x}^\nu = 0 \tag{4.5.11}$$

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{(1 - \frac{2m}{r})} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right] \tag{4.5.12}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad \rightarrow \quad \frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\dot{\varphi}^2 \tag{4.5.13}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\varphi}^2 \tag{4.5.14}$$

$$\begin{aligned}
&\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\varphi}^2 \\
&\theta(\lambda = \lambda_0) = \frac{\pi}{2} \\
&\dot{\theta}(\lambda = \lambda_0) = 0
\end{aligned} \tag{4.5.15}$$

$$\theta(\lambda) \equiv \frac{\pi}{2} \tag{4.5.16}$$

$$g_{\alpha\mu}\xi^\mu u^\alpha = const \tag{4.5.17}$$

$$g_{\alpha\mu}k^\mu u^\alpha = const_1 \rightarrow \left(1 - \frac{2m}{r}\right)\dot{t} = E \qquad g_{\alpha\mu}m^\mu u^\alpha = const_2 \rightarrow r^2\sin^2\theta\dot{\varphi} = L \tag{4.5.18}$$

$$const_1 = -E \quad const_2 = L \tag{4.5.19}$$

$$i = \frac{E}{(1 - \frac{2m}{r})} \qquad \dot{\varphi} = \frac{L}{r^2} \tag{4.5.20}$$

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{t})} = 0 \rightarrow \left(1 - \frac{2m}{r}\right)\dot{t} = const_1 \qquad \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(\dot{\varphi})} = 0 \rightarrow r^2 \sin^2\theta\dot{\varphi} = const_2 \tag{4.5.21}$$

$$g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2 = -1 \tag{4.5.22}$$

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2 \tag{4.5.23}$$

$$g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2m}{r}\right)i^2 + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r}\right)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2 = 0 \tag{4.5.24}$$

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = E^2 \tag{4.5.25}$$

4.6 Orbits

4.7 Kinematical Tests of General Relativity

5 Gravitational Waves

5.1 Pertubative Approach

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \tag{5.1.1}$$

$$|h_{\mu\nu}| \ll |g^0_{\mu\nu}| \tag{5.1.2}$$

$$g^{\mu\nu} = g^0\mu\nu - h^{\mu\nu} + O(h^2) \tag{5.1.3}$$

$$h^{\mu\nu} \equiv g^0\mu\alpha g^0\nu\beta h_{\alpha\beta} \tag{5.1.4}$$

$$(g^{0\ \mu\nu} - h^{\mu\nu})(g^0_{\nu\alpha} + h_{\nu\alpha}) = \delta^\mu_\alpha + O(h^2). \tag{5.1.5}$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{5.1.6}$$

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} \tag{5.1.7}$$

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2}g^{\gamma\alpha}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{5.1.8}$$

$$\begin{aligned}
\Gamma^\gamma_{\beta\mu}\left(g_{\mu\nu}\right) = &\quad \frac{1}{2}\left(g^{0\gamma\alpha} - h^{\gamma\alpha}\right)\left[\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) + \left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right)\right] \\
= &\quad \frac{1}{2}g^{0\gamma\alpha}\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) + \frac{1}{2}g^{0\gamma\alpha}\left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right) \\
- &\quad \frac{1}{2}h^{\gamma\alpha}\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) + O(h^2) \\
\equiv &\quad \Gamma^\gamma_{\beta\mu}\left(g^0\right) + \delta\Gamma^\gamma_{\beta\mu}\left(g^0, h\right) + O(h^2),
\end{aligned} \tag{5.1.9}$$

$$\delta\Gamma^\gamma_{\beta\mu}\left(g^0, h\right) = \frac{1}{2}g^{0\gamma\alpha}\left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right) - \frac{1}{2}h^{\gamma\alpha}\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) \tag{5.1.10}$$

