Solving Gaussian Integrals

In this section we will show how to solve some Gaussian integrals, first of all, lets look to the simple one: $I = \int\limits_{-\infty}^{\infty} e^{-x^2} dx$.

$$I^2 = \int\limits_{-\infty}^{\infty} e^{-x^2} dx \cdot \int\limits_{-\infty}^{\infty} e^{-x^2} dx = \int\limits_{-\infty}^{\infty} e^{-x^2} dx \cdot \int\limits_{-\infty}^{\infty} e^{-y^2} dy$$

We now introduce an other 2 variables $r^2 = x^2 + y^2$ and $\theta = \arctan \frac{x}{y}$ hence:

$$I^2 = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int\limits_{0}^{\infty} \int\limits_{0}^{2\pi} re^{-r^2} drd\theta = \int\limits_{0}^{2\pi} d\theta \int\limits_{0}^{\infty} re^{-r^2} dr$$

Let be $u = -r^2 \implies 2rdr = -du$, so we have:

$$I^2 = -\pi \int\limits_{0}^{-\infty} e^u du = \pi$$

$$\int\limits_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

To find $\int\limits_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$, we make a simple substitution $u = \frac{(x-\mu)}{\sqrt{2}\sigma}$:

$$\int\limits_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int\limits_{-\infty}^{\infty} \sqrt{2}\sigma e^{-u^2} du = \quad = \sqrt{2\pi}\sigma.$$

To find the expected value of x we need to solve the integral $\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} xe^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$ and again we will use a simple substitution $\frac{(x-\mu)^2}{2\sigma^2} = u^2 \implies udu = \frac{1}{\sqrt{2\pi}\sigma}(xdx - \mu dx)$, so we have:

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} xe^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \mu dx + \int\limits_{-\infty}^{\infty} ue^{-u^2} du$$

We already solve the first integral, it is equal μ, the second one it is just 0 because is an integral of odd function. Finally we have:

$$< x >= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} x e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \mu$$