Chebyshev's inequality

In this section we are going to proof the Chebyshev's inequality, this is a very important results that leads us to understand the relation between probability and frequency of an event. For the continuous case, let be $P(x)$ the probability distribution of a random variable X with mean μ and variance σ^2, we know that:

$$P(a \leq X \leq b) = \int_b^a P(x)dx$$

$$\int_{-\infty}^{\infty} x \cdot P(x)dx = \mu$$

$$\int_{-\infty}^{\infty} x^2 \cdot P(x)dx = \sigma^2 + \mu^2$$

Let N, $\epsilon \in \mathbb{R}$, so we have:

$$P(\mid X - \mu \mid \geq N\sigma) = \int_{|X-\mu| \geq N\sigma} P(x)dx \qquad \text{Since } \frac{|X-\mu|}{N\sigma} \geq 1 :$$

$$P(\mid X - \mu \mid \geq N\sigma) = \int_{|X-\mu| \geq N\sigma} P(x)dx \leq \int_{|X-\mu| \geq N\sigma} \frac{(X-\mu)^2}{N^2\sigma^2} \cdot P(x)dx \leq$$

$$\frac{1}{N^2\sigma^2} \int_{-\infty}^{\infty} (X-\mu)^2 \cdot P(x)dx = \frac{1}{N^2}$$

Let $\epsilon = N\sigma$ so:

$$P(\mid X - \mu \mid \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

$$P(\mid X - \mu \mid < \epsilon) > 1 - \frac{\sigma^2}{\epsilon^2}$$

For the discrete case