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(h_{\mu\rho;\nu} + h_{\nu\rho;\mu} - h_{\mu\nu;\rho}\right) = \frac{1}{2}g^{0\lambda\rho}\left(h_{\mu\rho;\nu} + h_{\nu\rho;\mu} - h_{\mu\nu;\rho}\right) + O(h^2) \tag{5.1.11}$$

$$\begin{aligned}
R_{\mu\nu}\left(g_{\mu\nu}\right) = &\quad R^0_{\mu\nu}\left(g^0\right) + \frac{\partial}{\partial x^\alpha}\delta\Gamma^\alpha_{\mu\nu}\left(g^0, h\right) - \frac{\partial}{\partial x^\nu}\delta\Gamma^\alpha_{\mu\alpha}\left(g^0, h\right) \\
+ &\quad \Gamma^\alpha_{\sigma\alpha}\left(g^0\right)\delta\Gamma^\sigma_{\mu\nu}\left(g^0, h\right) + \delta\Gamma^\alpha_{\sigma\alpha}\left(g^0, h\right)\Gamma^\sigma_{\mu\nu}\left(g^0\right) \\
- &\quad \Gamma^\alpha_{\sigma\nu}\left(g^0\right)\delta\Gamma^\sigma_{\mu\alpha}\left(g^0, h\right) - \delta\Gamma^\alpha_{\sigma\nu}\left(g^0, h\right)\Gamma^\sigma_{\mu\alpha}\left(g^0\right) + O(h^2) \\
= &\quad R^0_{\mu\nu}\left(g^0\right) + \delta\Gamma^\alpha_{\mu\nu;\alpha}\left(g^0, h\right) - \delta\Gamma^\alpha_{\mu\alpha;\nu}\left(g^0, h\right) + O(h^2) \\
\equiv &\quad R^0_{\mu\nu}\left(g^0\right) + \delta R_{\mu\nu}\left(g^0, h\right) + O(h^2)
\end{aligned} \tag{5.1.12}$$

$$\begin{aligned}
T = &\quad g^{\mu\nu}T_{\mu\nu} = \left(g^{0\mu\nu} - h^{\mu\nu}\right)\left(T^0_{\mu\nu} + \delta T_{\mu\nu}\right) + O(h^2) \\
= &\quad g^{0\mu\nu}T^0_{\mu\nu} - h^{\mu\nu}T^0_{\mu\nu} + g^{0\mu\nu}\delta T_{\mu\nu} + O(h^2) \equiv T^0 + \delta T
\end{aligned} \tag{5.1.13}$$

$$\begin{aligned}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) &= T^0_{\mu\nu}+\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}+h_{\mu\nu}\right)\left(T^0+\delta T\right)\\ &= \left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right)+\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2)\end{aligned} \tag{5.1.14}$$

$$R_{\mu\nu}\left(g^0\right)=\frac{8\pi G}{c^4}\left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right) \tag{5.1.15}$$

$$\delta\Gamma^\alpha_{\mu\nu;\alpha}\left(g^0,h\right)-\delta\Gamma^\alpha_{\mu\alpha;\nu}\left(g^0,h\right)=\frac{8\pi G}{c^4}\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2) \tag{5.1.16}$$

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\quad |h_{\mu\nu}|\ll 1 \tag{5.1.17}$$

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{2}\eta^{\lambda\rho}\left[h_{\rho\mu,\nu}+h_{\rho\nu,\mu}-h_{\mu\nu,\rho}\right]+O(h^2) \tag{5.1.18}$$

$$\begin{aligned}&\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}-\frac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu}+O(h^2)\\ =\ &\frac{1}{2}\left\{-\Box_F\, h_{\mu\nu}+\left[\frac{\partial^2}{\partial x^\lambda\partial x^\mu}h^\lambda_\nu+\frac{\partial^2}{\partial x^\lambda\partial x^\nu}h^\lambda_\mu-\frac{\partial^2}{\partial x^\mu\partial x^\nu}h^\lambda_\lambda\right]\right\}+O(h^2)\end{aligned} \tag{5.1.19}$$

$$\left\{\Box_F h_{\mu\nu}-\left[\frac{\partial^2}{\partial x^\lambda\partial x^\mu}h^\lambda_\nu+\frac{\partial^2}{\partial x^\lambda\partial x^\nu}h^\lambda_\mu-\frac{\partial^2}{\partial x^\mu\partial x^\nu}h^\lambda_\lambda\right]\right\}=-\frac{16\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T\right) \tag{5.1.20}$$

