Exercise

- Let be the probability distribution $P(m \mid pn) = \frac{n!}{m!(n-m)!} \cdot p^m q^{n-m}$ where p is a parameter $\in [0,1]$ and $p+q = 1$. Find the expected values $< m >$ and $< m^2 >$.

Solution:

$$< m >= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot m p^m q^{n-m}$$

We know that $mp^m = p\frac{\partial}{\partial p} p^m$, so we have:

$$< m >= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot p\frac{\partial}{\partial p} p^m q^{n-m} = p\frac{\partial}{\partial p} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot p^m q^{n-m}$$

Using the binomial expansion $\sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot p^m q^{n-m} = (p+q)^n$:

$$< m >= p\frac{\partial}{\partial p}(p+q)^n = np(p+q)^{n-1}$$

$$p+q = 1 \implies < m >= np$$

$$< m^2 >= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot m^2 p^m q^{n-m}$$

We know that $m^2 p^m = p^2 \frac{\partial^2}{\partial p^2} p^m$, so we have:

$$< m >= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot p^2 \frac{\partial^2}{\partial p^2} p^m q^{n-m} = p^2 \frac{\partial^2}{\partial p^2} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot p^m q^{n-m} =$$
$$p^2 \frac{\partial^2}{\partial p^2}(p+q)^n$$

$$< m^2 >= n^2 p^2 + np(1-p)$$