Exercise

- The motion of a drunk particle can be described as series of independent steps of length a. Each step makes an angle θ with the z axis, with a probability density $P(\theta) = (1+\cos\theta)/\pi$ while the angle ϕ is uniformly distributed between 0 and 2π. The drunk particle starts at origin and make a large number of steps N. Find the expectation values: $E[z]$, $E[x^2]$, $E[y^2]$ and $E[z^2]$.

Solution:

We can write:

$$E[z] = E[a\cos\theta_1 + a\cos\theta_2 + ... + a\cos\theta_N] = aN\cdot E[\cos\theta]$$

$$E[\cos\theta] = \int_0^\pi \frac{1}{\pi}\cos\theta\cdot(1+\cos\theta)d\theta = \frac{1}{2}$$

$$E[z] = \frac{aN}{2}$$

Using spherical coordinates we know:

$$x_i = a\sin\theta_i\cos\phi_i$$

We know that:

$$E[x^2] = E[(\sum_{i=1}^N x_i)^2] = E[\sum_{i=1}^N x_i \sum_{j=1}^N x_j] = E[\sum_{i=j}^N \sum_{i\neq j} x_i\cdot x_j + \sum_{i=j} x_i^2]$$

Since x_j and x_i are independents we have:

$$E[x^2] = \sum_{i=j}^N \sum_{i\neq j} E[x_i]\cdot E[x_j] + \sum_{i=j} E[x_i^2]$$

by geometrical and symmetrical arguments $E[x_i] = 0$, so we have:

$$E[x^2] = \sum_{i=j} E[x_i^2] = N\cdot E[x_i^2] \text{ and } x_i^2 = a^2 \sin^2\theta_i \cos^2\phi_i$$

$$E[x_i^2] = a^2 \int_0^{\pi} \frac{1}{\pi} \sin^2\theta \cdot (1+\cos\theta) d\theta \int_0^{2\pi} \frac{1}{2\pi} cos^2\phi d\phi = \frac{a^2}{4}$$

$$E[x^2] = \frac{N\cdot a^2}{4}$$

By symmetry:

$$E[y^2] = \frac{N\cdot a^2}{4}$$

For z we have:

$$E[z^2] = \sum_{i=j}^{N} \sum_{i\neq j} E[z_i]\cdot E[z_j] + \sum_{i=j} E[z_i^2] = N(N-1)\cdot E[z]^2 + NE[z_i^2]$$

We know that $E[z] = \frac{aN}{2}$ and $E[z_i^2] = a^2 \int_0^{\pi} \frac{1}{\pi} \cos^2\theta \cdot (1+\cos\theta) d\theta = \frac{a^2}{2}$ so:

$$E[z^2] = \frac{N(N+1)\cdot a^2}{4}$$