$$x^{\alpha\prime}=x^\alpha+\epsilon^\alpha(x) \tag{5.1.21}$$

$$\frac{\partial x^{\alpha\prime}}{\partial x^\mu}=\delta^\alpha_\mu+\frac{\partial\epsilon^\alpha}{\partial x^\mu} \tag{5.1.22}$$

$$\begin{aligned}g_{\mu\nu} &= g'_{\alpha\beta}\frac{\partial x^{\alpha\prime}}{\partial x^\mu}\frac{\partial x^{\beta\prime}}{\partial x^\nu}=\left(\eta_{\alpha\beta}+h'_{\alpha\beta}\right)\left(\delta^\alpha_\mu+\frac{\partial\epsilon^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu+\frac{\partial\epsilon^\beta}{\partial x^\nu}\right)\\ &= \eta_{\mu\nu}+h'_{\mu\nu}+\epsilon_{\nu,\mu}+\epsilon_{\mu,\nu}+O(h^2)\end{aligned} \tag{5.1.23}$$

$$h'_{\mu\nu}=h_{\mu\nu}-\left(\epsilon_{\nu,\mu}+\epsilon_{\mu,\nu}\right) \tag{5.1.24}$$

$$\Gamma^\lambda=g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=0 \tag{5.1.25}$$

$$\begin{aligned}g^{\mu\nu}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho}\left[h_{\rho\mu,\nu}+h_{\rho\nu,\mu}-h_{\mu\nu,\rho}\right]+O(h^2)\\ &= \frac{1}{2}\eta^{\lambda\rho}\left\{h^\nu{}_{\rho,\nu}+h^\mu{}_{\rho,\mu}-h^\nu{}_{\nu,\rho}\right\}+O(h^2)\end{aligned} \tag{5.1.26}$$

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=\eta^{\lambda\rho}\left(h^\mu_{\rho,\mu}-\frac{1}{2}h^\nu_{\nu,\rho}\right)+O(h^2) \tag{5.1.27}$$

$$\frac{\partial}{\partial x^\mu}h^\mu{}_\rho=\frac{1}{2}\frac{\partial}{\partial x^\rho}h \tag{5.1.28}$$

$$h=\eta^{\mu\nu}h_{\mu\nu}\equiv h^\nu_\nu \tag{5.1.29}$$

$$\begin{cases} \Box_F h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \\ \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h \end{cases} \tag{5.1.30}$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{5.1.31}$$

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \\ \frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0 \end{cases} \tag{5.1.32}$$

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = 0 \\ \frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0 \end{cases} \tag{5.1.33}$$

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h \tag{5.1.34}$$

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{5.1.35}$$

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu} = 0 \tag{5.1.36}$$

$$\mathcal{G}\left(\vec{x} - \vec{x}'\right) \equiv \frac{4G}{c^5}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right] \tag{5.1.37}$$

$$\bar{h}_{\mu\nu}(\vec{x}) = \int_\Omega T_{\mu\nu}(\vec{x}')\mathcal{G}(\vec{x} - \vec{x}')d^4x' \tag{5.1.38}$$

$$\frac{\partial}{\partial x^\mu}\left[\mathcal{G}\left(\vec{x} - \vec{x}'\right)\right] = -\frac{\partial}{\partial x'^\mu}\left[\mathcal{G}\left(\vec{x} - \vec{x}'\right)\right] \tag{5.1.39}$$

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu}(\vec{x}) = \int_\Omega T^{\mu\nu}(\vec{x}')\frac{\partial}{\partial x^\mu}\mathcal{G}(\vec{x} - \vec{x}')d^4x' = -\int_\Omega T^{\mu\nu}(\vec{x}')\frac{\partial}{\partial x'^\mu}\mathcal{G}(\vec{x} - \vec{x}')d^4x' \tag{5.1.40}$$

$$\begin{gathered} \int_\Omega T^{\mu\nu}(\vec{x}\ ')\frac{\partial}{\partial x'^\mu}\mathcal{G}(\vec{x} - \vec{x}')\ d^4x' = \\ \int_\Omega d^4x'\frac{\partial}{\partial x'^\mu}\left[T^{\mu\nu}(\vec{x}\ ')\mathcal{G}(\vec{x} - \vec{x}')\right] - \int_\Omega d^4x'\left[\frac{\partial}{\partial x'^\mu}T^{\mu\nu}(\vec{x}\ ')\right]\mathcal{G}(\vec{x} - \vec{x}')d^4x'. \end{gathered} \tag{5.1.41}$$

