Variance of a Normal Distribution

In this section we will prove that if $\frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ is a probability distribution then your variance is given by σ^2. Using the definition of Var(x):

$$Var(x) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

We now introduce two variables: $u = x - \mu$ and $a = \frac{1}{2\sigma^2}$. We know that:

$$Var(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} u^2 e^{-au^2} du$$

$$\frac{d}{da} e^{-au^2} = -u^2 e^{-au^2} dx$$

$$Var(x) = -\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \frac{d}{da} e^{-au^2} du = -\frac{1}{\sqrt{2\pi\sigma}} \frac{d}{da} \int_{-\infty}^{\infty} e^{-au^2} du$$

$$\text{Lets call } I = \int_{-\infty}^{\infty} e^{-au^2} du \implies I^2 = \int_{-\infty}^{\infty} e^{-au^2} du \int_{-\infty}^{\infty} e^{-av^2} dv$$

We now introduce an other 2 variables $r^2 = u^2 + v^2$ and $\theta = \arctan \frac{v}{u}$ hence:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(u^2+v^2)} dxdy = \int_{0}^{\infty} \int_{0}^{2\pi} re^{-ar^2} drd\theta = \int_{0}^{2\pi} d\theta \int_{0}^{\infty} re^{-ar^2} dr$$

Let be $h = -ar^2 \implies 2ardr = -dh$, so we have:

$$I^2 = -\pi \int_{0}^{-\infty} \frac{1}{a} e^h dh = \frac{\pi}{a}$$

$$\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$$

So we have:

$$Var(x) = -\frac{1}{\sqrt{2\pi}\sigma}\frac{d}{da}\int_{-\infty}^{\infty} e^{-au^2}du = -\frac{1}{\sqrt{2\pi}\sigma}\frac{d}{da}\sqrt{\frac{\pi}{a}} = \frac{a^{-\frac{3}{2}}}{2\sqrt{2}\sigma} = \frac{(2\sigma^2)^{\frac{3}{2}}}{2\sqrt{2}\sigma}$$

$$Var(x) = \sigma^2$$

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