$$\int_\Omega d^4x'\frac{\partial}{\partial x'^\mu}\left[T^{\mu\nu}(\vec{x}')\mathcal{G}\left(\vec{x} - \vec{x}'\right)\right] = \int_{\partial\Omega}\left[T^{\mu\nu}(\vec{x}')\mathcal{G}\left(\vec{x} - \vec{x}'\right)\right]dS_\mu \tag{5.1.42}$$

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu}(\vec{x}) = 0 \tag{5.1.43}$$

6 Isolated Stationary Object

7 Kerr Solution and Spacetime

8 Compact Stars

9 Black Hole Thermodynamics

Part VI

Quantum Field Theory

1 Preposition: Mathematical & Physical

1.1 Notation & Convention

We take natrual units that

$$c = \hbar = 1 \tag{1.1.1}$$

Covariant and contravariant derivatives are therefore

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (-\frac{\partial}{\partial t}, \nabla) \tag{1.1.2}$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \nabla) \tag{1.1.3}$$

with

$$\partial^m u x^\nu = g^{\mu\nu} \tag{1.1.4}$$

We take the Lagrangian to have the dimension of energy scale.
The signature is taken $(-,+,+,+)$, which gives the metric in a linearly inertial frame

$$\eta^{\mu\nu} = \eta_{\mu\nu} = diag(-1,1,1,1) \tag{1.1.5}$$

1.2 Lie Group Theory

Group

A group $(G, \circ)$ is a set G, together with a binary operation $\circ$ defined on G, that satisfies the following axioms

Rotations 2D

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{1.2.1}$$

$$R_\theta = \mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1.2.2}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.2.3}$$

$$1^2 = 1, \quad i^2 = -1, \quad 1i = i1 = i \tag{1.2.4}$$

$$R_\theta = \cos(\theta) + i\sin(\theta) = \cos(\theta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(\theta)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{1.2.5}$$

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\theta)a - \sin(\theta)b \\ \sin(\theta)a + \cos(\theta)b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} \tag{1.2.6}$$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} R_y = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} R_z = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.7}$$

$$R_z(\theta)\vec{v} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} \tag{1.2.8}$$

Rotations 3D
Lie Algebra and BCH Formula

$$g(\epsilon) = I + \epsilon X \tag{1.2.9}$$

$$h(\theta) = (I + \epsilon X)(I + \epsilon X)(I + \epsilon X)... = (I + \epsilon X)^k \tag{1.2.10}$$

$$g(\theta) = I + \frac{\theta}{N} X \tag{1.2.11}$$

$$h(\theta) = \lim_{N\to\infty} (I + \frac{\theta}{N} X)^N \tag{1.2.12}$$

$$h(\theta) = \lim_{N\to\infty} (I + \frac{\theta}{N} X)^N = e^{\theta X} \tag{1.2.13}$$

$$\mathrm{e}^X \circ \mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots} \tag{1.2.14}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.15}$$

$$(J_i)_{jk} = -\epsilon_{ijk} \tag{1.2.16}$$

$$[J_i, J_j] = \epsilon_{ijk} J_k \tag{1.2.17}$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.18}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{1.2.19}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2.20}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{1.2.21}$$

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.2.22}$$

$$R \to R' = S^{-1} R S \tag{1.2.23}$$

$$J^2_{3-\mathrm{dim}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad J^2_{2-\mathrm{dim}} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \tag{1.2.24}$$

Poincaré Group

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{1.2.25}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{1.2.26}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{1.2.27}$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \tag{1.2.28}$$

$$[N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+ \tag{1.2.29}$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk}N_k^- \tag{1.2.30}$$

$$[N_i^+, N_j^-] = 0 \tag{1.2.31}$$

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \quad K_i = M_{0i} \tag{1.2.32}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{1.2.33}$$

Spinor
BCH Formula

1.3 Field and Principle of Action

1.4 Integrals

2 Spin 0